

**Universität  
Stuttgart**

**Fachbereich  
Mathematik**

---

A Least Squares Functional for Solving Inverse  
Sturm-Liouville Problems

Norbert Röhl

---

**Preprint 2005/004**



**Universität Stuttgart**  
**Fachbereich Mathematik**

---

A Least Squares Functional for Solving Inverse  
Sturm-Liouville Problems

Norbert Röhl

---

**Preprint 2005/004**

Fachbereich Mathematik  
Fakultät Mathematik und Physik  
Universität Stuttgart  
Pfaffenwaldring 57  
D-70569 Stuttgart

**E-Mail:** [preprints@mathematik.uni-stuttgart.de](mailto:preprints@mathematik.uni-stuttgart.de)

**WWW:** <http://www.mathematik.uni-stuttgart.de/preprints>

ISSN **1613-8309**

© Alle Rechte vorbehalten. Nachdruck nur mit Genehmigung des Autors.  
L<sup>A</sup>T<sub>E</sub>X-Style: Winfried Geis, Thomas Merkle

## 1. INTRODUCTION

The one dimensional inverse Sturm-Liouville problem and methods for its numerical solution are very well studied [9, 12, 2]. Still there is no general purpose algorithm, which efficiently satisfies all needs.

Recently Brown and Weikard [3] proved, that an integrable potential on a simply connected finite tree is uniquely determined by the generalized Dirichlet-to-Neumann map. Numerical studies by Schapotschnikow [13] suggest that a variational approach is most suited for solving the inverse problem numerically. He successfully applied the approach of Brown et al. [2], but found the convergence speed not to be satisfactory in this case.

We present a variational algorithm which proves to be very robust under noisy input, does not require any special tuning or additional input besides the partial spectra, and is quite efficient in the class of variational methods. In section 2 we will define the functional and prove the absence of true local minima, which could trap our minimization procedure. Numerical examples, the optimal choice of the weights and the performance under noisy input will be discussed in section 3. Section 4 is concerned with implementation details and compares this algorithm to related approaches, and section 5 finally contains the proof of the linear independence of the squares of eigenfunctions.

## 2. DEFINITION AND PROPERTIES OF THE FUNCTIONAL

We consider the Sturm-Liouville equation

$$(SL) \quad -u'' + q(x)u = \lambda u$$

on  $[0, 1]$  with  $q(x) \in L^2([0, 1], \mathbb{R})$  real, and separated boundary conditions

$$(\alpha\beta) \quad u(0) \cos \alpha + u'(0) \sin \alpha = 0, \quad u(1) \cos \beta + u'(1) \sin \beta = 0.$$

For the asymptotics of the eigenvalues  $(\lambda_n)_{n=0}^{\infty}$  w.r.t. boundary conditions  $(\alpha\beta)$  there are three cases (see eg. [14, 5, 4, 10]):

$$(2.3) \quad \lambda_n = \begin{cases} \pi^2 n^2 - \frac{2 \sin(\beta-\alpha)}{\sin \alpha \sin \beta} + \int_0^1 q \, ds + a_n & \text{if } \sin \alpha \sin \beta \neq 0 \\ \pi^2 (n + 1/2)^2 + \frac{2 \cos \alpha \cos \beta}{\sin(\beta-\alpha)} + \int_0^1 q \, ds + a_n & \text{if } \sin \alpha \sin \beta = 0 \wedge \sin(\beta - \alpha) \neq 0 \\ \pi^2 (n + 1)^2 + \int_0^1 q \, ds + a_n & \text{if } \sin \alpha = \sin \beta = 0 \end{cases}$$

where  $(a_n) \in l^2$ .

It is a classical result, that two spectra determine the potential uniquely. Fix three angles  $\alpha$ ,  $\beta$ , and  $\gamma$  with  $\sin(\beta - \gamma) \neq 0$ . Let  $\lambda_{q,i,n}, g_{q,i,n}$ ,  $i \in \{1, 2\}$  denote the eigenvalues and eigenfunctions of  $L^2$ -norm 1 with respect to the boundary conditions  $(\alpha\beta)$  resp.  $(\alpha\gamma)$ .

**Theorem 2.1** (Borg [1], Levinson [6]). *Given potentials  $Q, q \in L^1[0, 1]$  with*

$$\lambda_{Q,i,n} = \lambda_{q,i,n} \quad \text{for all } n \in \mathbb{N}, \quad i = 1, 2,$$

*then  $Q = q$ .*

Suppose we are given (partial) spectral data  $\lambda_{Q,i,n}$  with  $(i, n)$  in  $I \subseteq \{1, 2\} \times \mathbb{N}$  of an unknown potential  $Q$ . For a trial potential  $q$  and positive weights  $\omega_{i,n}$ , we define the functional

$$(2.4) \quad G(q) = \sum_{(n,i) \in I} \omega_{i,n} (\lambda_{q,i,n} - \lambda_{Q,i,n})^2.$$

If  $I$  is infinite and the sequence  $(\omega_{n,i})$  is summable, this converges because of the asymptotics of the eigenvalues (2.3)

$$G(q) = \sum_{(n,i) \in I} \omega_{n,i} \left( \int_0^1 Q - \int_0^1 q + a_{i,n} \right)^2 < \infty,$$

where  $a_{i,n} \in l^2, i = 1, 2$ .

If moreover  $I = \{1, 2\} \times \mathbb{N}$ , the functional is zero if and only if  $q = Q$  (by theorem 2.1). In the case of partial knowledge of the spectrum, a solution of  $G(q) = 0$  includes all information given about the unknown potential  $Q$ . A recent result of Marletta and Weikard [8] guarantees that increasing the number of eigenvalues and minimizing  $G(q)$  brings us (in a certain weak sense) arbitrarily close to  $Q$ . With sufficiently strong a-priori conditions on the boundedness of  $q - q_0$ , even  $H^n$  convergence for a fixed  $n$  holds.

An important question is now, for which sequences  $(\lambda_{Q,i,n})$  there is a  $q$  with  $G(q) = 0$ , i.e.  $\lambda_{q,i,n} = \lambda_{Q,i,n}$ . It is easy to see with the classical methods [14], that it is necessary for the eigenvalues to interlace

$$\lambda_{Q,1,n} < \lambda_{Q,2,n} < \lambda_{Q,1,n+1}.$$

It is also known that if we choose either

- (i)  $\sin \alpha, \sin \beta, \sin \gamma \neq 0$  (Levitan [7]) or
- (ii)  $\alpha = \beta = 0, \sin \gamma \neq 0$  (Dahlberg Trubowitz [4]),

then the interlacing property in connection with the correct asymptotics is sufficient for the existence of a potential  $q$  with  $G(q) = 0$ .

The most interesting feature of this functional is, that all its critical points are at global minima. To prove this, we first compute the derivative of  $G$ . The derivative of  $\lambda_{q,i,n}$  w.r.t.  $q$  in direction  $h$  is

$$\dot{\lambda}_{q,i,n}[h] = \int_0^1 h g_{q,i,n}^2$$

(see [10] for a nice proof). Thus the derivative of  $G$  is given by

$$\dot{G}[h](q) = 2 \sum_{(n,i) \in I} \int_0^1 \omega_{n,i} (\lambda_{q,i,n} - \lambda_{Q,i,n}) g_{q,i,n}^2 h \, dx.$$

We note that if  $n\omega_{n,i}$  is summable, then the function

$$2 \sum_{(n,i) \in I} (n+1) \omega_{n,i} (\lambda_{q,i,n} - \lambda_{Q,i,n}) \frac{1}{n+1} g_{q,i,n}^2$$

is in  $H^1$  because  $\|g_{q,i,n}^2\|_{H^1} = O(n)$  [14] and  $\lambda_{q,i,n} - \lambda_{Q,i,n} = O(1)$ .

Theorem 5.1 below shows that the eigenfunctions  $g_{q,i,n}^2$  are linearly independent in  $H^1$ . This immediately implies the essential convexity of the functional:

**Theorem 2.2.** *The functional  $G$  has no local minima at  $q$  with  $G(q) > 0$ , i.e.*

$$\dot{G}[h](q) = 0 \quad \forall h \iff G(q) = 0.$$

Thus a conjugate gradient algorithm is an effective method to compute zeros of  $G$ , as we will also observe in the examples.

Finally, we want to address the question, which of the infinitely many  $q$  with  $G(q) = 0$  our algorithm will select. Let us first look at a related functional, which actually was also our first try. It is given by

$$\tilde{G}(q) = \sum_{(n,i) \in I} (\lambda_{q,i,n} - \lambda_{Q,i,n} + \int_0^1 (Q - q) dx)^2,$$

with derivative

$$\dot{\tilde{G}}[h](q) = 2 \sum_{(n,i) \in I} \int_0^1 (\lambda_{q,i,n} - \lambda_{Q,i,n}) (g_{q,i,n}^2 - 1) h dx.$$

In case the mean of  $Q$  is known, this functional works as well as the other. Its gradient flow leaves the mean of  $q$  constant by construction. The function  $g_{q,i,n}^2 - 1$  is the gradient of  $\lambda_{q,i,n} - \int_0^1 q dx$  and the direction of strongest increase of  $\lambda_{q,i,n}$  which leaves  $\int_0^1 q dx$  fixed.

Choosing  $\alpha = \beta = 0$  and  $\gamma = \pi/2$ , the asymptotics of the squared eigenfunctions are given by

$$(2.5) \quad g_{q,i,n}^2 = \begin{cases} 1 - \cos((2n+2)\pi x) + O(n^{-1}) & , i = 1 \\ 1 - \cos((2n+1)\pi x) + O(n^{-1}) & , i = 2 \end{cases}.$$

Hence the functions  $\{g_{q,i,n}^2 - 1 | n \in \mathbb{N}, i = 1, 2\} \cup \{1\}$  are almost orthogonal for large  $n$ . Now, the derivative of our functional (2.4) can be written as

$$\dot{G}[h](q) = 2 \sum_{(n,i) \in I} \int_0^1 \omega_{n,i} (\lambda_{q,i,n} - \lambda_{Q,i,n}) ((g_{q,i,n}^2 - 1) + 1) h dx,$$

and the difference  $\lambda_{q+c,i,n} - \int (q+c) dx = \lambda_{q,i,n} - \int q dx$  is invariant under adding a constant function  $c$  to  $q$ . It follows that a gradient flow of our functional  $G$  leaves  $\lambda_{q,i,n} - \int_0^1 q dx$  with  $(i, n) \notin I$  almost invariant for  $n$  large enough.

Since also a conjugate gradient descent is just an approximation of the gradient flow, in practice we do get some little changes in the higher eigenvalues. A similar argument holds for the case  $\sin \alpha, \sin \beta, \sin \gamma \neq 0$ , but there the asymptotics for  $i = 1, 2$  are equal. So we still have asymptotic orthogonality in  $n$ , but can not separate  $i = 1$  and  $i = 2$ .

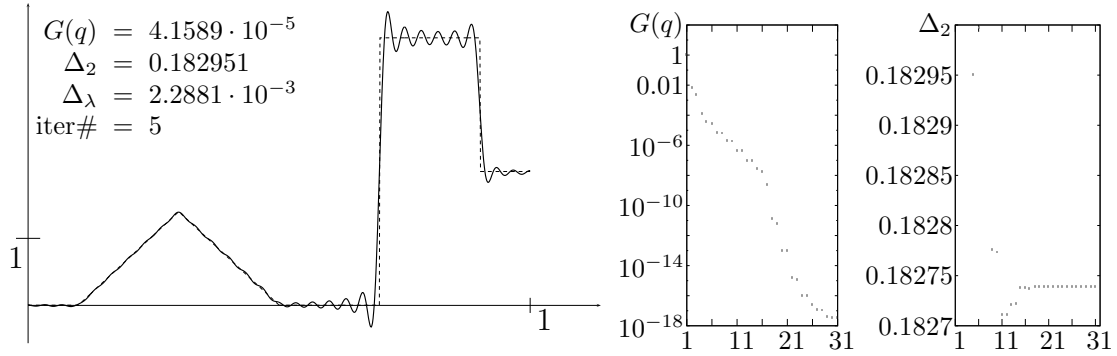


FIGURE 1. Plot using  $\omega_{n,i} = 1$ . The right two graphs show  $G(q)$  and  $\Delta_2$  over the number of iterations.

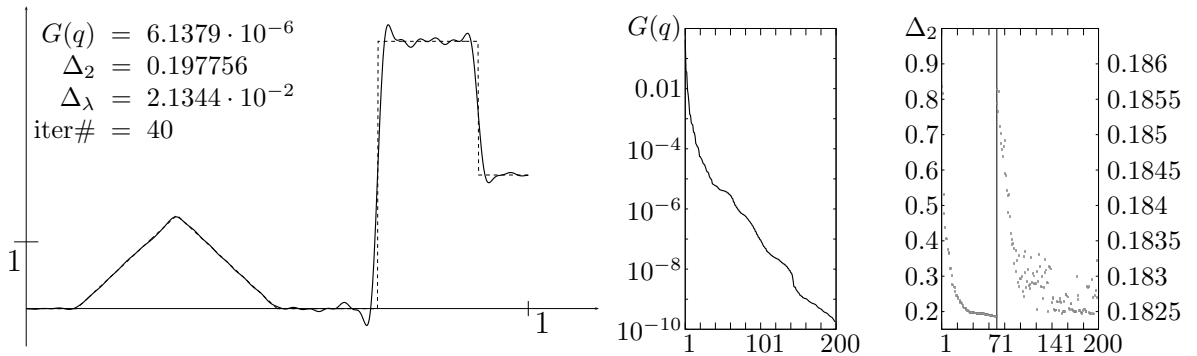


FIGURE 2. Plot using  $\omega_{n,i} = (n+1)^{-2}$ .

### 3. NUMERICAL EXAMPLES

The first natural question is the optimal choice of the constants  $\omega_{n,i}$ . A short look at the asymptotics (2.3) may suggest something like  $\omega_{n,i} = (n+1)^{-2}$ . But on the other hand we have  $\lambda_{q+c,i,n} = \lambda_{q,i,n} + c$  for all  $n$ . This observation corresponds to choosing  $\omega_{n,i} = 1$  and yields the fastest convergence in all examples.

Figures 1 and 2 were computed with  $\omega_{n,i} = 1$  resp.  $\omega_{n,i} = (n+1)^{-2}$ . For all plots (except 2, 3) we chose the lowest number of iterations after which the plot stabilizes, while usually already the second iteration reveals the global structure of the potential. By default we use 30 pairs of eigenvalues,  $\omega_{n,i} = 1$ , and the boundary conditions  $\alpha = \beta = 0$ ,  $\gamma = \pi/2$ . We give the current value of the functional  $G(q)$  as well as the  $L^2$  error  $\Delta_2 = \|q - Q\|_2$ , and the maximum of the difference of the eigenvalues  $\Delta_\lambda = \max_{(n,i) \in I} \{|\lambda_{q,i,n} - \lambda_{Q,i,n}|\}$ . In addition if there is a visible difference, we also plot the original potential  $Q$  using a dashed line.

We see that both  $G(q)$  and  $\Delta_2$  converge much faster for  $\omega_{n,i} = 1$ . With  $\omega_{n,i} = (n+1)^{-2}$  on the other hand,  $q$  moves through smoother functions (as in figure 2). In contrast to the optical impression these are in general worse approximations w.r.t.  $G(q)$  and  $\Delta_2$ . But finally  $q$ , after around 100 iterations, will also converge to the function shown in figure 1.



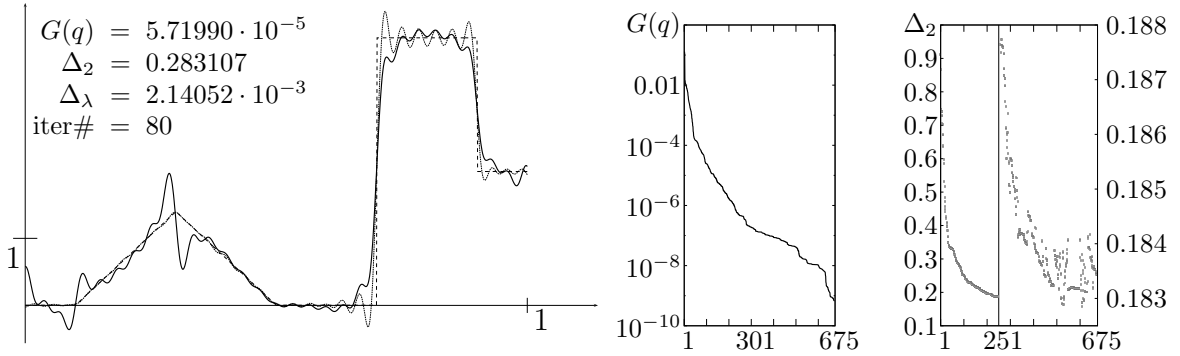


FIGURE 3. Plot with boundary conditions  $\alpha = \beta = \pi/4$ ,  $\gamma = -\pi/4$ . The light graph shows the 620th iteration, where the iteration stabilizes.

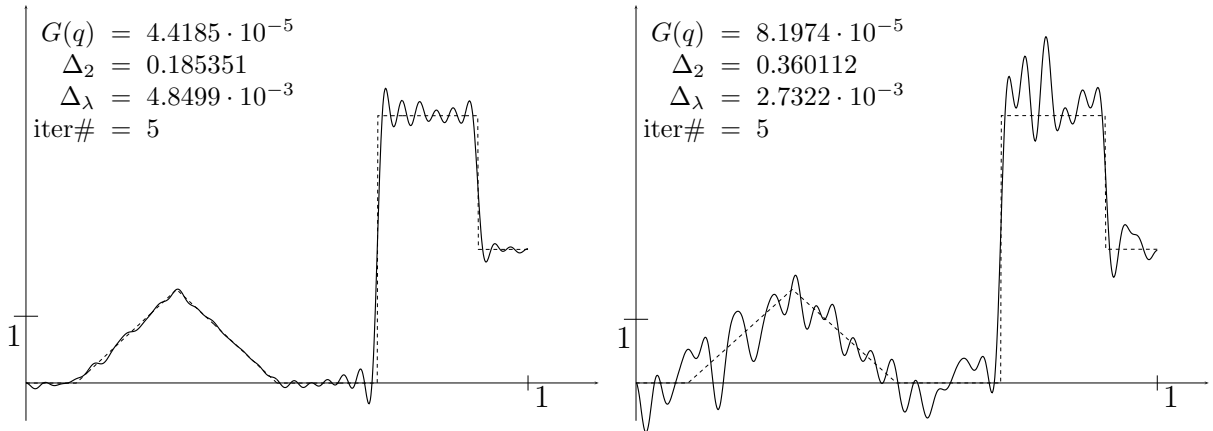


FIGURE 4. Plots with added white noise of absolute value smaller or equal 0.01 resp. 0.1.

From the good average convergence of  $G(q)$  in both cases, we can tell, that our algorithm also in practice does not get near a local minimum, as proved in theorem 2.2.

The optimal choice of boundary conditions can already be guessed from the asymptotics of the squared eigenfunctions (2.5). Choosing  $\alpha = \beta = 0$  and  $\gamma = \pi/2$ , the functions  $g_{q,i,n}^2 - 1$  are almost orthogonal for large  $n$  and  $i = 1, 2$  (c.f. (2.5)). In contrast, if  $\sin \alpha$ ,  $\sin \beta$ , and  $\sin \gamma$  are all non zero,  $g_{q,1,n}^2 - 1$  and  $g_{q,2,n}^2 - 1$  will get close for large  $n$ . Therefore, in the latter case, the algorithm will converge much slower. This can also clearly be observed in numerical examples, like in figure 3. But also there, after approximately 620 iterations, we finally will get a solution which is close to figure 1.

Another important aspect for applications is stability against noise in the given spectral data. In figure 4 we computed two examples with random noise  $|\tilde{\lambda}_{Q,i,n} - \lambda_{Q,i,n}| \leq r$ , with  $r = 0.01$  and  $r = 0.1$ , respectively. It is remarkable, that the convergence speed in  $G(q)$  is not significantly affected by the random noise; both examples reach  $G(q) \sim 10^{-18}$  in 30 iterations.

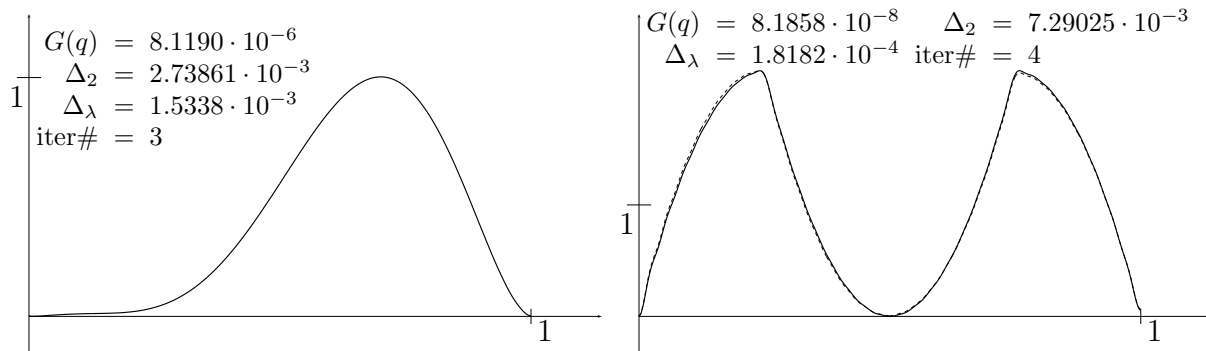


FIGURE 5. Continuous examples.

For testing the robustness, we also fed the functional with the following random sequence

9.99742, 11.6265, 14.4527, 23.9247, 26.2413, 31.091, 40.6658, 48.1088, 53.5093, 60.9088,

which is far from the generic asymptotics. Convergence of  $G(q)$  is slow but steady. In log scale the graph of  $G(q)$  over the number of iteration looks similar to those given above. From  $G(q) = 36287$  initially, we get down to  $G(q) = 3.56528 \cdot 10^{-9}$  in 450 iterations.

Finally figure 5 shows the results for the other examples of [2, 12].

#### 4. IMPLEMENTATION AND COMPARISON

The implementation of the conjugate gradient minimization algorithm is straight forward (see eg. [11]), but our application crucially depends on a reliable eigenvalue problem solver. After some trial and error we settled for d02kdf from the Numerical Algorithms Group (NAG). For standard numerical routines we used the GNU Scientific Library.

Another important point is the internal representation of the functions. We used cubic splines on 2000 intervals of equal length. Since all high level routines, like the eigenvalue solver, only sample these functions on some points, the details of the representation are not so important. But since very many values are needed, this is the part in the algorithm, where speed optimizations are most valuable.

Compared to the variational algorithm by Brown et al. [2], which uses other spectral data, our functional is more expensive to compute, because they only have to solve initial value problems. On the other hand they often need about 10-100 times as many iterations to get similar results. Another plus on our side is, that even in presence of noise, we know to get closer to our goal each step and do not have to regularize the process by limiting the number of iterations.

The method of Rundell and Sacks [12] uses the same spectral data as our algorithm and is without any doubt much faster, but needs the mean  $\int_0^1 Q dx$  as additional input, which has to be guessed from the (partial) spectral data. Our method does this automatically. In the case of figure 1 for example, the error of the mean is  $2 \cdot 10^{-3}\%$  and using 5 pairs of eigenvalues we still only get an error of about 0.2%. This is probably better than an independent algorithm for computing the mean value from spectral data could do.

## 5. LINEAR INDEPENDENCE OF SQUARED EIGENFUNCTIONS

Define the Wronskian  $[f, g] = fg' - f'g$  and the bilinear form

$$\begin{aligned} \Gamma : H^1([0, 1], \mathbb{R})^2 &\longrightarrow \mathbb{R} \\ (f, g) &\longmapsto \int_0^1 [f, g] \, dx \end{aligned}$$

which is bounded by

$$|\Gamma(f, g)| \leq \|f\|_{H^1} \|g\|_{H^1}, \text{ i.e. } \|\Gamma(f, \cdot)\| = \|f\|_{H^1}.$$

(We use the definition  $\|f\|_{H^1} = \sqrt{\|f\|_{L^2}^2 + \|f'\|_{L^2}^2}$  with distributional derivatives.) In particular  $\Gamma$  is continuous on  $H^1$ . We have the following rules for the Wronskian:

- (i)  $[fg, hj] = gj[f, h] + fh[g, j] = fj[g, h] + gh[f, j]$
- (ii) If the functions  $f_1$  and  $f_2$  fulfill the condition

$$f_i(a) \cos \alpha + f'_i(a) \sin \alpha = 0 \quad i = 1, 2,$$

then  $[f_1, f_2](a) = 0$ .

- (iii) For two arbitrary solutions  $f_1$  and  $f_2$  of the equation (SL) with eigenvalue parameters  $\lambda_1$  and  $\lambda_2$  we have

$$[f_1, f_2]' = f_1 f_2'' - f_2 f_1'' = (\lambda_1 - \lambda_2) f_1 f_2.$$

In this section we are only talking about a single  $q$  and therefore drop it from the index. Let  $s_{i,n}$  and  $c_{i,n}$  be the solutions of the differential equation (SL) for the eigenvalue parameter  $\lambda_{i,n}$  and initial values

$$\begin{aligned} s_{i,n}(1) &= \sin \beta, & c_{i,n}(1) &= \sin \gamma, \\ s'_{i,n}(1) &= -\cos \beta, & c'_{i,n}(1) &= -\cos \gamma. \end{aligned}$$

It is well known and easy to see that the Wronskian of these solutions is constant and we can compute its value at 1:

$$[s_{i,n}, c_{i,n}] = -\sin \beta \cos \gamma + \cos \beta \sin \gamma = \sin(\gamma - \beta).$$

The normalized eigenfunctions are  $g_{1,n} = s_{1,n}/\|s_{1,n}\|_2$  and  $g_{2,n} = c_{2,n}/\|c_{2,n}\|_2$ . Now we prove the central lemma, which builds on the ideas of a similar result for the Dirichlet case in the book of Pöschel and Trubowitz[10]. A result in the same spirit can also already be found in the paper of Borg [1].

**Lemma 5.1.** *Given three angles  $\alpha$ ,  $\beta$ , and  $\gamma$  with  $\sin(\beta - \gamma) \neq 0$  and denote the  $L^2$ -normalized eigenfunctions of the Sturm-Liouville equation with boundary conditions  $(\alpha\beta)$  and  $(\alpha\gamma)$  by  $g_{i,n}$ ,  $i = 1, 2$ . With  $s_{i,n}$ ,  $c_{i,n}$  as defined above we have the following relations for the squared eigenfunctions for all  $i, j \in \{1, 2\}$  and  $m, n \in \mathbb{N}$ :*

- (i)  $\Gamma(g_{i,n}^2, g_{i,m}^2) = 0$
- (ii)  $\Gamma(c_{i,n} s_{i,n}, g_{j,m}^2) = (-1)^i \sin(\gamma - \beta) \delta_{n,m} \delta_{i,j}$

*Proof.* i)

$$\Gamma(g_{i,n}^2, g_{i,m}^2) = 2 \int_0^1 g_{i,n} g_{i,m} [g_{i,n}, g_{i,m}] \, dx$$

If  $n = m$ , the term clearly vanishes. If  $n \neq m$ , we get

$$\Gamma(g_{i,n}^2, g_{i,m}^2) = \frac{2}{\lambda_{i,n} - \lambda_{i,m}} \int_0^1 [g_{i,n}, g_{i,m}]' [g_{i,n}, g_{i,m}] dx = \frac{1}{\lambda_{i,n} - \lambda_{i,m}} [g_{i,n}, g_{i,m}]^2 \Big|_0^1 = 0,$$

because  $g_{i,n}$  and  $g_{i,m}$  satisfy the same boundary conditions (rule ii).

ii)

$$\Gamma(c_{i,n} s_{i,n}, g_{j,m}^2) = \int_0^1 (c_{i,n} g_{j,m} [s_{i,n}, g_{j,m}] + s_{i,n} g_{j,m} [c_{i,n}, g_{j,m}]) dx$$

If  $i = j = 1, m = n$ , the first term vanishes and we are left with

$$\int_0^1 s_{1,m} g_{1,m} [c_{1,m}, g_{1,m}] = \int_0^1 g_{1,m}^2 [c_{1,m}, s_{1,m}] = -\sin(\gamma - \beta),$$

and if  $i = j = 2, m = n$ , the second term vanishes and we get

$$\int_0^1 c_{2,m} g_{2,m} [s_{2,m}, g_{2,m}] = \int_0^1 g_{2,m}^2 [s_{2,m}, c_{2,m}] = +\sin(\gamma - \beta).$$

If  $(i, n) \neq (j, m)$  we compute

$$\begin{aligned} \Gamma(c_{i,n} s_{i,n}, g_{j,m}^2) &= \frac{1}{\lambda_{i,n} - \lambda_{j,m}} \int_0^1 ([c_{i,n}, g_{j,m}]' [s_{i,n}, g_{j,m}] + [s_{i,n}, g_{j,m}]' [c_{i,n}, g_{j,m}]) dx = \\ &= \frac{1}{\lambda_{i,n} - \lambda_{j,m}} [s_{i,n}, g_{j,m}] [c_{i,n}, g_{j,m}] \Big|_0^1 = 0 \end{aligned}$$

We note that  $\lambda_{i,n} - \lambda_{j,m} \neq 0$  in this case, because the eigenfunctions satisfy the same boundary conditions at 0 and different boundary conditions at 1.  $\square$

**Theorem 5.2.** *With the notations of the above lemma, the set of squared eigenfunctions*

$$\{g_{i,n}^2 | (i, n) \in \{1, 2\} \times \mathbb{N}\}$$

*is linearly independent in  $H^1$ .*

*Proof.* Suppose for some fixed  $(i, n)$  we have

$$g_{i,n}^2 = \sum_{k \in \mathbb{N}} a_k g_k^2$$

in  $H^1$ , where  $a_k \in \mathbb{R}$  and  $g_k = g_{j_k, m_k}$  with  $(j_k, m_k) \neq (i, n)$ . But this would imply

$$(-1)^i \sin(\gamma - \beta) = \Gamma(c_{i,n} s_{i,n}, g_{i,n}^2) = \Gamma\left(c_{i,n} s_{i,n}, \sum_{k \in \mathbb{N}} a_k g_k^2\right) = \sum_{k \in \mathbb{N}} \Gamma(c_{i,n} s_{i,n}, a_k g_k^2) = 0$$

$\square$

#### ACKNOWLEDGEMENTS

The author wishes to thank Ian Knowles and Rudi Weikard for drawing his attention to this topic and for the interesting discussions during his time at the University of Alabama at Birmingham.

## REFERENCES

- [1] G. Borg. Eine Umkehrung der Sturm-Liouvilleschen Eigenwertaufgabe. Bestimmung der Differentialgleichung durch die Eigenwerte. *Acta Math.*, 78:1–96, 1946.
- [2] B. M. Brown, V. S. Samko, I. W. Knowles, and M. Marletta. Inverse spectral problem for the Sturm-Liouville equation. *Inverse Problems*, 19(1):235–252, 2003.
- [3] B. M. Brown and R. Weikard. A borg-levinson theorem for trees. *Preprint* [www.math.uab.edu/rudi/](http://www.math.uab.edu/rudi/), 2004.
- [4] B. E. J. Dahlberg and E. Trubowitz. The inverse Sturm-Liouville problem. III. *Comm. Pure Appl. Math.*, 37(2):255–267, 1984.
- [5] E. L. Isaacson, H. P. McKean, and E. Trubowitz. The inverse Sturm-Liouville problem. II. *Comm. Pure Appl. Math.*, 37(1):1–11, 1984.
- [6] N. Levinson. The inverse Sturm-Liouville problem. *Mat. Tidsskr. B.*, 1949:25–30, 1949.
- [7] B. M. Levitan. *Inverse Sturm-Liouville problems*. VSP, Zeist, 1987. Translated from the Russian by O. Efimov.
- [8] M. Marletta and R. Weikard. Weak stability for an inverse sturm-liouville problem with finite spectral data and complex potential. *Preprint* [www.math.uab.edu/rudi/](http://www.math.uab.edu/rudi/), 2005.
- [9] J. R. McLaughlin. Analytical methods for recovering coefficients in differential equations from spectral data. *SIAM Rev.*, 28(1):53–72, 1986.
- [10] J. Pöschel and E. Trubowitz. *Inverse spectral theory*, volume 130 of *Pure and Applied Mathematics*. Academic Press Inc., Boston, MA, 1987.
- [11] W. H. Press, S. A. Teukolsky, W. T. Vetterling, and B. P. Flannery. *Numerical recipes in C*. Cambridge University Press, Cambridge, second edition, 1992.
- [12] W. Rundell and P. E. Sacks. Reconstruction techniques for classical inverse Sturm-Liouville problems. *Math. Comp.*, 58(197):161–183, 1992.
- [13] P. Schapotschnikow. Inverse sturm-liouville problems on trees. a variational approach. Master’s thesis, School of Computer Science, Cardiff University, UK, 2004.
- [14] E. C. Titchmarsh. *Eigenfunction expansions associated with second-order differential equations. Part I*. Second Edition. Clarendon Press, Oxford, 1962.

Norbert Röhrli

**E-Mail:** [norbert.roehrl@mathematik.uni-stuttgart.de](mailto:norbert.roehrl@mathematik.uni-stuttgart.de)

Fachbereich Mathematik, IADM, University of Stuttgart, 70511 Stuttgart, Germany



## Erschienenene Preprints ab Nummer 2004/001

Komplette Liste: <http://www.mathematik.uni-stuttgart.de/preprints>

- 2004/001 *Walk, H.:* Strong Laws of Large Numbers by Elementary Tauberian Arguments.
- 2004/002 *Hesse, C.H., Meister, A.:* Optimal Iterative Density Deconvolution: Upper and Lower Bounds.
- 2004/003 *Meister, A.:* On the effect of misspecifying the error density in a deconvolution problem.
- 2004/004 *Meister, A.:* Deconvolution Density Estimation with a Testing Procedure for the Error Distribution.
- 2004/005 *Efendiev, M.A., Wendland, W.L.:* On the degree of quasiruled Fredholm maps and nonlinear Riemann-Hilbert problems.
- 2004/006 *Dippon, J., Walk, H.:* An elementary analytical proof of Blackwell's renewal theorem.
- 2004/007 *Mielke, A., Zelik, S.:* Infinite-dimensional hyperbolic sets and spatio-temporal chaos in reaction-diffusion systems in  $\mathbb{R}^n$ .
- 2004/008 *Exner, P., Linde, H., Weidl T.:* Lieb-Thirring inequalities for geometrically induced bound states.
- 2004/009 *Ekholm, T., Kovarik, H.:* Stability of the magnetic Schrödinger operator in a waveguide.
- 2004/010 *Dillen, F., Kühnel, W.:* Total curvature of complete submanifolds of Euclidean space.
- 2004/011 *Afendikov, A.L., Mielke, A.:* Dynamical properties of spatially non-decaying 2D Navier-Stokes flows with Kolmogorov forcing in an infinite strip.
- 2004/012 *Pöschel, J.:* Hill's potentials in weighted Sobolev spaces and their spectral gaps.
- 2004/013 *Dippon, J., Walk, H.:* Simplified analytical proof of Blackwell's renewal theorem.
- 2004/014 *Kühnel, W.:* Tight embeddings of simply connected 4-manifolds.
- 2004/015 *Kühnel, W., Steller, M.:* On closed Weingarten surfaces.
- 2004/016 *Leitner, F.:* On pseudo-Hermitian Einstein spaces.
- 2004/017 *Förster, C., Östensson, J.:* Lieb-Thirring Inequalities for Higher Order Differential Operators.
- 2005/001 *Mielke, A.; Schmid, F.:* Vortex pinning in super-conductivity as a rate-independent model
- 2005/002 *Kimmerle, W.; Luca, F., Raggi-Cárdenas, A.G.:* Irreducible Components of the Burnside Ring
- 2005/003 *Höfert, C.; Kimmerle, W.:* On Torsion Units of Integral Group Rings of Groups of Small Order
- 2005/004 *Röhrl, N.:* A Least Squares Functional for Solving Inverse Sturm-Liouville Problems