Irreducible Components of the Burnside Ring

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Abstract
In this paper, we prove that each component of the Burnside ring of a finite group is the solvable component of the Burnside ring of a Weyl subgroup of its corresponding group, and we give some applications.

1 Preliminaries
Throughout this paper, $G$ is a finite group. Its Burnside ring $B(G)$ is the Grothendieck ring of the category of finite left $G$-sets. This is the free abelian group on the isomorphism classes of transitive left $G$-sets of the form $G/H$ for subgroups $H$ of $G$, two such subsets being identified if their stabilizers $H$ are conjugated in $G$. The addition and multiplication are given by the disjoint union and Cartesian product, respectively.

For a subgroup $H$ of $G$ we write $[H]$ for its conjugacy class. We write $V(G)$ for the set of conjugacy classes of subgroups of $G$ and we give it the partial order in which $[H] \leq [K]$ if some conjugate of $H$ lies in $K$.

In general, given a finite group $G$ to which one associates in a natural way some other algebraic object (such as the Burnside ring $B(G)$, or the character ring, or the cohomology with coefficients in some fixed field, etc.), one asks whether or not one can recover the group from knowledge of the algebraic object associated to it. The general answer to such a question is no and the Burnside ring is no exception. For the particular case of the Burnside rings the specific question whether two finite groups having the same Burnside ring must be isomorphic was raised by Yoshida in [9] and the first counterexamples were found by Thévenaz in [8] (see also [1] for a counterexample involving $p$-groups). If one imposes further restrictions on the groups, then the answer can become
affirmative, an instance of this occurring in [7] where it is shown that the answer to the above question is yes if we further know that both groups are abelian or Hamiltonian.

Knowing that, in general, the Burnside ring does not determine the group, one tries to find other invariants of the group which can be recovered from its Burnside ring. Very little is known in this direction. For example, it is not even known that if $G$ is simple and $G_1$ is some other group such that $B(G) \cong B(G_1)$, then $G$ is isomorphic with $G_1$. A finer invariant closely related to the Burnside ring is the so-called table of marks. This is the square matrix of order $n = \#V(G)$ whose entries are $\#[(G/H)^K]$, where $[H]$ and $[K]$ are elements of $V(G)$ and for a $G$-set $X$ and a subgroup $H$ of $G$ we use $X^H$ for the $G$-set of fixed points of $X$ under the action of $H$. It is known that the above integer is well defined; i.e., does not depend on the particular representatives $H$ and $K$ for $[H]$ and $[K]$, respectively. The morphism

$$
\nu_K : B(G) \rightarrow \prod_{[H] \in V(G)} \mathbb{Z}
$$

given by $\nu_K(X) = |X^K|$ for all $G$-sets $X$ is called the mark corresponding to $K$.

Kimmerle (see [2, Satz 7.5]) showed that the table of marks of $G$ determines the composition factors for $G$. This result was rediscovered in [4]. In particular, if $G$ is simple and $G_1$ is some other group having the same table of marks as $G$, then $G_1$ is isomorphic with $G$. As we have just said, this result is not known if one replaces the table of marks by the Burnside ring. It is known that the table of marks determines the Burnside ring but there is no known method to read a table of marks out of a Burnside ring.

In this paper, we prove a structure theorem for the Burnside ring. It is known that the Burnside ring is a product of blocks, each block being of the form $B(G)e_H$, where $e_H$ is an idempotent associated to a perfect subgroup of $G$. Recall that $H$ is perfect if $H$ has no proper normal subgroups $K$ such that the quotient group $H/K$ is solvable. When $H = 1$ the block is called the principal block. Our main result shows (a little bit more than) that every block $B(G)e_H$ of $B(G)$ is isomorphic to the principal block $B(W_G H)e_1$ of the Burnside ring of the Weyl group $N_G H / H$ of $H$. Such a result can be useful when dealing with isomorphisms of Burnside rings, which we illustrate by a couple of applications.

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2 Notation

Let $G$ be a finite group and $\Pi$ be a subset of all the prime numbers. We write $B_\Pi(G) = B(G) \otimes \mathbb{Z}_\Pi$ for the Burnside ring with coefficients in $\mathbb{Z}_\Pi$, where $\mathbb{Z}_\Pi$ is the ring of rational numbers $a/b$, with a positive integer $b$ whose prime factors lie in $\Pi$. For a subgroup $H$ of $G$ we write $O^\Pi(H)$ for the smallest normal subgroup $K$ of $H$ such that $H/K$ is solvable of order coprime to the primes in $\Pi$. We put $P_\Pi(G) = \{ [H] \in V(G) \mid O^\Pi(H) = H \}$. For a subgroup $H$ of $G$ we let $e_{G,H}$ be the primitive idempotent in the ghost ring

$$
\Omega(G) = \prod_{[H] \in V(G)} \mathbb{Z}
$$

of $G$ corresponding to $H$; i.e., of the form $e_{G,H} = (\delta_{HK})_{[K]}$, where $\delta_{HK} = 1$ if $[H] = [K]$ and 0 otherwise. Finally, for $H \in P_\Pi(G)$ we let $e^\Pi_{G,H}$ be the primitive idempotent of $B(G)$ corresponding to $H$, which by a result of Yoshida (see [10]) is given by

$$
e^\Pi_{G,H} = \sum_{[K] \in V(G) \atop O^\Pi(K) = H} e_{G,K}.$$
We also use the standard notation, namely that for \( g \in G \) and \( H \leq G \) we write \( g^H \) for \( gHg^{-1} \) and \( N_G H \) for the normalizer of \( H \) in \( G \). Finally, to simplify the notation, for a subgroup \( H \) of \( G \) we denote its Weyl group \( N_G H = H \) in \( G \) by \( W_G \).

\[ \text{3 Blocks of the Burnside ring} \]

Our main result is the following.

**Theorem 3.1.** We have

\[
B_\Pi(W_G H) e_{W_G H, 1} = B_\Pi(G) e_{G, 1} = \oplus_{[K] \leq V(G)} \mathbb{Z} \Pi G/K e_{G, H}^{II},
\]

In particular,

\[
B_\Pi(G) e_{G, 1} = \oplus_{[K] \leq V(G)} \mathbb{Z} \Pi G/K
\]

**Proof.** We first prove that the sum in formula (1) is direct. Assume that an equation of the form

\[
\sum_{[K] \leq V(G)} \lambda_K G/K e_{G, H}^{II} = 0
\]

holds with some coefficients \( \lambda_K \) not all zero. Let \( [L] \in V(G) \) be maximal with \( [O^H(L)] = [H] \) and \( \lambda_L \neq 0 \). We apply the mark \( \nu_L \) to the above equation (3) and get \( \lambda_L \nu_L(G/L) = 0 \) because \( \nu_L(e_{G, H}^{II}) = 1 \). Thus, \( \lambda_L = 0 \), which is a contradiction. Hence, the sum is direct.

We now only need to prove that the left hand side of (1) is contained in the right hand side of it. Notice that the relation

\[
e_{G, H}^{II} = \sum_{O^H(K) \leq H} b_K G/K,
\]

holds with some coefficients \( b_K \), so for any subgroup \( T \leq G \) we have

\[
G/T e_{G, H}^{II} = \sum_{O^H(K) \leq H} c_K G/K e_{G, H}^{II},
\]

with some other coefficients \( c_K \). Thus, it is enough to check that if the containment \( [O^H(K)] < [H] \) holds, then \( G/K e_{G, H}^{II} = 0 \). Assume therefore that \( G/K e_{G, H}^{II} \neq 0 \), take \( L \leq G \) such that \( \nu_L(G/K e_{G, H}^{II}) \neq 0 \) and note that

\[
0 = \nu_L(G/K e_{G, H}^{II}) = \nu_L(G/K) \nu_L(e_{G, H}^{II}).
\]

Hence, \( [O^H(L)] = [H] \) and \( [L] \leq [K] \), therefore

\[
[H] \leq [O^H(L)] \leq [O^H(K)] < [H].
\]

This contradiction proves the stated equality.

Applying the equality (1) at \( H = 1 \) we get

\[
B_\Pi(G) e_{G, 1}^{II} = \oplus \mathbb{Z} \Pi G/K e_{G, H}^{II}.
\]

By applying the mark \( \nu_K \) with a solvable subgroup \( K \) to this last equality, we get \( G/K e_{G, 1}^{II} = G/K \).

We now consider the morphism

\[
B_\Pi(W_G H) e_{W_G H, 1}^{II} = \oplus_{[O^H(K)] = [H]} \mathbb{Z} \Pi W_G H/K e_{G, 1}^{II} = \oplus \mathbb{Z} \Pi G/K e_{G, 1}^{II},
\]
induced by
\[ f \left( \frac{W_G H}{K/H} \right) = G \prod_{K \leq G} \frac{G}{K} \]
and extended linearly. It is easy to see that this morphism is well defined. On the other hand, it sends a basis into a basis since if \( G/Ke \cong G/Le \), then \( |K| = |L| \). Thus, \( K = aLa^{-1} \) holds with some \( a \in G \), therefore \( H = O^\Pi(K) = aO^\Pi(L)a^{-1} = aHa^{-1} \), which implies that \( a \in N_G H \), leading to the conclusion that
\[ \frac{W_G H}{K/H} = \frac{W_G H}{L/H}. \]

The above argument shows that \( f \) is an isomorphism of abelian groups. We now check that this is a ring isomorphism as well. It clearly suffices to check it for the elements of a base. We have
\[ f \left( \frac{W_G H}{K/H} \times \frac{W_G H}{L/H} \right) = \sum_{a \in K \setminus W_G H \setminus L} f \left( \frac{W_G H}{K \cap a L/H} \right), \tag{4} \]
where in the above formula (4) we used \( K \) and \( L \) for \( K/H \) and \( L/H \), respectively, and the last sum above equals
\[ \sum_{a \in K \setminus N_G H \setminus L} \frac{G}{K \cap a L} \prod_{K \leq a} G/H. \tag{5} \]

We now see that if
\[ \frac{G}{K \cap a L} \prod_{K \leq a} G/H \neq 0, \]
then there exists a subgroup \( T \leq G \) such that \( O^\Pi(T) = H \) and \( T \leq K \cap a L \leq a L \). In this case, \( aH = H \), therefore \( a \in N_G H \), and so
\[ \sum_{a \in K \setminus N_G H \setminus L} \frac{G}{K \cap a L} \prod_{K \leq a} G/H = \sum_{a \in K \setminus G/L} \frac{G}{K \cap a L} \prod_{K \leq a} G/H = \frac{G}{K} \times G \prod_{K \leq L} G/H. \tag{6} \]

Now (4)–(6) complete the proof of Theorem 3.1. \( \Box \)

We will denote by \( B_S(G) \) the solvable component of the Burnside ring of \( G \).

4 Normal subgroups and the Burnside ring

Throughout this section, we assume that \( G \) and \( G_* \) are two groups such that their Burnside rings are isomorphic. Let \( \sigma \) be an isomorphism of \( B(G) \) onto \( B(G_*) \). Building on work of Nicolson [5] and extending results of Kimmerle and Roggenkamp from [3] in which the group of automorphisms of the Burnside ring of a finite group was analyzed, in [6] it is shown that one may assume that this isomorphism is normalized; i.e., that \( \sigma(G/1) = G_*/1 \).

We give a generalization of these results for the components of the Burnside ring.

Given an isomorphism \( \theta : B_S(G) \to B_S(G') \) we can extend it to an isomorphism of the corresponding restricted ghost rings \( \theta : \Omega_S(G) \to \Omega_S(G') \), where
\[ \Omega_S(G) = \prod_{\substack{|H| \leq V(G) \\text{H solvable}}} \mathbb{Z}. \]

**Theorem 4.1.** Let \( G \) and \( G' \) be finite groups, and \( \theta : B_S(G) \to B_S(G') \) a normalized isomorphism. For any solvable subgroup \( D \) of \( G \), let \( D' \) denote a subgroup of \( G' \) such that \( \theta(e_{G,D}) = e_{G',D'} \). Let \( V \) be a solvable subgroup of \( G \). Then \( V' \) is solvable, \(|V'| = |V|\), \(|N_G(V')| = |N_G(V)|\), and \( \theta(G/V) = G'/V' + \sum_{T \in S_V} a_T G'/T \) where \( S_V \) is the family of solvable subgroups \( T \) of \( G' \) such that \( |T| \) is a proper divisor of \(|U|\).
Claim 2]. Hence, there exists a subgroup follows now by replacing \( V = N \) that contains a normal subgroup of index \( p \) and \( \sigma_p(U) \) is normalized and let.

Throughout the following proposition and its proof replace \( G \) by \( G = V \).

Using (8) with \( e \) maps \( U \) \( \sigma(V) \) to \( \sigma(V) \) preserves the ordering. In general, the answer is no, and our first result here illustrates the obstruction to \( \sigma \), preserving the ordering.

Throughout the following proposition and its proof \( p \) denotes a prime number.

**Proposition 4.2.** Let \( \sigma : B(G) \longrightarrow B(G) \) be an isomorphism of Burnside rings. Assume that \( \sigma \) is normalized and let \( \sigma \) denote the induced bijection from \( V(G) \) to \( V(G) \). Let \( U \leq V \leq G \) be such that \( U \) is maximal in \( V \) of index \( p \) and \( [\sigma_p(U)] \not\subseteq [\sigma_p(V)] \). Assume further that both \( U \) and \( V \) sit inside the block corresponding to the same perfect group \( P \). Then the number \# \{ \{U \mid \sigma_p(U) \subseteq V\} \) is a multiple of \( p \).

**Proof.** By Theorem 3.1 we may assume that \( V \) is solvable; i.e., that \( P = 1 \) for if not we may replace \( G \) by \( W_G \).

Applying \( \sigma \) to the well known formula

\[
\sigma(e_{G;V}) = e_{G;V}.
\]

Applying \( \sigma \) to the well known formula

\[
e_{G;V} = \frac{1}{|N_{G;V}|} G_{V} - \beta(U;V) G_{U} + \sum_{[L_1 \leq [V]]} a_{L_1} G_{L_1}/L_1,
\]

where \( \beta(U;V) = \# \{ \{U \mid \sigma_p(U) \subseteq V\} \) (see [10]) we arrive at

\[
\sigma(e_{G;V}) = \frac{1}{|N_{G;V}|} \sigma(G/V) - \beta(U;V) \sigma(G/U) + \sum_{[L_1 \leq [V]]} a_{L_1} \sigma(G/L).
\]

Using (8) with \( \sigma \) instead of \( V \) we get

\[
e_{G;V} = \frac{1}{|N_{G;V}|} G_{V}/V + \sum_{[L_1 \leq [V]]} b_{L_1} G_{L_1}/L_1.
\]

Using 4.1 and (7)–(10) together we arrive at

\[
\frac{\beta(U;V)}{p} G_{V}/U + \sum_{[L_1 < [V]]} c_{L_1} G_{L_1}/U.
\]

In this last equation we used the known fact that \( |N_{G;V}/V| = |N_{G}/V| \). From (11) we immediately see that if \( [U_1] \not\subseteq [V_1] \), then the coefficient of \( G_{V}/U \) must be an integer and therefore \( p \mid \beta(U;V) \).}

An isomorphism \( \sigma \) from \( B(G) \) onto \( B(G) \) is called completely normalized if for each perfect subgroup \( P \) the subgroup \( P \) of \( G \) is again perfect. We record the following corollaries.

**Corollary 4.3.** Suppose that \( \sigma : B(G) \longrightarrow B(G) \) is a completely normalized isomorphism. Assume that \( V \) and \( N \) are subgroups of \( G \) sitting inside the block of the same perfect subgroup \( P \); that is, \( P \leq H \leq N/G \) and \( H/P \) is solvable holds for \( H \in \{V, N\} \). Assume further that \( N \) is normal in \( G \) and \( N \leq V \). Then \( N \leq V \).

**Proof.** By Theorem 3.1, it follows that we may assume that \( V \) is solvable (otherwise, we may replace \( G \) by \( W_G \) and both \( V \) and \( N \) by \( V/P \) and \( N/P \), respectively). Let \( p \) be some prime such that \( V/N \) contains a normal subgroup of index \( p \). Since \( V \) itself contains a normal subgroup of index \( p \), it follows that the number of maximal subgroups of index \( p \) in \( V \) is not 1 (mod \( p \)) (see [3, Claim 2]). Hence, there exists a subgroup \( U \) of \( G \) such that \( N \leq U \leq V \), the index of \( U \) in \( V \) is \( p \) and \( \{U \mid \sigma(U) \subseteq V\} \) is not a multiple of \( p \). Proposition 4.2 shows that \( U \leq V \). Corollary 4.3 follows now by replacing \( V \) with \( U \) and by induction.

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Note that each isomorphism between Burnside rings maps the principal blocks in to each other. Thus, an immediate consequence of Corollary 4.3 is the following result.

**Corollary 4.4.** $B(G)$ determines the lattice of the solvable normal subgroups of $G$.

As final application, we describe the structure of the automorphism group of $B(G)$ for a general finite group $G$. Let $B_1, \ldots, B_m$ be representatives of the isomorphism types of the blocks of $B(G)$ with multiplicity $k_i$. Clearly,

$$\text{Aut}(B(G)) = \prod_{i=1}^{m} \text{Aut}(B_i) \rtimes S_{k_i}.$$ 

Denote by $P_i$ a perfect subgroup of $G$ corresponding to $B_i$. By Theorem 3.1, we know that the $B_i$ are the principal blocks of $N_i := N_G(P_i)/P_i$. Hence, we can apply the results on automorphisms of Burnside rings of finite solvable groups (see [3] and [6]). Using [5, Proof of Prop.3.4] and [3, 2.3 and 2.4], we get

$$\text{Aut}(B_i) = \text{Aut}_n(B_i) \rtimes X,$$

where $\text{Aut}_n(B_i)$ denotes the group of normalized automorphisms of the block $B_i$ and $X$ depends on the normal subgroup structure of $N_G(P_i)/P_i$ as follows:

a) If $N_i \not\cong C_2 \times C_2 \times O$, where $O$ is a group of odd order, then

$$X \cong C_2^{n_i},$$

with $n_i = a_i + b_i$, where

- $a_i$ is the number of odd primes $p$ such that $N_i$ has a unique subgroup of order $p$, and
- $b_i = 1$ if $N_i$ has a central subgroup of order 2 which is contained in each subgroup of order 4 of $N_i$, or which is the only subgroup of $N_i$ of order 2; otherwise, $b_i = 0$.

b) If $N_i \cong C_2 \times C_2 \times O$, then

$$\text{Aut}(B_i) = S_4 \times \text{Aut } O \text{ and } \text{Aut } O = \text{Aut}_n(B_i) \rtimes C_2^{a_i},$$

where $a_i$ denotes again the number of odd primes $p$ such that $N_i$ has a unique subgroup of order $p$.

Thus, the determination of $\text{Aut}(B(G))$ is reduced to the determination of normalized automorphisms of the principal block of the Burnside rings of its Weyl groups. These automorphisms induce on the other hand automorphisms of the subgroup lattice of the solvable normal subgroups of these Weyl groups, cf. Corollary 4.4. For computational aspects of the determination of such normalized automorphisms, we refer the reader to [3].

### References


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