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Preprint 2004/003

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Fachbereich Mathematik
Universität Stuttgart
Pfaffenwaldring 57
D-70 569 Stuttgart

E-Mail: preprints@mathematik.uni-stuttgart.de
WWW: http://www.mathematik/uni-stuttgart.de/preprints

ISSN **1613-8309**

1 Introduction

Deconvolution density estimation has become a widely studied topic. Lots of articles (Devroye (1989), Fan (1991), Fan (1993), Hesse (1999), Liu and Taylor (1990), Stefanski and Carroll (1990)) were published. The basic problem is the estimation of a probability density f based on contamined observations $Y_1, \ldots, Y_n$. Mathematically spoken, identically distributed random variables $X_1, \ldots, X_n$ with probability density f and also identically distributed random variables $\varepsilon_1, \ldots, \varepsilon_n$ with probability density g which represent the error or the contamination are given. Furthermore, the random variables $X_1, \varepsilon_1, \ldots, X_n, \varepsilon_n$ are independent. The random variables $X_1, \ldots, X_n$ whose density f shall be estimated cannot be observed directly, but only the contamined data $Y_1, \ldots, Y_n$ defined by

$$Y_j = X_j + \varepsilon_j , \qquad \forall j \in \{1, \ldots, n\}$$

can be used for the construction of the estimator. So the density h of Y_j equals $h = f * g$, i.e. the convolution of the densities f and g. Now it is aimed to find an estimator $\hat{f}_n$ of the density f based on the observations $Y_1, \ldots, Y_n$. Deterministic stipulations of nonparametric character like conditions refering to the asymptotic behaviour of the Fourier-transform for the density f are made. These can be expressed by the definition of a density class $\mathcal{F}$ and the stipulation $f \in \mathcal{F}$. In the classical approach of deconvolution estimation , the error density g is supposed to be exactly known and therefore it may be used for the construction of the estimator. An essential condition is a non-vanishing Fourier-transform of the error density $\psi_g(t) \neq 0$ for all $t \in \mathbb{R}$ (see Devroye (1989)). Commonly, the error density g is assumed to be a normal distribution density or a commonly smooth density with a polynomial asymptotic behaviour of the Fourier-transform. In the literaure described above, consistent estimators are constructed in the case of a known error density. That means the MISE (= mean integrated square error) converges to zero if the number of observations n tends to infinity. The rates of convergence of the MISE has intensively been studied. In the case of a normal distributed error, the rates which are achieved by the constructed estimators are very slow (logarithmic rates), but they are optimal as it is proved in Fan (1993).
However, in lots of practical work, the assumption of a perfectly known error density g is not always realistic. So, there are papers of Efromovich (1997) and Neumann (1997), in which the error distribution is assumed to be unknown but can be estimated based on additional empirical data which directly hail from the error distribution.
In this paper, we consider the situation of an unknown error density without any further observations. In Hesse (1999), it is mentioned that lots of work is left to do for this situation. So several densities can occure as the error density. We introduce the set $\mathcal{G}$ of all densities that can possibly be the error density. In section 2, We give theorem 1 that answers the question how the MISE will asymptotically behave if the used error density is misspecified. In section 3, we will investigate some useful properties of the distance $d_{\mathcal{F}}$ which is important in theorem 1. In section 4, we focus on the consequences of theorem 1 and will give some examples. Even a surprisingly fatal result is derived about the effects a misspecification may cause. In section 5, we derive some important rules for everybody using deconvolution estimation from the theory of the previous sections. These rules shall avoid the disasters caused by a misspecification. In section 6, we give the proofs of the lemmas and theorems of this paper.

2 Asymptotic behaviour

The most general shape of the deconvolution density estimator may be

$$\hat{f}_n(x) := \frac{1}{2\pi} \int \exp(-itx) L_n(t) \frac{1}{n} \sum_{j=1}^{n} \exp(itY_j)/\psi_g(t)\, dt \tag{2.1}$$

as it is mentioned in Neumann (1997), for example. ψ_y denotes the Fourier-transform of a function y. One has to stipulate that ψ_g vanishes nowhere and that $L_n/\psi_g \in L_2(\mathbb{R})$, the set of all absolute square integrable functions. By this, we can be sure that the estimator (2.1) is in $L_2(\mathbb{R})$, otherwise we cannot consider the MISE of this estimator. The usual selection of L_n is $\psi_K(\cdot/\omega_n)$ with a kernel function K and a bandwidth sequence $(\omega_n)_n$ which is chosen with respect to the error density. A widely used kernel is the sync kernel $K(x) := \frac{\sin(x)}{\pi x}$ with Fourier-transform $\psi_K(t) = 1_{[-1,1]}(t)$. In this case, the estimator (2.1) is

$$\hat{f}_n(x) := \frac{1}{2\pi} \int_{-\omega_n}^{\omega_n} \exp(-itx) \frac{1}{n} \sum_{j=1}^{n} \exp(itY_j)/\psi_g(t)\, dt. \tag{2.2}$$

Now, we consider the situation of a misspecified error density. This means, a false error density $\tilde{g}$ is used instead of the real error density g. However, a misspecification may occure so that the false error density $\tilde{g}$ is not even a density. This can theoretically be avoided, but if numerical effects are included, an arbitrarily small perturbation of g in the L_1-sense can push g out of the sets of all densities: Just imagine we have $\tilde{g} = g + \eta \cdot 1_{[0,1]}$ with some small $\eta > 0$. Then $\tilde{g}$ is no density because it does not integrate to one. Therefore, we introduce a function $\xi : \mathbb{R} \to \mathbb{C}$ with $\inf\limits_{t \in [-R,R]} |\xi(t)| > 0$ for all $R > 0$ which replaces the error density's Fourier-transform. So we have the deconvolution estimator (2.1) with misspecified error density

$$\hat{f}_n(x) = \frac{1}{2\pi} \int \exp(-itx) L_n(t) \frac{1}{n} \sum_{j=1}^{n} \exp(itY_j)/\xi(t)\, dt. \tag{2.3}$$

The density class $\mathcal{F}$ which the density f should be a member of can be chosen respecting the only condition that $\mathcal{F}$ is a subset of $L_2(\mathbb{R})$. Then we can state

Lemma 1 *Let $\hat{f}_n$ be the estimator defined in (2.3) and let $\inf\limits_{t \in [-R,R]} |\xi(t)| > 0$ for all $R > 0$ and $L_n/\xi \in L_2(\mathbb{R})$ hold. Then for the real error density g, the supreme MISE of $\hat{f}_n$ equals*

$$\begin{aligned} &\sup_{f \in \mathcal{F}} E_{f,g} \, \|\hat{f}_n - f\|^2_{L_2(\mathbb{R})} \\ &= \frac{1}{2\pi} \sup_{f \in \mathcal{F}} \int \left(\left| \frac{L_n(t)}{\xi(t)} \right|^2 \cdot \frac{1 - |\psi_{f*g}(t)|^2}{n} + \left| L_n(t) \frac{\psi_f(t)\psi_g(t)}{\xi(t)} - \psi_f(t) \right|^2 \right) dt. \end{aligned}$$

So lower bounds of the MISE are

$$\frac{1}{2\pi} \sup_{f \in \mathcal{F}} \int |\psi_f(t)|^2 \left| L_n(t) \frac{\psi_g(t)}{\xi(t)} - 1 \right|^2 dt \text{ and } \frac{1}{2\pi} \sup_{f \in \mathcal{F}} \int \left| \frac{L_n(t)}{\xi(t)} \right|^2 \cdot \frac{1 - |\psi_{f*g}(t)|^2}{n} dt$$

and an upper bound is

$$\frac{1}{2\pi} \sup_{f \in \mathcal{F}} \int \left| \frac{L_n(t)}{\xi(t)} \right|^2 \cdot \frac{1 - |\psi_{f*g}(t)|^2}{n} dt + \frac{1}{2\pi} \sup_{f \in \mathcal{F}} \int \left| L_n(t) \frac{\psi_f(t)\psi_g(t)}{\xi(t)} - \psi_f(t) \right|^2 dt$$

If ξ is the Fourier-transform of a density then it suffices to stipulate that ξ vanishes nowhere.

Lemma 2 *We assume: The selected function ξ is the Fourier-transform of a density, the Fourier-transforms of all densities in $\mathcal{G}$ vanish nowhere, in the case of a correct specification of the error density, i.e. $\xi = \psi_g$, the supremum of the MISE converges to zero (uniform consistency). Then*

$$\sup_{f \in \mathcal{F}} \int |\psi_f(t)|^2 |L_n(t) - 1|^2 \, dt \overset{n \to \infty}{\longrightarrow} 0 \tag{2.4}$$

and

$$\sup_{f\in\mathcal{F}} \int\limits_{-R}^{R} \left| L_n(t)\frac{\psi_g(t)}{\xi(t)} - \frac{\psi_g(t)}{\xi(t)} \right|^2 |\psi_f(t)|^2 \, dt \overset{n\to\infty}{\longrightarrow} 0 \tag{2.5}$$

hold for all $R > 0$ and for all possible g and

$$\frac{1}{n} \int \left| \frac{L_n(t)}{\xi(t)} \right|^2 dt \overset{n\to\infty}{\longrightarrow} 0. \tag{2.6}$$

Let us return to the MISE in the case of misspecification. If ξ is not the Fourier-transform of a density in $\mathcal{G}$, then there is no correct specification. However, as we want to continue to study the general case, we will stipulate (2.5) or (2.6) if necessary. In order to state the theorem we need another lemma

Lemma 3 *The law*

$$\frac{1}{2\pi} \sup_{f\in\mathcal{F}} \int\limits_{-R}^{R} |\psi_f(t)|^2 \left| \frac{\psi_g(t)}{\xi(t)} - 1 \right|^2 dt \overset{R\to\infty}{\longrightarrow} \frac{1}{2\pi} \sup_{f\in\mathcal{F}} \int |\psi_f(t)|^2 \left| \frac{\psi_g(t)}{\xi(t)} - 1 \right|^2 dt.$$

is valid. The tending $\overset{R\to\infty}{\longrightarrow}$ is meant as convergence if

$$\frac{1}{2\pi} \sup_{f\in\mathcal{F}} \int |\psi_f(t)|^2 \left| \frac{\psi_g(t)}{\xi(t)} - 1 \right|^2 dt < +\infty$$

and as divergence to infinity otherwise.

Now we can formulate the theorem

Theorem 1 *Let $\hat{f}_n$ be the estimator defined by (2.3) and $\inf\limits_{|t|\le T} |\xi(t)| > 0$ and $L_n/\xi \in L_2(\mathbb{R})$ $n \in \mathbb{N}$ hold for every $T > 0$ and for all $n \in \mathbb{N}$. Condition (2.5) holds. Then*
(a) The estimator sequence $(\sup\limits_{f\in\mathcal{F}} E_{f,g} \, \|\hat{f}_n - f\|^2_{L_2(\mathbb{R})})_{n\in\mathbb{N}}$ possesses no accumulation point which is smaller than $\frac{1}{2\pi} \sup\limits_{f\in\mathcal{F}} \int |\psi_f(t)|^2 \left| \frac{\psi_g(t)}{\xi(t)} - 1 \right|^2 dt$.
(b) If in addition (2.4), (2.6) and $|L_n(t)| \le 1$, $\forall t \in \mathbb{R}$, $\forall n \in \mathbb{N}$ hold, then $(\sup\limits_{f\in\mathcal{F}} E_{f,g} \, \|\hat{f}_n - f\|^2_{L_2(\mathbb{R})})_{n\in\mathbb{N}}$ converges or diverges to $\frac{1}{2\pi} \sup\limits_{f\in\mathcal{F}} \int |\psi_f(t)|^2 \left| \frac{\psi_g(t)}{\xi(t)} - 1 \right|^2 dt$.

(An undefined integral is to be taken over the whole real line.) The additional condition $|L_n(t)| \le 1, \forall t \in \mathbb{R} \forall n \in \mathbb{N}$ seems to be realistic. It is fulfilled if $L_n = \psi_K(\cdot/\omega_n)$ with K being a density or the sync kernel.
Whenever one wants to keep the consistency of the deconvolution estimator in the case of correct error density selection, the supremum of the MISE tends to the distance

$$\frac{1}{2\pi} \sup_{f\in\mathcal{F}} \int |\psi_f(t)|^2 \left| \frac{\psi_g(t)}{\xi(t)} - 1 \right|^2 dt.$$

So the theorem makes all attempts to improve the asymptotic quality of the estimator by changing the bandwidth sequence or the kernel function fail. Even if a deterioration of the convergence rate in the case of correct specification is accepted while the pure consistency is kept, one is not able to change something about the convergence or divergence of the MISE. So this distance decides about the robustness of the deconvolution estimator. Because of its importance, we will illustrate the distance in the following section.

3 Distance $d_{\mathcal{F}}$

If numerical aspects are not considered ξ can be seen as a density's Fourier-transform $\xi = \psi_{\tilde{g}}$ as explained in section 2. Then we derive the distance $d_{\mathcal{F}}$ defined by

$$\begin{aligned} d_{\mathcal{F}} \quad : \quad \mathcal{G} \times \mathcal{G} &\longrightarrow \mathbb{R} \cup \{+\infty\} \\ (g, \tilde{g}) &\longmapsto \tfrac{1}{2\pi} \sup_{f \in \mathcal{F}} \int \left| \tfrac{\psi_g(t)}{\psi_{\tilde{g}}(t)} - 1 \right|^2 |\psi_f(t)|^2 \, dt \end{aligned} \tag{3.1}$$

with $\mathcal{G}$ being the set defined in section 1. Theorem 1 says that the supremum of the MISE with $f \in \mathcal{F}$ tends to $d_{\mathcal{F}}(g, \tilde{g})$ if $\tilde{g}$ is used in the construction of the estimator instead of the real error density g. Notice that this distance of two densities can also be $+\infty$. Furthermore the distance is not symmetric, i.e. in general we have $d_{\mathcal{F}}(g, \tilde{g}) \neq d_{\mathcal{F}}(\tilde{g}, g)$. So $d_{\mathcal{F}}$ is no metric. However common properties of the intuitive expression distance can be derived:

- $d_{\mathcal{F}}$ is positive semidefinite, i.e.

$$\begin{aligned} d_{\mathcal{F}}(g, \tilde{g}) \geq 0 \qquad & \forall g, \tilde{g} \in \mathcal{G} \\ d_{\mathcal{F}}(g, g) = 0 \qquad & \forall g \in \mathcal{G} \end{aligned}$$

  The first law follows from the non negative integrand in (3.1), the second one is elementary.

- This leads to the question whether $d_{\mathcal{F}}$ is positive definite. As the integrand is non negative, we have

$$\begin{aligned} d_{\mathcal{F}}(g, \tilde{g}) = 0 \quad &\Leftrightarrow \quad \left| \tfrac{\psi_g(t)}{\psi_{\tilde{g}}(t)} - 1 \right|^2 |\psi_f(t)|^2 = 0 \text{ for almost all } t \in \mathbb{R}, \forall f \in \mathcal{F} \\ & \qquad \textit{As the integrand is continuous:} \\ &\Leftrightarrow \quad \left| \tfrac{\psi_g(t)}{\psi_{\tilde{g}}(t)} - 1 \right|^2 |\psi_f(t)|^2 = 0 \quad \forall t \in \mathbb{R}, \forall f \in \mathcal{F} \\ &\Leftrightarrow \quad \psi_g(t) = \psi_{\tilde{g}}(t) \vee \psi_f(t) = 0 \quad \forall t \in \mathbb{R}, \forall f \in \mathcal{F} \end{aligned}$$

  Hence, $d_{\mathcal{F}}$ is positive definite if and only if

$$\psi_g(t) = \psi_{\tilde{g}}(t) \vee \psi_f(t) = 0 \quad \forall t \in \mathbb{R}, \forall f \in \mathcal{F} \Longrightarrow g = \tilde{g}.$$

  This is a very weak condition. The membership of at least one f with a non-vanishing Fourier-transform (normal distribution density or double exponential density, for example) in $\mathcal{F}$ suffices to ensure positive definiteness because the identity of the Fourier-transforms ($\psi_g = \psi_{\tilde{g}}$) implies the identity of the densities ($g = \tilde{g}$) by a famous result of probability theory. If $d_{\mathcal{F}}$ is positive definite then any misspecification of the error density destroys the consistency of the deconvolution estimator i.e. the supremum of the MISE will not tend to zero.

- Assume $g, \tilde{g} \in \mathcal{G}$ with $|\psi_g(t)| \geq |\psi_{\tilde{g}}(t)| \quad \forall t \in \mathbb{R}$. Then we receive

$$d_{\mathcal{F}}(g, \tilde{g}) \geq d_{\mathcal{F}}(\tilde{g}, g) \tag{3.2}$$

Proof:

$$\begin{aligned} d_{\mathcal{F}}(g,\tilde{g}) &= \tfrac{1}{2\pi} \sup_{f\in\mathcal{F}} \int \left| \tfrac{\psi_g(t)}{\psi_{\tilde{g}}(t)} - 1 \right|^2 |\psi_f(t)|^2 \, dt \\ &= \tfrac{1}{2\pi} \sup_{f\in\mathcal{F}} \int \left| 1 - \tfrac{\psi_{\tilde{g}}(t)}{\psi_g(t)} \right|^2 \underbrace{\left| \frac{\psi_g(t)}{\psi_{\tilde{g}}(t)} \right|^2}_{\geq 1, \forall t} |\psi_f(t)|^2 \, dt \\ &\geq \tfrac{1}{2\pi} \sup_{f\in\mathcal{F}} \int \left| 1 - \tfrac{\psi_{\tilde{g}}(t)}{\psi_g(t)} \right|^2 |\psi_f(t)|^2 \, dt \\ &= d_{\mathcal{F}}(\tilde{g},g). \end{aligned}$$

(3.2) is important for the discussion in the next section.

4 Selection of the error density

In order to calculate a deconvolution estimator (2.3) we have to choose an error density $\tilde{g}$ and $\xi = \psi_{\tilde{g}}$. The set $\mathcal{G}$ consists of more than one density, so a misspecification is possible. We assume that there is no a priori weight of the error densities but every density in $\mathcal{G}$ is equal before the empirical data become known. (3.2) may help if one Fourier-transform $|\psi_g(t)|$ is larger or equal to another $|\psi_{\tilde{g}}(t)|$ for all $t \in \mathbb{R}$. Then density g should be prefered. But all densities cannot be compared by the stipulation of (3.2). This is the case in the following very important example:

Assume $\mathcal{G}$ consists of two densities, the normal density g_N with Fourier-transform $\psi_{g_N}(t) = \exp(-x^2/2)$ and the double exponential density $g_L(x) = \frac{1}{2}\exp(-|x|)$ with Fourier-transform $\psi_{g_L}(t) = \frac{1}{1+t^2}$. These are the usual examples for supersmooth and smooth densities. The only stipulations made refering to the density class $\mathcal{F}$ are the membership of the Laplace density in $\mathcal{F}$ and the uniform $L_2(\mathbb{R})$-boundedness of $\mathcal{F}$, i.e. there is a constant $c > 0$ with $\sup_{f\in\mathcal{F}} \int |f(t)|^2 dt \leq c$.

- We assume that the real error density is g_L but g_N is mistakenly used in the deconvolution estimator. For the asymptotic bias we have

$$\begin{aligned} d_{\mathcal{F}}(g_L, g_N) &= \tfrac{1}{2\pi} \sup_{f\in\mathcal{F}} \int \left| \tfrac{\psi_{g_L}(t)}{\psi_{g_N}(t)} - 1 \right|^2 |\psi_f(t)|^2 \, dt \\ &\geq \tfrac{1}{2\pi} \int \left| \tfrac{(1+t^2)^{-1}}{\exp(-t^2/2)} - 1 \right|^2 |\psi_{g_L}(t)|^2 \, dt \\ &= \tfrac{1}{2\pi} \int \left| \tfrac{\exp(t^2/2)}{1+t^2} - 1 \right|^2 \left| \tfrac{1}{1+t^2} \right|^2 dt \\ &= \tfrac{1}{2\pi} \int \left| \tfrac{\exp(t^2/2) - 1 - t^2}{(1+t^2)^2} \right|^2 dt \end{aligned}$$

The integrand is a fraction with the numerator tending to infinity with a exponential rate while the denominator diverges to infinity in 8th power. So the integrand itself diverges to infinity. Therefore the integral does not exist as a finite number and we get

$$d_{\mathcal{F}}(g,\tilde{g}) = +\infty.$$

That is a disaster! The supremum of the MISE tends to infinity. The more observations are used the worse the result of the estimation becomes. Even an estimator which is based on ignoring the contamination has got a better asymptotic quality than the deconvolution estimator. As huge sample sizes are commonly used in density estimation, the misspecification like this can cause a totally false result.

- Now we consider the opposite situation. g_N is the real error density and g_L is misspecified. We receive

$$
\begin{aligned}
d_{\mathcal{F}}(g_N, g_L) &= \tfrac{1}{2\pi} \sup_{f \in \mathcal{F}} \int \left| \tfrac{\psi_{g_N}(t)}{\psi_{g_L}(t)} - 1 \right|^2 |\psi_f(t)|^2 \, dt \\
&= \tfrac{1}{2\pi} \sup_{f \in \mathcal{F}} \int \underbrace{\left| \frac{t^2+1}{\exp(t^2/2)} - 1 \right|^2}_{\substack{\to 1 \text{ for } |t| \to \infty \text{ and continuous} \\ \Rightarrow \text{ upper bounded to } S < \infty}} |\psi_f(t)|^2 \, dt \\
&\leq \tfrac{S}{2\pi} \sup_{f \in \mathcal{F}} \int |\psi_f(t)|^2 \, dt \\
&= S \sup_{f \in \mathcal{F}} \|f\|^2_{L_2(\mathbb{R})} \, dt \\
&< +\infty.
\end{aligned}
$$

  So in this case the supremum of the MISE is no sequence tending to zero but it possesses at least a finite upper bound.

So if one has to choose if the normal density or the double exponential density has to be used in deconvolution estimation without any a priori knowledge refering to the error density, then the double exponential density should be selected. Another aspect emphasizes this choice: In the case of correct selection the MISE converges to zero with an algebraic rate with a double exponential error density while the MISE converges to zero with a logarithmic rate with a normal error density, see Fan (1993).
Now, a further example

(a) The stipulations to $\mathcal{F}$ are the same as in the previous context, $\mathcal{G}$ consists of normal distribution densities with an exactly known variance σ^2 but an unknown mean $\mu \in \mathbb{R}$. So $\#\mathcal{G}$ is infinity. For the asymptotic quality we receive

$$
\begin{aligned}
d_{\mathcal{F}}(g, \tilde{g}) &= \frac{1}{2\pi} \sup_{f \in \mathcal{F}} \int \left| \frac{\exp(i\mu t - (1/2) \cdot \sigma^2 t^2)}{\exp(it\tilde{\mu} - (1/2) \cdot \sigma^2 t^2)} - 1 \right|^2 |\psi_f(t)|^2 dt \\
&= \frac{1}{2\pi} \sup_{f \in \mathcal{F}} \int \underbrace{|\exp(i(\mu - \tilde{\mu})t) - 1|^2}_{\leq 4} |\psi_f(t)|^2 dt \\
&\leq 4 \cdot \sup_{f \in \mathcal{F}} \|f\|^2_{L_2(\mathbb{R})}.
\end{aligned}
$$

  So the supremum of the MISE is upper bounded and cannot tend to infinity. If the $|\psi_f|$ with $f \in \mathcal{F}$ are uniformly upper bounded to a square integrable function ζ, we can derive that small distance between μ and $\tilde{\mu}$ causes a small asymptotic bias using dominated convergence here.

(b) Now, we assume μ is exactly known but the variance of the normal density is misspecified. So we have

$$d_{\mathcal{F}}(g,\tilde{g}) = \frac{1}{2\pi}\sup_{f\in\mathcal{F}}\int\left|\frac{\exp(i\mu t-(1/2)\cdot\sigma^2t^2)}{\exp(i\mu t-(1/2)\cdot\tilde{\sigma^2}t^2)}-1\right|^2|\psi_f(t)|^2dt$$

$$=\frac{1}{2\pi}\sup_{f\in\mathcal{F}}\int\left|\exp((1/2)(\tilde{\sigma^2}-\sigma^2)t^2)-1\right|^2|\psi_f(t)|^2dt.$$

In the case $\tilde{\sigma^2}>\sigma^2$, $\left|\exp((1/2)(\tilde{\sigma^2}-\sigma^2)t^2)-1\right|^2$ diverges to infinity with an exponential rate for $|t|\to\infty$. It suffices that a density like g_L with $\psi_{g_L}(t)=\frac{1}{1+t^2}$ is in $\mathcal{F}$ and we have $d_{\mathcal{F}}(g,\tilde{g})=\infty$. Now, if $\tilde{\sigma^2}<\sigma^2$, then $\left|\exp((1/2)(\tilde{\sigma^2}-\sigma^2)t^2)-1\right|^2$ possesses the upper bound 1. So the uniform boundedness of the density class $\mathcal{F}$ in L_2-sense suffices to prove that $d_{\mathcal{F}}(g,\tilde{g})<\infty$. So we see that a too large selected variance is more dangerous than a too small one.
We also notice in this example that $d_{\mathcal{F}}$ is not $L_1(\mathbb{R})$-continuous, this means $\int|g_n(t)-g(t)|dt\to 0$ does not imply $d_{\mathcal{F}}(g,g_n)\to 0$, in general. Assume g_n to be the normal density with mean 0 and variance σ_n^2 and g the normal density with mean 0 and variance σ^2 and $\sigma_n^2\downarrow\sigma^2$. Then we have $\int|g_n(t)-g(t)|dt\to 0$, $d_{\mathcal{F}}(g,g_n)=\infty$ for each $n\in\mathbb{N}$ under realistic conditions to $\mathcal{F}$ as seen before and $d_{\mathcal{F}}(g,g)=0$. So $d_{\mathcal{F}}(g,g_n)\not\to d_{\mathcal{F}}(g,g)$. Hence, even an arbitrarily small difference between g and $\tilde{g}$ can cause a completely wrong result.

5 Consequences

We summarize and derive the following rules from the previous sections. They are interesting for any user of deconvolution density estimation.

- A supersmooth error density (for example normal density) causes more trouble than a smooth density (for example double exponential density) if a misspecification can possibly occure.
- If the error density is a normal density the variance should be chosen rather too small than too large.

6 Proofs

Proof of lemma 1:
Consider the risk

$$\sup_{f\in\mathcal{F}} E_{f,g}\,\|\hat{f}_n - f\|^2_{L_2(\mathbb{R})}$$

$$= \sup_{f\in\mathcal{F}} E_{f,g}\,\|\tfrac{1}{2\pi}\int \exp(-it\cdot)L_n(t)\tfrac{1}{n}\sum_{j=1}^{n}\exp(itY_j)/\xi(t)\,dt - f\|^2_{L_2(\mathbb{R})}$$

(Parseval identity)

$$= \tfrac{1}{2\pi}\sup_{f\in\mathcal{F}} E_{f,g}\int\left|L_n(t)\tfrac{1}{n}\sum_{j=1}^{n}\exp(itY_j)/\xi(t)\,dt - \psi_f(t)\right|^2 dt$$

(Fubini's theorem)

$$= \tfrac{1}{2\pi}\sup_{f\in\mathcal{F}}\int E_{f,g}\left|L_n(t)\tfrac{1}{n}\sum_{j=1}^{n}\exp(itY_j)/\xi(t)\,dt - \psi_f(t)\right|^2 dt$$

$$= \tfrac{1}{2\pi}\sup_{f\in\mathcal{F}}\int\Big\{var_{f,g}\big(L_n(t)\tfrac{1}{n}\sum_{j=1}^{n}\exp(itY_j)/\xi(t)\big) + \left|E_{f,g}L_n(t)\tfrac{1}{n}\sum_{j=1}^{n}\exp(itY_j)/\xi(t) - \psi_f(t)\right|^2\Big\}dt$$

$$= \tfrac{1}{2\pi}\sup_{f\in\mathcal{F}}\int\left(\left|\tfrac{L_n(t)}{\xi(t)}\right|^2\cdot\tfrac{1-|\psi_{f*g}(t)|^2}{n} + \left|L_n(t)\tfrac{\psi_f(t)\psi_g(t)}{\xi(t)} - \psi_f(t)\right|^2\right)dt$$

$\blacksquare$

Proof of lemma 2:
It follows from lemma 1 using the uniform L_2-consistency on the one hand

$$\Rightarrow \quad \tfrac{1}{2\pi}\sup_{f\in\mathcal{F}}\int|\psi_f(t)|^2\,|L_n(t)-1|^2\,dt \quad \stackrel{n\to\infty}{\longrightarrow}\ 0\ \text{, so (2.4)}$$

($R > 0$ arbitrary)

$$\Rightarrow \quad \tfrac{1}{2\pi}\sup_{f\in\mathcal{F}}\int_{-R}^{R}|\psi_f(t)|^2\,|L_n(t)-1|^2\,dt \quad \stackrel{n\to\infty}{\longrightarrow}\ 0$$

As $\inf_{|t|\le R}|\xi(t)|^2 > 0$ and $\xi(t)\neq 0, \forall t\in\mathbb{R}$ is valid because of the stipulations (ξ is a density in $\mathcal{G}$) and the continuity of ξ, we conclude

$$\sup_{|t|\le R}\left|\frac{\psi_g(t)}{\xi(t)}\right|^2 \le \frac{1}{\inf_{|t|\le R}|\xi(t)|^2} < \infty.$$

Hence

$$\frac{1}{2\pi}\sup_{f\in\mathcal{F}}\int_{-R}^{R}\sup_{|\tau|\le R}\left|\frac{\psi_g(\tau)}{\xi(\tau)}\right|^2|\psi_f(t)|^2\left|L_n(t)-1\right|^2\,dt \quad \overset{n\to\infty}{\longrightarrow}\ 0$$

$$\geq \frac{1}{2\pi}\sup_{f\in\mathcal{F}}\int_{-R}^{R}\left|L_n(t)\frac{\psi_g(t)}{\xi(t)}-\frac{\psi_g(t)}{\xi(t)}\right|^2|\psi_f(t)|^2\,dt$$

$$\geq 0$$

So we receive (2.5)

$$\frac{1}{2\pi}\sup_{f\in\mathcal{F}}\int_{-R}^{R}\left|L_n(t)\frac{\psi_g(t)}{\xi(t)}-\frac{\psi_g(t)}{\xi(t)}\right|^2|\psi_f(t)|^2\,dt \overset{n\to\infty}{\longrightarrow} 0$$

for all $R>0$.

As the second condition of consistency one gets

$$\frac{1}{2\pi}\sup_{f\in\mathcal{F}}\int\left|\frac{L_n(t)}{\xi(t)}\right|^2\cdot\frac{1-|\psi_{f*g}(t)|^2}{n}dt \qquad \overset{n\to\infty}{\longrightarrow}\ 0$$

$$\geq \frac{1}{2\pi}\int_{|t|\geq T}\left|\frac{L_n(t)}{\xi(t)}\right|^2\cdot\frac{1-|\psi_{f*g}(t)|^2}{n}dt$$

$$\geq 0$$

for an arbitrary $T>0$ and an arbitrary density $f\in\mathcal{F}$. As $f*g$ is a density we get according to the results of probability theory $\psi_{f*g}(t)=1 \Leftrightarrow t=0$ and $|\psi_{f*g}(t)|\leq 1, \forall t\in\mathbb{R}$. Finally, it follows from the continuity of ψ_{f*g} that $\inf\limits_{|t|\geq T}(1-|\psi_{f*g}(t)|^2)>0$. So

$$\int_{|t|\geq T}\left|\frac{L_n(t)}{\xi(t)}\right|^2\cdot\frac{1-|\psi_{f*g}(t)|^2}{n}dt \overset{n\to\infty}{\longrightarrow} 0$$

$$\geq \frac{1}{n}\underbrace{\inf_{|t|\geq T}(1-|\psi_{f*g}(t)|^2)}_{\text{independent of } n}\int_{|t|\geq T}\left|\frac{L_n(t)}{\xi(t)}\right|^2 dt$$

$$\geq 0$$

$\Rightarrow \frac{1}{n}\int\limits_{|t|\geq T}\left|\frac{L_n(t)}{\xi(t)}\right|^2 dt \overset{n\to\infty}{\longrightarrow} 0$ for each $T>0$.

We choose $f\in\mathcal{F}$ arbitrarily and determine $T>0$ so that $|\psi_f(t)|>\frac{1}{2}$ for all t with $|t|\leq T$. This is possible because $\psi_f(0)=1$ and ψ_f is continuous.

$$\sup_{f\in\mathcal{F}} \int |\psi_f(t)|^2 \left|L_n(t)-1\right|^2 dt \quad \overset{n\to\infty}{\longrightarrow} 0$$

$$\geq \int_{-T}^{T} |\psi_f(t)|^2 \left|L_n(t)-1\right|^2 dt$$

$$\geq \frac{1}{2}\int_{-T}^{T} \left|L_n(t)-1\right|^2 dt$$

That implies the convergence of the functional sequence $(L_n-1)_n$ in $L_2([-T,T])$ and hence its boundedness. As 1 seen as function in this space possesses the (finite) norm $\sqrt{2T}$, the sequence $(L_n)_n$ is bounded in the $L_2([-T,T])$-norm.

Notice that the stipulation $\inf_{|t|\leq T}|\xi(t)|>0$ is valid following from the condition that the Fourier-transform of densities in $\mathcal{G}$ vanishes nowhere. Then (2.6)

$$\frac{1}{n}\int_{-T}^{T}\left|\frac{L_n(t)}{\xi(t)}\right|^2 dt \overset{n\to\infty}{\longrightarrow} 0,$$

hence

$$\frac{1}{n}\int \left|\frac{L_n(t)}{\xi(t)}\right|^2 dt$$
$$= \frac{1}{n}\int_{-T}^{T}\left|\frac{L_n(t)}{\xi(t)}\right|^2 dt + \frac{1}{n}\int_{|t|\geq T}\left|\frac{L_n(t)}{\xi(t)}\right|^2 dt \overset{n\to\infty}{\longrightarrow} 0.$$

■

Proof of lemma 3:

We have to consider two cases

1st case: $\frac{1}{2\pi}\sup_{f\in\mathcal{F}}\int |\psi_f(t)|^2 \left|\frac{\psi_g(t)}{\xi(t)}-1\right|^2 dt < +\infty$

$$\sup_{f\in\mathcal{F}}\int_{-R}^{R} |\psi_f(t)|^2 \left|\frac{\psi_g(t)}{\xi(t)}-1\right|^2 dt$$

is monotonically increasing refering to R and bounded by the proposed limit

$$\frac{1}{2\pi}\sup_{f\in\mathcal{F}}\int |\psi_f(t)|^2 \left|\frac{\psi_g(t)}{\xi(t)}-1\right|^2 dt.$$

On the other hand

$$\frac{1}{2\pi}\sup_{f\in\mathcal{F}}\int |\psi_f(t)|^2 \left|\frac{\psi_g(t)}{\xi(t)}-1\right|^2 dt$$

has to be the infimum because for each $\epsilon>0$ there is a $\tilde{f}(\epsilon)\in\mathcal{F}$ so that

$$\sup_{f\in\mathcal{F}} \int |\psi_f(t)|^2 \left|\frac{\psi_g(t)}{\xi(t)}-1\right|^2 dt \quad \le \int |\psi_{\tilde f}(t)|^2 \left|\frac{\psi_g(t)}{\xi(t)}-1\right|^2 dt \,+\, \epsilon/2$$

$$(\exists R(\tilde f(\epsilon),\epsilon)>0 :) \qquad \le \int_{-R(\tilde f(\epsilon),\epsilon)}^{R(\tilde f(\epsilon),\epsilon)} |\psi_{\tilde f}(t)|^2 \left|\frac{\psi_g(t)}{\psi_{\xi(t)}}-1\right|^2 dt \,+\, \epsilon/2 \,+\, \epsilon/2$$

$$\le \sup_{f\in\mathcal{F}} \int_{-R(\tilde f(\epsilon),\epsilon)}^{R(\tilde f(\epsilon),\epsilon)} |\psi_f(t)|^2 \left|\frac{\psi_g(t)}{\xi(t)}-1\right|^2 dt \,+\, \epsilon$$

<u>2nd case</u>: $\frac{1}{2\pi}\sup_{f\in\mathcal{F}} \int |\psi_f(t)|^2 \left|\frac{\psi_g(t)}{\xi(t)}-1\right|^2 dt = \infty$

There is a sequence of densities $(f_n)_n$ so that

$$\frac{1}{2\pi}\int |\psi_{f_n}(t)|^2 \left|\frac{\psi_g(t)}{\xi(t)}-1\right|^2 dt$$

for $n\to\infty$ tends to infinity. For every f_n a positive number R_n can be found so that

$$\frac{1}{2\pi}\int_{-R_n}^{R_n} |\psi_{f_n}(t)|^2 \left|\frac{\psi_g(t)}{\xi(t)}-1\right|^2 dt \text{ and hence } \frac{1}{2\pi}\sup_{f\in\mathcal{F}}\int_{-R_n}^{R_n} |\psi_f(t)|^2 \left|\frac{\psi_g(t)}{\xi(t)}-1\right|^2 dt$$

diverge to infinity for $n\to\infty$. Because of the increasing monotonicity refering to R we get

$$\sup_{f\in\mathcal{F}}\int_{-R}^{R} |\psi_f(t)|^2 \left|\frac{\psi_g(t)}{\xi(t)}-1\right|^2 dt \overset{R\to+\infty}{\longrightarrow} \infty.$$

■

Proof of theorem 1:

ad (a): $\sup_{f\in\mathcal{F}} E_{f,g}\,\|\hat f_n - f\|^2_{L_2(\mathbb{R})}$ can be lower bounded (lemma 1) by $\frac{1}{2\pi}\sup_{f\in\mathcal{F}} \int |\psi_f(t)|^2 \left|L_n(t)\frac{\psi_g(t)}{\xi(t)}-1\right|^2 dt$. Let us consider a sequence of inequalities for this upper bound

$$\frac{1}{2\pi}\sup_{f\in\mathcal{F}} \int |\psi_f(t)|^2 \left|L_n(t)\frac{\psi_g(t)}{\xi(t)}-1\right|^2 dt$$

$$\ge \frac{1}{2\pi}\sup_{f\in\mathcal{F}} \int_{-R}^{R} |\psi_f(t)|^2 \left|L_n(t)\frac{\psi_g(t)}{\xi(t)}-1\right|^2 dt$$

$$= \frac{1}{2\pi}\sup_{f\in\mathcal{F}} \int_{-R}^{R} |\psi_f(t)|^2 \left|L_n(t)\frac{\psi_g(t)}{\xi(t)}-\frac{\psi_g(t)}{\xi(t)}+\frac{\psi_g(t)}{\xi(t)}-1\right|^2 dt$$

$$= \frac{1}{2\pi} \sup_{f\in\mathcal{F}} \{ \int_{-R}^{R} |\psi_f(t)|^2 \left| L_n(t) \tfrac{\psi_g(t)}{\xi(t)} - \tfrac{\psi_g(t)}{\xi(t)} \right|^2 \, dt$$
$$+ 2 \cdot Re \int_{-R}^{R} |\psi_f(t)|^2 \left(L_n(t) \tfrac{\psi_g(t)}{\xi(t)} - \tfrac{\psi_g(t)}{\xi(t)} \right) \overline{\left(\tfrac{\psi_g(t)}{\xi(t)} - 1 \right)} \, dt$$
$$+ \int_{-R}^{R} \left| \tfrac{\psi_g(t)}{\xi(t)} - 1 \right|^2 |\psi_f(t)|^2 \, dt \}$$

(Cauchy-Schwarz inequality)

$$\geq \frac{1}{2\pi} \sup_{f\in\mathcal{F}} \{ \int_{-R}^{R} |\psi_f(t)|^2 \left| L_n(t) \tfrac{\psi_g(t)}{\xi(t)} - \tfrac{\psi_g(t)}{\xi(t)} \right|^2 \, dt$$
$$-2 \cdot \left(\int_{-R}^{R} |\psi_f(t)|^2 \left| L_n(t) \tfrac{\psi_g(t)}{\xi(t)} - \tfrac{\psi_g(t)}{\xi(t)} \right|^2 \, dt \right)^{1/2} \cdot \left(\int_{-R}^{R} \left| \tfrac{\psi_g(t)}{\xi(t)} - 1 \right|^2 |\psi_f(t)|^2 \, dt \right)^{1/2}$$
$$+ \int_{-R}^{R} \left| \tfrac{\psi_g(t)}{\xi(t)} - 1 \right|^2 |\psi_f(t)|^2 \, dt \}$$

$$= \frac{1}{2\pi} \sup_{f\in\mathcal{F}} \{ \left(\int_{-R}^{R} |\psi_f(t)|^2 \left| L_n(t) \tfrac{\psi_g(t)}{\xi(t)} - \tfrac{\psi_g(t)}{\xi(t)} \right|^2 \, dt \right)^{1/2}$$
$$- \left(\int_{-R}^{R} |\psi_f(t)|^2 \left| \tfrac{\psi_g(t)}{\xi(t)} - 1 \right|^2 \, dt \right)^{1/2} \}^2$$

$$\geq \frac{1}{2\pi} \{ \left(\sup_{f\in\mathcal{F}} \int_{-R}^{R} |\psi_f(t)|^2 \left| L_n(t) \tfrac{\psi_g(t)}{\xi(t)} - \tfrac{\psi_g(t)}{\xi(t)} \right|^2 \, dt \right)^{1/2}$$
$$- \left(\sup_{f\in\mathcal{F}} \int_{-R}^{R} |\psi_f(t)|^2 \left| \tfrac{\psi_g(t)}{\xi(t)} - 1 \right|^2 \, dt \right)^{1/2} \}^2$$

It follows from (2.5) that $(\sup_{f\in\mathcal{F}} E_{f,g} \|\hat{f}_n - f\|^2_{L_2(\mathbb{R})})_{n\in\mathbb{N}}$ cannot possess any accumulation point that is smaller than $\frac{1}{2\pi} \sup_{f\in\mathcal{F}} \int_{-R}^{R} |\psi_f(t)|^2 \left| \frac{\psi_g(t)}{\xi(t)} - 1 \right|^2 dt$ for every $R > 0$. Because of lemma 3 there can be no accumulation point smaller than $\frac{1}{2\pi} \sup_{f\in\mathcal{F}} \int |\psi_f(t)|^2 \left| \frac{\psi_g(t)}{\xi(t)} - 1 \right|^2 dt$. So (a) is shown.

ad (b): We just have to look at the case $\frac{1}{2\pi} \sup_{f\in\mathcal{F}} \int |\psi_f(t)|^2 \left| \frac{\psi_g(t)}{\xi(t)} - 1 \right|^2 dt < \infty$, in the other case everything is shown by (a) because a sequence of real numbers with a lower bound without any finite accumulation points diverges to $+\infty$ (theorem of Bolzano-Weierstrass). The risk $(\sup_{f\in\mathcal{F}} E_{f,g} \|\hat{f}_n - f\|^2_{L_2(\mathbb{R})})_{n\in\mathbb{N}}$ can be upper bounded according to Lemma 1 by

$$\frac{1}{2\pi} \sup_{f\in\mathcal{F}} \int \left| \frac{L_n(t)}{\xi(t)} \right|^2 \cdot \underbrace{\frac{1 - |\psi_{f*g}(t)|^2}{n}}_{\leq 1/n} dt + \frac{1}{2\pi} \sup_{f\in\mathcal{F}} \int \left| L_n(t) \frac{\psi_f(t)\psi_g(t)}{\xi(t)} - \psi_f(t) \right|^2 dt.$$

Condition (2.6) which can be used in addition now implies that the first part of the sum in the term above converges to 0. For the second part we have found a lower bound in the proof of (a) and we can derive an upper bound as well

$$\frac{1}{2\pi}\sup_{f\in\mathcal{F}}\int|\psi_f(t)|^2\left|L_n(t)\tfrac{\psi_g(t)}{\xi(t)}-1\right|^2dt$$

$$=\frac{1}{2\pi}\sup_{f\in\mathcal{F}}\int|\psi_f(t)|^2\left|L_n(t)\tfrac{\psi_g(t)}{\xi(t)}-L_n(t)+L_n(t)-1\right|^2dt$$

$$=\frac{1}{2\pi}\sup_{f\in\mathcal{F}}\int|\psi_f(t)|^2\Big(\left|L_n(t)\tfrac{\psi_g(t)}{\xi(t)}-L_n(t)\right|^2+2\,Re\Big(\big(L_n(t)\tfrac{\psi_g(t)}{\xi(t)}-L_n(t)\big)\cdot\big(L_n(t)-1\big)\Big)+|L_n(t)-1|^2\Big)dt$$

$$\le\frac{1}{2\pi}\sup_{f\in\mathcal{F}}\int|\psi_f(t)|^2\Big(\left|L_n(t)\tfrac{\psi_g(t)}{\xi(t)}-L_n(t)\right|^2+2\left|L_n(t)\tfrac{\psi_g(t)}{\xi(t)}-L_n(t)\right|\cdot|L_n(t)-1|+|L_n(t)-1|^2\Big)dt$$

(Cauchy-Schwarz inequality)

$$\le\frac{1}{2\pi}\sup_{f\in\mathcal{F}}\Big\{\int|\psi_f(t)|^2\left|L_n(t)\tfrac{\psi_g(t)}{\xi(t)}-L_n(t)\right|^2dt$$
$$+2\left(\int|\psi_f(t)|^2\left|L_n(t)\tfrac{\psi_g(t)}{\xi(t)}-L_n(t)\right|^2dt\right)^{1/2}\left(\int|\psi_f(t)|^2|L_n(t)-1|^2dt\right)^{1/2}$$
$$+\int|\psi_f(t)|^2|L_n(t)-1|^2dt\Big\}$$

$$=\frac{1}{2\pi}\sup_{f\in\mathcal{F}}\left\{\left(\int|\psi_f(t)|^2\left|L_n(t)\tfrac{\psi_g(t)}{\xi(t)}-L_n(t)\right|^2dt\right)^{1/2}+\left(\int|\psi_f(t)|^2|L_n(t)-1|^2dt\right)^{1/2}\right\}^2$$

$$\le\frac{1}{2\pi}\left\{\sup_{f\in\mathcal{F}}\left(\int|\psi_f(t)|^2\left|L_n(t)\tfrac{\psi_g(t)}{\xi(t)}-L_n(t)\right|^2dt\right)^{1/2}+\sup_{f\in\mathcal{F}}\left(\int|\psi_f(t)|^2|L_n(t)-1|^2dt\right)^{1/2}\right\}^2$$

(Because of $|L_n(t)|\le 1$ we get)

$$\le\frac{1}{2\pi}\left\{\sup_{f\in\mathcal{F}}\left(\int|\psi_f(t)|^2\left|\tfrac{\psi_g(t)}{\xi(t)}-1\right|^2dt\right)^{1/2}+\sup_{f\in\mathcal{F}}\left(\int|\psi_f(t)|^2|L_n(t)-1|^2dt\right)^{1/2}\right\}^2$$

Because of (2.4) one can conclude

$$\sup_{f\in\mathcal{F}}\left(\int|\psi_f(t)|^2|L_n(t)-1|^2dt\right)^{1/2}$$
$$=\left(\sup_{f\in\mathcal{F}}\int|\psi_f(t)|^2|L_n(t)-1|^2dt\right)^{1/2}\stackrel{n\to\infty}{\longrightarrow}0.$$

So we have shown the boundedness of the sequence $(\sup_{f\in\mathcal{F}}E_{f,g}\,\|\hat{f}_n-f\|^2_{L_2(\mathbb{R})})_{n\in\mathbb{N}}$ and the fact that this sequence cannot possess any accumulation points which are larger than

$$\sup_{f\in\mathcal{F}}\int|\psi_f(t)|^2\left|\frac{\psi_g(t)}{\xi(t)}-1\right|^2dt.$$

According to (a) the sequence can also possess no accumulation points which are smaller than this term. So there is exactly one accumulation point and so the sequence converges to this term according to results of elementary analysis (Bolzano-Weierstrass). ■

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Alexander Meister
Pfaffenwaldring 57
70569 Stuttgart
Germany
E-Mail: `meistear@mathematik.uni-stuttgart.de`

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