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Strong Laws of Large Numbers by Elementary Tauberian Arguments

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Abstract. For Kolmogorov's strong law of large numbers an alternative short proof is given which weakens Etemadi's condition of pairwise independence. The argument uses the known – and elementary – equivalence of (Cesàro) C_1- and C_2-summability for one-sided bounded sequences. Also other strong laws of large numbers are established, partially via Borel summability.
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Key words: strong law of large numbers, Tauberian theorem, C_1- and C_2-summability, Borel summability, covariance, maximal-correlation coefficient

1 INTRODUCTION

A sequence $(X_n)_{n\in\mathbb{N}}$ of real random variables having finite expectations EX_n obeys a so-called strong law of large numbers, if

$$\frac{1}{n}\sum_{k=1}^{n}(X_k - EX_k) \to 0 \text{ almost surely (a.s.),} \tag{1}$$

i.e.,

$$C_1 - \lim(X_n - EX_n) = 0 \text{ a.s.}$$

For the classical strong law of large numbers of Kolmogorov, which concerns independent identically distributed (integrable) real random variables, Etemadi [6] gave an elementary proof together with a generalization to pairwise independence. He used Chebyshev's inequality and the Borel-Cantelli lemma in the context of suitable subsequences of partial sums. In this paper we give an alternative short proof of the result, under a weakened independence assumption (Theorem 2), using the monotone convergence theorem in series form and the well-known equivalence of (Cesàro) C_1- and C_2-summability for one-sided bounded sequences which can be proved in an elementary way. The core of the argument leads to a general theorem which generalizes several known results on strong laws of large numbers for square integrable random variables (Theorem 1). A corresponding argument using C_1 and Borel summability yields a strong law of large numbers in a context slightly more general than second order stationarity (Theorem 3) and further under the classical Cramér-Leadbetter condition. A comparison with related results is given.

2 STRONG LAWS OF LARGE NUMBERS

First we formulate a rather general result.

Theorem 1 *Let $(X_n)_{n\in\mathbb{N}}$ be a sequence of square integrable real random variables satisfying*

$$\inf(X_n - EX_n) > -\infty \ \ (a.s.) \tag{2}$$
$$(e.g., X_n \geq 0,\ \sup EX_n < \infty),$$
$$\sum_{n=1}^{\infty} \frac{\mathrm{Var}(X_1 + \ldots + X_n)}{n^3} < \infty. \tag{3}$$

Then (X_n) obeys a strong law of large numbers, i.e., (1) holds.

Remark 1 a) With covariance $\Gamma(i,j) := E\Big[(X_i - EX_i)(X_j - EX_j)\Big]$, $i, j \in \mathbb{N}$, one has

$$\mathrm{Var}(X_1 + \ldots + X_n) = \sum_{i,j\in\{1,\ldots,n\}} \Gamma(i,j). \tag{4}$$

Thus Theorem 1 generalizes the discrete time version of the almost sure stability theorem in Loève [21], section 37.7, A, in so far as (2) and (3) weaken the assumptions

$$|X_n| \leq c < \infty, \ \ \sum \frac{1}{n^3} \sum_{i,j\in\{1,\ldots,n\}} |\Gamma(i,j)| < \infty$$

there. A sufficient condition for (3) together with (4) is

$$\sum_{i,j\in\mathbb{N}} \frac{|\Gamma(i,j)|}{(i+j)^2} < \infty.$$

For, with

$$d_n := \sum_{i,j\in\{1,\ldots,n\}} |\Gamma(i,j)|$$

and by partial summation,

$$\begin{aligned}
\sum_{n=1}^{\infty} \frac{d_n}{n^3} &= d_1 \sum_{k=1}^{\infty} \frac{1}{k^3} + \sum_{n=1}^{\infty} \left(\sum_{k=n+1}^{\infty} \frac{1}{k^3} \right) (d_{n+1} - d_n) \\
&\leq 2|\Gamma(1,1)| + \sum_{n=1}^{\infty} \frac{1}{n^2} \sum_{j=1}^{n+1} |\Gamma(n+1,j)| \\
&\leq 16 \sum_{n,j\in\mathbb{N}} \frac{|\Gamma(n,j)|}{(n+j)^2} < \infty.
\end{aligned}$$

b) Theorem 1 also generalizes Theorem 1 of Etemadi [7], mainly in so far as (3) weakens the assumptions

$$EX_iX_j \leq EX_iEX_j,\ j > i,$$

$$\sum \frac{\mathrm{Var}(X_n)}{n^2} < \infty \tag{5}$$

there. For these two conditions imply (3) via

$$\begin{aligned}\sum_{n=1}^{\infty} \cdot \frac{1}{n^3} \mathrm{Var}\left(\sum_{k=1}^{n} X_k\right) &\leq \sum_{n=1}^{\infty} \frac{1}{n^3}\left(\sum_{k=1}^{n} \mathrm{Var}(X_k)\right) \\ &= \sum_{k=1}^{\infty}\left(\sum_{n=k}^{\infty} \frac{1}{n^3}\right) \mathrm{Var}(X_k) < \infty.\end{aligned}$$

c) Note that the Rademacher-Menchoff theorem (Rademacher [25], Menchoff [22]; see Révész [26], § 3.2, and, in a generalization, Loève [21], section 36, and Stout [29], Theorem 3.7.2) states a.s. convergence of $\sum (X_n - EX_n)/n$ and thus (1) for a sequence $(X_n)_{n\in\mathbb{N}}$ of square integrable pairwise uncorrelated real random variables under the condition

$$\sum_{n=1}^{\infty} \frac{\mathrm{Var}(X_n)}{n^2} (\log\ n)^2 < \infty.$$

The proof of Theorem 1 is based on the following deterministic Lemma 1, which is a classic Tauberian theorem and can be proved in an elementary way (see section 3).

Lemma 1 *Let the sequence* $(c_n)_{n\in\mathbb{N}}$ *of real numbers be bounded from below. If*

$$\sum_{n=1}^{\infty} \frac{(c_1 + \ldots + c_n)^2}{n^3} < \infty \tag{6}$$

or only

$$\frac{1}{n^2} \sum_{k=1}^{n} (c_1 + \ldots + c_k) \to 0, \tag{7}$$

then

$$\frac{1}{n} \sum_{i=1}^{n} c_i \to 0. \tag{8}$$

The following theorem can be easily deduced from Theorem 1, by usual truncation and use of the well-known Lemma 2. It contains Etemadi's [6] as well as Cohn's [4] generalizations of the classical strong law of large numbers of Kolmogorov [18] (Loève [21], section 17). Its formulation uses the maximal-correlation coefficient $\kappa = \kappa(X,Y)$ (Hirschfeld [13], Gebelein [10]) defined for a pair (X,Y) of random variables by

$$\begin{aligned}\kappa(X,Y): \quad &= \sup\left\{\frac{E(\tilde{X}-E\tilde{X})(\tilde{Y}-E\tilde{Y})}{\operatorname{Var}(\tilde{X})^{1/2}\operatorname{Var}(\tilde{Y})^{1/2}};\ \tilde{X}\in\mathcal{L}_2(\mathcal{F}(X)), \tilde{Y}\in\mathcal{L}_2(\mathcal{F}(Y))\right\}\\ &= \sup\left\{\frac{E\tilde{X}\tilde{Y}-E\tilde{X}E\tilde{Y}}{(E\tilde{X}^2)^{1/2}(E\tilde{Y}^2)^{1/2}}; \tilde{X}\in\mathcal{L}_2(\mathcal{F}(X)), \tilde{Y}\in\mathcal{L}_2(\mathcal{F}(Y))\right\}\end{aligned}$$

where $0/0 := 0$, $\mathcal{F}(X)$ denotes the σ-algebra generated by X and $\tilde{X}\in\mathcal{L}_2(\mathcal{F}(X))$ denotes an $\mathcal{F}(X)$-$\mathcal{B}$-measurable square integrable real random variable (compare also Kolmogorov and Rozanov [19]). $\kappa(X,Y)=0$ means independence of X and Y.

Theorem 2 *Let $(X_n)_{n\in\mathbb{N}}$ be a sequence of identically distributed integrable real random variables. If $\kappa_{i,j} := \kappa(X_i, X_j)$, satisfies*

$$\sum_n \sup_m \kappa_{m,m+n} < \infty \tag{9}$$

or only

$$\sup_{j\in\{2,3,\dots\}} \sum_{i=1}^{j-1} \kappa(j-i,j) < \infty, \tag{10}$$

then (1) holds.

Remark 2 a) For $\kappa_{i,j} = 0$ $(i,j\in\mathbb{N})$, i.e., pairwise independence of the $X_n's$, Theorem 2 yields Etemadi's ([6], [7]) generalization of Kolmogorov's strong law of large numbers from independence to pairwise independence.
b) For a pair (X,Y) of random variables let

$$\begin{aligned}\phi(X,Y) &:= \sup\left\{\tfrac{|P(A\cap B)-P(A)P(B)|}{P(A)}; A\in\mathcal{F}(X), B\in\mathcal{F}(Y)\right\},\\ \lambda(X,Y) &:= \sup\left\{\tfrac{|P(A\cap B)-P(A)P(B)|}{P(A)^{1/2}P(B)^{1/2}}; A\in\mathcal{F}(X), B\in\mathcal{F}(Y)\right\}.\end{aligned}$$

Then

$$\kappa(X,Y) \le 2\phi(X,Y)^{1/2}$$

(Ibragimov [14], p. 351) and

$$\lambda(X,Y)\le\kappa(X,Y)\le 3000\lambda(X,Y)(1-\log\lambda(X,Y))$$

(Bradley [2], p. 168, and Bradley and Bric [3], p. 337). Setting

$$\begin{aligned}\phi_n^* &:= \sup_m \phi(X_m, X_{m+n}),\\ \lambda_n^* &:= \sup_m \lambda(X_m, X_{m+n}),\\ \kappa_n^* &:= \sup_m \kappa(X_m, X_{m+n}) \ (= \sup_m \kappa_{m,m+n})\end{aligned}$$

for a sequence $(X_n)_{n\in\mathbb{N}}$ of random variables, we immediately obtain that

$$\sum \phi_n^{*1/2} < \infty \text{ or } \sum \lambda_n^* |\log \lambda_n^*| < \infty$$

implies $\sum \kappa_n^* < \infty$, i.e., (9), which implies condition (10).
c) For a sequence $(X_n)_{n\in\mathbb{N}}$ of random variables let $\mathcal{F}_1^m = \mathcal{F}(X_1, \ldots, X_m)$, $\mathcal{F}_{m+n} = \mathcal{F}(X_{m+n}, X_{m+n+1}, \ldots)$ be the σ-algebras generated by $X_1, \ldots, X_m$ and $X_{m+n}, X_{m+n+1}, \ldots)$ respectively, and let

$$\phi(n) := \sup_m \sup_{A\in\mathcal{F}_1^m, B\in\mathcal{F}_{m+n}} \frac{|P(A\cap B) - P(A)P(B)|}{P(A)}.$$

$\phi(n) \to 0$ $(n \to \infty)$ means that $(X_n)_{n\in\mathbb{N}}$ is ϕ-mixing. Cohn [4] showed that for a sequence $(X_n)_{n\in\mathbb{N}}$ of identically distributed integrable real random variables (1) holds if $\sum \phi_n^{1/2} < \infty$ (see also Iosifescu and Theodorescu [15], p. 19). This result is a consequence of Theorem 2, because $\sum \phi_n^{1/2} < \infty$ implies $\sum \phi_n^{*1/2} < \infty$ and thus (9) by b).

Lemma 2 *Let X be an integrable nonnegative random variable and set*

$$X^{[n]} := XI_{[X\le n]}, \quad n \in \mathbb{N}$$

(I denoting an indicator function). Then

$$\sum_{n=1}^{\infty} \frac{E(X^{[n]^2})}{n^2} < \infty.$$

Remark 3 Theorem 1, Lemma 1, and Remark 1 can be generalized from arithmetic means to weighted means. Let $\alpha_n \in [0,1), n \in \mathbb{N}$, satisfy $\alpha_n \to 0$, $\sum \alpha_n = \infty$, and set

$$\beta_n := \frac{1}{(1-\alpha_n)\ldots(1-\alpha_n)}, \quad \gamma_n := \alpha_n\beta_n,$$

thus $\beta_n = 1+\gamma_1+\ldots+\gamma_n \uparrow \infty$. The special case $\alpha_n = 1/(n+1)$ leads to $\beta_n = n+1, \gamma_n = 1$. Replace assumption (3) in Theorem 1 by

$$\sum_{n=1}^{\infty} \frac{\gamma_n \operatorname{Var}(\gamma_1 X_1 + \ldots + \gamma_n X_n)}{\beta_n^3} < \infty,$$

assumptions (6) and (7) in Lemma 1 by

$$\sum_{n=1}^{\infty} \frac{\gamma_n(\gamma_1 c_1 + \ldots + \gamma_n c_n)^2}{\beta_n^3} < \infty$$

and

$$\frac{1}{\beta_n} \sum_{k=1}^{n} \gamma_k \frac{1}{\beta_k} \left(\sum_{j=1}^{k} \gamma_j c_j \right) \to 0,$$

respectively, and assumption (5) in Remark 1b by

$$\sum_{n=1}^{\infty} \alpha_n^2 \operatorname{Var}(X_n) < \infty.$$

Then the assertion (1) in Theorem 1, also in context of Remark 1 b, has to be replaced by

$$\frac{1}{\beta_n} \sum_{k=1}^{n} \gamma_k (X_k - EX_k) \to 0 \text{ a.s.}, \tag{11}$$

and the assertion (8) in Lemma 1 has to be replaced by

$$\frac{1}{\beta_n} \sum_{k=1}^{n} \gamma_k c_k \to 0$$

(as to the latter, compare Karamata [17], with review in Zentralblatt Math. 19, pp. 341, 342). Relation (11) is of interest in stochastic approximation (see, e.g., Ljung, Pflug, and Walk [20], I.1, I.2, with further references).

Condition (2) of one-sided boundedness in Theorem 1 can be avoided by sharpening now the condition on Γ appearing in (3) together with (4). The proof uses another Tauberian argument.

Proposition 1 *Let $(X_n)_{n\in\mathbb{N})}$ be a sequence of square integrable real random variables satisfying*

$$\sum_{k=0}^{\infty} \sum_{j=k}^{\infty} e^{-k^2/(6j)} j^{-3/2} |\Gamma(j+k, j)| < \infty, \tag{12}$$

where $\Gamma(i,j)$ is defined by (4). Then (1) holds.

The following theorem is a consequence of Proposition 1 and comprehends the case of second order stationarity where $\Gamma(i,j)$ only depends on the difference $i-j$.

Theorem 3 *Let $(X_n)_{n\in\mathbb{N}}$ be a sequence of square integrable real random variables satisfying*

$$|\Gamma(i,j)| \le r(|i-j|), i, j \in \mathbb{N} \tag{13}$$

with

$$\sum_{k=1}^{\infty} \frac{r(k)}{k} < \infty, \tag{14}$$

e.g., with

$$r(k) = O\left(\frac{1}{\log k (\textit{loglog } k)^{1+\delta}}\right) \tag{15}$$

for some $\delta > 0$. Then (1) holds.

Remark 4 In the case of second order stationarity with autocovariance function $R(k) := \Gamma(j+k,k)$, via the spectral measure of (X_n), Gaposhkin [9] established (1) under the (in some sense) weakest possible condition that

$$\sum_{k=2}^{\infty} \frac{R(k)}{k \log k} \log\log k \;\text{ converges},$$

where $R(k)$ also may be replaced by $\overline{R}(k) := (R(1)+\ldots+R(k))/k$. It is well-known and can easily be proved via Lemma 1 that for a second order stationary process (X_n) the condition $\overline{R}(n) \to 0 \;\; (n \to \infty)$ is necessary and sufficient for

$$E\left|\frac{1}{n}\sum_{k=1}^{n}(X_k - EX_k)\right|^2 \to 0 \; (n \to \infty).$$

Remark 5 Noticing

$$\sum_{i,j\in\{m+1,\ldots,n\}} |\Gamma(i,j)| + \sum_{i,j\in\{n+1,\ldots,N\}} |\Gamma(i,j)| \le \sum_{i,j\in\{m+1,\ldots,N\}} |\Gamma(i,j)|$$

for $m < n < N$, from Serfling [28] (see also Stout [29], Theorem 3.7.3) one obtains that the condition

$$\sum_{i,j\in\{1,\ldots,n\}} |\Gamma(i,j)| \le cn^2(\log n \log\log n)^{-2}, n \in \mathbb{N} \tag{16}$$

for some $c \in \mathbb{R}_+$, is sufficient for (1). $|\Gamma|$ may be replaced by the positive part Γ_+. In the special case (13), the conditions

$$\sum_{j=1}^{k} |r(j)| = O\left(k(\log k \log\log k)^{-2}\right)$$

(compare Stout [29], Theorem 3.7.4 with proof) and

$$r(k) = O((\log k \log\log k)^{-2})$$

yielding (16), are stronger than (14) and (15), respectively.
The proof of Proposition 1 is based on the deterministic Lemma 4 below. This lemma will be proved by Lemma 3, which is a consequence of a classic and deep Tauberian theorem on Borel summability, but can be shown in an elementary way (see section 3).

Lemma 3 *Let the sequence $(d_n)_{n\in\mathbb{N}_0}$ of real numbers satisfy*

$$\sum_{n=1}^{\infty} n^{1/2}(d_n - d_{n-1})^2 < \infty. \tag{17}$$

If

$$e^{-\lambda}\sum_{j=0}^{\infty}\frac{1}{j!}\lambda^j d_j \to 0 \ \ (\lambda\to\infty), \tag{18}$$

then

$$d_n \to 0 \ \ (n\to\infty). \tag{19}$$

Relation (18) denotes the so-called Borel summability of the series $d_0 + \sum_{k=0}^{\infty}(d_{k+1} - d_0)$ or of the sequence (d_n) (see Hardy [11], p. 80, or Zeller and Beekmann [30], p. 134).

Lemma 4 *Let the sequence $(c_n)_{n\in\mathbb{N}}$ of real numbers satisfy*

$$\sum_{n=1}^{\infty} n^{-3/2}c_n^2 < \infty. \tag{20}$$

If

$$\int_0^{\infty} t^{-1}|e^{-t}\sum_{k=1}^{\infty}\frac{t^k}{k!}c_k|^2 dt < \infty \tag{21}$$

or only

$$\frac{1}{\lambda}\int_0^{\lambda} e^{-t}\sum_{k=1}^{\infty}\frac{t^k}{k!}c_k dt \to 0 \ \ (\lambda\to\infty), \tag{22}$$

then

$$\frac{1}{n}\sum_{i=1}^{n} c_i \to 0 \ \ (n\to\infty), \tag{23}$$

Remark 6 As in the proof of Theorem 3 given in section 3, one can conclude from Proposition 1 that the condition

$$|\Gamma(i,j)| \le c\frac{i^\alpha + j^\alpha}{1+|i-j|^\beta}, \ i,j\in\mathbb{N}, \tag{24}$$

for some $c\in\mathbb{R}_+, 0 < 2\alpha < \beta < 1$ is sufficient for (11) and thus (1). This condition for the strong law of large numbers was established by Cramér and Leadbetter [5], p. 94. It seems to be an open problem whether the condition can be relaxed to $0 < \alpha < \beta < 1$. Ninness [24] in his argument does not verify completely the conditions of Lemma 1 there. The relaxed condition yields

$$\sum_{i,j\in\{1,\dots,n\}} |\Gamma(i,j)| \le c^* n^{2+(\alpha-\beta)}, n\in\mathbb{N}$$

for some $c^*\in\mathbb{R}_+$ (see Ninness [24], p. 219, and (3), via (4)). Under the additional assumption (2), now (1) is obtained by Theorem 1.

Remark 7 In a straightforward way Theorems 1 and 3 and their proofs can be transferred to the case of stochastic processes $\{X(t); t \in \mathbb{R}_+ (\text{or } \mathbb{R})\}$ that are continuous in squared mean, where sums are replaced by integrals. Especially one obtains a generalization of Loève's [21] continuous time version of his result mentioned in Remark 1 a.

3 PROOFS

Assumption (6) in Lemma 1 implies

$$\frac{1}{n^3}\sum_{k=1}^{n}(c_1+\ldots+c_k)^2 \to 0$$

by the Kronecker lemma and thus assumption (7) by the Cauchy-Schwarz inequality. (7) is equivalent to

$$\frac{1}{\binom{n+1}{2}}\sum_{k=1}^{n}(n+1-k)c_k \to 0, \tag{25}$$

i.e., C_2-summability of the sequence (c_n) to 0, while the assertion (8) means C_1-summability of (c_n) to 0 (see Hardy [11], p. 96, p. 7, or Zeller and Beekmann [30] p. 100, p. 104). With

$$(1-s)\sum_{k=1}^{\infty} c_k s^{k-1} \to 0 \ \ (s \uparrow 1), \tag{26}$$

i.e., Abel summability of (c_n) to 0, one has (8) $\Longrightarrow$ (25) $\Longrightarrow$ (26) for general sequences (c_n) of real numbers (see Hardy [11], p. 7, Theorems 43 and 55, or Zeller and Beekmann [30], p. 110, 53 I and 55 II). For one-side bounded sequences (c_n) also the converse holds. In this situation, in contrast to the implication (26)$\Longrightarrow$(8) (see Hardy [11], Theorem 97, or Zeller and Beekmann [30], 55 IV, or Feller [8], XIII.5, Theorem 5) the implication (25) (or (7)) $\Longrightarrow$ (8) can be proved in an elementary way. To make the paper self-contained, we give the proof of Lemma 1, i.e., of (7) $\Longrightarrow$ (8) for(c_n) bounded from below, according to Mordell [23] and Boas [1], proof of Theorem 4b there (in these papers with $\phi(n) = n^2$, $\psi(n) = -1$ and 1, respectively), compare also Zeller and Beekmann [30], p. 117, for further references.

Proof of Lemma 1. It suffices to conclude (8) from (7) under the assumption $c_n \geq -c \ \ (n \in \mathbb{N}), c \in \mathbb{R}_+$. Set

$$t_n := \sum_{k=1}^{n} c_k, \quad w_n := \sum_{k=1}^{n} t_k, \ n \in \mathbb{N}.$$

For $k \in \{1, \ldots, n\}$ one has

$$\left.\begin{array}{ll} \sum\limits_{j=1}^{k} & (t_{n+j}-t_n) \\ \sum\limits_{j=0}^{k-1} & (t_n - t_{n-j}) \end{array}\right\} \geq -c\sum_{j=1}^{k} j \geq -k^2 c,$$

thus

$$\begin{array}{rcl} w_{n\pm k} & \geq & w_n \pm kt_n - k^2 c, \\ \pm\dfrac{t_n}{n} & \leq & \dfrac{w_{n\pm k} - w_n}{kn} + \dfrac{k}{n}c. \end{array} \tag{27}$$

One notices

$$\begin{array}{rcl} \sigma_n & := & \max\{|w_j|; j = 1, \ldots, 2n\} = o(n^2) \ \ (\text{by (7)}), \\ \delta_n & := & 1 + \lfloor\sqrt{\sigma_n}\rfloor = o(n), \end{array}$$

($\lfloor\ \rfloor$ denoting integer part). Now from (27) with $k = \delta_n$ ($< n$ for n sufficiently large), one obtains

$$\limsup(\pm\frac{t_n}{n}) \leq \lim\left(\frac{2\sigma_n}{n\delta_n} + \frac{\delta_n}{n}c\right) = 0,$$

i.e. $t_n/n \to 0$, i.e., (8). □

Proof of Theorem 1. Using the monotone convergence theorem in series form, we obtain

$$\begin{array}{rcl} E\displaystyle\sum_{n=1}^{\infty}\frac{|\sum_{j=1}^{n}(X_j - EX_j)|^2}{n^3} & = & \displaystyle\sum_{n=1}^{\infty}\frac{E|\sum_{j=1}^{n}(X_j - EX_j)|^2}{n^3} \\ & = & \displaystyle\sum_{n=1}^{\infty}\frac{\mathrm{Var}(X_1 + \ldots X_n)}{n^3} \\ & < & \infty \ \ (\text{by (2.2)}). \end{array}$$

Thus

$$\sum_{n=1}^{\infty}\frac{(\sum_{j=1}^{n}(X_j - EX_j))^2}{n^3} < \infty \text{ a.s.}$$

By (2) the sequence $(X_n - EX_n)$ is bounded from below a.s. Now Lemma 1 yields the assertion. □

To make the paper self-contained we repeat the following well-known proof.

Proof of Lemma 2.

$$\begin{array}{rcl} \displaystyle\sum_{n=1}^{\infty}\frac{E(X^{[n]^2})}{n^2} & = & \displaystyle\sum_{n=1}^{\infty}\sum_{i=1}^{n}\frac{1}{n^2}\int\limits_{(i-1,i]} t^2 P_X(dt) \\ & = & \displaystyle\sum_{i=1}^{\infty}\int\limits_{(i-1,i]} t^2 P_X(dt)\sum_{n=i}^{\infty}\frac{1}{n^2} \\ & \leq & \displaystyle\sum_{i=1}^{\infty}\frac{2}{i}\int\limits_{(i-1,i]} t^2 P_X(dt) \\ & \leq & 2EX < \infty. \end{array}$$

□

Proof of Theorem 2. Assume $X_n \geq 0$ without loss of generality. The first step is well-known. Set

$$X_n^* := X_n^{[n]} := X_n I_{[X_n \leq n]}, \quad n \in \mathbb{N}.$$

Because of

$$\sum_{i=1}^{\infty} P\left[X_i \neq X_i^*\right] = \sum_{i=1}^{\infty} P\left[X_1 > i\right] \leq EX_1 < \infty,$$

a.s. $X_i = X_i^*$ from some index on (by the Borel-Cantelli lemma; Loève [21], sections 16,17). Therefore and because of

$$EX_n^* = EX_1^{[n]} \to EX_1 < \infty$$

(by the monotone convergence theorem), it suffices to show

$$\frac{1}{n}\sum_{i=1}^{n}(X_i^* - EX_i^*) \to 0 \text{ a.s.} \tag{28}$$

In the second step we notice

$$X_n^* \geq 0, \ EX_n^* \leq EX_1 < \infty,$$

further,

$$\begin{aligned}
& \sum_{n=1}^{\infty} \frac{\operatorname{Var}(X_1^* + \ldots + X_n^*)}{n^3} \\
\leq \ & \sum_{n=1}^{\infty} \frac{\operatorname{Var}(X_1^*) + \ldots + \operatorname{Var}(X_n^*)}{n^3} + 2\sum_{n=1}^{\infty}\frac{1}{n^3}\sum_{j=2}^{n}\sum_{i=1}^{j-1}\kappa(j-i,j)\sqrt{\operatorname{Var}(X_{j-i})}\sqrt{\operatorname{Var}(X_j)} \\
\leq \ & \sum_{n=1}^{\infty}\frac{EX_n^{*^2}}{n^2}\left(1 + 2\sup_{j\in\{2,3,\ldots\}}\sum_{i=1}^{j-1}\kappa(j-i,j)\right) \\
< \ & \infty \text{ (by Lemma 2 and (10)).}
\end{aligned}$$

Thus (28) follows by Theorem 1. □

Proof of Lemma 3. We use the abbreviation

$$v_{n,k} := e^{-n}\frac{n^k}{k!} \ (n \in \mathbb{N}, k \in \mathbb{N}_0).$$

By Hölder's inequality, for arbitrary $M \in \mathbb{N}$ we obtain

$$\sum_{k=M+1}^{\infty} v_{n,k}|d_k - d_n| \leq \left(\sum_{k=M+1}^{\infty} v_{n,k}|d_k - d_n|^4\right)^{1/4}$$

noticing

$$\sum_{k=0}^{\infty} v_{n,k} = 1. \tag{29}$$

We now modify an argument of Jurkat [16], p. 280. We set

$$u_j := d_j - d_{j-1},\ j \in \mathbb{N}_0$$

and obtain

$$\begin{aligned} |d_l - d_i|^2 &= \left| \sum_{j=i+1}^{l} u_j \right|^2 \\ &\leq \sum_{j=i+1}^{l} j^{-1/2} \sum_{j=i+1}^{l} j^{1/2} u_j^2 \\ &\quad \text{(by the Cauchy-Schwarz inequality)} \\ &\leq 2\frac{l-i}{l^{1/2}+i^{1/2}} \sum_{j=i+1}^{l} j^{1/2} u_j^2 \quad (1 \leq i < l), \end{aligned} \tag{30}$$

thus for $n > M(\in \mathbb{N})$

$$\begin{aligned} &\left(\sum_{k=M+1}^{\infty} v_{n,k} |d_k - d_n| \right)^4 \\ \leq\ & 4 \sum_{k=M+1}^{\infty} v_{n,k} \frac{|k-n|^2}{n} \left(\sum_{j=M+1}^{\infty} j^{1/2} u_j^2 \right)^2 \\ \leq\ & 4 \left(\sum_{j=M+1}^{\infty} j^{1/2} u_j^2 \right)^2, \end{aligned} \tag{31}$$

the latter because of

$$\sum_{k=0}^{\infty} v_{n,k}(k-n)^2 = n,\ n \in \mathbb{N}.$$

For arbitrary $\varepsilon > 0$, by (31) and (17),

$$\sum_{k=M+1}^{\infty} v_{n,k} |d_k - d_n| < \varepsilon$$

if M is sufficiently large and $n > M$. Further for each $k \in \mathbb{N}_0$

$$v_{n,k} \to 0, \qquad v_{n,k}|d_n| \to 0 \ \ (n \to \infty),$$

because (17) yields $d_n - d_{n-1} = o(n^{-1/4})$, and thus $d_n = o(n^{3/4})$. Therefore

$$\sum_{k=0}^{\infty} v_{n,k} |d_k - d_n| \to 0 \quad (n \to \infty).$$

This together with (18) and (29) yields (19). □

Lemma 3 can also be obtained by a classic and deep Tauberian theorem of R. Schmidt [27] (see Hardy [11], p. 225, p. 312, and Zeller and Beekmann [30], 66 X). It states that (19) is implied by (18) together with

$$\liminf (d_n - d_m) \geq 0 \tag{32}$$

where

$$m \to \infty, n > m, \frac{n-m}{m^{1/2}} \to 0.$$

(32) is weaker than (17), because of (30).

Proof of Lemma 4. Set $c_0 := 0$ and

$$d_j := \frac{1}{j+1} \sum_{k=0}^{j} c_k, \ j \in \mathbb{N}_0. \tag{33}$$

Then

$$\begin{aligned} & \sum_{n=1}^{\infty} n^{1/2} (d_n - d_{n-1})^2 \\ \leq \ & 2 \sum_{n=1}^{\infty} n^{-3/2} c_n^2 + 2 \sum_{n=1}^{\infty} n^{-7/2} \left(\sum_{k=1}^{n} c_k \right)^2 \\ \leq \ & 2 \sum_{n=1}^{\infty} n^{-3/2} c_n^2 + 2 \sum_{n=1}^{\infty} n^{-5/2} \sum_{k=1}^{n} c_k^2. \end{aligned}$$

Changing the order of summation, we obtain (17) by (20). From (21) we obtain

$$\frac{1}{\lambda} \int_0^{\lambda} |e^{-t} \sum_{k=1}^{\infty} \frac{t^k}{k!} c_k|^2 dt \to 0 \quad (\lambda \to \infty)$$

via partial integration as in the proof of the classical Kronecker lemma (see, e.g., Loève [21], section 17), and then (22) by the Cauchy-Schwarz inequality. Thus it suffices to assume (22). This assumption means that the integral Cesàro limit of the Borel transformation

$$\lambda \to e^{-\lambda} \sum_{k=0}^{\infty} \frac{\lambda^k}{k!} c_k, \ \lambda \geq 0$$

of $(c_k)_{k\in\mathbb{N}_0}$ is 0. As is well known, Cesàro transformation and Borel transformation are commutative for sequences whose Borel transformation exists (see Zeller and Beekmann [30], p. 139). In fact

$$\int_0^\lambda \frac{1}{k!} t^k e^{-t} dt = e^{-\lambda} \sum_{j=k+1}^\infty \frac{1}{j!} \lambda^j, \ k \in \mathbb{N}_0, \lambda > 0$$

(proof by differentiation), thus for the considered sequence $(c_k)_{k\in\mathbb{N}_0}$

$$\begin{aligned} & \frac{1}{\lambda} \int_0^\lambda \sum_{k=0}^\infty \frac{1}{k!} t^k e^{-t} c_k dt \\ = \ & e^{-\lambda} \sum_{k=0}^\infty \sum_{j=k+1}^\infty \frac{1}{j!} \lambda^{j-1} c_k \\ = \ & e^{-\lambda} \sum_{j=1}^\infty \left(\sum_{k=0}^{j-1} c_k \right) \frac{1}{j!} \lambda^{j-1} \\ = \ & e^{-\lambda} \sum_{j=0}^\infty \frac{1}{j!} \lambda^j \left(\frac{1}{j+1} \sum_{k=0}^{j} c_k \right), \ \lambda > 0. \end{aligned}$$

Therefore (22) and (18) with (33) are equivalent.
Now we use Lemma 3. With (33), from (18) and (17) we obtain (19), i.e., the assertion (23). □

Proof of Proposition 1. We assume $EX_n = 0, \ n \in \mathbb{N}$, without loss of generality. Condition (12) together with $\Gamma(j,j) = EX_j^2$ immediately yields

$$\sum n^{-3/2} EX_n^2 < \infty \tag{34}$$

and thus

$$\sum n^{-3/2} X_n^2 < \infty \text{ a.s.} \tag{35}$$

We shall show

$$I := \int_0^\infty t^{-1} E|e^{-t} \sum_{k=1}^\infty \frac{t^k}{k!} X_k|^2 dt < \infty \tag{36}$$

and thus

$$\int_0^\infty t^{-1} |e^{-t} \sum_{k=1}^\infty \frac{t^k}{k!} X_k|^2 dt < \infty \text{ a.s.} \tag{37}$$

Apparently

$$\begin{aligned} I &= \int_0^\infty t^{-1}e^{-2t}\sum_{j,l\in\mathbb{N}}\frac{t^j}{j!}\frac{t^l}{l!}\Gamma(l,j)dt \\ &\leq 2\int_0^\infty t^{-1}e^{-2t}\sum_{k=0}^\infty\sum_{j=1}^\infty\frac{t^{2j}}{j!}\frac{t^k}{(j+k)!}|\Gamma(j+k,j)|dt \\ &= 2\sum_{k=0}^\infty s_k \end{aligned}$$

with

$$s_k=\sum_{j=1}^\infty(2j+k-1)!\ \ 2^{-(2j+k)}\frac{1}{j!}\frac{1}{(j+k)!}|\Gamma(j+k,j)|,\ \ k\in\mathbb{N}_0,$$

via the Gamma function. Now, by Stirling's formula, we obtain

$$\begin{aligned} s_k \leq&\ \ c_1\sum_{j=1}^\infty\frac{(2j+k)^{2j+k}}{j^j(j+k)^{j+k}\sqrt{j}(j+k)}2^{-(2j+k)}|\Gamma(j+k,j)| \\ =&\ \ c_1\sum_{j=1}^\infty\frac{(1+k/(2j))^{2j+k}}{(1+k/j)^{j+k}\sqrt{j}(j+k)}|\Gamma(j+k,j)| \end{aligned}$$

with suitable $c_1>0$. By (34) we have

$$s_0<\infty \tag{38}$$

We notice for $0<s<\infty$

$$\begin{aligned} 0>\ h\ (s)\ &:=(\frac{2}{s}+1)\ \ln\ (1+\frac{s}{2})-(\frac{1}{s}+1)\ \ln\ (1+s) \\ &=\begin{cases}-(s/4)+o(s) & (s\to 0)\\ -\ln\ 2+o(1) & (s\to\infty),\end{cases} \end{aligned}$$

thus, for suitable $c>0$,

$$\frac{(1+s/2)^{(2/s)+1}}{(1+s)^{(1/s)+1}}\leq\begin{cases} e^{-cs} & ,\ \ 0<s\leq 1\\ e^{-c} & ,\ \ s\geq 1.\end{cases}$$

Noticing convexity of h we can choose $c=1/6$. Then, with $s=k/j$ $(k\in\mathbb{N},j\in\mathbb{N})$ we obtain

$$\begin{aligned} s_k\ &\leq\ c_1\left[e^{-ck}\sum_{j=1}^k j^{-3/2}|\Gamma(j+k,j)|+\sum_{j=k+1}^\infty e^{-ck^2/j}j^{-\frac{3}{2}}|\Gamma(j+k,j)|\right] \\ &=\ s_k'+s_k''. \end{aligned}$$

From (34) we obtain

$$\begin{aligned} |\Gamma(j+k,j)| &\leq |\Gamma(j+k,j+k)|^{1/2}|\Gamma(j,j)|^{1/2} \\ &= O((j+k)^{3/2}) \end{aligned}$$

and thus

$$\sum_{k=1}^{\infty} s_k' < \infty. \tag{39}$$

By (12) we have

$$\sum_{k=1}^{\infty} s_k'' < \infty \tag{40}$$

(38), (39), and (40) yield (36). Finally from (35) together with (37) we obtain (1) by Lemma 4. □

Proof of Theorem 3. We use Proposition 1. It is enough to show

$$\sum_{k=1}^{\infty} \int_k^{\infty} e^{-k^2/(6t)} t^{-3/2} dt \;\; r(k) < \infty$$

(because of (13) and piecewise monotonicity of the integrand). But this follows from

$$\int_k^{\infty} e^{-k^2/(6t)} t^{-3/2} dt = \int_0^{k} e^{-v/6} v^{-1/2} dv \; k^{-1} \;\; \text{and (14).}$$

□

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