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Preprint 2005/001

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ISSN **1613-8309**

Vortex pinning in super-conductivity as a rate-independent model

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22 December 2004

1 Introduction

For superconductors of type II the phenomenon of vortex pinning plays an important role in technological applications. Several models have been proposed for this effect, see [Bos94, KHS63, Bea64]. In [QGL99, Pri96] some of these models are analysed. In this work we want to add to these analytical studies for the particular two-dimensional model proposed in [Cha00].

For this model $\Omega \subset \mathbb{R}^2$ be a bounded Lipschitz domain and denote by $\tilde{H} : \Omega \to \mathbb{R}$ the magnetic field perpendicular to the plane. The vortex tube density $\omega : \Omega \to \mathbb{R}$ is related to $\tilde{H}$ via the constitutive relation

$$\omega = \tilde{A}(\tilde{H}) := \alpha \tilde{H} - \operatorname{div}(\beta \nabla \tilde{H}),$$

where α and β are material parameters and $\lambda = \sqrt{\beta/\alpha}$ is called the penetration depth. The modelling assumption in [Cha00] is now that the vortex tubes will not move if the modulus of the induced current $J = \nabla \tilde{H} \in \mathbb{R}^2$ is smaller than a critical value J_{c} and that they move immediately if $|J| = J_{\mathrm{c}}$. The movement is then described by a mobility function $m : \Omega \to \mathbb{R}$ which plays the role of a Lagrange multiplier. The full problem has then the following form:

$$\left.\begin{array}{c} \partial_t \omega = \operatorname{div}(m \nabla \tilde{H}) \quad \text{with } \omega = \tilde{A}(\tilde{H}), \\ m \geq 0, \quad J_{\mathrm{c}} - |\nabla \tilde{H}| \geq 0, \quad (J_{\mathrm{c}} - |\nabla \tilde{H}|)m = 0 \end{array}\right\} \ \text{in } [0,T] \times \Omega, \tag{1.1}$$
$$\tilde{H}(0,\cdot) = H_0 \quad \text{on } \Omega \quad \text{and} \quad \tilde{H}(t,x) = H_{\mathrm{ext}}(t) \quad \text{on } [0,T] \times \partial\Omega.$$

The first equation expresses the conservation of the vortex-tube density which is driven by the current J. The second line contains the variational inequalities which model the pinning as an activated process.

The aim of this work is to rewrite the problem in an energetic formulation which provides a much easier approach to the existence and uniqueness theory. As the main

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unknown, we use $H = \tilde{H} - GH_{\mathrm{ext}}(t)$, where $G : \Omega \to \mathbb{R}$ is defined in (2.1), choose the state space $X = \mathrm{H}^1_0(\Omega)$. We define the energy functional $\mathcal{E} : [0, T] \times X \to \mathbb{R}$ via

$$\mathcal{E}(t, H) = \int_\Omega \frac{1}{2} A(H)(x)H(x) - \alpha H_{\mathrm{ext}}(t)H(x)\,\mathrm{d}x$$

and a dissipation functional for $v = \partial_t H$ via

$$\Psi(v) = \sup\left\{ \int_\Omega A(\hat{H})(x)v(x)\,\mathrm{d}x \;\middle|\; \hat{H} \in \mathrm{H}^1_0(\Omega), |\nabla\hat{H}| \le J_{\mathrm{c}} \right\}. \tag{1.2}$$

Here A denotes the self-adjoint operator $A : \mathrm{H}^1_0(\Omega) \to \mathrm{H}^{-1}(\Omega), H \mapsto \tilde{A}(H)$. By definition Ψ is 1-homogeneous, i.e.,

$$\forall\, \lambda \ge 0\; \forall\, v \in X : \quad \Psi(\lambda v) = \lambda\Psi(v), \tag{1.3}$$

and convex. This implies the triangle inequality

$$\forall\, v_1, v_2 \in X : \quad \Psi(v_1 + v_2) \le \Psi(v_1) + \Psi(v_2). \tag{1.4}$$

Note that $\Psi(H_1 - H_0)$ has the physical dimension of an energy and can be interpreted as the minimal amount of energy dissipated due to vortex movement when changing the state from H_0 to H_1.

We show that (1.1) is formally equivalent to the differential inclusion

$$0 \in \partial\Psi(\partial_t H) + \mathrm{D}\mathcal{E}(t, H) \subset X^*, \tag{1.5}$$

where $\partial\Psi(v)$ is the set-valued subdifferential defined in (2.6). Moreover, the differential inclusion is equivalent to the following *energetic formulation*:

For all $t \in [0, T]$ we have

$$\text{(S)}\quad \mathcal{E}(t, H(t)) \le \mathcal{E}(t, \hat{H}) + \Psi(\hat{H} - H(t)) \quad \text{for all } \hat{H} \in X \tag{1.6}$$

$$\text{(E)}\quad \mathcal{E}(t, H(t)) + \int_0^t \Psi(\partial_t H(t))\mathrm{d}t = \mathcal{E}(0, H(0)) - \int_0^t \int_\Omega \partial_\tau H_{\mathrm{ext}}(\tau)H(\tau, x)\,\mathrm{d}x\,\mathrm{d}\tau.$$

Under the simple assumption $H_{\mathrm{ext}} \in \mathrm{C}^1([0, T], \mathbb{R})$ we show that (1.5) and (1.6) have, for each $H(0) = H_0 \in \mathrm{H}^1_0(\Omega)$ which satisfy (S) at time 0, a unique solution $H \in \mathrm{C}^{\mathrm{Lip}}([0, T], X)$.

The reformulation of problem (1.1) into (1.5) and (1.6) will be discussed in Section 2. In Section 3 we provide a self-contained existence and uniqueness proof which is a slight generalization of the theory in [MT04]. It is based on time-discretization and the incremental minimization problem

$$\mathcal{E}(t_k, H) + \Psi(H - H_{k-1}) \to \underset{H \in X}{\mathrm{minimal}}.$$

We believe that the simplicity of the approach will allow for several generalizations such that more general models in super-conductivity can be studied.

2 Reformulation of the model

We denote by ${}_*\langle \cdot, \cdot \rangle_X$ the duality between the dual $X^* = \mathrm{H}^{-1}(\Omega)$ and $X = \mathrm{H}^1_0(\Omega)$. By the general assumption that $\alpha, \beta \in (0, \infty)$ are fixed, we see that $\tilde{A}(H) = \alpha H - \nabla \cdot (\beta \nabla H)$ defines a self-adjoint operator

$$A : \begin{cases} X & \to & X^*, \\ H & \mapsto & \tilde{A}(H), \end{cases}$$

i.e. ${}_*\langle A(H_2), H_1 \rangle_X = {}_*\langle A(H_1), H_2 \rangle_X$. In fact, we may also assume $\alpha \in \mathrm{L}^\infty(\Omega)$ and $\beta \in \mathrm{L}^\infty(\Omega, \mathbb{R}^{2\times 2}_{\mathrm{sym}})$ with $\alpha, \beta \geq \delta > 0$ for some $\delta > 0$. We also define the auxiliary function $G \in \mathrm{H}^1(\Omega)$ via

$$\tilde{A}(G) = 0 \text{ in } \Omega \quad \text{and} \quad G|_{\partial\Omega} \equiv 1. \tag{2.1}$$

The choice was done such that for $H = \tilde{H} - GH_{\mathrm{ext}}$ with $\tilde{H}(t, x) = H_{\mathrm{ext}}(t)$ for $x \in \partial\Omega$ we have

$$H(t, x) = 0 \quad \text{for } (t, x) \in [0, T] \times \partial\Omega \quad \text{and} \quad \tilde{A}(\partial_t \tilde{H}) = \tilde{A}(\partial_t H).$$

With this definition the first equation in (1.1) can be written in weak form as

$$-{}_*\langle A(\partial_t H), \hat{H} \rangle_X = \int_\Omega m \nabla (H + GH_{\mathrm{ext}}) \cdot \nabla \hat{H} \mathrm{d}x \quad \text{for all } \hat{H} \in X \tag{2.2}$$

The conditions involving the Lagrange multiplier (or mobility factor) can be written more precisely in terms of convex analysis. For this introduce the set

$$\mathcal{C} = \{\, \hat{H} \in X \mid |\nabla \hat{H}| \leq J_{\mathrm{c}} \text{ a.e. in } \Omega \,\} \subset X. \tag{2.3}$$

Obviously, $\mathcal{C}$ is closed, convex and bounded. Note that $0 \in \mathcal{C}$, but $\mathcal{C}$ has empty interior in X. We define the set-valued normal cone $\mathrm{N}_{\mathcal{C}}$ via

$$\mathrm{N}_{\mathcal{C}}(H) := \begin{cases} \{\, v^* \in X^* \mid {}_*\langle v^*, H - \hat{H} \rangle_X \geq 0 \quad \text{for all} \quad \hat{H} \in \mathcal{C} \,\} & \text{for } H \in \mathcal{C}, \\ \emptyset & \text{for } H \notin \mathcal{C}. \end{cases}$$

With this we postulate the following differential inclusion:

$$-A(\partial_t H) \in \mathrm{N}_{\mathcal{C}}\big(H + (G - 1)H_{\mathrm{ext}}(t)\big) \subset X^* \quad \text{for a.e. } t \in [0, T]. \tag{2.4}$$

Proposition 2.1 *If $\tilde{H} \in \mathrm{W}^{1,1}([0, T], \mathrm{H}^1(\Omega))$ is a solution of* (1.1)*, then $H = \tilde{H} - GH_{\mathrm{ext}}$ solves* (2.4)*.*

Proof: We first rewrite (1.1) by eliminating the Lagrange multiplicator m . For $\tilde{H} \in \mathrm{H}^1(\Omega)$ we set

$$\mathcal{M}(\tilde{H}) := \left\{ v^* \in \mathrm{H}^{-1}(\Omega) \;\middle|\; \begin{array}{c} \exists m \in \mathrm{L}^\infty(\Omega) : m \geq 0 \text{ and } m(J_{\mathrm{c}} - |\nabla \tilde{H}|) = 0 \text{ a.e.}, \\ {}_*\langle v^*, \varphi \rangle_X = \int_\Omega m \nabla \tilde{H} \cdot \nabla \varphi \mathrm{d}x \text{ for all } \varphi \in X \end{array} \right\} \tag{2.5}$$

if $|\nabla \tilde{H}| \leq J_{\mathrm{c}}$ a.e. and $\mathcal{M}(\tilde{H}) := \emptyset$ else. Note that for each constant h we have $\mathcal{M}(\tilde{H}) = \mathcal{M}(\tilde{H} - h)$. With this definition (1.1) takes the form $-\tilde{A}(\partial_t \tilde{H}) \in \mathcal{M}(\tilde{H} - H_{\mathrm{ext}}(t))$.

Using $\tilde{H} = H + GH_{\text{ext}}$ and $\tilde{A}(G) = 0$, we see that the assertion holds if we are able to show that $\mathcal{M}(H) \subset \mathrm{N}_{\mathcal{C}}(H)$ for all $H \in X$. For $H \notin \mathcal{C}$ we have $\mathcal{M}(H) = \mathrm{N}_{\mathcal{C}}(H) = \emptyset$. Thus, assume $H \in \mathcal{C}$ and take $v^* \in \mathcal{M}(H)$, we then have to show

$$_*\langle v^*, H - \hat{H}\rangle_X \geq 0 \quad \text{for all} \quad \hat{H} \in \mathcal{C}.$$

By the definition of $\mathcal{M}(H)$ there exists $m \in \mathrm{L}^\infty(\Omega)$ with $m \geq 0$ and $m(J_{\text{c}} - |\nabla H|) \geq 0$ a.e. and

$$_*\langle v^*, H - \hat{H}\rangle_X = \int_\Omega m\nabla H \cdot (\nabla H - \nabla \hat{H})\mathrm{d}x.$$

In the last integral the integrand is in fact pointwise nonnegative a.e.. In fact, if $m(x) = 0$ this is obvious, and if $m(x) > 0$ then $|\nabla H| = J_{\text{c}}$ which implies

$$\nabla H \cdot (\nabla H - \nabla \hat{H}) = |\nabla H|^2 - \nabla H \cdot \nabla \hat{H} \geq (J_{\text{c}})^2 - J_{\text{c}}|\nabla \hat{H}| \geq 0,$$

since $\hat{H} \in \mathcal{C}$. Thus, we have $_*\langle v^*, H - \hat{H}\rangle_X \geq 0$ as desired. ■

In fact, we believe that the problems (1.1) and (2.4) are equivalent. However, so far we were unable to prove $\mathcal{M}(H) = \mathrm{N}_{\mathcal{C}}(H)$ in general.

It is now easy to reformulate (2.4) in several ways by using the Legendre transform, see [Vis94, Mon93, MT04]. Introduce the convex characteristic function $\mathcal{X}_{\mathcal{C}}$ via $\mathcal{X}_{\mathcal{C}}(H) = 0$ for $H \in \mathcal{C}$ and ∞ else and its Legendre-Fenchel transform $\mathcal{X}_{\mathcal{C}}^* = \mathcal{L}\mathcal{X}_{\mathcal{C}}$ via

$$(\mathcal{L}\mathcal{X}_{\mathcal{C}})(v^*) = \sup\{\, _*\langle v^*, \varphi\rangle_X - \mathcal{X}_{\mathcal{C}}(\varphi) \mid \varphi \in X\,\}.$$

Moreover, define the subdifferential ∂f for any convex function $f : Y \to \mathbb{R} \cup \{\infty\}$ via

$$\partial f(y) = \{\, v^* \in Y^* \mid \forall \hat{y} \in Y : \ f(\hat{y}) \geq f(y) + \langle v^*, \hat{y} - y\rangle \,\}, \tag{2.6}$$

where Y will be either X or X^*. Then, the following standard relations hold:

$$\begin{array}{ll} (a) & \mathrm{N}_{\mathcal{C}}(H) = \partial \mathcal{X}_{\mathcal{C}}(H), \\ (b) & v^* \in \partial \mathcal{X}_{\mathcal{C}}(H) \Leftrightarrow H \in \partial \mathcal{X}_{\mathcal{C}}^*(v^*). \end{array}$$

Using (a) and (b) we see that (2.4) is equivalent to $H + (G-1)H_{\text{ext}} \in \partial \mathcal{X}_{\mathcal{C}}^*(-A\partial_t H)$: Exploiting the symmetry $\mathcal{C} = -\mathcal{C}$ and applying A we arrive at

$$-(AH - \alpha H_{\text{ext}}) \in A\big(\partial \mathcal{X}_{\mathcal{C}}^*(A\partial_t H)\big) \subset X^*, \tag{2.7}$$

where we have used $\tilde{A}G = 0$ and $\tilde{A}1 = \alpha$.

Lemma 2.2 *Let $\Psi : X \to [0, \infty)$ be defined via $\Psi(v) = \sup\{\, _*\langle AH, v\rangle_X \mid H \in \mathcal{C}\,\}$, then $\Psi(v) = \mathcal{X}_{\mathcal{C}}^*(Av)$ and $\partial\Psi(v) = A\partial\mathcal{X}_{\mathcal{C}}^*(Av)$ for all $v \in X$.*

Proof: By this definition we easily find $\mathcal{X}_{\mathcal{C}}^*(v^*) = \sup\{\, _*\langle v^*, H\rangle_X \mid H \in \mathcal{C}\,\}$. Thus we have $\Psi(v) = \mathcal{X}_{\mathcal{C}}^*(Av)$ and the result for the subdifferential follows from the chainrule and $A = A^*$. ■

Finally we define the energy functional

$$\mathcal{E}(t, H) = \frac{1}{2}\, _*\langle AH, H\rangle_X - \int_\Omega \alpha H(x) H_{\text{ext}}(t)\mathrm{d}x$$

and obtain the main result of this section, since $\mathrm{D}\mathcal{E}(t, H) = AH - \alpha H_{\text{ext}}$.

Proposition 2.3 *Equation* (2.7) *is equivalent to*

$$0 \in \partial\Psi(\partial_t H) + \mathrm{D}\mathcal{E}(t,H) \quad \textit{for a.e. } t \in [0,T]. \tag{2.8}$$

Using the rate-independence, which is the same as the 1-homogeneity of Ψ (see (1.3)), and the triangle inequality for Ψ in (1.4) it is easy to see that (2.8) is equivalent to the two conditions

$$\left.\begin{array}{ll} (\mathrm{S})_{\mathrm{loc}} & \langle \mathrm{D}\mathcal{E}(t,H), v\rangle + \Psi(v) \geq 0 \quad \text{for all } v \in X, \\ (\mathrm{E})_{\mathrm{loc}} & \langle \mathrm{D}\mathcal{E}(t,H), \partial_t H\rangle + \Psi(\partial_t H) = 0. \end{array}\right\} \tag{2.9}$$

Since $\mathcal{E}(t,\cdot) : X \to \mathbb{R}$ is also convex, we arrive at the energetic formulation

$$\begin{array}{ll} (\mathrm{S}) & \mathcal{E}(t,H(t)) \leq \mathcal{E}(t,\hat{H}) + \Psi(\hat{H} - H(t)) \quad \text{for all } \hat{H} \in X, \\ (\mathrm{E}) & \mathcal{E}(t,H(t)) + \int_0^t \Psi(\partial_t H(\tau))\,\mathrm{d}\tau = \mathcal{E}(0,H(0)) - \int_0^t \int_\Omega \partial_t H_{\mathrm{ext}}(\tau)\alpha H(\tau,x)\,\mathrm{d}x\,\mathrm{d}\tau \end{array}$$

The stability condition (S) has the obvious interpretation, that a state $H(t)$ can only occur if for no other state $\hat{H}$ we can release more energy than is dissipated by the moving vortices. Obviously, $(\mathrm{S})_{\mathrm{loc}}$ is the same as $0 \in \partial\Psi(0) + \mathrm{D}\mathcal{E}(t,H(t))$. Using Lemma 2.2 we find

$$\partial\Psi(0) = A\mathcal{C} = \{\, AH \mid H \in \mathcal{C} \,\} \subset X^* \tag{2.10}$$

and thus, $(\mathrm{S})_{\mathrm{loc}}$, and hence (S), is equivalent to $A^{-1}\mathrm{D}\mathcal{E}(t,H(t)) = A^{-1}(AH - \alpha H_{\mathrm{ext}}) = H + (G-1)H_{\mathrm{ext}} \in \mathcal{C}$. This is of course the condition $|\nabla\widetilde{H}| \leq J_{\mathrm{c}}$.

The energy balance (E) just states that the total stored energy $\mathcal{E}(t,H(t))$ at time t is the initial energy plus the work of the boundary conditions through the external field H_{ext} minus the dissipated energy.

For more exact proofs of these equivalences we refer to [MT04].

3 Existence and Uniqueness

To formulate the main result most conveniently we recall $\partial\Psi(0) = A\mathcal{C}$.

Theorem 3.1 *Let* $H_{\mathrm{ext}} \in \mathrm{C}^1([0,T])$ *and* H_0 *be given with* $H_0 + (G-1)H_{\mathrm{ext}}(0) \in \mathcal{C}$*. Then,* (2.8) *has a unique solution* $H \in \mathrm{C}^{\mathrm{Lip}}([0,T],X)$ *with* $H(0) = H_0$*.*

This result is a special case of several well-established theories. In fact, we simplified the problem by assuming C^1 smoothness of H_{ext} which would not be necessary. However, in rate-independent systems we may always rescale time to gain smoothness. For instance, combining Thm 3.1 and Prop. 3.5 in [Kre99] proves our result. Moreover, in [Vis94] or [Mon93] corresponding results can be found. Nevertheless, we find it worthwhile to provide an independent short proof which is based on the energetic formulation (S) and (E), and thus is closer to the underlying physics. We follow the more general approach in [MT99, MT04], however we have to work around their hypothesis $\Psi(v) \geq c\|v\|$ which is not true in our situation.

We introduce the set $\mathcal{S}(t)$ of stable states at time t via

$$\mathcal{S}(t) = \{\, H \in X \mid \mathcal{E}(t,H) \leq \mathcal{E}(t,\hat{H}) + \Psi(\hat{H} - H) \text{ for all } \hat{H} \in X \,\}.$$

The condition (S) is equivalent to $H(t) \in \mathcal{S}(t)$. As seen at the end of Section 2 we have

$$\mathcal{S}(t) = (1-G)H_{\text{ext}}(t) + \mathcal{C},$$

which shows that $\mathcal{S}(t)$ is a closed, convex, bounded set, which depends smoothly on $t \in [0,T]$.

Proof: [of Theorem 3.1] The proof relies on time discretization. For $n \in \mathbb{N}$ subdivide $[0,T]$ equidistantly into 2^n intervals via $t_k^n = kT/2^n$ for $k = 0,1,\dots,2^n$. We let $H_0^n = H_0$ and define H_k^n iteratively via

$$H_{k+1}^n = \arg\min \left\{ \mathcal{E}(t_{k+1}^n, H) + \Psi(H - H_k^n) \;\middle|\; H \in X \right\}.$$

Since $\mathcal{E}$ is strictly convex, the minimizer exists and is unique. Moreover, we have

(A) $H_k^n \in \mathcal{S}(t_k^n)$ for $n \in \mathbb{N}$ and $k \in \{0,1,\dots,2^n\}$,

(B) $\mathcal{E}(t_k^n, H_k^n) + \Psi(H_k^n - H_{k-1}^n) \le \mathcal{E}(t_{k-1}^n, H_{k-1}^n) + \int_{t_{k-1}^n}^{t_k^n} \partial_s \mathcal{E}(s, H_{k-1}^n)\,\mathrm{d}s$

To see (A) simply use that (i) H_k^n is a minimizer and that (ii) Ψ satisfies the triangle inequality:

$$\begin{aligned}
&\mathcal{E}(t_k^n, \hat{H}) + \Psi(\hat{H} - H_k^n) \\
&\quad = \mathcal{E}(t_k^n, \hat{H}) + \Psi(\hat{H} - H_{k-1}^n) + \Psi(\hat{H} - H_k^n) - \Psi(\hat{H} - H_{k.1}^n) \\
&\quad \overset{\text{(i)}}{\ge} \mathcal{E}(t_k^n, H_k^n) + \Psi(H_k^n - H_{k-1}^n) + \Psi(\hat{H} - H_k^n) - \Psi(\hat{H} - H_{k-1}^n) \\
&\quad \overset{\text{(ii)}}{\ge} \mathcal{E}(t_k^n, H_k^n).
\end{aligned}$$

For (B) we again use that H_k^n is a minimizer

$$\mathcal{E}(t_k^n, H_k^n) + \Psi(H_k^n - H_{k-1}^n) \le \mathcal{E}(t_k^n, H_{k-1}^n) = \mathcal{E}(t_{k-1}^n, H_{k-1}^n) + \int_{t_{k-1}^n}^{t_k^n} \partial_s \mathcal{E}(s, H_{k-1}^n)\,\mathrm{d}s.$$

The stability in (A) is equivalent to

$$_*\langle \mathrm{D}\mathcal{E}(t_k^n, H_k^n), v\rangle_X + \Psi(v) \ge 0 \quad \text{for all } v \in X, \tag{3.1}$$

and the minimization property shows that for $v = H_k^n - H_{k-1}^n$ equality holds. Thus, we have

$$\begin{aligned}
&_*\langle A(H_k^n - H_{k-1}^n), H_k^n - H_{k-1}^n\rangle_X \\
&\quad = {}_*\langle \mathrm{D}\mathcal{E}(t_k^n, H_k^n) - \mathrm{D}\mathcal{E}(t_k^n, H_{k-1}^n), H_k^n - H_{k-1}^n\rangle_X \\
&\quad \overset{(3.1)}{=} -\Psi(H_k^n - H_{k-1}^n) - {}_*\langle \mathrm{D}\mathcal{E}(t_{k-1}^n, H_{k-1}^n), H_k^n - H_{k-1}^n\rangle_X \\
&\qquad\quad - \int_{t_{k-1}^n}^{t_k^n} {}_*\langle \partial_s \mathrm{D}\mathcal{E}(s, H_{k-1}^n), H_k^n - H_{k-1}^n\rangle_X\,\mathrm{d}s \\
&\quad \overset{(3.1)}{\le} 0 + \|H_k^n - H_{k-1}^n\|_X \|\partial_t H_{\text{ext}}\|_{\mathrm{C}^0} \|\alpha\|_{X^*} (t_k^n - t_{k-1}^n).
\end{aligned}$$

Since the operator A is positive definite, we obtain the a priori Lipschitz bound

$$\|H_k^n - H_{k-1}^n\|_X \le \mathrm{C}_1 |t_k^n - t_{k-1}^n|.$$

We now define the piecewise linear interpolants $H^n : [0,T] \to X$ with $H^n(t_k^n) = H_k^n$, then we know $\| \partial_t H^n(t)\|_X \le \mathrm{C}_1$ for a.a. $t \in [0,T]$. Thus, the Arzelà-Ascoli theorem for $\mathrm{C}^0([0,T],X)$ yields a subsequence (not renumbered) and a limit function $H : [0,T] \to X$ such that for all $t \in [0,T]$ we have $H^n(t) \rightharpoonup H(t)$ in X as $n \to \infty$, where $\rightharpoonup$ denotes weak convergence. Moreover H is Lipschitz continuous with $\|\partial_t H(t)\| \le \mathrm{C}_1$ a.e. in [0,T].

Keeping $t^* = k^*T/2^{n_*}$ fixed, then for all $n \ge n_*$ we have $H^n(t^*) \in \mathcal{S}(t^*)$. Since $\mathcal{S}(t^*)$ is closed and convex we conclude $H(t^*) \in \mathcal{S}(t^*)$. Since $\big\{ k^*T/2^{n_*} \in [0,T] \;\big|\; n_* \in \mathbb{N} \text{ and } k^* \in \{0,\dots,2^{n_*}\} \big\}$ is dense in $[0,T]$ and since $H : [0,T]$ is Lipschitz continuous, we conclude $H(t) \in \mathcal{S}(t)$ for all $t \in [0,T]$.

Finally we consider the energy equation. Let t^* be as above and add the discrete energy estimates (B) for $n = n_*$ over $k = 1,\dots,k^*$. Note that in the case $k = 1$ we use the fact that $H_0 = H_0^n$ lies in $\mathcal{S}(0)$. We find

$$\mathcal{E}(t^*, H^{n_*}(t^*)) + \int_0^{t^*} \Psi(\partial_t H^{n_*}(\tau))\,\mathrm{d}\tau \le \mathcal{E}(0,H_0) - \int_0^{t^*} \partial_t H_{\mathrm{ext}}(\tau) \int_\Omega \alpha\, \overline{H}^{n_*}(\tau)\,\mathrm{d}x\,\mathrm{d}\tau \quad (3.2)$$

where H^{n_*} is the piecewise linear interpolant from above while $\overline{H}^{n_*}$ is the piecewise constant interpolant with $\overline{H}^{n_*}(t) = H_{k-1}^n$ for $t \in [t_{k-1}^n, t_k^n)$. The right-hand side is weakly continuous and on the left-hand side $\mathcal{E}(t,\cdot)$ is convex and continuous and hence weakly lower semicontinuous. It remains us to show the following lemma.

Lemma 3.2 *Assume the sequence $(H^n)_{n\in\mathbb{N}}$ as above, then*

$$\int_0^t \Psi(\partial_t H(\tau))\,\mathrm{d}\tau \le \liminf_{n\to\infty} \int_0^t \Psi(\partial_t H^n(\tau))\,\mathrm{d}\tau\,.$$

Proof: The sequence $(H^n)_{n\in\mathbb{N}}$ is bounded in $\mathrm{C}^{\mathrm{Lip}}([0,T],X) = \mathrm{W}^{1,\infty}([0,T],\mathrm{H}^1(\Omega))$, which is continuously embedded into the Hilbert space $\mathcal{H} = \mathrm{H}^1([0,T],X)$. Thus, the sequnce converges weakly in $\mathcal{H}$ to the limit H cocnstructed above. For this note, that the sequence is aalso bounded in $\mathcal{H}$ and hence it has a weakly converging subsequence. Since $\mathcal{H}$ is compactly embedded in $\mathcal{Y} = \mathrm{L}^2([0,T],\mathrm{L}^2(\Omega)) = \mathrm{L}^2([0,T]\times\Omega)$ this subsequence converges strongly in $\mathcal{Y}$. However, the convergence invoked from the Arzelà-Ascoli theorem also implies strong converge in $\mathcal{Y}$. Thus, the weak limit in $\mathcal{H}$ is unique and equal to H.

We now define the functional $\mathcal{I} : \mathcal{H} \to \mathbb{R}$ via $\mathcal{I}(H) := \int_0^t \Psi(\partial_t H(\tau))\,\mathrm{d}\tau$. Since $\Psi : \mathrm{H}^1 \to [0,\infty)$ is convex we get immediately the convexity of $\mathcal{I}$. Further the upper estimate $\Psi(v) \le C\|v\|_{\mathrm{H}^1}$ implies the strong continuity of $\mathcal{I}$. Together with convexity this implies sequential weak lower semicontinuity of $\mathcal{I}$ on $\mathcal{H}$, which is the desired result. ■

Hence we can go to the limit in (3.2) and find

$$0 \ge m(t) \quad \text{where}$$
$$m(t) := \mathcal{E}(t,H(t)) + \textstyle\int_0^t \Psi(\partial_t H(\tau))\,\mathrm{d}\tau - \mathcal{E}(0,H_0) + \int_0^{t^*} \partial_t H_{\mathrm{ext}}(\tau) \int_\Omega \alpha H(\tau,x)\,\mathrm{d}x\,\mathrm{d}\tau.$$

This provides one side of the energy balance.

As H is Lipschitz, we can differentiate m and obtain, after a cancellation, $\dot m(t) = {}_*\langle \mathrm{D}\mathcal{E}(t,H(t)), \partial_t H\rangle_X + \Psi(\partial_t H(t))$ which is nonnegative by the stability of $H(t)$. Thus, $m(t) \le 0$, $m(0) = 0$ and $\dot m(t) \ge 0$ imply $m \equiv 0$. Thus, we have established (E) as well.

Finally we have to show uniqueness which follows again from the variational inequalities (2.9). Let H_j, $j = 1, 2$ be two solutions, then for each v we have by subtracting $(\mathrm{S})_{\mathrm{loc}}$ from $(\mathrm{E})_{\mathrm{loc}}$

$$_*\langle \mathrm{D}\mathcal{E}(t, H_j), \partial_t H_j - v\rangle_X + \Psi(\partial_t H_j) - \Psi(v) \leq 0.$$

Testing with $v = \partial_t H_{3-j}$ and adding both inequalities gives

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\, _*\langle A(H_1 - H_2), H_1 - H_2\rangle_X = {}_*\langle \mathrm{D}\mathcal{E}(t, H_1) - \mathrm{D}\mathcal{E}(t, H_2), \partial_t(H_1 - H_2)\rangle_X \leq 0.$$

If $H_1(0) = H_2(0)$, this implies $H_1(t) = H_2(t)$ and uniqueness is established. ∎

It should be noted that the theory in Section 7 of [MT04] can be generalized to prove strong convergence with

$$\|H^n(t) - H(t)\|_X \leq C(\tau_n)^{1/2} \quad \text{with } \tau_n = \frac{1}{2^n}.$$

It might be also possible to establish convergence of the order $(\tau_n)^1$, see in [Mie04, Section 4.4] and the references there.

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