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SIMPLIFIED ANALYTICAL PROOF OF BLACKWELL'S RENEWAL THEOREM [1]

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Abstract

Blackwell's renewal theorem in probability theory deals with the asymptotic behavior of an expected number of renewals. A proof is given which combines the measure theoretic and the Fourier analytic access with considerably simpler single steps.

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Key words and phrases: Blackwell's renewal theorem, Arzelà-Ascoli theorem, Fourier transform

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1 Introduction

In classical renewal theory the partial sum sequence $(X_1 + \ldots + X_n)$ for independent indentically distributed (i.i.d.) nonnegative random variables $X_1, X_2, \ldots$ is called renewal process and is interpreted as the sequence of random renewal epochs (random arrival times of customers) in a technical system (at a server). The trivial case $P[X_1 = 0] = 1$ is excluded, i.e., $0 < EX_1 \leq \infty$ is assumed. The renewal function V on $\mathbb{R}$ defined by

$$V(a) := \begin{cases} 1 + E\sup\{k \in \mathbb{N} : X_1 + \ldots + X_k \leq a\}, & a \geq 0 \\ 0, & a < 0 \end{cases}$$

satisfies

$$V = \sum_{n=0}^{\infty} F^{n*} < \infty$$

where F^{n*} is the n-fold convolution of the distribution function F of X_1, and $F^{0*} := F_0 := I_{\mathbb{R}_+}$ (I denoting an indicator function). There is of central importance in renewal theory the asymptotic behaviour of $V(a) - V(a-h)$, the mean number of renewals in the time interval $(a-h, a]$, for $a \to \infty$ with arbitrary fixed $h > 0$. The case of an arithmetic distribution of X_1, i.e., concentration of the distribution on $\{0, \lambda, 2\lambda, \ldots\}$ for some $\lambda > 0$, has been treated by Erdös, Feller and Pollard (1949), see also Feller (1968), Ch. XIII. The case of a nonarithmetic distribution of X_1 has been treated by Blackwell (1948, 1953). Especially in the this case different proofs, also for the extension to i.i.d. real random variables X_n with $EX_1 > 0$ have been given, partially with restriction to the case $EX_1 < \infty$. Among others, Smith (1954) used Wiener's theory of Tauberian theorems in summability theory, Feller and Orey (1961) used Fourier analysis, Walk (1975) used Laplace transforms. Feller (1971), in Section XI.2, mainly used measure theory together with selection principles, Lindvall (1977) used the probabilistic coupling method. Besides the monographs of Feller (1968, 1971), we mention Breiman (1968), Woodroofe (1982), Alsmeyer (1991) and Asmussen (2003) with further references.

In this paper we combine the measure theoretic and the Fourier analytic access and considerably simplify the single steps. We use the selection principle only once (Arzelà-Ascoli theorem), avoid Feller's (1971) intermediate argument leading from Blackwell's renewal theorem to the key renewal theorem, and use (and prove) only a weakened version of a uniqueness theorem of Beurling (1945) and Choquet and Deny (1960). The Fourier analytic part consists, according to Kac (1965), of "a few lines". We first show Blackwell's renewal theorem for the case $EX_1^3 < \infty$ and then obtain asymptotic denseness of the support of the renewal measure and Blackwell's renwal theorem in the general case $0 < EX_1 \leq \infty$.

2 Blackwell's Renewal Theorem

For a distribution function (or difference of distribution functions) H and a measurable function $z : \mathbb{R} \to \mathbb{R}$ bounded on bounded intervals we shall use the notation

$$(H * z)(x) := \int_{\mathbb{R}} z(x-t)\, H(dt), \quad x \in \mathbb{R}.$$

Assume a nonnegative real random variable X_1 with $0 < EX_1 \leq \infty$. In other words, its distribution function $F : \mathbb{R} \to [0,1]$ satisfies $F(x) = 0$ for $x < 0$ and $F(0) < 1$. Then the renewal function $V : \mathbb{R} \to \mathbb{R}$ is given by $V = \sum_{n=0}^{\infty} F^{n*}$ where F^{n*} is the n-fold convolution of F ($F^{n*} = F^{(n-1)*} * F$), and $F^{0*} := F_0$. Assume further that the distribution Q of X_1 is nonarithmetic, i.e., that Q is not concentrated on $\{0, \lambda, 2\lambda, \ldots\}$ for any $\lambda > 0$. This means, for its Fourier-Stieltjes transform,

$$\widehat{Q}(u) := \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{-iux}\, Q(dx) \neq 1 \quad \text{for all } u \neq 0. \tag{1}$$

Theorem 1 (Blackwell's Renewal Theorem) *For any fixed $h > 0$ the renewal function fulfills*

(2) $$V(a) - V(a-h) \to \frac{h}{EX_1} \quad (a \to \infty),$$

where $1/EX_1 := 0$ in the case $EX_1 = \infty$.

Proof of Theorem 1 in the special case $\mathbf{EX_1^3} < \infty$.

1st step. Let $h = 1$ without loss of generality. For fixed $c \in (0,1)$ set

$$z(t) := \begin{cases} 0, & t \le -c \\ \frac{t+c}{c}, & -c \le t \le 0 \\ 1, & 0 \le t \le 1 \\ \frac{-t+1+c}{c}, & 1 \le t \le 1+c \\ 0, & t \ge 1+c \end{cases}$$

and $w := V * z \ge V * I_{[0,1)} = V - V(\cdot - 1)$. Then

(3) $$(F_0 - F) * w = (F_0 - F) * V * z = z \text{ on } \mathbb{R}$$

and thus

(4) $$\underset{x \in \mathbb{R}_+}{\forall} \quad \int_{\mathbb{R}_+} w(x-t)(1-F(t))\,dt = \int_{[0,x]} z(s)\,ds.$$

For each $h' > 0$ the function $V - V(\cdot - h')$ is bounded because of $V * (F_0 - F) = I_{\mathbb{R}_+}$ and $F_0 - F \ge (1 - F(h''))I_{[0,h'')}$ with $1 - F(h'') > 0$ for $h'' > 0$ sufficiently small. Therefore and because z is Lipschitz continuous with bounded support, one obtains boundedness of w and

$$|w(x') - w(x)| \le \int |z(x'-t) - z(x-t)|\,V(dt) \le c^*|x'-x|$$

for some constant $c^* < \infty$ and all $x, x' \in \mathbb{R}$ with $|x - x'| \leq 1$, thus Lipschitz continuity of w.

Let $(\tau_{n'})$ be an arbitrary sequence in $\mathbb{R}$ with $\tau_{n'} \to \infty$ such that

$$w(\tau_{n'}) \to \limsup_{s\to\infty} w(s) =: \alpha.$$

Then the sequence $w(\cdot + \tau_{n'})$ is equibounded and equicontinuous, thus by the Arzelà-Ascoli theorem (see, e.g., Feller (1971), p. 270, or Yosida (1968), Section III.3, as references) a subsequence (τ_n) and a bounded continuous function g exist such that $w(\cdot + \tau_n) \to g$ uniformly on bounded intervals, where $0 \leq g \leq g(0)$. g is even Lipschitz continuous, by Lipschitz continuity of w. By the dominated convergence theorem one has

$$\underset{x\in\mathbb{R}}{\forall} \quad \int_{\mathbb{R}} w(x + \tau_n - t)\, d(F_0 - F)(t) \to \int_{\mathbb{R}} g(x - t)\, d(F_0 - F)(t) \quad (n \to \infty).$$

On the other hand, by (3),

$$\underset{x\in\mathbb{R}}{\forall} \quad \int_{\mathbb{R}} w(x + \tau - t)\, d(F_0 - F)(t) = z(x + \tau) \to 0 \quad (\tau \to \infty).$$

Thus

$$\underset{x\in\mathbb{R}}{\forall} \quad \int_{\mathbb{R}} g(x - t)\, d(F_0 - F)(t) = 0, \quad \text{i.e.,} \quad (F_0 - F) * g = 0, \tag{5}$$

which is equivalent to $F * g = g$ on $\mathbb{R}$.

2nd step. It will be shown that (5) has no other solution than a constant (special version of a uniqueness theorem of Beurling (1945) and Choquet and Deny (1960)), thus

$$g(x) = g(0) = \alpha, \; x \in \mathbb{R}. \tag{6}$$

Because of $\int_{\mathbb{R}_+} (1 - F(t))\, dt = EX_1 < \infty$, from (5) one obtains

$$\underset{x\in\mathbb{R}}{\forall} \quad \int_{\mathbb{R}} g(x - t)(F_0(t) - F(t))\, dt = const < \infty. \tag{7}$$

It remains to show uniqueness of the solution $g(x) = EX_1/\,const$ of (7). It suffices to show that for a given integrable function $p : \mathbb{R} \to \mathbb{R}$ with Fourier transform $\widehat{p}(u) := \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{-iut} p(t)\,dt \neq 0$ for all $u \in \mathbb{R}$ and additionally $\int t^2 |p(t)|\,dt < \infty$, the equation

$$\forall_{x\in\mathbb{R}} \quad \int_{\mathbb{R}} m(x-t)p(t)\,dt = 0$$

for continuous bounded real-valued m has only the solution $m(x) \equiv 0$. The function $p = F_0 - F$ satisfies both conditions, because Q is non-arithmetic and thus, by partial integration, $\int e^{-iut} p(t)\,dt = (1 - \widehat{Q}(u))/(iu) \neq 0$ and because $EX_1^3 < \infty$. A simple Fourier analytic argument from Tauberian theory, due to Kac (1965), is used. Let $\varphi \in C^2(\mathbb{R})$, not vanishing identically, with compact support, and let $\psi := \varphi/\widehat{p}$. By the second condition on p, one has $\widehat{p} \in C^2(\mathbb{R})$, thus $\psi \in C^2(\mathbb{R})$ with compact support. Then $\widehat{\varphi}$ and $\widehat{\psi}$ are integrable (see, e.g., Hewitt and Stromberg (1965), Exercise (21.61)). One has (with classic convolution $\star$ for functions)

$$\widehat{\varphi}(t) = (p \star \widehat{\psi(-\cdot)})(-t), \quad t \in \mathbb{R},$$

and

$$\begin{aligned} \int_{\mathbb{R}} \widehat{\varphi}(t)m(t)\,dt &= \int_{\mathbb{R}} \int_{\mathbb{R}} \widehat{\psi}(t+s)p(s)\,ds\,m(t)\,dt \\ &= \int_{\mathbb{R}} \widehat{\psi}(x) \int_{\mathbb{R}} m(x-s)p(s)\,ds\,dx = 0. \end{aligned}$$

Replacing φ by $\varphi(\cdot - \alpha)$, $\alpha \in \mathbb{R}$, one obtains

$$\forall_{\alpha\in\mathbb{R}} \quad \int m(t)\widehat{\varphi}(t)e^{-i\alpha t}\,dt = 0,$$

thus, by uniqueness of the Fourier transform,

$$\forall_{t\in\mathbb{R}} \quad m(t)\widehat{\varphi}(t) = 0.$$

Because $\widehat{\varphi}$ is an entire function not vanishing identically and thus with at most a countable number of zeros, the continuous function m vanishes identically.

3rd step. Using (4) and $w(\cdot + \tau_n) \to g$ (pointwise) one obtains, by Fatou's lemma,

$$\begin{aligned}\int_{\mathbb{R}_+} g(x-t)(1-F(t))dt &\leq \lim_{n\to\infty} \int_{\mathbb{R}_+} w(x+\tau_n-t)(1-F(t))\,dt \\ &= \int_{\mathbb{R}_+} z(s)\,ds = 1+c, \quad x \in \mathbb{R}.\end{aligned}$$

Then, by (6), or only by

$$g(x) \to g(0) = \alpha \quad (x \to -\infty) \tag{8}$$

and thus $g(x-t) \to \alpha$ $(x \to -\infty)$ for each $t \in \mathbb{R}$ together with Fatou's lemma once more, one has

$$\alpha \int_{\mathbb{R}_+} (1-F(t))\,dt \left(\leq \liminf_{x\to-\infty} \int g(x-t)(1-F(t))\,dt \right) \leq 1+c,$$

i.e.,

$$\alpha \leq \frac{1+c}{EX_1}.$$

Thus, for $c \to 0$,

$$\limsup_{a\to\infty}(V(a)-V(a-1)) \leq \frac{1}{EX_1.}$$

4th step. Analogously to steps 1–3, for the case $EX_1 < \infty$, instead of the majorant z of $I_{[0,1)}$ using a corresponding minorant, one obtains

$$\liminf_{a\to\infty}(V(a)-V(a-1)) \geq \frac{1}{EX_1}.$$

Thus the assertion $V(a)-V(a-1) \to 1/EX_1$ is obtained. □

Proof of Theorem 1 in the general case $\mathbf{EX_1} \leq \infty$. The proof differs from the proof in the special case only in the 2nd step and in use only of (8) instead of (6) in the 3rd step.

2nd step. The aim is to show (8). (5) implies $F^{n*} * g = g$ $(n \in \mathbb{N})$ and

$$g = \left(\sum_{n=1}^{\infty} \frac{F^{n*}}{2^n} \right) * g.$$

Noticing $\max_{x\in\mathbb{R}} g(x) = g(0)$ and continuity of g, one now obtains

$$(9) \qquad g(-x) = g(0) \quad \text{for all } x \in \operatorname{supp} \sum_{n=1}^{\infty} \frac{Q^{n*}}{2^n} = \operatorname{supp} \sum_{n=1}^{\infty} Q^{n*}.$$

Choose a continuous strictly increasing function $q : [0,1] \to \mathbb{R}_+$ such that $q(0) = 0,\ q(1) \leq 1$ and $\int_{\mathbb{R}_+} t^2 q(1 - F(t))\, dt < \infty$ and define a distribution function $\widetilde{F} : \mathbb{R} \to \mathbb{R}$ by $\widetilde{F}(t) = 0$ for $t < 0$ and $q(1 - F(t)) = 1 - \widetilde{F}(t)$ for $t \geq 0$. Then the probability distributions Q and $\widetilde{Q}$ (the latter belonging to $\widetilde{F}$) are equivalent, where $\int_{\mathbb{R}_+} t^3\, d\widetilde{F}(t) < \infty$. Further the renewal measures $\sum_{n=0}^{\infty} Q^{n*}$ and $\sum_{n=0}^{\infty} \widetilde{Q}^{n*}$ (the latter with renewal function $\widetilde{V}$) are equivalent and have the same support. By Theorem 1 in the special case $EX_1^3 < \infty$, for each $h' > 0$ one has

$$\widetilde{V}(a) - \widetilde{V}(a - h') \to \frac{h'}{\int_{\mathbb{R}_+} t\, d\widetilde{F}(t)} > 0 \quad (a \to \infty).$$

Thus $\operatorname{supp} \sum_{n=1}^{\infty} Q^{n*} = \operatorname{supp} \sum_{n=1}^{\infty} \widetilde{Q}^{n*}$ is asymptotically dense, i.e.,

$$\operatorname{dist}\left(x, \operatorname{supp} \sum_{n=1}^{\infty} Q^{n*}\right) \to 0 \quad (x \to \infty).$$

This together with (9) and uniform continuity of g yields (8). $\square$

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