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Abstract

We describe and construct here pseudo-Hermitian structures θ without torsion (i.e. with transversal symmetry) whose Webster-Ricci curvature tensor is a constant multiple of the exterior differential $d\theta$. We call these structures pseudo-Hermitian Einstein. We show that any pseudo-Hermitian Einstein structure can locally be derived from a Kähler-Einstein manifold.

1 Introduction

CR-geometry is a $|2|$-graded parabolic geometry on a smooth manifold M^n. Its underlying Weyl-structures are the pseudo-Hermitian forms θ. CR-geometry is closely related to conformal geometry via the Fefferman construction. For conformal structures, there is the notion of being conformally Einstein, that means there is a Riemannian metric in the conformal class which is Einstein. In terms of tractors the conformal Einstein condition can be expressed through the existence of a parallel standard tractor (cf. [Gov04], [Lei05]). The concept of parallel standard tractors works for CR-geometry as well. One can define in this case that a pseudo-Hermitian structure with parallel standard CR-tractor, whose first 'slot' is a constant real function, is Einstein.

However, we do not use here tractor calculus to define the Einstein condition for a pseudo-Hermitian structure. Instead, we say a pseudo-Hermitian structure θ is Einstein if and only if its torsion vanishes and the Webster-Ricci curvature is a constant multiple of the exterior differential $d\theta$. The two definitions of pseudo-Hermitian Einstein spaces coincide.

We will proceed as follows. In section 2 we introduce the notions that we use here for pseudo-Hermitian geometry, in particular, Webster curvature. In section 3 we consider pseudo-Hermitian structures with transversal symmetry and define the Einstein condition. In section 4 we compare the pseudo-Hermitian geometry of θ with the Riemannian geometry of the induced metric g_θ. In section 5 we derive the natural Riemannian submersion of a transversally symmetric pseudo-Hermitian space. We will see that the Ricci tensor of the base space of the Riemannian submersion determines the Webster-Ricci tensor of the transversally symmetric pseudo-Hermitian space. Finally, in section 6 we find construction principles for pseudo-Hermitian Einstein spaces taking off with a Kähler-Einstein space. We will see that these construction principles generate locally all pseudo-Hermitian Einstein structures.

2 Pseudo-Hermitian structures

We fix here in brief some notations for pseudo-Hermitian structures. Threreby, we follow mainly the notations of [Bau99]. More material on pseudo-Hermitian geometry can be found e.g. in [Lee86], [Lee88], [CS00] [Cap01] and [CG02].

With a CR-structure on a smooth manifold M^n of odd dimension $n = 2m + 1$ we mean here a pair (H, J), which consists of

1. a contact distribution H in TM of codimension 1 and
2. a complex structure J on H, i.e. $J^2 = -id|_H$, subject to the integrability conditions $[JX, Y] + [X, JY] \in \Gamma(H)$ and

$$J([JX, Y] + [X, JY]) - [JX, JY] + [X, Y] = 0$$

for all $X, Y \in \Gamma(H)$.

The conditions that the distribution H is contact and the complex structure J is integrable ensures that (H, J) determines a $|2|$-graded parabolic geometry on M (cf. e.g. [CS00]). In particular, the (infinitesimal) automorphism group of (M, H, J) is finite dimensional.

A nowhere vanishing 1-form $\theta \in \Omega(M)$ is called a pseudo-Hermitian structure on the CR-manifold (M, H, J) if

$$\theta|_H \equiv 0 .$$

Then we call (M,H,J,θ) a pseudo-Hermitian space. Since the distribution H is contact, the 1-form θ is necessarily a contact form. Such a contact form θ exist on (M,H,J) if and only if M is orientable. Furthermore, two pseudo-Hermitian structures θ and $\tilde{\theta}$ on (M,H,J) differ only by multiplication with a real nowhere vanishing function $f \in C^\infty(M)$:

$$\tilde{\theta} = f \cdot \theta \ .$$

We consider now the exterior differential $d\theta$ of a pseudo-Hermitian structure. This 2-form is non-degenerate on H, i.e.

$$(d\theta)^m|_H \neq 0$$

and the 2-tensor

$$L_\theta(\cdot,\cdot) := d\theta(\cdot, J\cdot)$$

is symmetric and non-degenerate on H. If L_θ is positive definite the pseudo-Hermitian structure θ is called strictly pseudoconvex. In general, the 2-tensor L_θ has signature (p,q) on H. The conditions

$$T \lrcorner\ \theta \equiv 1 \qquad \text{and} \qquad T \lrcorner\ d\theta \equiv 0$$

uniquely determine a vector field T on M. This T is called a Reeb vector field. For convenience, we set $J(T) = 0$.

To a pseudo-Hermitian structure θ on (M,H,J) (with arbitrary signature for L_θ) belongs a canonical covariant derivative

$$\nabla^W : \Gamma(TM) \longrightarrow \Gamma(T^*M \otimes TM) \ ,$$

which is called the Tanaka-Webster connection. It is uniquely determined by the following conditions:

1. The connection ∇^W is metric with respect to the non-degenerate symmetric 2-tensor

   $$g_\theta := L_\theta + \theta \circ \theta$$

   on M, i.e.

   $$\nabla^W g_\theta = 0 \ ,$$

   and

2. its torsion $Tor^W(X,Y) := \nabla^W_X Y - \nabla^W_Y X - [X,Y]$ satisfies

   $$Tor^W(X,Y) = L_\theta(JX,Y)\cdot T \qquad \text{for all } X,Y \in \Gamma(H) \quad \text{and}$$
   $$Tor^W(T,X) = -\tfrac{1}{2}([T,X] + J[T,JX]) \qquad \text{for all } X \in \Gamma(H) \ .$$

In addition, for this connection it holds

$$\nabla^W \theta = 0 \qquad \text{and} \qquad \nabla^W \circ J = J \circ \nabla^W \ .$$

The curvature operator of the connection ∇^W is defined in the usual manner:

$$R^{\nabla^W}(X,Y) = [\nabla^W_X, \nabla^W_Y] - \nabla^W_{[X,Y]} \ .$$

The $(4,0)$-curvature tensor R^W is given for $X,Y,Z,V \in TM$ by

$$R^W(X,Y,Z,V) := g_\theta(R^{\nabla^W}(X,Y)Z,V) \ .$$

This curvature tensor has the symmetry properties

1. $R^W(X,Y,Z,V) = -R^W(Y,X,Z,V) = -R^W(X,Y,V,Z)$,

2. $R^W(X,Y,JZ,V) = -R^W(X,Y,Z,JV)$.

We have not listed here the Bianchi type identities. We just note that the Bianchi identities for R^{∇^W} do not (formally) look like those for the Riemannian curvature tensor. We will come back to this point later.

There is also the notion of Ricci curvature for a pseudo-Hermitian structure. It is called the Webster-Ricci curvature tensor and can be defined as follows. Let

$$(e_\alpha, Je_\alpha)_{\alpha=1,\ldots,m}$$

be a local orthonormal frame of L_θ on H and $\varepsilon_\alpha := g_\theta(e_\alpha, e_\alpha)$. Then it is defined

$$Ric^W(X,Y) := \frac{i}{2}\sum_{\alpha=1}^{m} \varepsilon_\alpha R^W(X,Y,e_\alpha,Je_\alpha)\ .$$

The Webster-Ricci curvature is skew-symmetric with values in the purely imaginary numbers $i\mathbb{R}$. And the Webster scalar curvature is

$$scal^W := \frac{i}{2}\sum_{\alpha=1}^{m} \varepsilon_\alpha Ric^W(e_\alpha, Je_\alpha)\ .$$

The function $scal^W$ on (M,H,J,θ) is real.

3 Transversal symmetry

Let (M,H,J) be a CR-manifold. A vector field $T \neq 0$ is called a transversal symmetry of (H,J) if it is not tangential to the subbundle H (i.e. it is transversal to H) and if the flow of T consists (at least locally for small parameters) of CR-automorphisms, i.e. the distribution H is preserved and $\mathcal{L}_T J = 0$, or equivalently

$$[T,X] + J[T,JX] = 0 \qquad \text{for all } X \in \Gamma(H)\ .$$

Now let θ be a non-degenerate pseudo-Hermitian structure on (M,H,J) and let T be the corresponding Reeb vector field determined by

$$\theta(T) \equiv 1 \qquad \text{and} \qquad T \lrcorner\, d\theta \equiv 0\ .$$

Obviously, the Reeb vector field to θ is a transversal symmetry of (H,J) if and only if the torsion part $Tor^W(T,X)$ of the Tanaka-Webster connection ∇^W vanishes for all vector fields $X \in \Gamma(H)$. Equivalently, it is true to say that T is a transversal symmetry if and only if T is a Killing vector field for the metric g_θ, i.e.

$$\mathcal{L}_T g_\theta = 0\ .$$

This uses the fact that

$$\mathcal{L}_T J = 0 \qquad \text{and} \qquad \mathcal{L}_T \theta = 0$$

for the case when T is a transversal symmetry.

The above observations suggest the following notation. We say that a non-degenerate pseudo-Hermitian structure θ on a CR-manifold (M,H,J) is transversally symmetric if its Reeb vector field T is a transversal symmetry of (H,J). In short, we say θ is a (TSPH)-structure on (M,H,J).

We extend our notations here further and say that θ is a pseudo-Hermitian Einstein structure on (M,H,J) if and only if θ is transversally symmetric and the Webster-Ricci curvature Ric^W is a constant multiple of $d\theta$, i.e.

$$Ric^W = -i\frac{scal^W}{2m}\cdot d\theta \qquad \text{and} \qquad Tor^W(T,X) = 0$$

for all $X \in \Gamma(H)$. In this case (M,H,J,θ) is called a pseudo-Hermitian Einstein space (cf. [Lee88]).

4 Comparision between ∇^W and ∇^{g_θ} and their curvature tensors

We calculate in this section the endomorphism

$$D^\theta := \nabla^W - \nabla^{g_\theta},$$

where ∇^{g_θ} denotes the Levi-Civita connection of g_θ, and derive comparision formulas for the Riemannian and the Webster curvature tensors. We will restrict this discussion to the transversally symmetric case. So let θ be a (TSPH)-structure on (M, H, J). A straightforward calculation shows that the covariant derivative

$$\nabla^W - \frac{1}{2} d\theta \cdot T + \frac{1}{2}(\theta \otimes J + J \otimes \theta)$$

is metric and has no torsion with respect to g_θ. We conclude that it is the Levi-Civita connection of g_θ and we have as comparision tensor

$$D^\theta := \nabla^W - \nabla^{g_\theta} = \frac{1}{2}\left(d\theta \cdot T - (\theta \otimes J + J \otimes \theta)\right) \ .$$

Another straightforward calculation shows that

$$\begin{aligned} R^{\nabla^W}(X,Y)Z &= R^{g_\theta}(X,Y)Z - \frac{1}{2}\nabla^{g_\theta}_Z d\theta(X,Y) \cdot T - \frac{1}{2} d\theta(X,Y) \cdot J(Z) \\ &+ \frac{1}{4} d\theta(Y,Z) \cdot J(X) - \frac{1}{4} d\theta(X,Z) \cdot J(Y) \\ &+ \frac{1}{4}\theta(Z) \cdot \theta(X) \cdot Y - \frac{1}{4}\theta(Z) \cdot \theta(Y) \cdot X \ . \end{aligned}$$

This is the comparision of the curvature tensors. The formula immediately shows that (in the transversally symmetric case!) the Webster curvature R^{∇^W} resp. R^W satisfies the first Bianchi identity of the style of a Riemannian curvature tensor, i.e. it holds

$$R^{\nabla^W}(X,Y)Z + R^{\nabla^W}(Y,Z)X + R^{\nabla^W}(Z,X)Y = 0 \ .$$

This is our main observation for all future considerations here to come.

Lemma 1. *Let θ be a (TSPH)-structure on (M, H, J). Then the Webster curvature tensor R^W satisfies*

$$R^W(X,Y,Z,V) + R^W(Y,Z,X,V) + R^W(Z,X,Y,V) = 0$$

for all $X, Y, Z, V \in TM$. In particular, it holds

$$\begin{aligned} &R^W(X,Y,Z,V) = R^W(Z,V,X,Y) \qquad \textit{and} \\ &R^W(X,JY,JZ,V) = R^W(JX,Y,Z,JV) \ . \end{aligned}$$

Using these derived symmetry properties of the Webster curvature for the particular case of transversal symmetry, we obtain the following comparision between the Riemannian Ricci tensor and the Webster-Ricci tensor. Let

$$(e_\alpha, J\alpha)_{\alpha=1,\dots,m} = (e_i)_{i=1,\dots,2m}$$

denote a local orthonormal frame of H in TM. It is

$$\begin{aligned} Ric^{g_\theta}(X,Y) &= \sum_{i=1}^{2m} \varepsilon_i R^{g_\theta}(X, e_i, e_i, Y) + R^{g_\theta}(X,T,T,Y) \\ &= \sum_{\alpha=1}^{m} \varepsilon_\alpha R^{g_\theta}(X, e_\alpha, e_\alpha, Y) + \sum_{\alpha=1}^{m} R^{g_\theta}(X, Je_\alpha, Je_\alpha, Y) + R^{g_\theta}(X,T,T,Y) \end{aligned}$$

and

$$\begin{aligned} Ric^W(X,Y) &= \frac{i}{2}\sum_\alpha \varepsilon_\alpha R^W(X,Y,e_\alpha,Je_\alpha) \\ &= \frac{i}{2}\sum_\alpha \varepsilon_\alpha R^W(X,e_\alpha,e_\alpha,JY) + \frac{i}{2}\sum_\alpha \varepsilon_\alpha R^W(JY,Je_\alpha,Je_\alpha,X) \\ &= \frac{i}{2}\sum_i \varepsilon_i R^W(X,e_i,e_i,JY) \end{aligned}$$

These formulas combined and the fact that $R^{g_\theta}(X,T)T = -\frac{1}{4}X$ for $X \in H$ gives

$$\begin{aligned} &Ric^{g_\theta}(X,Y) = 2iRic^W(X,JY) - \tfrac{1}{2}g_\theta(X,Y), \\ &Ric^W(T,X) = 0, \qquad Ric^W(T,T) = 0 \end{aligned}$$

and

$$Ric^{g_\theta}(T,X) = 0\ , \qquad Ric^{g_\theta}(T,T) = \frac{m}{2}g_\theta(T,T)\ ,$$

whereby $X, Y \in H$.

5 The natural Riemannian submersion of a (TSPH)-structure

We assume here that θ is a (TSPH)-structure on the CR-manifold (M,H,J) of dimension $n = 2m+1$. This implies that the Reeb vector field T to θ is Killing for the induced metric g_θ. At least locally, we can factorise through the integral curves of T on M and obtain a semi-Riemannian metric h_θ on a factor space, which has dimension $2m$. We describe this process here in more detail. In particular, we calculate the relation for the Ricci curvatures of the induced metric g_θ and the factorised metric h_θ.

Let θ be a (TSPH)-structure on (M,H,J) of signature (p,q). To every point in $p \in M$ exists a neigborhood (e.g. some small ball) $U \subset M$ and a map ϕ_U such that ϕ_U is a diffeomorphism between M and the $\mathbb{R}^n$, and moreover, it holds $d\phi_U(T) = \frac{\partial}{\partial x_1}$, that is the first standard coordinate vector in $\mathbb{R}^n$. This implies that there exists a smooth submersion

$$\pi_U : U \subset M \to N \subset \mathbb{R}^{2m}$$

such that for all $v \in N$ the inverse image $\pi_U^{-1}(v)$ consists of an entire integral curve of T through some point in U. Since T is a Killing vector field, the expression

$$h_\theta(X,Y) := g_\theta(\pi_U^{-1}X, \pi_U^{-1}Y)$$

is uniquely defined for any $X, Y \in TN$ and gives rise to a smooth metric tensor on N of dimension $2m$ of signature (p,q). In particular, the map

$$\pi_U : (U, g_\theta) \to (N, h_\theta)$$

is a smooth Riemannian submersion. The construction is naturally derived from θ only (and some chosen neighborhood U). The distribution H in TU is horizontal for this submersion (i.e. orthogonal to the vertical).

For simplicity, we assume now that

$$\pi : (M, g_\theta) \to (N, h_\theta)$$

is globally a smooth Riemannian submersion, whereby the inverse images are the integral curves of the Reeb vector field T to a (TSPH)-structure θ on M with CR-structure (H,J). Since the complex structure J acts on H and T is an infinitesimal automorphism of J, the complex structure can be uniquley projected to a smooth endomorphism on N, which we also denote by J and which satisfies $J^2 = -id|_N$. Since J is integrable on H, the endomorphism J is integrable on N as well, i.e. J is a complex structure on N. In fact, J is a Kähler structure on (N, h_θ), i.e.

$$\nabla^{h_\theta} J = 0 \ .$$

The latter fact can be seen with the comparision tensor D^θ. It is

$$\begin{aligned}(\nabla^{g_\theta}_X J)(Y) &= \nabla^{g_\theta}_X (J(Y)) - J\nabla^{g_\theta}_X Y \\ &= \nabla^W_X (JY) - J\nabla^W_X Y - \frac{1}{2} d\theta(X, J(Y)) \cdot T \\ &= -\frac{1}{2} g_\theta(X,Y) \cdot T\end{aligned}$$

and

$$Vert_\pi \nabla^{g_\theta}_X (J(Y)) = -\frac{1}{2} g_\theta(Y,X) \cdot T \ .$$

This implies

$$\nabla^{h_\theta} J = 0$$

on N.

Altogether, we know yet that a (TSPH)-space (M, H, J, θ) gives rise (locally) in a natural manner to a $(2m)$-dimensional Kähler space (N, h_θ, J). We use now the well known formulas for the Ricci tensor of a Riemannian submersion to calculate Ric^{h_θ} (cf. [ONe66]). The application of the standard formulas results to

$$\begin{aligned}&Ric^{g_\theta}(T,T) = \tfrac{m}{2} g_\theta(T,T), \qquad Ric^{g_\theta}(T,X) = 0 \qquad \text{and} \\ &Ric^{h_\theta} = Ric^{g_\theta} + \tfrac{1}{2} g_\theta(X,Y) \ .\end{aligned}$$

Now using the above result for the Ricci tensor of g_θ with respect to the Webster-Ricci curvature, we obtain

$$Ric^{h_\theta}(X,Y) = 2iRic^W(X, JY)$$

for all $X, Y \in TN \cong H$. In words, this result basically says that the Ricci-Webster curvature of a (TSPH)-structure is the Ricci curvature of the base of the natural submersion.

6 Description and construction of pseudo-Hermitian Einstein spaces

We give here explicit constructions of pseudo-Hermitian Einstein spaces with arbitrary Webster scalar curvature. On the other side, we show that locally these construction principles generate all pseudo-Hermitian Einstein structures. So we gain a locally complete description.

Let (M, H, J, θ) be a pseudo-Hermitian Einstein space with arbitrary signature (p, q), i.e. it holds

$$Ric^W = -i\frac{scal^W}{2m} \cdot d\theta \qquad \text{and} \qquad Tor^W(T, X) = 0$$

for all $X \in \Gamma(H)$. Moreover, we assume for simplicity that θ generates globally a smooth Riemannian submersion

$$\pi : (M, g_\theta) \to (N, h_\theta) \ .$$

With the relation for the Ricci tensors from the end of the last section we obtain

$$Ric^{h_\theta} = \frac{scal^W}{m} d\theta(\cdot, J\cdot) = \frac{scal^W}{m} h_\theta \ .$$

This shows that the base space of the natural submersion to the (TSPH)-structure θ is a Kähler-Einstein space of scalar curvature

$$scal^{h_\theta} = 2 \cdot scal^W \ .$$

We conclude that a pseudo-Hermitian Einstein space (M, H, J, θ) of dimension $n = 2m + 1$ determines uniquely (at least locally) a Kähler-Einstein manifold (N, h_θ, J) of dimension $2m$ and signature (p, q).

We want to show now that there is a unique way back (up to gauge transformations) to obtain a pseudo-Hermitian Einstein space from any (simply connected) Kähler-Einstein space (with signature (p, q)). To start with, let (N, h, J) be a Kähler-Einstein space with $scal^h > 0$ and let $P(N)$ be the $\mathrm{U}(n)$-reduction of the orthonormal frame bundle to (N, h). Then it is

$$\mathcal{S}(-1) := P(N) \times_{det} S^1$$

the principal S^1-fibre bundle over N, which is associated to the anti-canonical line bundle $\mathcal{O}(-1)$ of the Kähler manifold (N, J). The Levi-Civita connection to h_θ induces a connection form ρ on the anti-canonical S^1-bundle $\mathcal{S}(-1)$ with values in $i\mathbb{R}$. For its curvature we have

$$\Omega^\rho(\pi^{-1}_{\mathcal{S}(-1)} X, \pi^{-1}_{\mathcal{S}(-1)} Y) = iRic^h(X, JY), \qquad X, Y \in TN \ .$$

At first, we see from this formula that the horizontal spaces of $(\mathcal{S}(-1), \rho)$ generate a contact distribution H of codimension 1 in $T\mathcal{S}(-1)$ and the horizontal lift of the complex structure J to H produces a non-degenerate (integrable) CR-structure (H, J) on $\mathcal{S}(-1)$. Secondly, we see that

$$\theta := i \frac{2m}{scal^h} \rho$$

is a pseudo-Hermitian structure on $M := \mathcal{S}(-1)$ furnished with the CR-structure (H, J). The Reeb vector field T on the pseudo-Hermitian space

$$(\mathcal{S}(-1), H, J, \theta)$$

is by construction transversally symmetric. And, since $d\theta = \pi^*_{\mathcal{S}(-1)} h(J\cdot, \cdot)$ on H, the base space of the corresponding submersion is again the Kähler-Einstein space (N, h, J) that we started with. For that reason, we know that the Webster-Ricci curvature to θ must be given by

$$2iRic^W(X, JY) = \pi^*_{\mathcal{S}(-1)} Ric^h(X, Y), \qquad X, Y \in H \ .$$

Since h is Einstein, we can conclude that the pseudo-Hermitian space

$$(\mathcal{S}(-1), H, J, \theta)$$

is Einstein as well with Webster-Ricci curvature

$$Ric^W = -i \frac{scal^h}{4m} \cdot d\theta \ .$$

For the inverse construction on the Kähler-Einstein space (N, h, J), the choice of $\theta = i \frac{2m}{scal^h} \rho$ as pseudo-Hermitian 1-form is not unique. One might replace θ by $\hat{\theta} := \theta + df$ for some smooth function f on $\mathcal{S}(-1)$. We obtain again a pseudo-Hermitian Einstein structure on $\mathcal{S}(-1)$. However, it is straightforward to prove that there is a diffeomorphism (gauge transformation) on $\mathcal{S}(-1)$, which transforms $\theta + df$ to θ, i.e. there is an isomorphism of pseudo-Hermitian structures. Since (locally) $\theta + df$ is the most general choice of a 1-form whose exterior differential projects to $h(J\cdot, \cdot)$ on N, we see that our construction exhaust locally all pseudo-Hermitian Einstein structures with positive Webster scalar curvature.

For the case of negative Webster scalar curvature note that if (M, H, J, θ) with signature (p, q) has positive Webster scalar curvature $scal^W > 0$ then $(M, H, J, -\theta)$ has negative Webster scalar curvature $-scal^W$ and the base space of the natural submersion is $(N, -h_\theta, J)$, which is Kähler-Einstein with reversed signature (q, p). We conclude that any pseudo-Hermitian Einstein structure with $scal^W \neq 0$ can be realised locally on $(\mathcal{S}(-1), H, J)$ either by $\theta = i\frac{2m}{scal^h}\rho$ or $-\theta$.

As we have seen above a Webster-Ricci flat pseudo-Hermitian space (M, H, J, θ) gives rise to a Ricci-flat Kähler space. Again we aim to find an inverse construction. So let (N, h, J) be a simply connected Ricci-flat Kähler space. We set $\omega := h(\cdot, J\cdot)$, This is the Kähler form. Since N is simply connected, there exists a potential 1-form γ on N with $d\gamma = \omega$. The S^1-principal fibre bundle $\mathcal{S}(-1)$ has a Levi-Civita connection form ρ with values in $i\mathbb{R}$ which is flat, i.e. $d\rho = 0$. We set

$$\theta := \rho - \gamma$$

on $\mathcal{S}(-1)$. Obviously, it holds

$$\pi_{\mathcal{S}(-1)*} d\theta = -\omega \ ,$$

i.e. θ is a contact form on $\mathcal{S}(-1)$ and the distribution H in $T\mathcal{S}(-1)$, which is given by $\theta|_H \equiv 0$ is contact as well. We denote the lift of J to the distribution H again by J and find that (H, J) is an integrable CR-structure on $M := \mathcal{S}(-1)$. Moreover, θ is a pseudo-Hermitian structer on $(\mathcal{S}(-1), H, J)$. As the construction of $(\mathcal{S}(-1), H, J, \theta)$ is done, it becomes clear that locally around every point of $(\mathcal{S}(-1), g_\theta)$, the base of the natural Riemannian submersion is a subset of the Ricci-flat space (N, h, J). We conclude that

$$2iRic^W(X, Y) = Ric^h(X, Y) = 0$$

for all $X, Y \in H$, i.e. the pseudo-Hermitian space

$$(\mathcal{S}(-1), H, J, \theta)$$

over a simply connected and Ricci-flat Kähler space (N, h, J), where $\theta = \rho - \gamma$ and $d\gamma = \omega$ is the Kähler form, is Webster-Ricci flat.

In the Webster Ricci-flat construction, the pseudo-Hermitian form θ can be replaced by $\hat{\theta} = \rho - \gamma + df$, where f is some smooth function on $\mathcal{S}(-1)$. This is the most general 1-form on $\mathcal{S}(-1)$ with $\pi_{\mathcal{S}(-1)*} d\hat{\theta} = -\omega$. However, again it is not difficult to see that θ and $\hat{\theta} = \theta + df$ are 'gauge equivalent' on $\mathcal{S}(-1)$, i.e. they are isomorphic as pseudo-Hermitian structures. We conclude that with our construction we found (locally) the most general form of a Webster-Ricci flat pseudo-Hermitian space. We summarise our results.

Theorem 1. *Let (N, h, J) be a Kähler-Einstein space of dimension $2m$ and signature (p, q) with scalar curvature $scal^h$.*

1. *If $scal^h > 0$ then the anti-canonical S^1-principal bundle*

$$\mathcal{S}(-1) = P(N) \times_{det} S^1$$

with induced CR-structure (H, J) and connection 1-form

$$\theta := i\frac{2m}{scal^h}\rho \ ,$$

where ρ is the Levi-Civita connection to h, is a pseudo-Hermitian Einstein space of signature (p, q) with $scal^W = \frac{1}{2}scal^h > 0$.

2. *If $scal^h < 0$ then $\mathcal{S}(-1)$ with induced CR-structure (H, J) and connection 1-form $\theta := -i\frac{2m}{scal^h}\rho$ is a pseudo-Hermitian Einstein space of signature (p, q) with $scal^W = \frac{1}{2}scal^h < 0$.*

3. *If $scal^h = 0$ and N is simply connected with Kähler form $\omega = d\gamma$ then $(\mathcal{S}(-1), H, J)$ with pseudo-Hermitian structure $\theta = \rho - \gamma$ of signature (p, q) is Webster-Ricci flat.*

Locally, any pseudo-Hermitian Einstein space (M, H, J, θ) is isomorphic to one of these three models according to the sign of the Webster scalar curvature $scal^W$.

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