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Abstract

We derive Lieb-Thirring inequalities for the Riesz means of eigenvalues of order $\gamma \geq 3/4$ for a fourth order operator in arbitrary dimensions. We also consider some extensions to polyharmonic operators, and to systems of such operators, in dimensions greater than one. For the critical case $\gamma = 1 - 1/(2l)$ in dimension $d=1$ with $l \geq 2$ we prove the inequality $L^0_{l,\gamma,d} < L_{l,\gamma,d}$, which holds in contrast to current conjectures.

0. Introduction

0.1. **Known facts.** Consider for $l \geq 1$ and $d \in \mathbb{N}$ the polyharmonic operator $(-\Delta)^l + V$ in $L^2(\mathbb{R}^d)$, where V is a real-valued function. For suitable V the negative spectrum of this operator is discrete. The Lieb-Thirring inequalities are estimates on the negative eigenvalues of the form [1]

$$\operatorname{tr}\left((-\Delta)^l + V\right)_-^\gamma \leq L_{l,\gamma,d} \int_{\mathbb{R}^d} V_-^{\gamma+\kappa}(x)\, dx, \quad V \in L^{\gamma+\kappa}(\mathbb{R}^d), \tag{0.1}$$

which holds for certain $\gamma \geq 0$ with a constant $L_{l,\gamma,d}$, depending only on l, d and γ. Here and in the following we use the abbreviations

$$\kappa = \kappa(d,l) := \frac{d}{2l}, \quad \nu = \nu(d,l) := 1 - \frac{d}{2l}.$$

This type of inequalities was introduced by Lieb and Thirring in [15]. They proved that (0.1) holds in the case $l = 1$ for all $\gamma > \max(0, \nu)$ with a finite constant $L_{l,\gamma,d}$. Their argument can easily be extended to all $l \geq 1$. On the other hand it is known that (0.1) fails for $\gamma = 0$ if $d = 2l$ and for $0 \leq \gamma < \nu$ if $d < 2l$. In the critical case $\gamma = 0, d > 2l$ the bound (0.1) exists and is for $l = 1$ known as the Cwikel-Lieb-Rosenblum inequality, see [4, 14, 20] and also [3, 13]. The existence of $L_{l,\gamma,d}$ in the remaining critical case $d < 2l, \gamma = \nu$ was verified by Netrusov and Weidl for *integer* values of l in [21, 19]. Hence, the cases of existence for bounds of type (0.1) with $\gamma \geq 0$ are completely settled for integer l, while for non-integer l only the case $2l > d, \gamma = \nu$ is still open.

For sufficiently regular potentials $V \in L^{\gamma+\kappa}(\mathbb{R}^d)$ the inequalities (0.1) are accompanied by the Weyl type asymptotic formula

$$\begin{aligned} \lim_{\alpha \to +\infty} \frac{1}{\alpha^{\gamma+\kappa}} \operatorname{tr}\left((-\Delta)^l + \alpha V\right)_-^\gamma &= \lim_{\alpha \to +\infty} \frac{1}{\alpha^{\gamma+\kappa}} \iint_{\mathbb{R}^d \times \mathbb{R}^d} \left(|\xi|^{2l} + \alpha V\right)_-^\gamma \frac{dx\, d\xi}{(2\pi)^d} \\ &= L^{cl}_{l,\gamma,d} \int_{\mathbb{R}^d} V_-^{\gamma+\kappa}\, dx\,, \end{aligned} \tag{0.2}$$

where the so-called classical constant $L^{cl}_{l,\gamma,d}$ is defined by

$$L^{cl}_{l,\gamma,d} = \frac{\Gamma(\gamma+1)\Gamma(\kappa+1)}{2^d \pi^{d/2} \Gamma(l\kappa+1)\Gamma(\kappa+\gamma+1)}, \quad \gamma \geq 0\,. \tag{0.3}$$

Formula (0.2) can be closed to *all* potentials $V \in L^{\gamma+\kappa}(\mathbb{R}^d)$ if the bound (0.1) holds.

Furthermore we consider the Lieb-Thirring constant for the ground state, that is the smallest constant $L^0_{l,\gamma,d}$ which fulfils

$$\varkappa_0^\gamma \leq L^0_{l,\gamma,d} \int_{\mathbb{R}^d} V_-^{\gamma+\kappa}\, dx \tag{0.4}$$

[1] Here and below we use the notion $2x_- := |x| - x$ for the negative part of variables, functions, Hermitian matrices or self-adjoint operators.

for all $V \in L^{\gamma+\kappa}(\mathbb{R}^d)$, where $-\varkappa_0$ is the ground state of $(-\Delta)^l + V$. In the case $d < 2l$ with $\gamma = \nu$ the value of $L^0_{l,\nu,d}$ is given by

$$L^0_{l,\nu,d} = \frac{\pi\kappa}{\sin(\pi\kappa)} L^{cl}_{l,0,d} = \frac{1}{\nu} L^{cl}_{l,\nu,d}, \tag{0.5}$$

see [19]. It is interesting to compare the value of the sharp constant $L_{l,\gamma,d}$ in (0.1) with the values of $L^{cl}_{l,\gamma,d}$ and $L^0_{l,\gamma,d}$. In view of (0.2) and (0.4) we immediately obtain that

$$\max(L^{cl}_{l,\gamma,d}, L^0_{l,\gamma,d}) \leq L_{l,\gamma,d} \tag{0.6}$$

for all l, d and γ. One of the sparse results on exact values of $L_{l,\gamma,d}$ is due to Lieb and Thirring. In [15] they obtained for d=l=1, using the Buslaev-Faddeev-Zakharov trace formulae [2, 5], that

$$L_{l,\gamma,d} = L^{cl}_{l,\gamma,d} \tag{0.7}$$

for $\gamma = 3/2 + n$ with $n \in \mathbb{N}_0$. In [1] Aizenman and Lieb found an argument, how to prove (0.7) in $d = 1$ for all $\gamma \geq 3/2$. Applying a "lifting" argument with respect to dimension, Laptev and Weidl finally succeeded in [11] to prove (0.7) for all $d \in \mathbb{N}$ and $\gamma \geq 3/2$ in the case $l = 1$. In fact, their result is even more general, and is obtained for infinite-dimensional systems of Schrödinger operators.

However, in the case $l > 1$ no sharp constants are known, not even in dimension $d = 1$. In the paper [17] an attempt was made to prove, that (0.7) holds for $d = 1, l = 2$ and $\gamma \geq 7/4$. The constant appearing in [17], in the trace formula for the Riesz mean of order $7/4$, is precisely the classical, but whether the equality (0.7) holds true or not in that case is still open.

The only other case where the sharp value of $L_{l,\gamma,d}$ is presently known, is d=l=1 with critical $\gamma = 1/2$, for which in [9] it was proven by Hundertmark, Lieb and Thomas that

$$L_{1,1/2,1} = L^0_{1,1/2,1} = 2\, L^{cl}_{1,1/2,1} = 1/2\,. \tag{0.8}$$

For the remaining cases the values of the Lieb-Thirring constants constitute an interesting open problem.

It shall be mentioned, that at least for the case $l = 1$ there exists a conjecture about the value of $L_{l,\gamma,d}$, which is due to Lieb and Thirring [15]. The conjecture is, that for each dimension d there exists a unique $\gamma_c(d)$ so that

$$L_{1,\gamma,d} = L^{cl}_{1,\gamma,d} \qquad \text{for} \quad \gamma \geq \gamma_c(d) \quad \text{and} \tag{0.9}$$

$$L_{1,\gamma,d} = L^0_{1,\gamma,d} \qquad \text{for} \quad \gamma \leq \gamma_c(d). \tag{0.10}$$

Comparing this with the results above one sees, that (0.9) is proven to hold with $\gamma_c(d) \leq 3/2$ for all $d \in \mathbb{N}$, where (0.10) is still open, but supported by (0.8).

0.2. **Main results of this paper.** In section 1 we follow the idea of [10] and extend the argumentation in dimension one to the case $l > 1$, which leads to (non-sharp) inequalities for this case. See Theorem 1.1 for the special case of the biharmonic operator $\partial^4 + V$, and Theorem 1.6 for the general case. Our results also apply to systems of operators of the above kind, an issue raised in the paper [12], as well as to non-integer l. We also discuss an extension of [21] to the case $l = 2$, see subsection 1.3. In section 2 we prove the inequality

$$L^0_{l,\nu,1} < L_{l,\nu,1}$$

for integer $l \geq 2$, which holds in contrast to equality (0.8). This answers a question posed in section 2.8 of [19] and, in particular, shows that the conjecture (0.10) does not apply to higher order operators. In section 3 we lift the results from section 1 to higher dimensions, see especially Theorem 3.3.

1. LIEB-THIRRING INEQUALITIES FOR RIESZ MEANS OF EIGENVALUES FOR POLYHARMONIC OPERATORS IN DIMENSION ONE

1.1. **Notation and auxiliary material.**

Let $\mathcal{G}$ be a separable Hilbert space with norm $||\cdot||_{\mathcal{G}}$ and scalar product $\langle\cdot,\cdot\rangle_{\mathcal{G}}$. Further, let $\mathbf{0}_{\mathcal{G}}$ respectively $\mathbf{1}_{\mathcal{G}}$ be the zero respectively identity operator on $\mathcal{G}$, and $\mathcal{B}(\mathcal{G})$ be the Banach space of bounded operators on $\mathcal{G}$. The Hilbert space $\mathcal{H} := L^2(\mathbb{R}^d, \mathcal{G})$ is the space of all measurable functions $u : \mathbb{R}^d \to \mathcal{G}$ such that

$$||u||^2_{\mathcal{H}} := \int_{\mathbb{R}^d} ||u(x)||^2_{\mathcal{G}}\, dx < \infty.$$

The scalar product in $\mathcal{H}$ is given by

$$\langle u, v\rangle_{\mathcal{H}} := \int_{\mathbb{R}^d} \langle u(x), v(x)\rangle_{\mathcal{G}}\, dx, \quad \text{for } u, v \in \mathcal{H}.$$

The space $L^2(\mathbb{R}^d, \mathcal{G})$ is naturally isomorphic to $L^2(\mathbb{R}^d)\otimes\mathcal{G}$, and we will make no distinction between them. We shall denote by Φ the Fourier transform unitary on $L^2(\mathbb{R}^d)$. For simplicity of notation, whenever $u \in L^2(\mathbb{R}^d, \mathcal{G})$ we further let $\hat{u} := (\Phi \otimes \mathbf{1}_{\mathcal{G}})\, u$. The Sobolev space $H^l(\mathbb{R}^d, \mathcal{G})$, for $l > 0$, is the subset of $L^2(\mathbb{R}^d, \mathcal{G})$ defined by

$$H^l\left(\mathbb{R}^d, \mathcal{G}\right) := \left\{ u \in L^2(\mathbb{R}^d, \mathcal{G}) : \left(1 + |\xi|^2\right)^{l/2} \hat{u}(\xi) \in L^2\left(\mathbb{R}^d, \mathcal{G}\right) \right\}.$$

The space $H^l(\mathbb{R}^d, \mathcal{G})$, equipped with the scalar product

$$\langle u, v\rangle_{H^l(\mathbb{R}^d,\mathcal{G})} := \int_{\mathbb{R}^d} \left(1 + |\xi|^2\right)^l \langle \hat{u}(\xi), \hat{v}(\xi)\rangle_{\mathcal{G}}\, d\xi,$$

is a Hilbert space. As in the scalar case $\mathcal{G} = \mathbb{C}$ one sees that if $l \in \mathbb{N}$, then

$$H^l\left(\mathbb{R}^d, \mathcal{G}\right) = \left\{ u \in L^2(\mathbb{R}^d, \mathcal{G}) : \partial^\alpha u \in L^2\left(\mathbb{R}^d, \mathcal{G}\right), \quad |\alpha| \leq l \right\}.$$

Obviously, for $l > 0$, the quadratic form

$$h[u, u] := \int_{\mathbb{R}^d} |\xi|^{2l}\, ||\hat{u}(\xi)||^2_{\mathcal{G}}\, d\xi$$

is semibounded from below and closed on the form-domain $H^l(\mathbb{R}^d, \mathcal{G}) \subset L^2(\mathbb{R}^d, \mathcal{G})$. It is associated with the self-adjoint operator $(-\Delta)^l \otimes \mathbf{1}_{\mathcal{G}}$ on $H^{2l}(\mathbb{R}^d, \mathcal{G})$.

Let $V : \mathbb{R}^d \to \mathcal{B}(\mathcal{G})$ be an operator-valued function, for which $V(x) = (V(x))^*$ for a.e. $x \in \mathbb{R}^d$, satisfying:

$$||V(\cdot)||_{\mathcal{B}(\mathcal{G})} \in L^p\left(\mathbb{R}^d\right) \tag{1.1}$$

with some finite p with

$$\begin{array}{ll} p \geq 1 & \text{if } d < 2l, \\ p > 1 & \text{if } d = 2l, \\ p \geq d/2l & \text{if } d > 2l. \end{array}$$

Then the form

$$v[u, u] := \int_{\mathbb{R}^d} \langle Vu, u\rangle_{\mathcal{G}}\, dx$$

is well-defined on $H^l\left(\mathbb{R}^d, \mathcal{G}\right)$ and

$$|v[u,u]| \leq C\left(\int_{\mathbb{R}^d} \|V\|^p_{\mathcal{B}(\mathcal{G})}\,dx\right)^{1/p} \|u\|^2_{H^l(\mathbb{R}^d,\mathcal{G})}. \tag{1.2}$$

This follows from analogs of the standard Sobolev imbedding theorems which hold in the scalar case. For instance, in case $d > 2l$, this follows from Hölder's inequality and the imbedding $H^l\left(\mathbb{R}^d, \mathcal{G}\right) \hookrightarrow L^q\left(\mathbb{R}^d, \mathcal{G}\right)$, $q \leq q^* := \frac{2d}{d-2l}$. Moreover, for all $\epsilon > 0$ there exists a constant $C(\epsilon, V)$ such that

$$|v[u,u]| \leq \epsilon h[u,u] + C(\epsilon, V)\int_{\mathbb{R}^d} \|u\|^2_{\mathcal{G}}\,dx, \quad u \in H^l\left(\mathbb{R}^d, \mathcal{G}\right). \tag{1.3}$$

This is also a version of the corresponding inequality which is well-known in the scalar case, when $\mathcal{G} = \mathbb{C}$. It follows that the form

$$h[u,u] + v[u,u]$$

is semibounded from below and closed on $H^l\left(\mathbb{R}^d, \mathcal{G}\right)$. It induces a self-adjoint operator

$$Q := (-\Delta)^l \otimes \mathbf{1}_{\mathcal{G}} + V \tag{1.4}$$

in $\mathcal{H} = L^2\left(\mathbb{R}^d, \mathcal{G}\right)$.
More precise conditions guaranteeing V to be a weak Hardy weight, stated in terms of capacities, are given in [16].

If V satisfies the condition (1.1) and if $V(x) \in S_\infty(\mathcal{G})$ for a.e. $x \in \mathbb{R}^d$, the negative spectrum of the operator Q is discrete and might accumulate only to 0. In other words, the operator Q_- is compact in $\mathcal{H} = L^2\left(\mathbb{R}^d, \mathcal{G}\right)$. This can be proven as follows. We clearly may assume $V \leq 0$, by the minimax principle, and put $W := \sqrt{-V}$. By the Birman-Schwinger principle, for $\varkappa > 0$, the number $N_-(-\varkappa, Q)$ of eigenvalues of Q less than $-\varkappa$ equals the number $N_+(1, B_W(\varkappa))$ of eigenvalues greater than 1 of the Birman-Schwinger operator

$$B_W(\varkappa) := W\left((-\Delta)^l \otimes \mathbf{1}_{\mathcal{G}} + \varkappa\right)^{-1} W$$

on $L^2\left(\mathbb{R}^d, \mathcal{G}\right)$. One sees that $B_W(\varkappa) = S_W S_W^*$, where

$$S_W := W\left(\Phi^* \otimes \mathbf{1}_{\mathcal{G}}\right)\left(|\xi|^{2l} + \varkappa\right)^{-1/2}.$$

Thus the claim follows by compactness of S_W on $L^2\left(\mathbb{R}^d, \mathcal{G}\right)$.

1.2. **Estimates of Riesz means for the biharmonic operator in d = 1.** In this section we obtain the following:

Theorem 1.1. *Let* $V : \mathbb{R} \to \mathcal{B}(\mathcal{G})$ *be an operator-valued function satisfying* $V(x) = (V(x))^*$ *and* $V(x) \in S_1(\mathcal{G})$ *for a.e.* $x \in \mathbb{R}$ *and such that* $\operatorname{tr} V_-(\cdot) \in L^1(\mathbb{R})$*. Then the following inequality holds true:*

$$\operatorname{tr}\left(\partial^4 \otimes \mathbf{1}_{\mathcal{G}} + V\right)_-^{3/4} \leq \frac{3^{3/4}}{4}\int_{\mathbb{R}} \operatorname{tr} V_-(x)\,dx. \tag{1.5}$$

The original proof of the analog of Theorem 1.1 for Schrödinger operators $-\partial^2 + V$ was given in the paper by Hundertmark, Lieb and Thomas [9]. Here we follow closely the argument in the proof of the same statement given by Hundertmark, Laptev and Weidl in [10].

For the proof of the theorem we need to introduce some auxiliary results on the notion of "majorization". Let A be a compact operator on a separable Hilbert space H. Let us denote

$$\|A\|_n := \sum_{j=1}^{n} \sqrt{\lambda_j(A^*A)}, \tag{1.6}$$

where $(\lambda_j(A^*A))_j$ is the sequence of the eigenvalues of A^*A in non-increasing order according to their multiplicities. Then by Ky-Fan's inequality (see for instance [8]) the functionals $\|\cdot\|_n$ are norms on $S_\infty(H)$, and for any unitary operator $\mathcal{U}$ in H we have

$$\|\mathcal{U}^* A\,\mathcal{U}\|_n = \|A\|_n.$$

We shall need the following definition and lemma, which were stated in [10].

Definition 1.2. *Let A, B be any two compact operators on H. We say that A majorizes B, written $B \prec A$, if*

$$\|B\|_n \le \|A\|_n \quad \textit{for all} \quad n \in \mathbb{N}.$$

Lemma 1.3. *Let A be a non-negative compact operator on H, $\{\mathcal{U}(\omega)\}_{\omega\in\Omega}$ a weakly measurable family of unitary operators on H, and μ a probability measure on Ω. Then the operator*

$$B := \int_\Omega \mathcal{U}^*(\omega) A\, \mathcal{U}(\omega)\, d\mu(\omega)$$

is majorized by the operator A.

Proof. This follows immediately from Ky-Fan's inequality:

$$\|B\|_n \le \int_\Omega \|\mathcal{U}^*(\omega) A \mathcal{U}(\omega)\|_n \, d\mu(\omega) = \mu(\Omega)\|A\|_n = \|A\|_n, \quad n \in \mathbb{N}.$$

□

By the minimax principle we may assume V non-positive and put $W := \sqrt{-V}$. We shall have use of the following family of operators on $\mathcal{H} = L^2(\mathbb{R}, \mathcal{G})$:

$$\mathcal{L}_\epsilon := W\left[\epsilon^3\left(\partial^4 + \epsilon^4\right)^{-1} \otimes \mathbf{1}_{\mathcal{G}}\right] W,$$
$$\tilde{\mathcal{L}}_\epsilon := W\left[a\epsilon\left(-\partial^2 + \epsilon^2 b\right)^{-1} \otimes \mathbf{1}_{\mathcal{G}}\right] W,$$

for $0 < \epsilon < \infty$. Furthermore, let us define $\tilde{\mathcal{L}}_0 := A$, where A is the non-negative compact operator having integral-kernel $A(x,y) := \frac{a}{2\sqrt{b}} W(x)\, W(y)$. The positive constants a and b will be specified later. The following result is almost identical to a lemma proven in [10].

Lemma 1.4. *The operator $\tilde{\mathcal{L}}_\epsilon$ is majorized by $\tilde{\mathcal{L}}_{\epsilon'}$,*

$$\tilde{\mathcal{L}}_\epsilon \prec \tilde{\mathcal{L}}_{\epsilon'}$$

for all $0 \le \epsilon' < \epsilon$.

Proof. We shall use the majorization Lemma 1.3. Introduce a family of probability measures μ_ϵ on $\mathbb{R}$ by $\mu_0 := \delta_0$, the Dirac measure, and

$$\frac{d\mu_\epsilon}{d\xi} = \frac{\epsilon\sqrt{b}}{\pi}\,\frac{1}{\xi^2 + \epsilon^2 b} =: g_\epsilon(\xi), \quad \epsilon > 0,$$

where $d\xi$ denotes the Lebesgue measure. Furthermore, let $\{\mathcal{U}(\xi)\}_{\xi\in\mathbb{R}}$ be the unitary multiplication operators in $L^2(\mathbb{R},\mathcal{G})$ defined by $(\mathcal{U}(\xi)u)(x) = e^{-i\xi x}u(x)$. We then see that

$$\tilde{\mathcal{L}}_\epsilon = \int_{\mathbb{R}} \mathcal{U}^*(\xi) A\,\mathcal{U}(\xi)\, d\mu_\epsilon, \tag{1.7}$$

for any $0 \le \epsilon < \infty$. It follows from Lemma 1.3 and (1.7) that $\tilde{\mathcal{L}}_\epsilon \prec \tilde{\mathcal{L}}_0$. Since the Fourier transform of g_ϵ is given by

$$\Phi g_\epsilon(\xi) = \frac{1}{\sqrt{2\pi}}\, e^{-\epsilon\sqrt{b}\,|\xi|}, \quad \text{for } \epsilon > 0,$$

it follows that for any $0 < \epsilon' < \epsilon$

$$g_\epsilon = g_{\epsilon'} * g_{\epsilon-\epsilon'}. \tag{1.8}$$

Using the relation (1.8) in (1.7) as well as the group property of the unitary operators $\mathcal{U}(\xi)$ it is now seen that

$$\tilde{\mathcal{L}}_\epsilon = \int_{\mathbb{R}} \mathcal{U}^*(\eta)\tilde{\mathcal{L}}_{\epsilon'}\,\mathcal{U}(\eta) g_{\epsilon-\epsilon'}(\eta)\, d\eta \prec \tilde{\mathcal{L}}_{\epsilon'}, \tag{1.9}$$

where the last subordination follows from Lemma 1.3. This completes the proof. □

We are now in the position of proving the above theorem.

Proof of Theorem 1.1. First note that if we put $a := \frac{b+\sqrt{b^2+1}}{2}$, then for any $b > 0$

$$\mathcal{L}_\epsilon \le \tilde{\mathcal{L}}_\epsilon. \tag{1.10}$$

This follows immediately from the computation

$$\begin{aligned}
\langle \mathcal{L}_\epsilon u, u\rangle_{\mathcal{H}} &= \int_{\mathbb{R}} \frac{\epsilon^3}{\xi^4+\epsilon^4} \|(\Phi\otimes \mathbf{1}_{\mathcal{G}})\, W u\,(\xi)\|^2_{\mathcal{G}}\, d\xi \\
&\le \int_{\mathbb{R}} \frac{a\epsilon}{\xi^2+\epsilon^2 b} \|(\Phi\otimes \mathbf{1}_{\mathcal{G}})\, W u\,(\xi)\|^2_{\mathcal{G}}\, d\xi \\
&= \langle \tilde{\mathcal{L}}_\epsilon u, u\rangle_{\mathcal{H}}, \quad u \in L^2(\mathbb{R},\mathcal{G}),
\end{aligned}$$

which holds in view of the scalar inequality

$$\frac{\epsilon^2}{\xi^4+\epsilon^4} \le \frac{a}{\xi^2+\epsilon^2 b}.$$

Here Φ denotes the Fourier transform on $L^2(\mathbb{R})$. For $E > 0$ let us define

$$\mathcal{K}_E := \frac{1}{E^{3/4}}\mathcal{L}_{E^{1/4}} = W\left[(\partial^4 + E)^{-1}\otimes \mathbf{1}_{\mathcal{G}}\right] W. \tag{1.11}$$

Denote by $(-E_j)_j$ the negative eigenvalues of the operator $\partial^4 + V$, and by $(\lambda_j(T))_j$ the eigenvalues of a non-negative compact operator T, enumerated according to their multiplicities in non-decreasing respectively non-increasing order. By the Birman-Schwinger principle

$$1 = \lambda_j(\mathcal{K}_{E_j}). \tag{1.12}$$

Multiplying the identity (1.12) by $E_j^{3/4}$ and summing over j we get from (1.10) and the minimax principle

$$\sum_j E_j^{3/4} = \sum_j \lambda_j(\mathcal{L}_{E_j^{1/4}}) \le \sum_j \lambda_j(\tilde{\mathcal{L}}_{E_j^{1/4}}). \tag{1.13}$$

The interesting point now is that although the trace of $\tilde{\mathcal{L}}_\epsilon$ is independent of ϵ, by Lemma 1.4 the partial traces $\sum_{j\le n}\lambda_j(\tilde{\mathcal{L}}_\epsilon)$ are monotone decreasing in ϵ for any $n\in\mathbb{N}$. It follows that

$$\sum_{j\le n}\lambda_j(\tilde{\mathcal{L}}_{E_j^{1/4}}) \le \sum_{j\le n}\lambda_j(\tilde{\mathcal{L}}_{E_n^{1/4}}) \le \sum_{j\le n}\lambda_j(\tilde{\mathcal{L}}_0), \quad \text{for all } n\in\mathbb{N}. \tag{1.14}$$

The first inequality above follows from this monotonicity by induction over $n\in\mathbb{N}$, the second from the monotonicity directly. Combining (1.13) and (1.14) gives

$$\sum_j E_j^{3/4} \le \operatorname{tr}\tilde{\mathcal{L}}_0,$$

where

$$\operatorname{tr}\tilde{\mathcal{L}}_0 = \frac{a}{2\sqrt{b}}\int \operatorname{tr} V_-(x)\,dx = \frac{b+\sqrt{b^2+1}}{4\sqrt{b}}\int \operatorname{tr} V_-(x)\,dx. \tag{1.15}$$

Minimizing the right hand side of (1.15) with respect to b leads to the choice $b := 1/\sqrt{3}$, and an evaluation of the expression completes the proof. $\square$

Applying the Aizenman-Lieb argument from [1], we obtain the following corollary:

Corollary 1.5. *Let* $V:\mathbb{R}\to\mathcal{B}(\mathcal{G})$ *be an operator-valued function satisfying* $V(x)=(V(x))^*$ *and* $V(x)\in S_1(\mathcal{G})$ *for a.e.* $x\in\mathbb{R}$ *and such that* $\operatorname{tr} V_-(\cdot)\in L^{\gamma+\frac14}(\mathbb{R})$, *for some* $\gamma\ge 3/4$. *Then the following inequality holds true:*

$$\operatorname{tr}\left(\partial^4\otimes\mathbf{1}_{\mathcal{G}}+V\right)_-^{\gamma} \le \frac{4}{3^{1/4}\sqrt{2}}\,L^{cl}_{2,\gamma,1}\int_{\mathbb{R}}\operatorname{tr}\,(V_-(x))^{\gamma+\frac14}\,dx. \tag{1.16}$$

Remark. A numerical calculation yields $\frac{4}{3^{1/4}\sqrt{2}}\approx 2.149$.

Proof. First note that for $\gamma>3/4$

$$\int_0^\infty t^{\gamma-\frac74}\,(t+\lambda)_-^{3/4}\,dt = \lambda_-^{\gamma}\,B\left(\gamma-\frac34,\frac74\right),$$

where $B(x,y)=\frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}$ is the Beta-function. Let E_Q be the spectral measure associated with the self-adjoint operator $Q=\partial^4\otimes\mathbf{1}_{\mathcal{G}}+V$ and denote by $(-\mu_j(x))_j$ the negative eigenvalues of the operator $V(x)$. Since

$$\operatorname{tr} Q_-^{\gamma} = \operatorname{tr}\int_{\mathbb{R}}\lambda_-^{\gamma}\,dE_Q(\lambda)$$

we obtain

$$
\begin{aligned}
&B\left(\gamma-\frac{3}{4},\frac{7}{4}\right)\,\mathrm{tr}\,Q_-^{\gamma}=\mathrm{tr}\,\left\{\int_{\mathbb{R}}dE_Q(\lambda)\int_0^\infty t^{\gamma-\frac{7}{4}}\,(t+\lambda)_-^{3/4}\;dt\right\}\\
&=\mathrm{tr}\,\left\{\int_0^\infty dt\,t^{\gamma-\frac{7}{4}}\int_{\mathbb{R}}dE_Q(\lambda)(t+\lambda)_-^{3/4}\right\}=\int_0^\infty t^{\gamma-\frac{7}{4}}\,\mathrm{tr}\,(t+Q)_-^{3/4}\,dt\\
&\le\frac{3^{3/4}}{4}\int_0^\infty t^{\gamma-\frac{7}{4}}\int_{\mathbb{R}}\mathrm{tr}\,\left(t+V(x)\right)_-\,dx\,dt\\
&=\frac{3^{3/4}}{4}\int_{\mathbb{R}}dx\int_0^\infty t^{\gamma-\frac{7}{4}}\sum_{j=1}^{\infty}\left(t-\mu_j(x)\right)_-\;dt\\
&=\frac{3^{3/4}}{4}\,B\left(\gamma-\frac{3}{4},2\right)\int_{\mathbb{R}}\mathrm{tr}\,\left(V_-(x)\right)^{\gamma+\frac{1}{4}}\;dx.
\end{aligned}
$$

It follows that

$$
\mathrm{tr}\,\left(\partial^4\otimes\mathbf{1}_{\mathcal{G}}+V\right)_-^{\gamma}\le\frac{3^{3/4}}{4}\,\frac{B\left(\gamma-\frac{3}{4},2\right)}{B\left(\gamma-\frac{3}{4},\frac{7}{4}\right)}\int_{\mathbb{R}}\mathrm{tr}\,\left(V_-(x)\right)^{\gamma+\frac{1}{4}}\;dx.
$$

The proof ends by noting that

$$
\begin{aligned}
L_{2,\gamma,1}^{\mathrm{cl}}&=\frac{\Gamma(\gamma+1)\,\Gamma(5/4)}{2\sqrt{\pi}\,\Gamma(3/2)\,\Gamma(\gamma+5/4)}=\frac{\Gamma(7/4)\,\Gamma(5/4)}{2\sqrt{\pi}\,\Gamma(3/2)\,\Gamma(2)}\cdot\frac{\Gamma(\gamma+1)\,\Gamma(2)}{\Gamma(\gamma+5/4)\,\Gamma(7/4)}\\
&=\frac{3\sqrt{2}}{16}\cdot\frac{\Gamma(\gamma+1)\,\Gamma(2)}{\Gamma(\gamma+5/4)\,\Gamma(7/4)}=\frac{3\sqrt{2}}{16}\cdot\frac{B\left(\gamma-\frac{3}{4},2\right)}{B\left(\gamma-\frac{3}{4},\frac{7}{4}\right)}.
\end{aligned}
$$

□

1.3. **Results by the method from Netrusov and Weidl** [21, 19]**.** This method is based on a special Neumann-bracketing technique which together with the Birman-Schwinger principle leads to rather implicit bounds for the Lieb-Thirring constants in the case $2l > d$ with critical $\gamma = 1 - \frac{d}{2l}$. In [6] a detailed analysis of the case $l = 2$, $d = 1$ was done, which yields for the corresponding Lieb-Thirring constant $L_{2,3/4,1}$ the estimate

(1.17) $$L_{2,3/4,1} < 2.129.$$

This estimate is much worse than (1.5). Its value is mainly, that it is also an upper estimate on the Lieb-Thirring constant $L^+_{2,\frac{3}{4},1}$ for the operator $\partial^4 + V$ in $L^2\big((0,\infty)\big)$ with *Neumann* conditions in zero. Notice, that Neumann conditions mean here, that the second and third derivative vanish at zero.

We remark furthermore that the unique negative eigenvalue $-\varkappa_+$ of the Neumann operator $\partial^4 - \delta_0$ in $L^2\big((0,\infty)\big)$, associated with the quadratic form

$$
h^+[u,u] := \|\partial^2 u\|^2 - |u(0)|^2, \quad u \in H^2\big((0,\infty)\big),
$$

fulfils

$$
\varkappa_+^{3/4} = \sqrt{2}.
$$

So for the half space problem the inequality

(1.18) $$\sqrt{2} \le L^+_{2,\frac{3}{4},1} < 2.129.$$

holds.

1.4. **Estimates of Riesz means for polyharmonic operators in d = 1.** It is not difficult to adapt the proof of Theorem 1.1 to polyharmonic operators of the form

$$(-\partial^2)^l \otimes \mathbf{1}_{\mathcal{G}} + V, \quad l > 1. \tag{1.19}$$

We obtain the following theorem:

Theorem 1.6. *Let* $V : \mathbb{R} \to \mathcal{B}(\mathcal{G})$ *be an operator-valued function satisfying* $V(x) = (V(x))^*$ *and* $V(x) \in S_1(\mathcal{G})$ *for a.e.* $x \in \mathbb{R}$ *and such that* $\operatorname{tr} V_-(\cdot) \in L^1(\mathbb{R})$*. Then the Riesz mean for the critical power* $\nu = 1 - 1/2l$ *of the polyharmonic operator* $(-\partial^2)^l + V$, $l > 1$, *satisfies the bound*

$$\operatorname{tr}\left(\left(-\partial^2\right)^l \otimes \mathbf{1}_{\mathcal{G}} + V\right)_-^{1-1/2l} \leq c_l \int \operatorname{tr} V_-(x)\, dx. \tag{1.20}$$

Here the constant c_l *is defined as*

$$c_l := \frac{1}{2l}\zeta_l^{l-1}, \tag{1.21}$$

where ζ_l *is the unique positive root of the equation*

$$(l-1) + lz - z^l = 0. \tag{1.22}$$

Proof. Define the following family of operators on $L^2(\mathbb{R}, \mathcal{G})$:

$$\mathcal{L}_\epsilon := W\left[\epsilon^{2l-1}\left(\left(-\partial^2\right)^l + \epsilon^{2l}\right)^{-1} \otimes \mathbf{1}_{\mathcal{G}}\right] W,$$
$$\tilde{\mathcal{L}}_\epsilon := \tilde{c}_l W\left[\epsilon\left(-\partial^2 + \epsilon^2\right)^{-1} \otimes \mathbf{1}_{\mathcal{G}}\right] W,$$

for $0 < \epsilon < \infty$. Here the constant $\tilde{c}_l$ is defined as

$$\tilde{c}_l := \sup_{x,y>0} \frac{y^{2l-2}\left(x^2+y^2\right)}{x^{2l}+y^{2l}}. \tag{1.23}$$

As before we may assume V non-positive and put $W := \sqrt{-V}$. In view of the scalar inequality

$$\frac{\epsilon^{2l-1}}{\xi^{2l}+\epsilon^{2l}} \leq \tilde{c}_l \frac{\epsilon}{\xi^2+\epsilon^2}, \tag{1.24}$$

we then see that

$$\mathcal{L}_\epsilon \leq \tilde{\mathcal{L}}_\epsilon.$$

We may therefore proceed similarly as in the proof above, to obtain

$$\operatorname{tr}\left(\left(-\partial^2\right)^l \otimes \mathbf{1}_{\mathcal{G}} + V\right)_-^{1-1/2l} \leq \frac{1}{2}\tilde{c}_l \int \operatorname{tr} V_-\, dx.$$

It remains only to prove that

$$\tilde{c}_l = \frac{1}{l}\zeta_l^{l-1}. \tag{1.25}$$

But a glance at the function f_l defined by

$$f_l(x,y) := \frac{y^{2l-2}\left(x^2+y^2\right)}{x^{2l}+y^{2l}}, \quad x, y > 0, \tag{1.26}$$

reveals that it attains constant values on the lines $y = \rho x$, $\rho > 0$. In fact,

$$f_l(x, \rho x) = \frac{\rho^{2l-2} + \rho^{2l}}{1 + \rho^{2l}} =: g(\rho), \quad \rho > 0. \tag{1.27}$$

A simple computation gives that

$$g'(\rho) = \frac{2\rho^{2l-3}}{(1+\rho^{2l})^2}\left((l-1) - \rho^{2l} + l\rho^2\right). \tag{1.28}$$

We see that g attains its maximal value in the critical points ρ_l given as the solution of

$$(l-1) + l\rho^2 - \rho^{2l} = 0.$$

The maximal value of the function g attained at the critical points ρ_l is seen to be

$$g(\rho_l) = \frac{1}{l}\rho_l^{2(l-1)}.$$

The theorem follows by putting $\zeta_l := \rho_l^2$. □

The Aizenman-Lieb [1] argument gives:

Corollary 1.7. *Let* $V : \mathbb{R} \to \mathcal{B}(\mathcal{G})$ *be an operator-valued function satisfying* $V(x) = (V(x))^*$ *and* $V(x) \in S_1(\mathcal{G})$ *for a.e.* $x \in \mathbb{R}$ *and such that* $\operatorname{tr} V_-(\cdot) \in L^{\gamma+\frac{1}{2l}}(\mathbb{R})$, *for some* $\gamma \geq 1 - 1/2l$, $l > 1$. *Then the following inequality holds true:*

$$\operatorname{tr}\left((-\partial^2)^l \otimes \mathbf{1}_{\mathcal{G}} + V\right)_-^{\gamma} \leq \frac{c_l}{L^{cl}_{l,1-1/2l,1}} L^{cl}_{l,\gamma,1} \int_{\mathbb{R}} \operatorname{tr}\left(V_-(x)\right)^{\gamma+\frac{1}{2l}}\, dx. \tag{1.29}$$

Here the constant c_l *is the same as in the above theorem.*

Proof. The proof is almost identical to that for the biharmonic operator. We put $\nu := 1 - 1/2l$ and note that

$$\int_0^\infty t^{\gamma-(1+\nu)}(t+\lambda)_-^{\nu}\, dt = \lambda_-^{\gamma}\, B(\gamma-\nu, 1+\nu).$$

We use this similarly as above to verify

$$\operatorname{tr}\left((-\partial^2)^l \otimes \mathbf{1}_{\mathcal{G}} + V\right)_-^{\gamma} \leq c_l \frac{B(\gamma-\nu, 2)}{B(\gamma-\nu, 1+\nu)} \int_{\mathbb{R}} \operatorname{tr}\left(V_-(x)\right)^{\gamma+\frac{1}{2l}}\, dx.$$

Finally we verify that

$$L^{cl}_{l,\gamma,1} = L^{cl}_{l,\nu,1} \cdot \frac{B(\gamma-\nu, 2)}{B(\gamma-\nu, 1+\nu)}.$$

□

2. Estimates of Lieb-Thirring constants from below in dimension one

In this section we prove the following result, where the emphasis is on the *strict* inequality (2.1).

Theorem 2.1. *For* $l \in \mathbb{N}$ *with* $l \geq 2$ *and* $\nu = 1 - \frac{1}{2l}$ *the inequality*

$$L^0_{l,\nu,1} < L_{l,\nu,1} \tag{2.1}$$

holds true.

We point out the difference to the case $l = 1$, where one has equality in (2.1). This difference originates in the fact, that the eigenfunction corresponding to the ground state of $(-\partial^2)^l - \delta_0$ has no zeros for $l = 1$, whereas it has zeros for all $l \geq 2$. The idea of the counterexample is to "hide" in such a zero a second δ-potential, which does not influence the previous ground state but produces a new eigenvalue, which can be chosen to imply the above inequality.

For the proof of the above theorem we consider at first the operator

$$H_l(c\delta_0) = (-\partial^2)^l - c\delta_0$$

with $l \in \mathbb{N}$ and $c \in (0, \infty)$, generated by the closure of the quadratic form

$$h_l(c\delta_0)[u, u] := \int_{\mathbb{R}} |\partial^l u|^2 \, dx - c|u(0)|^2 \quad \text{for } u \in C_0^\infty(\mathbb{R}).$$

Let us mention without proof, that the domain of $H_l(c\delta_0)$ consists of all functions $u \in W^{2,2l-1}(\mathbb{R}) \cap W^{2,2l}(\mathbb{R}\backslash\{0\})$ for which

$$\partial^{2l-1}u(0+) - \partial^{2l-1}u(0-) = (-1)^l \, c \, u(0).$$

As the computation in Appendix A shows, the operator $H_l(c\delta_0)$ has exactly one negative eigenvalue $-\varkappa$ which satisfies

$$\varkappa^\nu = L^0_{l,\nu,1} \, c. \tag{2.2}$$

Lemma 2.2. *For $l \in \mathbb{N}$ with $l \geq 2$ the eigenfunction u corresponding to the eigenvalue $-\varkappa$ of $H_l(c\delta_0)$ has at least one zero $x_0 \neq 0$.*

Proof. Let us assume that u has no zero. Then $(-\partial^2)^l u$ has, on the strength of the eigenvalue equation

$$(-\partial^2)^l u(x) = -\varkappa u(x) \quad \text{for} \quad x \in (-\infty, 0),$$

no zero in $(-\infty, 0)$ either. Because of

$$\partial^{2l-1}u(x) = \int_{-\infty}^{x} \partial^{2l}u(t) \, dt \quad \text{for} \quad x \in (-\infty, 0)$$

the same holds for the function $\partial^{2l-1}u$, and so on for all lower derivatives up to the second derivative u''. It therefore follows from continuity of u'' that

$$u'(0) = \int_{-\infty}^{0} u''(x) \, dx \neq 0. \tag{2.3}$$

On the other hand u is symmetric, which follows from the symmetry of the eigenvalue problem and the uniqueness (modulo a factor) of the eigenfunction. This together with the continuity of u' implies $u'(0) = 0$ in contradiction to (2.3). So u has a zero in $(-\infty, 0)$. □

One can indeed prove that u has countably many zeros, by computing u explicitly. But with the existence of one zero we are already able to prove Theorem 2.1:

Proof. Let $x_0 \neq 0$ be a zero of the eigenfunction u_1 corresponding to the unique negative eigenvalue $-\varkappa_1$ of $H_l(\delta_0) = (-\partial^2)^l - \delta_0$. Because of (2.2) we have

$$\varkappa_1^\nu = L^0_{l,\nu,1}. \tag{2.4}$$

For $\alpha > 1$ we consider the operator

$$H_l^\alpha = (-\partial^2)^l - \delta_0 - \alpha\delta_{x_0},$$

given by an appropriate quadratic form, which has exactly two negative eigenvalues. The latter follows from a standard variational argument. Obviously, u_1 is an eigenfunction of H_l^α corresponding to the eigenvalue $-\varkappa_1$. For the ground state of H_l^α, which we refer to as $-\varkappa_0$, the inequality

$$\varkappa_0^\gamma \geq L^0_{l,\gamma,1}\,\alpha \tag{2.5}$$

must hold. This is a consequence of the variational principle: If ϕ with $\|\phi\| = 1$ is the eigenfunction corresponding to the unique negative eigenvalue $-\tau$ of the operator $H_l(\alpha\delta_{x_0}) = (-\partial^2)^l - \alpha\delta_{x_0}$ and if $h_l(\alpha\delta_{x_0})$ and h_l^α are the quadratic forms associated with $H_l(\alpha\delta_{x_0})$ and H_l^α, then we have

$$\begin{aligned} -\tau = h_l(\alpha\delta_{x_0})[\phi,\phi] &= \|\partial^l\phi\|^2 - \alpha|\phi(x_0)|^2 \\ &\geq \|\partial^l\phi\|^2 - |\phi(0)|^2 - \alpha|\phi(x_0)|^2 = h_l^\alpha[\phi,\phi]. \end{aligned} \tag{2.6}$$

By the variational principle the lowest eigenvalue of H_l^α is lower or equal $-\tau$. Because of (2.2) we have $\tau^\gamma = L^0_{l,\gamma,1}\,\alpha$. Therefore (2.5) holds. Notice that $-\varkappa_1$ cannot be the ground state of the operator H_l^α if $\alpha > 1$.

Furthermore we get

$$\varkappa_0^\gamma > L^0_{l,\gamma,1}\,\alpha \tag{2.7}$$

if we choose $\alpha > 1$ in such a way, that $\phi(0) \neq 0$ holds. This is possible since $\phi(x) = cu_1\left(\alpha^{\frac{1}{2l-1}}(x - x_0)\right)$, for $x \in \mathbb{R}$ and some $c \in \mathbb{C}$. So for a proper $\alpha > 1$ we have $\phi(0) = cu_1\left(-\alpha^{\frac{1}{2l-1}}x_0\right) \neq 0$, since u_1 has only a countable set of zeros.

Thus, H_l^α has two negative eigenvalues $-\varkappa_0$ and $-\varkappa_1$, which fulfil the inequality

$$\varkappa_0^\gamma + \varkappa_1^\gamma > L^0_{l,\gamma,1}(1+\alpha). \tag{2.8}$$

Because of the Lieb-Thirring inequality, extended by a standard argument to δ-potentials, we have

$$\varkappa_0^\gamma + \varkappa_1^\gamma \leq L_{l,\gamma,1}(1+\alpha) \tag{2.9}$$

and therefore $L_{l,\gamma,1} > L^0_{l,\gamma,1}$. □

3. Lieb-Thirring inequalities for Riesz means of eigenvalues for polyharmonic operators in higher dimensions

In this section we apply the ideas of Laptev and Weidl from [11] to obtain results valid in dimensions greater than one.

Consider the following Weyl type asymptotics:

$$\begin{aligned} &\lim_{\alpha\to+\infty} \frac{1}{\alpha^{\gamma+\frac{d}{2l}}} \operatorname{tr}\left(\sum\nolimits_{j=1}^d \left(-\partial_j^2\right)^l + \alpha V\right)_-^\gamma = \\ &= \iint_{\mathbb{R}^d\times\mathbb{R}^d} \left(\sum\nolimits_{j=1}^d \xi_j^{2l} + V\right)_-^\gamma \frac{dx\,d\xi}{(2\pi)^d} = C_{l,\gamma,d}\int_{\mathbb{R}^d} V_-^{\gamma+\frac{d}{2l}}\,dx. \end{aligned}$$

We shall need the following lemma concerning the constants $C_{l,\gamma,d}$; the proof is basically a lengthy computation which shall not be presented here.

Lemma 3.1. *The constants* $C_{l,\gamma,d}$ *appearing in the above semi-classical limit obey the following identity*

$$C_{l,\gamma,d} = C_{l,\gamma+\frac{1}{2l},d-1}\cdot C_{l,\gamma,1}. \tag{3.1}$$

Furthermore, the constants $C_{l,\gamma,d}$ *are explicitly given by*

$$C_{l,\gamma,d} = \frac{1}{2(2\pi l)^d} B\left(\gamma+1, \frac{d}{2l}\right)\left(\frac{2\pi^{\frac{d-1}{2}}\Gamma\left(\frac{1}{2l}\right)}{\Gamma\left(\frac{ld+1-l}{2l}\right)}\right)^d, \tag{3.2}$$

where B *is the Beta-function.*

For the operator $(-\Delta)^l$ the ideas of Laptev and Weidl cannot be used directly, because there is no simple way to separate the variables in this case. Therefore we consider first the operator $\sum_{j=1}^d \left(-\partial_j^2\right)^l$.

Theorem 3.2. *Let* $V : \mathbb{R}^d \to \mathcal{B}(\mathcal{G})$ *be an operator-valued function satisfying* $V(x) = (V(x))^*$ *and* $V(x) \in S_1(\mathcal{G})$ *for a.e.* $x \in \mathbb{R}^d$ *and such that* $\operatorname{tr}(V_-(\cdot))^{\gamma+\frac{d}{2l}} \in L^1\left(\mathbb{R}^d\right)$, *for some* $\gamma \geq 1 - \frac{1}{2l}$, $l > 1$. *Then the following inequality holds true:*

$$\operatorname{tr}\left(\sum_{j=1}^d \left(-\partial_j^2\right)^l \otimes \mathbf{1}_{\mathcal{G}} + V\right)_-^\gamma \leq \left(\frac{c_l}{L^{cl}_{l,1-\frac{1}{2l},1}}\right)^d C_{l,\gamma,d} \int_{\mathbb{R}^d} \operatorname{tr}(V_-(x))^{\gamma+\frac{d}{2l}}\, dx. \tag{3.3}$$

In the case $l = 2$ *the constant on the right hand side can be replaced by* $\left(\frac{4}{3^{1/4}\sqrt{2}}\right)^d C_{2,\gamma,d}$.

Proof. Using Corollary 1.5 and Corollary 1.7 the result follows directly by applying the technique from [11], section 3. □

Because

$$\sum_{j=1}^d \left(-\partial_j^2\right)^l \otimes \mathbf{1}_{\mathcal{G}} \leq (-\Delta)^l \otimes \mathbf{1}_{\mathcal{G}} \tag{3.4}$$

in quadratic form sense, estimate (3.3) is also valid for the polyharmonic operator case with $\sum_{j=1}^d \left(-\partial_j^2\right)^l$ replaced by $(-\Delta)^l$. This follows from the minimax principle. Consequently we achieve

Theorem 3.3. *Let* V *and* γ *be as in Theorem 3.2. Then the following inequality holds true:*

$$\operatorname{tr}\left((-\Delta)^l \otimes \mathbf{1}_{\mathcal{G}} + V\right)_-^\gamma \leq \left(\frac{c_l}{L^{cl}_{l,1-\frac{1}{2l},1}}\right)^d C_{l,\gamma,d} \int_{\mathbb{R}^d} \operatorname{tr}(V_-(x))^{\gamma+\frac{d}{2l}}\, dx. \tag{3.5}$$

Again, for the biharmonic operator the constant on the right hand side can be replaced by $\left(\frac{4}{3^{1/4}\sqrt{2}}\right)^d C_{2,\gamma,d}$.

Remark. It is interesting to note that the proofs above may be modified as to include the case when the operator $\mathbf{1}_{\mathcal{G}}$ is replaced by some other operator A acting in $\mathcal{G}$. More precisely, let A be any self-adjoint operator acting in $\mathcal{G}$ which is positive, i.e.

$$\langle Ax, x\rangle_{\mathcal{G}} > 0, \quad \text{for all } 0 \neq x \in \mathcal{G}.$$

Then the inequality

$$\operatorname{tr}\left((-\Delta)^l \otimes A + V\right)_-^\gamma \leq \left(\frac{c_l}{L^{cl}_{l,1-1/2l,1}}\right)^d C_{l,\gamma,d} \int_{\mathbb{R}^d} \operatorname{tr} A^{-\frac{1}{2l}}(V_-(x))^{\gamma+\frac{d}{2l}}\, dx \tag{3.6}$$

is valid whenever the right-hand side is finite, for $\gamma > 1 - 1/2l$, $l > 1$. Note that if A is positive definite, i.e.

$$0 < m_A := \inf_{\|x\|=1} \langle Ax, x\rangle_{\mathcal{G}},$$

then the bound (3.6) is valid for any V satisfying the criteria listed in Theorem 3.3. The proof of (3.6) is basically the same as that of (3.5). In the estimate from above, the scalar inequality (1.24) is replaced by the operator-inequality

$$\frac{\epsilon^{2l-1}}{B^{2l} + \epsilon^{2l}} \leq \tilde{c}_l \frac{\epsilon}{B^2 + \epsilon^2},$$

valid for any positive self-adjoint operator B acting on $\mathcal{G}$. The majorization as well as the Aizenman-Lieb argument works out similar as before, as does the "lifting" to dimensions greater than one.
Note that the same technique, applied to the special case of the operator

$$-\frac{d^2}{dx^2} \otimes A + V \tag{3.7}$$

acting in $L^2(\mathbb{R}, \mathcal{G})$, implies that

$$\operatorname{tr}\left(-\frac{d^2}{dx^2} \otimes A + V\right)_-^{\gamma} \leq 2\, L^{\mathrm{cl}}_{1,\gamma,1} \int_{\mathbb{R}} \operatorname{tr} A^{-\frac{1}{2}} (V_-(x))^{\gamma+\frac{1}{2}}\, dx, \tag{3.8}$$

for any $\gamma \geq 1/2$. It is tantalizing to ask for the smallest bound in (3.8). Does it, as in case $A = \mathbf{1}_{\mathcal{G}}$ (see [11]), hold with the classical constant if we consider Riesz means of order $\gamma \geq 3/2$? This problem is still open.

APPENDIX A

Lemma A.1. *The unique negative eigenvalue $-\varkappa$ of the operator $H_l(c\delta_0) = (-\partial^2)^l - c\delta_0$ satisfies the identity*

$$\varkappa^{\nu} = L^0_{l,\nu,1} c. \tag{A.1}$$

Proof. At first we notice, that $(-\partial^2)^l u = -\varkappa u$ has the basic solutions

$$g_k(x) := \exp(r_k \sqrt[2l]{\varkappa}\, x) \quad \text{for} \quad k = 0, \ldots, 2l-1.$$

Here the r_k's are the $2l$ complex roots of the equation $r_k^{2l} = (-1)^{l+1}$. It holds

$$r_k = \exp\left(\frac{2k+1-l}{2l} i\pi\right) \quad \text{for} \quad k = 0, \ldots, 2l-1.$$

Notice that the roots are ordered so that r_0 to r_{l-1} have positive and r_l to r_{2l-1} have negative real parts. Therefore the functions g_0 to g_{l-1} are not square integrable on $(0, \infty)$, and neither are g_l to g_{2l-1} on $(-\infty, 0)$. Let us write $\mathfrak{g}(x) = \big(g_0(x), \ldots, g_{2l-1}(x)\big)$ and $\partial^k \mathfrak{g}(x) = \big(\partial^k g_0(x), \ldots, \partial^k g_{2l-1}(x)\big)$. Further let

$$\mathfrak{G}(x) := \begin{pmatrix} \mathfrak{g}(x) \\ \partial \mathfrak{g}(x) \\ \vdots \\ \partial^{2l-1}\mathfrak{g}(x) \end{pmatrix} \quad \text{and} \quad \mathfrak{E}(x) := \begin{pmatrix} 0 \\ \vdots \\ 0 \\ \mathfrak{g}(x) \end{pmatrix}.$$

Then we can formulate the conditions on the eigenfunction, which on each of the intervals $(-\infty, 0), (0, \infty)$ is a linear combination of the basic solutions, at the point 0 as follows:

$$\mathfrak{G}(0)\mathfrak{h} - \mathfrak{G}(0)\mathfrak{v} = (-1)^l c\mathfrak{E}(0)\mathfrak{v}.$$

Here $\mathfrak{v}, \mathfrak{h} \in \mathbb{C}^{2l}$ are the coefficients of the basic solutions on the left and right interval. Note, that the matrix $\mathfrak{G}(0)$ is invertible, because its determinant, the so called Wronskian determinant, is non-zero. Therefore the latter equation takes the form

$$\big(I + (-1)^l c \mathfrak{G}^{-1}(0)\, \mathfrak{E}(0)\big)\mathfrak{v} = \mathfrak{h}, \tag{A.2}$$

where I is the identity matrix. The inverse of the matrix

$$\mathfrak{G}(0) = \big[g_k^{(n)}(x)\big]_{\substack{n=0,\dots,2l-1\\ k=0,\dots,2l-1}} = \left[(r_0 \sqrt[2l]{\varkappa})^n \exp\left(\frac{nk}{l} i\pi\right)\right]_{\substack{n=0,\dots,2l-1\\ k=0,\dots,2l-1}}$$

is given by

$$\mathfrak{G}^{-1}(0) = \frac{1}{2l}\left[\frac{\exp\left(\frac{(2l-n)k}{l} i\pi\right)}{(r_0 \sqrt[2l]{\varkappa})^k}\right]_{\substack{n=0,\dots,2l-1\\ k=0,\dots,2l-1}}.$$

We furthermore get, with $\tau := (-1)^l c (2l)^{-1} (r_0 \sqrt[2l]{\varkappa})^{1-2l}$, that

$$(-1)^l c \mathfrak{G}^{-1}(0)\, \mathfrak{E}(0) = \tau \left[\exp\left(\frac{n}{l} i\pi\right)\right]_{\substack{n=0,\dots,2l-1\\ k=0,\dots,2l-1}} = \tau \begin{bmatrix} ye^T & ye^T \\ -ye^T & -ye^T \end{bmatrix},$$

where $y, e \in \mathbb{C}^l$ with $y := (\exp(\frac{0}{l} i\pi), \dots, \exp(\frac{l-1}{l} i\pi))^T$ and $e := (1, \dots, 1)^T$. Notice now, that we have $\mathfrak{v}_l = \dots = \mathfrak{v}_{2l-1} = 0$ and $\mathfrak{h}_0 = \dots = \mathfrak{h}_{l-1} = 0$, since the eigenfunction must be square integrable. Therefore, writing $\mathfrak{v} = \binom{\tilde{\mathfrak{v}}}{0}$, $\mathfrak{h} = \binom{0}{\tilde{\mathfrak{h}}}$, where $\tilde{\mathfrak{v}}, \tilde{\mathfrak{h}} \in \mathbb{C}^l$ we see that equation (A.2) has a non-trivial solution if and only if

$$(I + \tau y e^T)\tilde{\mathfrak{v}} = 0$$

has a non-trivial solution. But it is not difficult to see, that the latter holds if and only if

$$\tau e^T y + 1 = 0,$$

that is if

$$\varkappa^\nu = \left(\frac{(-1)^{(l+1)}}{2l r_0^{(2l-1)}} \sum_{k=0}^{l-1} \exp\left(\frac{k}{l} i\pi\right)\right) c = \frac{1}{2l \sin(\pi/(2l))} c = L^0_{l,\nu,1} c.$$

This completes the proof. □

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