Stuttgarter
Mathematische Berichte

An elementary analytical proof of Blackwell's renewal theorem

Jürgen Dippon, Harro Walk

Preprint 2004/006

Fakultät Mathematik und Physik
Universität Stuttgart

**Stuttgarter
Mathematische Berichte**

An elementary analytical proof of Blackwell's renewal theorem

Jürgen Dippon, Harro Walk

Preprint 2004/006

Fakultät Mathematik und Physik
Universität Stuttgart

Fakultät Mathematik und Physik
Universität Stuttgart
Pfaffenwaldring 57
D-70 569 Stuttgart

E-Mail: preprints@mathematik.uni-stuttgart.de
WWW: http://www.mathematik/uni-stuttgart.de/preprints

ISSN **1613-8309**

An Elementary Analytical Proof of Blackwell's Renewal Theorem [1]

Jürgen Dippon and Harro Walk

Fachbereich Mathematik

Universität Stuttgart, Germany

30th April 2004

Abstract

Blackwell's renewal theorem in probability theory deals with the asymptotic behaviour of an expected number of renewals. An analytical proof is given which combines a selection principle with a uniqueness lemma. The selection argument simplifies Feller's argument by using only Helly's selection theorem. The specialized Beurling or Choquet-Deny uniqueness theorem is proved by standard Fourier analytic tools.

[1] *Mathematics Subject Classification (2000):* Primary 60K05; Secondary 60E10.

Key words and phrases: Blackwell's renewal theorem, Helly's selection theorem, uniqueness theorems of Beurling and Choquet-Deny, Fourier transform

Address for correspondence: Jürgen Dippon, Institut für Stochastik und Anwendungen, Fachbereich Mathematik, Universität Stuttgart, 70550 Stuttgart, Germany. E-mail: dippon@mathematik.uni-stuttgart.de

1 Introduction

In classical renewal theory the partial sum sequence $(X_1 + \ldots + X_n)$ for independent indentically distributed (i.i.d.) nonnegative random variables $X_1, X_2, \ldots$ is called renewal process and is interpreted as the sequence of random renewal epochs (random arrival times of customers) in a technical system (at a sever). The trivial case $P[X_1 = 0] = 1$ is excluded, i.e., $0 < EX_1 \leq \infty$ is assumed. The renewal function V on $\mathbb{R}$ defined by

$$V(a) := \begin{cases} 1 + E \sup\{k \in \mathbb{N} : X_1 + \ldots + X_k \leq a\}, & a \geq 0 \\ 0, & a < 0 \end{cases}$$

satisfies

$$V = \sum_{n=0}^{\infty} F^{n*} < \infty$$

where F^{n*} is the n-fold convolution of the distribution function F of X_1 and $F^{0*} := F_0 := I_{\mathbb{R}_+}$ (I denoting an indicator function). There is of central importance in renewal theory the asymptotic behaviour of $V(a) - V(a-h)$, the mean number of renewals in the time interval $(a-h, a]$, for $a \to \infty$ with arbitrary fixed $h > 0$. The case of an arithmetic distribution of X_1, i.e., concentration of the distribution on $\{0, \lambda, 2\lambda, \ldots\}$ for some $\lambda > 0$, has been treated by Erdös, Feller and Pollard [7]. The case of a nonarithmetic distribution of X_1 has been treated by Blackwell [4]. Especially in the latter case different proofs, also for the extension to i.i.d. real random variables X_n with $EX_1 > 0$ have been given, partially with restriction to the case $EX_1 < \infty$. Among others, Smith [14] used Wiener's theory of Tauberian theorems in summability theory, Feller and Orey [10] used Fourier analysis, Walk [15] used Laplace transforms, Feller [9], Section XI.2, mainly used measure theory together with selection principles, Lindvall [13] used the probabilistic coupling method. We mention the monographs of Feller [8],[9], Alsmeyer [1] and Asmussen [2] with further references.

In this paper we give an elementary analytical proof of Blackwell's renewal theorem in the classical case of nonnegative X_n's. We simplify Feller's [9] reduction of the problem to a uniqueness lemma by use only of Helly's selection theorem. The uniqueness lemma, in different and more general forms, is due to Beurling [3] and Choquet and Deny [6]. We further give a Fourier analytic proof of the uniqueness lemma which in the special case $EX_1 < \infty$ is rather simple. Our proof of the renewal theorem can be extended to the case of real X_n's with $0 < EX_1 \leq \infty$.

The method to combine a selection principle with a uniqueness theorem for proving asymptotic renewal theorems has a predecessor in summability theory (see Zeller and Beekmann [17], 48 IV, with references).

2 Notations

We set

$$F_0(t) := \begin{cases} 1 & t \geq 0 \\ 0, & t < 0 \end{cases}$$

For the probability measure Q belonging to the distribution function F of X_1 we denote the Fourier-Stieltjes transform by $\widehat{Q}$, i.e.,

$$\widehat{Q}(u) := \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{-iux}\, Q(dx), \quad u \in \mathbb{R}.$$

This is essentially the characteristic function of Q. For a distribution function (or difference of distribution functions) H and a function $z : \mathbb{R} \to \mathbb{R}$ bounded on bounded intervals we set

$$(H * z)(x) := \int_{\mathbb{R}} z(x-t)\, H(dt), \quad x \in \mathbb{R}.$$

With numbers $0 \le a < b$, we use the functions $\chi_{a,b}$ with

$$\chi_{a,b}(v) := \begin{cases} \frac{1}{\sqrt{2\pi}}, & |v| \le a \\ \frac{1}{\sqrt{2\pi}} \frac{b-|v|}{b-a}, & a \le |v| \le b \\ 0, & |v| \ge b \end{cases}$$

and its Fourier transforms $\mathcal{F}(\chi_{a,b}) := \widehat{\chi_{a,b}}$ with

$$\widehat{\chi_{a,b}}(t) := \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \chi_{a,b}(v) e^{-itv}\, dv = \frac{1}{\pi(b-a)} \frac{\cos(at) - \cos(bt)}{t^2}, \quad t \in \mathbb{R}.$$

$\widehat{\chi_{0,2}}$ is the Fejér kernel which can also be written in the form

$$\widehat{\chi_{0,2}}(t) = \frac{1}{\pi} \left(\frac{\sin t}{t} \right)^2$$

(compare Wheeden and Zygmund [16], (9.11), and Hewitt and Stromberg [11], pp. 407, 408). We notice

$$\mathcal{F}(\chi_{a,b}(\cdot - c)) = \widehat{\chi_{a,b}} \cdot e^{-ic\cdot}, \quad \widehat{\chi_{0,b}} \ge 0 \quad \text{and} \quad \int_{\mathbb{R}} \widehat{\chi_{0,b}}(t)\, dt = 1.$$

In the case $EX_1 = \infty$ we set $1/EX_1 := 0$.

3 Blackwell's Renewal Theorem

Assume a nonnegative real random variable X_1 with $0 < EX_1 \le \infty$. In other words, its distribution function $F : \mathbb{R} \to [0,1]$ satisfies $F(x) = 0$ for $x < 0$, and $F(0) < 1$. Then the renewal function $V : \mathbb{R} \to \mathbb{R}$ is given by $V = \sum_{n=0}^{\infty} F^{n*}$ where F^{n*} is the n-fold convolution of F and $F^{0*} := F_0$. Assume further that the distribution Q of X_1 is nonarithmetic, i.e., that Q is not concentrated on $\{0, \lambda, 2\lambda, \ldots\}$ for any $\lambda > 0$. This means

$$\widehat{Q}(u) := \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{-iux}\, Q(dx) \ne 1 \quad \text{for all } u \ne 0. \tag{1}$$

Theorem 1 (Blackwell's Renewal Theorem). *For any fixed $h > 0$ the renewal function fulfills*

$$V(a) - V(a-h) \to \frac{h}{EX_1} \quad (a \to \infty). \tag{2}$$

Proof. It is well known that $(F_0 - F) * V = F_0$. Set $z := I_{[0,h)}$ for some fixed $h > 0$, and $w := V * r = V(\cdot) - V(\cdot - h)$. Then

$$(F_0 - F) * w = (F_0 - F) * V * z = z \quad \text{on } \mathbb{R} \tag{3}$$

and thus

$$\underset{x\in\mathbb{R}_+}{\forall} \quad \int_{\mathbb{R}_+} w(x-t)(1-F(t))\,dt = \int_{[0,x]} z(s)\,ds. \tag{4}$$

w is bounded. This can be concluded from $(F_0 - F) * V = 1$ and $F_0 - F \geq (1 - F(h'))I_{[0,h']}$ for h' sufficiently small. Then in all intervals of fixed finite length w can be represented as a difference of nondecreasing uniformly bounded functions. Thus in each fixed interval the functions $w(\cdot + \tau)$, $\tau \in \mathbb{R}$, can be represented as differences of nondecreasing uniformly bounded functions. For any sequence (τ'_k) in $\mathbb{R}$ with $\tau'_k \to \infty$ there is a subsequence (τ_j) of (τ'_k) such that $w(\cdot + \tau_j)$ converges pointwise to a bounded function $g : \mathbb{R} \to \mathbb{R}$, which has at most countably many discontinuity points x with $g(x)$ lying between $\lim_{s\uparrow x} g(s)$ and $\lim_{s\downarrow x} g(s)$. This follows by Helly's selection theorem (see, e.g., Feller [9], Section VIII.6, as a reference) and by Cantor's diagonal method for the at most countably many exception points in Helly's theorem. Then by the dominated convergence theorem we have

$$\underset{x\in\mathbb{R}}{\forall} \quad \int_{\mathbb{R}} w(x+\tau_j - t)\,d(F_0 - F)(t) \to \int_{\mathbb{R}} g(x-t)\,d(F_0 - F)(t) \quad (j \to \infty).$$

On the other hand, by (3),

$$\underset{x\in\mathbb{R}}{\forall} \quad \int_{\mathbb{R}} w(x+\tau - t)\,d(F_0 - F)(t) = z(x+\tau) \to 0 \quad (\tau \to \infty).$$

Thus

$$\forall_{x \in \mathbb{R}} \quad \int_{\mathbb{R}} g(x-t)\, d(F_0 - F)(t) = 0, \quad \text{i.e., } (F_0 - F) * g = 0, \tag{5}$$

which is equivalent to $F * g = g$ on $\mathbb{R}$. Then, by Lemma 1 in the following section, $g = const$ on the set of continuity points of g and therefore, by the above features of g, everywhere. We have to distinguish the cases of finite and infinite EX_1.

First case: $EX_1 = \int_{\mathbb{R}_+} (1 - F(t))\, dt < \infty$. By (4),

$$\int_{\mathbb{R}_+} w(\tau_j - t)(1 - F(t))\, dt = \int_{[0,\tau_j]} z(s)\, ds \to h \quad (j \to \infty).$$

Since $\lim_{j\to\infty} w(\tau_j - t) = const \in \mathbb{R}$ for each $t \in \mathbb{R}$, the dominated convergence theorem ensures that the left hand side integral converges to $const \cdot EX_1$. Thus $const = h/EX_1$.

Second case: $EX_1 = \int_{\mathbb{R}_+} (1 - F(t))\, dt = \infty$. By Fatou's lemma we deduce from (4)

$$\begin{aligned} \int_{\mathbb{R}_+} \varliminf_j w(\tau_j - t)\,(1 - F(t))\, dt &\le \varliminf_j \int_{\mathbb{R}_+} w(\tau_j - t)\,(1 - F(t))\, dt \\ &= \varliminf_j \int_{[0,\tau_j]} z(s)\, ds = 1. \end{aligned}$$

The left hand side above equals $const \cdot \int_{\mathbb{R}_+} (1 - F(t))\, dt$. Thus $const = 0$.

Therefore in each case we have $const = h/EX_1$. Thus for each sequence (τ'_k) in $\mathbb{R}$ with $\tau'_k \to \infty$ there is a subsequence (τ_j) with $w(\tau_j) \to h/EX_1$. Hence $V(x) - V(x-h) = w(x) \to h/EX_1$ as $x \to \infty$. $\square$

Remark. By essentially the same proof, one can show Blackwell's renewal theorem [5] in the extended case of a real random variable X_1 with nonarithmetic distribution and $0 < EX_1 \le \infty$. One notices that $V - V(\cdot - h)$ for fixed $h > 0$ remains bounded (see Feller [9], Section VI.10), further $-\infty < \int_{(-\infty,0)} (F_0(x) - F(x)\, dx \le 0 < \int_{[0,\infty)} (F_0(x) - F(x))\, dx \le \infty$.

4 Auxiliary Results

The following lemma is a specialized version of uniqueness theorems of Beurling [3] [(10) and Proposition on p. 134] and Choquet and Deny [6] [on measures on groups].

Lemma 1. *Suppose* (1). *If a bounded and measurable function* $g : \mathbb{R} \to \mathbb{R}_+$ *fulfills* (5), *then* g *is constant on the set of its continuity points (even on the set of its Lebesgue points).*

We give a Fourier analytic proof of Lemma 1. For the proof of Theorem 1 we use only the easier part concerning the continuity points. As to a measure theoretic proof of the equivalent (!) version with measurability of g replaced by continuity or uniform continuity we refer to Feller [9], p. 382, Corollary.

For the proof of Lemma 1 we use the following in essential known lemma on Fourier transforms.

Lemma 2. *Let* $f : \mathbb{R} \to \mathbb{C}$ *be L(ebesgue)-integrable and absolutely continuous with L-integrability of* $|f'|^2$. *Then its Fourier transform* $\mathcal{F}f := \widehat{f}$ *defined by*

$$\widehat{f}(t) := \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} f(v) e^{-itv}\, dv, \quad t \in \mathbb{R},$$

is L-integrable. More precisely,

$$\int_{\mathbb{R}} |\widehat{f}(t)|\, dt \leq 3 \cdot (2\pi)^{-1/6} \left(\int_{\mathbb{R}} |f(v)|\, dv \right)^{1/3} \left(\int_{\mathbb{R}} |f'(v)|^2\, dv \right)^{1/3}.$$

Proof of Lemma 2. For each $\delta > 0$ we have

$$\begin{aligned}
\int_{\mathbb{R}} |\widehat{f}(t)|\, dt &= \int_{|t| \leq \delta} |\widehat{f}(t)|\, dt + \int_{|t| > \delta} \frac{1}{|t|} |t| |\widehat{f}(t)|\, dt \\
&\leq 2\delta \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} |f(v)|\, dv + \sqrt{\frac{2}{\delta}} \sqrt{\int_{\mathbb{R}} t^2 |\widehat{f}(t)|^2\, dt} \\
&= 2\delta \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} |f(v)|\, dv + \sqrt{\frac{2}{\delta}} \sqrt{\int_{\mathbb{R}} |\widehat{f'}(t)|^2\, dt} \\
&= 2\delta \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} |f(v)|\, dv + \sqrt{\frac{2}{\delta}} \sqrt{\int_{\mathbb{R}} |f'(v)|^2\, dv}
\end{aligned}$$

according to Hewitt and Stromberg [11], (21.59)(b) and (21.52). Minimizing δ yields the assertion. $\square$

The proof of Lemma 1 given below is rather simple in the case $\int_{\mathbb{R}_+} x\,dF(x) < \infty$, i.e., $EX_1 < \infty$. There only the functions $\chi_{0,b}$ (of triangular form) and $\widehat{\chi_{0,b}}$ are used. For the proof in the general case additionally the functions $\chi_{a,b}$ with $a > 0$ sufficiently small (of trapezoidal form) and $\widehat{\chi_{a,b}}$ are used.

Proof of Lemma 1. In the first part we shall show that for each Lipschitz continuous complex-valued function φ with compact $\operatorname{supp}\varphi \not\ni 0$ one has

$$\int_{\mathbb{R}} \widehat{\varphi}(t) g(t)\,dt = 0. \tag{6}$$

$\widehat{\varphi}$ is L-integrable because of Lemma 2. We set

$$\psi := \frac{\varphi}{1-\widehat{Q}}\,.$$

and show that $\widehat{\psi}$ is L-integrable.

In the special case that $(EX_1 =) \int x\,dF(x) < \infty$ and thus $\widehat{Q}$ is continuously differentiable, we obtain L-integrability of $\widehat{\psi}$ by Lemma 2.

In the general case we notice that it suffices to show $(*)$: For each $c \neq 0$ there is an $a = a(c) \in (0, |c|)$ such that for each Lipschitz continuous complex-valued function φ with $\operatorname{supp}\varphi \subset (c-a, c+a)$ the function $\widehat{\psi}$ is L-integrable. For then an application of the Heine-Borel convering theorem and a suitable decomposition of φ with compact $\operatorname{supp}\varphi \not\ni 0$ yield the desired integrability result.

The proof of $(*)$ is motivated by an argument of Korevaar [12] in summability theory. Let $c > 0$ without loss of generality. Let $b \in (0, c)$. With $Q_s(B) := Q(-B)$, $B \in \mathcal{B}$, it holds

$$\int_{\mathbb{R}} |\left(Q_s * \mathcal{F}(\chi_{0,b}(\cdot - c))\right)(x)|\, dx = \int_{\mathbb{R}} |\left(Q_s * \left(\widehat{\chi_{0,b}} \cdot e^{-ic\cdot}\right)\right)(x)|\, dx < \int_{\mathbb{R}} \left(Q_s * \widehat{\chi_{0,b}}\right)(x)\, dx = 1. \tag{7}$$

To show strict inequality, suppose equality, then

$$Q_s * \left(\widehat{\chi_{0,b}} \cdot e^{-ic\cdot}\right) = Q_s * \widehat{\chi_{0,b}} \quad \text{L(ebesgue)-almost everywhere,}$$

then for L-almost all x

$$\widehat{\chi_{0,b}}(x-t)\, e^{-ic(x-t)} = \widehat{\chi_{0,b}}(x-t) \quad \text{for } Q_s\text{-almost all } t,$$

in contradiction to the assumption that Q is nonarithmetic. From (7) we obtain, via L-integrability of $u \mapsto \frac{1}{1+u^2}$ and the dominated convergence theorem, that also

$$\int_{\mathbb{R}} |\left(Q_s * \mathcal{F}(\chi_{a,b}(\cdot - c))\right)(x)|\, dx < 1$$

for $a \in (0,b)$ sufficiently small. Choose such an $a = a(c)$. We use the abbreviation $\chi = \chi_{a,b}(\cdot - c)$ and notice

$$\psi = \frac{\varphi}{1-\widehat{Q}} = \frac{\varphi}{1-\widehat{Q}\chi} = \varphi \sum_{n=0}^{\infty} \left(\widehat{Q}\chi\right)^n .$$

To show L-integrability of $\widehat{\psi}$ in the general case $EX_1 \leq \infty$, set $w := \widehat{\varphi} * \sum_{n=0}^{\infty} (Q_s * \widehat{\chi})^{n*}$. Then

$$\int_{\mathbb{R}} |w(x)|\, dx \leq \sum_{n=0}^{\infty} \int_{\mathbb{R}} |\left(\widehat{\varphi} * (Q_s * \widehat{\chi})^{n*}\right)(x)|\, dx \leq \sum_{n=0}^{\infty} \int_{\mathbb{R}} |\widehat{\varphi}(x)|\, dx \left(\int_{\mathbb{R}} |(Q_s * \widehat{\chi})(x)|\, dx\right)^n < \infty .$$

Thus w is L-integrable and

$$\frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} w(x) e^{iux}\, dx = \varphi(u) \cdot \sum_{n=0}^{\infty} \left(\widehat{Q}(u)\chi(u)\right)^n = \psi(u) \quad \text{for all } u \in \mathbb{R}.$$

This together with L-integrability of ψ yields $\widehat{\psi} = w$.

From the definition of ψ, by L-integrability of $\widehat{\psi}$ we obtain

$$\widehat{\varphi}(t) = \left((F_0 - F) * \widehat{\psi(-\cdot)}\right)(-t),$$

further by Fubini's theorem and (5)

$$\begin{aligned}\int_{\mathbb{R}} \widehat{\varphi}(t) g(t)\, dt &= \int_{\mathbb{R}} \int_{\mathbb{R}} \widehat{\psi}(s+t)\, d(F_0 - F)(s)\, g(t)\, dt \\ &= \int_{\mathbb{R}} \widehat{\psi}(x) \int_{\mathbb{R}} g(x-s)\, d(F_0 - F)(s)\, dx = 0.\end{aligned}$$

In the second part, we shall show that (6) also holds for each Lipschitz continuous complex-valued φ with compact $\operatorname{supp} \varphi \subset [-D, D]$ for some $D \in (0, \infty)$ and $\varphi(0) = 0$. Let L be a Lipschitz constant of φ. Then, if $\varepsilon > 0$ is sufficiently small, φ can be decomposed into a sum $\varphi_\varepsilon + \rho_\varepsilon$ of two Lipschitz continuous functions satisfying

$$\varphi_\varepsilon(v) = \begin{cases} \varphi(v), & \text{if } -\frac{\varepsilon}{2} \le v \le \frac{\varepsilon}{2} \\ \varphi(\varepsilon - v), & \text{if } \frac{\varepsilon}{2} < v < \varepsilon \\ \varphi(-\varepsilon - v), & \text{if } -\varepsilon < v < -\frac{\varepsilon}{2} \\ 0, & \text{if } |v| \ge \varepsilon \end{cases} \qquad \text{and} \qquad \rho_\varepsilon(v) = \begin{cases} 0, & \text{if } |v| \le \frac{\varepsilon}{2} \\ 0, & \text{if } |v| \ge D. \end{cases}$$

By the first part

$$\int_{\mathbb{R}} \widehat{\rho_\varepsilon}(t) g(t)\, dt = 0,$$

and by Lemma 2

$$\left|\int_{\mathbb{R}} \widehat{\varphi_\varepsilon}(t) g(t)\, dt\right| \le 3 \cdot (2\pi)^{-1/6} \sup_{t \in \mathbb{R}} |g(t)| \left(\int_{\mathbb{R}} |\varphi_\varepsilon(v)|\, dv\right)^{1/3} \left(2\varepsilon L^2\right)^{1/3} \to 0 \quad (\varepsilon \to 0).$$

This yields (6).

In the third part we shall prove the assertion. Choose any continuity or more generally Lebesgue points t^*, t^{**} of g. For $h > 0$ let $\varphi(h, \cdot) := \chi_{0,h} e^{it^{**}\cdot} - \chi_{0,h} e^{it^{*}\cdot}$. $\varphi(h, \cdot)$ is Lipschitz continuous with compact support and $\varphi(h, 0) = 0$. By the second part we have

$$0 = \int_{\mathbb{R}} \widehat{\varphi(h,\cdot)}(t) g(t)\, dt = \int_{\mathbb{R}} \widehat{\chi_{0,h}}(t - t^{**}) g(t)\, dt - \int_{\mathbb{R}} \widehat{\chi_{0,h}}(t - t^{*}) g(t)\, dt$$
$$\to g(t^{**}) - g(t^{*}) \quad (h \to 0).$$

As to the limit relation, which is elementary for continuity points t^*, t^{**}, see Wheeden and Zygmund [16], Ch. 9 with (9.9), (9.11), (9.13) and Exercise 12 for $p = \infty$. Thus $g(t^*) = g(t^{**})$. This shows that g is constant on its continuity set, even on its Lebesgue set. □

Remark. In the third step of the proof of Lemma 1, instead of $\chi_{0,h}$ one can use $\eta_{0,h}$ $(h > 0)$ with

$$\eta_{0,h}(v) = \frac{1}{\sqrt{2\pi}}\, e^{-\frac{v^2}{2h^2}}, \quad v \in \mathbb{R}.$$

In this case, the second step deals with a two times differentiable compex-valued function φ with $|\varphi''| \le L^* < \infty$ and $\varphi(0) = 0$, to be decomposed into a sum $\varphi_\varepsilon + \rho_\varepsilon$ such that $|\varphi_\varepsilon''| \le L^*$, $\varphi_\varepsilon(v) = 0$ for $\varepsilon \le |v| \le \frac{1}{\varepsilon}$, $\rho_\varepsilon(v) = 0$ for $|v| \le c\varepsilon$ and for $|v| \ge \frac{1}{c\varepsilon}$, with suitable $c \in (0, 1)$ depending on φ, and $\varepsilon > 0$ sufficiently small. Then one obtains $|\int_{\mathbb{R}} \widehat{\varphi_\varepsilon}(t) g(t)\, dt| \to 0$ $(\varepsilon \to 0)$ by $\widehat{\varphi_\varepsilon''}(t) = -t^2 \varphi_\varepsilon(t)$, $t \in \mathbb{R}$ (see Hewitt and Stromberg [11], (21.61)), without use of Lemma 2.

References

[1] G. Alsmeyer. *Erneuerungstheorie*, Teubner, 1991.

[2] S. Asmussen. *Applied Probability and Queues*, 2nd ed., Springer, 2003.

[3] A. Beurling. Un théorème sur les fonctions bornées et uniformément continues sur l'axe réel. *Acta Math.*, 77:127–136, 1945.

[4] D. Blackwell. A renewal theorem. *Duke Math. J.*, 15:145-150, 1948.

[5] D. Blackwell. Extension of a renewal theorem. *Pacific J. Math.*, 3:315-320, 1953.

[6] G. Choquet, J. Deny. Sur l'equation de convolution $\mu = \mu * \sigma$. *C. R. Acad. Sci. Paris*, 250:799–801, 1960.

[7] P. Erdös, W. Feller, H. Pollard. A theorem on power series. *Bull. Amer. Math. Soc.*, 55:201-204, 1949.

[8] W. Feller. *An Introduction to Probability Theory and Its Applications*, volume I, 3rd ed., Wiley, 1968.

[9] W. Feller. *An Introduction to Probability Theory and Its Applications*, volume II, 2nd ed., Wiley, 1971.

[10] W. Feller, S. Orey. A renewal theorem. *J. Math. Mech.*, 10:619-624, 1961.

[11] E. Hewitt, K. Stromberg. *Real and Abstract Analysis.* Springer, 1965.

[12] J. Korevaar. Distribution proof of Wiener's Tauberian theorem. *Proc. AMS*, 16:353–355, 1965.

[13] T. Lindvall. A probabilistic proof of Blackwell's renewal theorem. *Ann. Probab.*, 5:482-485, 1977.

[14] W.L. Smith. Asymptotic renewal theorems. *Proc. Roy. Soc. Edinb.*, A 64:9-48, 1954.

[15] H. Walk. Inverspositive Operatoren und Taubersätze in der Erneuerungstheorie. *Monatsh. Math.*, 79:333-346, 1975.

[16] R.L. Wheeden, A. Zygmund. *Measure and Integral*. Marcel Dekker, 1977.

[17] K. Zeller, W. Beekmann. *Theorie der Limitierungsverfahren*. Springer, 1970.

J¨urgen Dippon

Pfaffenwaldring 57

70569 Stuttgart

Germany

E-Mail: `dippon@mathematik.uni-stuttgart.de`

WWW: `http://www.isa.uni-stuttgart.de/LstStoch/Dippon`

Harro Walk

Pfaffenwaldring 57

70569 Stuttgart

Germany

E-Mail: `walk@mathematik.uni-stuttgart.de`

WWW: `http://www.isa.uni-stuttgart.de/LstStoch/Walk`

Erschienene Preprints ab Nummer 2004/001

Komplette Liste: `http://www.mathematik.uni-stuttgart.de/preprints`

2004/001 *Walk, H.*: Strong Laws of Large Numbers by Elementary Tauberian Arguments.

2004/002 *Hesse, C.H., Meister, A.*: Optimal Iterative Density Deconvolution: Upper and Lower Bounds.

2004/003 *Meister, A.*: On the effect of misspecifying the error density in a deconvolution problem.

2004/004 *Meister, A.*: Deconvolution Density Estimation with a Testing Procedure for the Error Distribution.

2004/005 *Efendiev, M.A., Wendland, W.L.*: On the degree of quasiruled Fredholm maps and nonlinear Riemann-Hilbert problems.

2004/006 *Dippon, J., Walk, H.*: An elementary analytical proof of Blackwell's renewal theorem.