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Stochastic Differential Equations Driven by Gaussian Processes with Dependent Increments*

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Abstract

We define an integral with respect to a class of centered Gaussian processes with dependent increments. Furthermore, we consider stochastic differential equations driven by such a process and discuss several examples. In the special case of a bilinear stochastic differential equation existence and uniqueness of the solution is proved. We derive a generalized Ornstein-Uhlenbeck process from an associated stochastic differential equation. Finally, several applications are presented.

1 Introduction

In the last ten years fractional Brownian motion B_t^H gained a lot of attention (e.g. [Be], [GrNo], [HuOk], [HuOkSa], [So]). Opposed to Brownian motion this Gaussian process has dependent increments. This is one of the reasons why it is interesting for applications such as in finance (e.g. [HuOk], [Be]) and network simulations (e.g. [No]). A disadvantage of fractional Brownian motion is that the shape of its covariance function $E(B_t^H B_s^H)$ depends on a single parameter, the Hurst parameter H, only. For example, this restricted flexibility in choosing the covariance function doesn't allow for modelling the noise term in a stochastic differential equations with a short range dependency by using a fractional Brownian motion. Therefore the authors defined a larger class of Gaussian processes with dependent increments, which contains the fractional Brownian motion, but is still capable to serve as an integrator in a useful stochastic integral.

Several authors suggested how to define a stochastic integral driven by fractional Brownian motion $\int_{\mathbb{R}} X_s dB_s^H$. To make it useful in the setting of stochastic differential equations, it is desirable that this stochastic integral has expectation value zero. This is property holds true, if the integral is defined by use of the Wick product ([Be], [HuOk]). The Wick product can be introduced by means of white noise distribution theory ([HiKuPoSt], [Ku]). This

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opens the possibility to differentate fractional Brownian motion in the Hida distribution sense, which has a lot of advantages in the treatment of the Wick product. For instance, one may define stochastic differential equations driven by the fractional Brownian motion and may solve bilinear equation (e.g. [HuOk]).

This approach was adopted by the authors of the present paper to treat stochastic differential equations driven by Gaussian processes with dependent increments. In the following section the class of Gaussian processes with dependent increments, which is used in this paper, is defined. Furthermore, useful results of white noise distribution theory involving Wick products and the so-called S-transform are sketched. In the third section a stochastic integral driven by a Gaussian process with dependent increments is defined, the bilinear stochastic differential equation driven by a Gaussian process with dependent increments is solved. In the fourth section some applications are presented. Remarks are formulated in the fifth section.

2 Gaussian processes and white noise calculus

2.1 The construction of the Schwartz space and its dual

Let $|\cdot|_0$ be the norm of $L^2(\mathbb{R})$. We sketch the construction of the Schwartz space $S(\mathbb{R})$ with the locally convex topology and its dual $S'(\mathbb{R})$ with the weak topology. Let $\langle\omega,\eta\rangle$ denote the bilinear pairing with $\omega\in S'(\mathbb{R})$ and $\eta\in S(\mathbb{R})$. It follows that $\langle\cdot,\cdot\rangle$ is the inner product of $L^2(\mathbb{R})$ if $\omega,\eta\in L^2(\mathbb{R})$. The following construction of the Schwartz space and its dual is presented, for instance, in [Ku], Chapter 3.2. Let $A := -\frac{d^2}{dx^2}+x^2+1$, so A is densely defined on $L^2(\mathbb{R})$. With the Hermite polynomial of degree n

$$H_n(x) := (-1)^n e^{x^2}\left(\frac{d}{dx}\right)^n e^{-x^2}$$

we define

$$e_n(x) := \frac{1}{\sqrt{\sqrt{\pi}2^n n!}}H_n(x)e^{\frac{-x^2}{2}}.$$

The functions $e_n(x)$ are eigenfunctions of A and the corresponding eigenvalue is $2n+2$, $n\in\mathbb{N}_0$. The operator A^{-1} is bounded on $L^2(\mathbb{R})$, especially A^{-p} is a Hilbert-Schmidt Operator for any $p>\frac{1}{2}$. Let for each $p\geq 0$, $|f|_p := |A^p f|_0$. The norm is given by the eigenvalues as

$$|f|_p = \left(\sum_{n=0}^{\infty}(2n+2)^{2p}\langle f,e_n\rangle^2\right)^{1/2}.$$

We define

$$S_p(\mathbb{R}) := \{f; f\in L^2(\mathbb{R}), |f|_p<\infty\}$$

and with these spaces we construct the Schwartz space $S(\mathbb{R})$ by $S(\mathbb{R}) = \cap_{p\geq 0}S_p(\mathbb{R})$. This construction leads to the Gel'fand triple $S(\mathbb{R})\subset L^2(\mathbb{R})\subset S'(\mathbb{R})$. Furthermore, we get the following continuous inclusion maps

$$S(\mathbb{R}) \subset S_p(\mathbb{R}) \subset L^2(\mathbb{R}) \subset S'_p(\mathbb{R}) \subset S'(\mathbb{R}).$$

Let $\mathcal{B}$ denote the Borel σ-algebra on $S'(\mathbb{R})$, i.e., the σ-algebra generated by the weak topology. One can show by the use of the Bochner-Minlos theorem that there is a unique Gaussian measure μ on $(S'(\mathbb{R}), \mathcal{B})$. The space $(S'(\mathbb{R}), \mathcal{B}, \mu)$ is called *white noise*, and the space (L^2) denotes $L^2(S'(\mathbb{R}), \mathcal{B}, \mu)$. The bilinear form $\langle \omega, f \rangle$ with $f \in L^2(\mathbb{R})$ and $\omega \in S'(\mathbb{R})$ is declared by

$$\lim_{k\to\infty} \langle \omega, \eta_k \rangle = \langle \omega, f \rangle$$

with $\eta_k \to f$ and $\{\eta_k\} \subset S(\mathbb{R})$. It is possible to show that $\langle \cdot, f \rangle = \int_{\mathbb{R}} f(s) dB_s$ is a random variable in (L^2) for all $f \in L^2(\mathbb{R})$. The random variable $\langle \cdot, f \rangle$ has expectation value zero and variance $|f|_0^2$.

2.2 A class of Gaussian processes with dependent increments

Suppose $m(\cdot,\cdot) : \mathbb{R}^2 \to \mathbb{R}$ such that for all $t \in \mathbb{R}$ the function $m(u,t) \in L^2(\mathbb{R})$. Define $v(s,t) = \int_{\mathbb{R}} m(u,t) m(u,s)\, du$. Then we have the stochastic process with $t \in \mathbb{R}$

$$B_t^v := \langle \cdot, m(u,t) \rangle = \int_{\mathbb{R}} m(u,t)\, dB_u,$$

with ordinary Brownian motion B_u. We choose $m(u,t)$ such that $\frac{d}{dt} m(u,t) \in S'_p(\mathbb{R})$ for all $t \in \mathbb{R}$ and for some $p \geq 0$. This property will be used later. The process B_t^v is a Gaussian process; its covariance function is given by

$$v(s,t) = \int_{S'(\mathbb{R})} B_s^v B_t^v \, d\mu = \int_{\mathbb{R}} m(u,s) m(u,t)\, du.$$

Now we show some properties of B_t^v which follows by supposed properties of $m(u,t)$. Let $m(u,0) \equiv 0$, hence $B_0^v = 0$, and B_t^v is a centered Gaussian process. The properties $v(t,t) \geq 0$ and $v(s,t) = v(t,s)$ are obvious. It is natural to request that B_t^v is pathwise continuous. This is ensured by the supposed continuity of the function $\langle \omega, m(u,\cdot) \rangle : \mathbb{R} \to \mathbb{R}$, $\omega \in S'(\mathbb{R})$. In the following we give some special instances of B_t^v.

Example 2.1 [Ordinary Brownian motion] The stochastic process $B_t^v = B_t$ is the ordinary Brownian motion if $m(u,t) = 1([0,t])(u)$, where $1([0,t])$ is the indicator function of the intervall $[0,t]$. This process has then the covariance function $v(s,t) = \min(t,s)$. This example is further discussed in [Ku], Chapter 3.1.

Example 2.2 [Fractional Brownian motion] We use the following operator $M_\pm^H$ to introduce fractional Brownian motion and later on its derivative. Define $M_\pm^H$ with $H \in (0,1)$ for $\eta \in S(\mathbb{R})$ as

$$(M_+^H \eta)(t) := \begin{cases} \frac{K_H}{\Gamma(H-1/2)} \int_{-\infty}^t \eta(s)(t-s)^{H-3/2}\, ds & \text{for } H > 1/2 \\ \eta(t) & \text{for } H = 1/2 \\ \frac{K_H (H-1/2)}{\Gamma(1/2-H)} \lim_{\epsilon\to 0} \int_\epsilon^\infty \frac{\eta(t)-\eta(t-s)}{(s)^{3/2-H}}\, ds & \text{for } H < 1/2 \end{cases}$$

and

$$(M_-^H\eta)(t) := \begin{cases} \frac{K_H}{\Gamma(H-1/2)} \int_{-\infty}^t \eta(s)(t-s)^{H-3/2}\, ds & \text{for } H > 1/2 \\ \eta(t) & \text{for } H = 1/2 \\ \frac{K_H(H-1/2)}{\Gamma(1/2-H)} \lim_{\epsilon\to 0} \int_\epsilon^\infty \frac{\eta(t)-\eta(t+s)}{(s)^{3/2-H}}\, ds & \text{for } H < 1/2 \end{cases}$$

with

$$K_H = \Gamma(H+1/2)\left(\frac{2H\Gamma(3/2-H)}{\Gamma(H+1/2)\Gamma(2-2H)}\right)^{1/2}$$

The operator $M_\pm^H$ is essentially the Riemann-Liouville fractional integral for $H > 1/2$ and the Marchaud fractional derivative for $H < 1/2$. For further information about these operators see [SaKiMa], Chapter 6, and [Be], Chapter 1.6. We just apply $M_\pm^H$ on functions $\eta \in S(\mathbb{R})$, and for indicator functions $1([0,t])$ for which $M_\pm^H$ is defined; see Chapter 1 of [Be]. The fractional Brownian motion B_t^H is a modification of $\int_\mathbb{R}(M_-^H 1((0,t)))(s)\, dB_s$. Another representation of the fractional Brownian motion up to modification is $B_t^H = \langle\cdot, M_-^H(1((0,t)))\rangle$. The covariance function is therefore $v(s,t) = 1/2(|t|^{2H}+|s|^{2H}-|s-t|^{2H})$. This example is further discussed in [Be].

Example 2.3 [A Gaussian process with short range dependency] Let B_t^s, $t \in \mathbb{R}$, be a centered Gaussian process with covariance function $v(s,t)$, such that $v(s,\cdot)$ has a global maximum and $\lim_{t\to\infty} v(s,t) = 0$ for all fixed $s \in \mathbb{R}$. Then we call B_t^s a short range Brownian motion. Let $m(u,t) = t^2\exp(-(u-t)^2)$, hence $v(s,t) = kt^2s^2\exp(-(t-s)^2/2)$ with a constant k. This process B_t^s is a short range Brownian motion. Later we derivate this process and define an integral driven by this process.

Having introduced a class of Gaussian processes with dependent increments we will define an integral $\int_\mathbb{R} X_s\, dB_s^v$ and a stochastic differential equation

$$X_t = \int_0^t X_s\, ds + \int_0^t X_s\, dB_s^v.$$

Before pursuing this goal some results of white noise analysis are summarized.

2.3 The construction of the Hida test and distribution space

In this section the construction of the Hida test and Hida distribution spaces (S) and $(S)^*$ are outlined. These spaces are used to derivate the Gaussian process B_t^v and to define the stochastic integral driven by these Gaussian processes. A more detailed description can be found e.g. in [Ku], Chapter 3.3, and in [Be], Chapter 5.3. Together with (L^2) they are also a Gel'fand triple $(S) \subset (L^2) \subset (S)^*$. Let $\langle\langle\Phi,\zeta\rangle\rangle$ denote the bilinearform of $\Phi \in (S)^*$ and $\zeta \in (S)$. For $f \in L^2(\mathbb{R}^n)$ we define the multiple Wiener Integral with respect to the ordinary Brownian motion

$$I_n(f) := n! \int_{\mathbb{R}^n} f(t_1,t_2,t_3,\dots,t_n)\, dB_{t_1} dB_{t_2} \dots dB_{t_n}.$$

The following proposition is the chaos decompostion of (L^2) (see [Be], Theorem 1.4.8).

Proposition 2.4 *For all* $F \in (L^2)$ *there is a unique sequence of* $(f_n)_{n\in\mathbb{N}_0}$ *such that* $f_n \in L^2(\mathbb{R}^n)$ *is symmetric and*

$$F = \sum_{n=0}^{\infty} I_n(f_n)$$

with convergence in (L^2).

We define the operator $\Gamma(A)$, A as above, on (L^2) to be

$$\Gamma(A)F = \sum_{n=0}^{\infty} I_n(A^{\otimes n} f_n)$$

with $F \in (L^2)$. Let $(S)_n = \Gamma(A)^n((L^2))$ with inner product

$$((F, G))_n := E[\Gamma(A)^{-n}F \cdot \Gamma(A)^{-n}G]$$

with $F, G \in (L^2)$. Now we set similarly as above $(S) := \cap_{n\in\mathbb{N}}(S)_n$. With $(S)'_n := (S)_{-n}$, we get $(S)^* := \cup_{n\in\mathbb{N}}(S)_{-n}$. The topologies of (S) and of $(S)^*$ are given by the projective limit topology and weak topology, respectively (see [Ku], Chapter 2.2). As before there are continuous inclusion maps

$$(S) \subset (S)_n \subset (L^2) \subset (S)_{-n} \subset (S)^*.$$

2.4 S-transform and Wick product

Now we introduce the S-transform from $(S)^*$ into the set of the functions from $S(\mathbb{R})$ to $\mathbb{R}$.

Remark 2.5 The image of the S-transform is discussed in [PoSt]. They proved that the S-transform as a mapping from $S(\mathbb{R})$ to $\mathbb{R}$ has some analytic properties.

We formulate with $I(\eta) := I_1(\eta) = \int_{\mathbb{R}} \eta(s)\, dB_s$ $\eta \in S(\mathbb{R})$:

Proposition 2.6 *For all* $\eta \in S(\mathbb{R})$ *the random variable* $\exp(I(\eta) - 1/2|\eta|_0^2)$ *is a Hida test function and the set*

$$\left\{ e^{I(\eta) - 1/2|\eta|_0^2} : \eta \in S(\mathbb{R}) \right\}$$

is total in (S).

The proof can be found in [Ku], Proposition 5.10.

Definition 2.7 *The* S*-transform of a Hida distribution* $\Phi \in (S)^*$ *is defined by*

$$(S\Phi)(\eta) := \langle\langle \Phi, e^{I(\eta) - 1/2|\eta|_0^2} \rangle\rangle, \quad \eta \in S(\mathbb{R}).$$

The S-transform is an injective mapping. This follows from the preceeding proposition (see [Ku], Proposition 5.10).

Proposition 2.8 *The S-transform of $\langle \cdot, f \rangle$ with $f \in S'(\mathbb{R})$ is given by*

$$S(\langle \cdot, f \rangle)(\eta) = \langle f, \eta \rangle$$

The proof is given in [Ku], Proposition 5.9. This proposition will be used to define the stochastic integral driven by Gaussian processes with dependent increments. For further details of the S-transform see [Ku], Chapter 5, or [PoSt].

Example 2.9 The S-transform of the ordinary Brownian motion B_t is

$$\begin{aligned} S(B_t)(\eta) &= \langle\langle B_t, e^{I(\eta)-1/2|\eta|_0^2} \rangle\rangle \\ &= \int_{S'(\mathbb{R})} B_t e^{I(\eta)-1/2|\eta|_0^2} \, d\mu \\ &= \int_{\mathbb{R}} 1([0,t])(u)\eta(u) \, du = \int_0^t \eta(u) \, du \end{aligned}$$

Example 2.10 We calculate the S-transform of the fractional Brownian motion B_t^H.

$$\begin{aligned} S(B_t^H)(\eta) &= \langle\langle B_t^H, e^{I(\eta)-1/2|\eta|_0^2} \rangle\rangle \\ &= \int_{S'(\mathbb{R})} B_t^H e^{I(\eta)-1/2|\eta|_0^2} \, d\mu \\ &= \int_{S'(\mathbb{R})} \int_{\mathbb{R}} (M_-^H 1((0,t)))(s) \, dB_s \, e^{I(\eta)-1/2|\eta|_0^2} \, d\mu \\ &= \int_0^t (M_+^H \eta)(s) \, ds \end{aligned}$$

where several steps like fractional integration by parts and the fact that the integrals are well-defined are used (see [Be], Theorem 1.6.8).

Example 2.11 For the short range Brownian motion B_t^s we get

$$\begin{aligned} S(B_t^s)(\eta) &= \langle\langle B_t^s, e^{I(\eta)-1/2|\eta|_0^2} \rangle\rangle \\ &= \int_{S'(\mathbb{R})} B_t^s e^{I(\eta)-1/2|\eta|_0^2} \, d\mu \\ &= \int_{\mathbb{R}} \eta(u) t^2 \exp(-(u-t)^2) \, du. \end{aligned}$$

An upper bound for $|(S\Phi)(\eta)|$ can be found in Theorem 8.2 of [Ku] as follows.

Proposition 2.12 *For $\Phi \in (S)^*$ numbers $K, a, \geq 0$ and $p \in \mathbb{R}$ exist such that for all $\eta \in S(\mathbb{R})$*

$$|(S\Phi)(\eta)| \leq K \exp\left(a|\eta|_p^2\right).$$

Now one can define the Wick product as follows (see [Ku], Chapter 8.4, page 92):

Definition 2.13 *The Wick product of two Hida distributions* Φ *and* Ψ *in* $(S)^*$*, denoted by* $\Phi \diamond \Psi$*, is the unique Hida distribution in* $(S)^*$ *such that* $S(\Phi \diamond \Psi)(\eta) = S(\Phi)(\eta)S(\Psi)(\eta)$ *for all* $\eta \in S(\mathbb{R})$*.*

The stochastic integral with dependent increments will be defined in terms of the white noise integral by use of the Wick product. The next proposition answers the question whether the Wick product is a continuous mapping (see [Ku], Chapter 8.4).

Proposition 2.14 *The mapping* $\cdot \times \cdot \to \cdot \diamond \cdot$ *from* $(S)^* \times (S)^* \to (S)^*$ *is continuous in the weak topology. The restriction to* $(S)_{-p}$ *is also Lipschitz continuous. So there exists a* $c > 0$ *such that for all* $\Psi, \Phi \in (S)_{-p}$ *there is a* $q > 0$ *such that:*

$$\| \Psi \diamond \Phi \|_{-q} \leq c \| \Psi \|_{-p} \| \Phi \|_{-p} .$$

Now we are prepared to calculate dB_t^v/dt in the $(S)^*$-sense and to define the stochastic integral $\int_{\mathbb{R}} X_t \, dB_t^v$.

3 Stochastic calculus with Gaussian processes with dependent increments

3.1 The white noise W_t^v

In this subsection we differentiate B_t^v. The following definition is taken from [Be], Chapter 5.3 and 5.4.

Definition 3.1 *Let* I *be an interval in* $\mathbb{R}$*. A mapping* $X : I \to (S)^*$ *is called a stochastic distribution process. A stochastic distribution process is called differentiable in the* $(S)^*$*-sense, if*

$$\lim_{h \to 0} \frac{X_{t+h} - X_t}{h}$$

exists in $(S)^*$*.*

The next theorem presents a criterion for differentiability (see [Be], Theorem 5.3.9). Let I be as above.

Theorem 3.2 *Let* $F : I \to S'(\mathbb{R})$ *be differentiable. Then* $\langle \cdot, F(t) \rangle$ *is a differentiable stochastic distribution process and*

$$\frac{d}{dt}\langle \cdot, F(t) \rangle = \langle \cdot, \frac{d}{dt} F(t) \rangle.$$

So we get a theorem for the derivative of the Gaussian process B_t^v in the $(S)^*$-sense.

Theorem 3.3 (Derivative of B_t^v)

$$W_t^v := \frac{d}{dt} B_t^v = \langle \cdot, \frac{d}{dt} m(u,t) \rangle$$

The S-transform of W_t^v is

$$S(W_t^v)(\eta) = \langle \frac{d}{dt} m(u,t), \eta(u) \rangle.$$

Proof. The proof is given by Theorem 3.2. □

With δ_t denoting the Dirac distribution we get

Example 3.4 [Ordinary Brownian motion] For the derivative of the ordinary Brownian motion B_t, we use $\frac{d}{dt} 1([0,t]) = \delta_t$ so $dB_t/dt =: W_t = \langle \cdot, \delta_t \rangle$, further $S(W_t)(\eta) = \eta(t)$ (see [Ku], Chapter 3.1).

Example 3.5 [Fractional Brownian motion] For $H \in (0,1)$ the fractional Brownian motion $B_\cdot^H : \mathbb{R} \to (S)^*$ is differentiable in the $(S)^*$-sense and

$$W_t^H := \frac{d}{dt} B_t^H = I(\delta_t \circ M_+^H)$$

and for all $\eta \in S(\mathbb{R})$

$$\langle \delta_t \circ M_+^H, \eta \rangle = (M_+^H \eta)(t).$$

The S-transform of W_t^H is given by

$$S(W_t^H)(\eta) = \langle\langle \frac{d}{dt} B_t^H, e^{I(\eta) - 1/2 |\eta|_0^2} \rangle\rangle = \frac{d}{dt} (S B_t^H)(\eta) = (M_+^H \eta)(t).$$

This example is presented in [Be], Chapter 5.

Example 3.6 [Short range Brownian motion] The derivative of B_t^s is obviously given by

$$\langle \cdot, \frac{d}{dt} t^2 \exp(-(t-u)^2) \rangle = \langle \cdot, 2t \exp(-(t-u)^2) + t^2(-2(u-t)) \exp(-(u-t)^2) \rangle.$$

The stochastic distribution process W_t^v is called white noise of B_t^v. With the white noise W_t^v we can define $\int_{\mathbb{R}} X_t \, dB_t^v$.

3.2 White noise integral and stochastic differential equations driven by B_t^v

We start with the definition of the white noise integral (see [Ku], Chapter 13). Suppose that X_t is a mapping from $\mathbb{R} \to (S)^*$.

Definition 3.7 *The stochastic distribution process X_t is white noise integrable, if there is $\Psi \in (S)^*$ such that, for all $\eta \in S(\mathbb{R})$, $(SX_\cdot)(\eta) \in L^1(\mathbb{R})$ and*

$$(S\Psi)(\eta) = \int_{\mathbb{R}} (SX_t)(\eta)\, dt.$$

This definition makes sense as the S-transform is injective. Now we formulate the definition of $\int_{\mathbb{R}} X_t\, dB_t^v$.

Definition 3.8 *The process X_t has the stochastic integral $\int_{\mathbb{R}} X_t\, dB_t^v$, if $X_t \diamond W_t^v$ is white noise integrable. So we have*

$$\int_{\mathbb{R}} X_t\, dB_t^v = \int_{\mathbb{R}} X_t \diamond W_t^v\, dt$$

This definition coincides in the case of the fractional Brownian motion with the definition of the fractional Ito integral (see Bender ([Be]), Øksendal and Hu [HuOk]).

The following theorem is highly inspired by Bender's theorem for fractional Ito integrals (see [Be], Chapter 5).

Theorem 3.9 *Let $a, b \in \mathbb{R}$ and let $X_\cdot : [a, b] \to (S)_{-p}$ be continuous for some $p \in \mathbb{N}$. Further let $W_\cdot^v : \mathbb{R} \to (S)_{-q}$ be continuous for some $q \in \mathbb{N}$. Then $\int_a^b X_t\, dB_t^v$ exists. Further for any sequence of tagged partitions $\tau_n = (\pi_k^{(n)}, t_k^{(n)})$ of $[a, b]$ with $\lim_{n\to\infty} \max\{|\pi_k - \pi_{k-1}|; k = 1, .., n\} = 0$, we have*

$$\lim_{n\to\infty} \sum_{k=0}^{n} X_{t_{k-1}^{(n)}} \diamond \left(B^v_{\pi_k^{(n)}} - B^v_{\pi_{k-1}^{(n)}} \right) = \int_a^b X_t\, dB_t^v$$

with limit in $(S)^*$.

Proof. For the integrability:

$$\begin{aligned} S\left(\int_a^b X_s\, dB_s^v\right)(\eta) &= \int_a^b S(X_s \diamond W_s^v)(\eta)\, ds \\ &= \int_a^b S(X_s)(\eta) S(W_s^v)(\eta)\, ds \\ &\le \max_s\{|S(X_s)(\eta)|\} \int_a^b S(W_s^v)(\eta)\, ds \\ &\le K \exp(a|\eta|_p^2) \int_a^b S(W_s^v)(\eta)\, ds < \infty \end{aligned}$$

From the continuity of W_t^v it follows by

$$\left|\langle \frac{d}{dt}m(u,t) - \frac{d}{dt}m(u,t_0), \eta(u)\rangle\right| \leq |\eta(u)|_q \left|\frac{d}{dt}m(u,t) - \frac{d}{dt}m(u,t_0)\right|_{-q}$$
$$= |\eta(u)|_q \| W_t^v - W_{t_0}^v\|_{-q}$$

with $t, t_0 \in [a,b]$, that $\langle \frac{d}{dt}m(u,t), \eta(u)\rangle$ is a continuous function in t. So the function $\langle m(u,t), \eta(u)\rangle$ is continuously differentiable on the interval $[a,b]$. It follows by a well-known result in real analysis that for all $\eta \in S(\mathbb{R})$ the continuous function $S(X_t)(\eta)$ is Riemann-Stieltjes integrable with respect to $\langle m(u,t), \eta(u)\rangle$, so the approximation is proved. □

The S-transform of $\int_{\mathbb{R}} X_t \, dB_t^v$ is therefore given by

$$S\left(\int_{\mathbb{R}} X_t \, dB_t^v\right)(\eta) = \int_{\mathbb{R}} S(X_t \diamond W_t^v)(\eta)\, dt = \int_{\mathbb{R}} (SX_t)(\eta)\langle \frac{d}{dt}m(u,t), \eta(u)\rangle \, dt.$$

Corollary 3.10 *Let $\int_0^t X_s \, dB_s^v$ exist and $\int_0^t X_s \, dB_s^v \in (L^2)$ than $E(\int_0^t X_s \, dB_s^v) = 0$.*

Proof. Let $\eta \equiv 0$, so $S(B_t^v)(0)$ is the expectation value of B_t^v and

$$\lim_{n\to\infty} S\left(\sum_{k=1}^{n} X_{t_{k-1}^{(n)}} \diamond (B^v_{\pi_k^{(n)}} - B^v_{\pi_{k-1}^{(n)}})\right)(0) = \lim_{n\to\infty} \sum_{k=1}^{n} S(X_{t_{k-1}^{(n)}})(0)\, S(B^v_{\pi_k^{(n)}} - B^v_{\pi_{k-1}^{(n)}})(0)$$
$$= 0,$$

which proofs the claim. □

Remark 3.11 The last property is necessary in order to justify that such processes are a sensible model for the stock price. In this case the additive stochastic part in the stochastic differential equation has expectation value zero. Otherwise one may imagine that the stochastic part of the stochastic differential equation has a drift and that might cause an arbitrage opportunity.

Example 3.12 We calculate

$$\int_0^t B_s^v \, dB_s^v = \int_0^t B_s^v \diamond W_s^v \, ds = \frac{1}{2}\left(B_t^v\right)^{\diamond 2},$$

where $(\cdot)^{\diamond 2}$ is the Wick square. By the Wick calculus presented in [Be], Chapter 1, or [Ku], Chapter 8, we get

$$\frac{1}{2}\left(B_t^v\right)^{\diamond 2} = \frac{1}{2}\left(B_t^v\right)^2 - \frac{1}{2}|m(u,t)|_0^2.$$

Note that this example coincides with the case of ordinary Brownian motion with the Ito-integral and with the case of fractional Brownian motion with the fractional Ito integral.

Now we can go on to stochastic differential equations driven by Gaussian processes with dependent increments.

Example 3.13 We want to solve the equation

$$dX_t = \mu X_t\, dt + \sigma X_t\, dB_t^v,$$

with constants μ and σ. This expression is declared by the integral equation

$$\begin{aligned} X_t &= \int_0^t \mu X_s\, ds + \int_0^t \sigma X_s\, dB_s^v. \\ &= \int_0^t (\mu X_s + \sigma X_s \diamond W_s^v)\, ds \\ &= \int_0^t X_s \diamond (\mu + \sigma W_s^v)\, ds, \end{aligned}$$

and the solution is given by

$$X_t = \exp^\diamond(\mu t + \sigma B_t^v) = \exp(\mu t + \sigma B_t^v - \frac{1}{2}\sigma^2 |m(u,t)|_0^2),$$

where $\exp^\diamond(X) = \sum_{k=0}^\infty X^{\diamond k}/k!$ and some Wick calculus is used in the last equation, see e.g. [Be], Chapter 1.

We consider the stochastic differential equation on $[0,T]$ $(T>0)$

$$dX_t = a(t,X_t)\, dt + b(t,X_t)\, dB_t^v.$$

It is defined in terms of the corresponding integral equation with $t \in [0,T]$

$$X_t = X_0 + \int_0^t a(s,X_s)\, ds + \int_0^t b(s,X_s)\, dB_s^v.$$

If this equation is regarded in the $(S)^*$-sense, it can be transformed to

$$X_t = X_0 + \int_0^t (a(s,X_s) + b(s,X_s) \diamond W_s^v)\, ds.$$

If we set $(a(s,X_s) + b(s,X_s) \diamond W_s^v) = f(s,X_s)$, we have an integral equation of the form

$$X_t = X_0 + \int_0^t f(s,X_s)\, ds$$

as a white noise integral equation, where $f(\cdot,\cdot)$ is a mapping $\mathbb{R} \times (S)^* \to (S)^*$. So we assume that $a(\cdot,\cdot)$ and $b(\cdot,\cdot)$ are also mappings of that kind.

We call a stochastic distribution process X_t weakly measurable if

$$\langle\langle X_\cdot, e^{I(\eta) - 1/2|\eta|_0^2} \rangle\rangle$$

is measurable for all $\eta \in S(\mathbb{R})$.

Definition 3.14 *A stochastic distribution process X_t is called a weak solution of the white noise integral equation on $[0,T]$, if it satisfies the following conditions:*

(a) X is weakly measurable.

(b) The stochastic distribution process $f(t,X_t)$ is white noise integrable.

(c) For each $\eta \in S(\mathbb{R})$ the equality holds for almost all $t \in [0,T]$

$$\langle\langle X_t, e^{I(\eta)-1/2|\eta|_0^2}\rangle\rangle = \langle\langle X_0, e^{I(\eta)-1/2|\eta|_0^2}\rangle\rangle + \int_0^t \langle\langle f(s,X_s), e^{I(\eta)-1/2|\eta|_0^2}\rangle\rangle\, ds.$$

We will need the following theorem on existence and uniqueness of solutions of white noise integral equations ([Ku], Th. 13.43).

Theorem 3.15 *Suppose f is function from $[0,T]\times(S)^*$ into $(S)^*$ satisfying the following conditions:*

(a) (Measurability condition) The function $f(s,X_s)$ $s\in[0,T]$, is weakly measurable for any weakly measurable function $X:[0,T]\to(S)^*$.

(b) (Lipschitz condition) For almost all $t\in[0,T]$,

$$|Sf(t,\Phi)(\eta) - Sf(t,\Psi)(\eta)| \le L(t,\eta)|S\Phi(\eta) - S\Psi(\eta)|,$$

for all $\eta\in S(\mathbb{R})$ and $\Psi,\Phi\in(S)^*$, where L is nonnegative and

$$\int_0^T L(t,\eta)\,dt \le K(1+|\eta|_p^2)$$

for some $K,p\ge 0$.

(c) (Growth condition) For almost all $t\in[0,T]$

$$|Sf(t,\Phi)(\eta)| \le \rho(t,\eta)(1+|S\Phi(\eta)|),$$

for all $\eta\in S(\mathbb{R})$ and $\Phi\in(S)^*$, where ρ is nonnegative and

$$\int_0^T \rho(t,\eta)\,dt \le K\exp\left(c|\eta|_p^2\right),$$

where K,p are the same as above and $c\ge 0$.

Then for any $X_0\in(S)^*$ the equation

$$X_t = X_0 + \int_0^t f(s,X_s)\,ds$$

has a unique weak solution X such that for all $\eta\in S(\mathbb{R})$

$$\operatorname*{ess\,sup}_{t\in[0,T]} |SX_t(\eta)| < \infty.$$

The last theorem will be used to prove the following theorem on existence and uniqueness of solutions of stochastic integral equations driven by Gaussian processes with dependent increments. There we suppose that $a(\cdot,\cdot) + b(\cdot,\cdot) \diamond W^v_\cdot$ satisfies the conditions of Kuo's theorem.

Theorem 3.16 (Existence and Uniqueness Theorem for bilinear SDE's) *Let $\sigma, \mu \in C([0,T])$ and $X_0 \in (S)^*$ and $t \in [0,T]$. Further let $m(u,t)$ such that there exist $K, p > 0$ satisfying*

$$\int_0^T \left| \langle \frac{d}{dt} m(u,t), \eta(u) \rangle \right| dt \leq K(1 + |A^p \eta|_0).$$

Then there exists a unique solution of

$$X_t = X_0 + \int_0^t (\mu(s) X_s + \sigma(s) X_s \diamond W^v_s)\, ds$$

which is given by

$$X_t = X_0 \exp^\diamond (\int_0^t \mu(s)\, ds + \int_0^t \sigma(s)\, dB^v_s).$$

Proof. We check the conditions of the existence and uniqueness Theorem 3.15. The measurability condition is met due to continuity of the Wick product and measurability of $dm(u,t)/dt$. Now we show the Lipschitz condition, for almost all $t \in [0,T]$ and and for all $\Phi, \Psi \in (S)^*$ and $\eta \in S(\mathbb{R})$

$$\begin{aligned}
&\Big| S(\mu(t)\Phi)(\eta) + S(\sigma(t)\Phi)(\eta) \, \langle \frac{d}{dt} m(u,t), \eta(u) \rangle \\
&\quad - S(\mu(t)\Psi)(\eta) - S(\sigma(t)\Psi)(\eta) \, \langle \frac{d}{dt} m(u,t), \eta(u) \rangle \Big| \\
&\leq (|\mu(t)| + |\sigma(t) \, \langle \frac{d}{dt} m(u,t), \eta(u) \rangle|) |S(\Phi)(\eta) - S(\Psi)(\eta)|.
\end{aligned}$$

So we get

$$(|\mu(t)| + |\sigma(t) \langle \frac{d}{dt} m(u,t), \eta(u) \rangle|) = L(t,\eta)$$

and the estimation for $\int_0^T L(t,\eta)\, dt$ is

$$\begin{aligned}
&\int_0^T (|\mu(t)| + |\sigma(t) \langle \frac{d}{dt} m(u,t), \eta(u) \rangle|)\, dt \\
&\leq \max\{|\mu(t)|\} T + \max\{|\sigma(t)|\} \int_0^T |\langle \frac{d}{dt} m(u,t), \eta(u) \rangle|\, dt.
\end{aligned}$$

This proves the Lipschitz condition. In order to check the growth condition we proceed in a similar manner

$$\begin{aligned}
&|S(\mu(t)\Phi)(\eta) + S(\sigma(t)\Phi)(\eta)\langle \frac{d}{dt}m(u,t), \eta(u)\rangle| \\
&\leq \left(\max_t\{|\mu(t)|\} + \max_t\{|\sigma_t|\}|\langle \frac{d}{dt}m(u,t), \eta(u)\rangle| \right) |S(\Phi)(\eta)| \\
&\leq \left(\max_t\{|\mu(t)|\} + \max_t\{|\sigma_t|\}|\langle \frac{d}{dt}m(u,t), \eta(u)\rangle| \right) (1 + |S(\Phi)(\eta)|).
\end{aligned}$$

Then $\big(\max_t\{|\mu(t)|\} + \max_t\{|\sigma_t|\}|\langle \frac{d}{dt}m(u,t), \eta(u)\rangle|\big) = \rho(t,\eta)$ and the condition for $\rho(t,\eta)$ is obviously satisfied. The solution is calculated by

$$\begin{aligned}
X_t &= \int_0^t (\mu(s) + \sigma(s)W_s^v) \diamond X_s \, ds \\
X_t &= \exp^{\diamond}(\int_0^t \mu(s)\, ds + \int_0^t \sigma(s)\, dB_s^v).
\end{aligned}$$

□

Remark 3.17 It is natural to ask why the bilinear case is considered only, whereas the existence and uniqueness theorem of Kuo is formulated even for a nonlinear situation. In the authors' point of view the theorem of Kuo is only applicable in the bilinear case. Note that the function f is a function from $\mathbb{R} \times (S)^* \to (S)^*$. But the motivation of the authors is to take a real vauled function $g : \mathbb{R}^2 \to \mathbb{R}$ with $|g(t,x)|^2 \leq K(1 + |x|^2)$, and to define the stochastic differential equation with this function. One has to explain such a real-valued function with elements of $(S)^*$. A further argument is that the Lipschitz condition fails with a nonlinear function g. Suppose $F, G \in (L^2)$ and a $\eta \in S(\mathbb{R})$ such that $|S(F-G)(\eta)| = 0$ and beside $|S(g(t,F) - g(t,G))(\eta)| \neq 0$. This is possible if $F - G$ is linearly independent to $g(t,F) - g(t,G)$. If we now demand the linear dependence above, we get with the property of a real valued function only the linear or constant case for g in x.

4 Special Gaussian processes with dependent increments

We introduce special Gaussian processes with dependent increments and show some properties of them. Before we state and prove

Lemma 4.1 *Let X_t be a (in the $(S)^*$-sense) continuously differentiable stochastic distribution processes and $a(t)$ a continuously differentiable real-valued function. Then the product rule*

$$\frac{d(a(t)X_t)}{dt} = \frac{da(t)}{dt}X_t + a(t)\frac{dX_t}{dt}$$

holds in the $(S)^*$-sense.

Proof. This obviously follows from

$$\begin{aligned}
&\lim_{h\to 0}\frac{a(t+h)X_{t+h}}{h}\\
= &\lim_{h\to 0}\left(\frac{a(t+h)X_{t+h}}{h}-\frac{a(t)X_{t+h}}{h}+\frac{a(t)X_{t+h}}{h}-\frac{a(t)X_t}{h}\right).
\end{aligned}$$

□

Now we discuss the ordinary Ornstein-Uhlenbeck process in the $(S)^*$-sense and then a generalized Ornstein-Uhlenbeck process is presented.

Example 4.2 [Ordinary Ornstein-Uhlenbeck process] We apply this to the centered Ornstein-Uhlenbeck process B_t^{OU}, the centered Ornstein Uhlenbeck process is a stationary centered Gaussian process with covariance function $v(s,t) = (\sigma^2/2\alpha)\exp(-\alpha|t-s|)$ with positiv constants α, σ, if the initial vaule has a normal B_0^{OU} distribution with mean zero and variance $(\sigma^2/2\alpha)$. So

$$B_t^{OU} := B_0^{OU}\exp(-\alpha t) + \langle\cdot, \sigma 1([0,t])\exp(-\alpha(t-u))\rangle,$$

so we get as derivative of the Ornstein-Uhlenbeck process

$$\frac{d}{dt}B_t^{OU} := B_0^{OU}(-\alpha)\exp(-\alpha t) + \langle\cdot, \sigma\frac{d}{dt}(1([0,t])\exp(-\alpha(t-u)))\rangle,$$

(see Karatzas and Shreve [KaSh], Chapter 5.6).

The integral and the stochastic differential equation with respect to the Ornstein Uhlenbeck process can be defined similarly as before. Note that now it is possible to derivate the Ornstein-Uhlenbeck process.

Example 4.3 [Generalized Ornstein-Uhlenbeck process] Let $\alpha, \sigma > 0$ and B_t^v be as in Theorem 3.9. The generalized Ornstein-Uhlenbeck process is the solution of the stochastic differential equation

$$dX_t = -\alpha X_t\,dt + \sigma\,dB_t^v,$$

which is the same as

$$\frac{X_t}{dt} = -\alpha X_t + \sigma W_t^v$$

in the $(S)^*$-sense. Note that by the existence and uniqueness theorem one can show with slight modifications that the solution is unique. We solve the corresponding non-disturbed equation $dX_t = -\alpha X_t\,dt$ and get the solution $X_t = C\exp(-\alpha t)$, we assume that the C is a stochastic distribution process and by the product rule the derivative becomes to

$$\frac{d(C_t\exp(-\alpha t))}{dt} = \frac{dC_t}{dt}\exp(-\alpha t) - \alpha C_t\exp(-\alpha t).$$

Comparing this with the stochastic differential equation we deduce

$$\frac{dC_t}{dt}\exp(-\alpha t) = \sigma W_t^v$$

and so $C_t = C_0 + \int_0^t \exp(\alpha s)\sigma W_s^v\, ds = C_0 + \sigma \int_0^t \exp(\alpha s)\, dB_s^v$. Thus the solution is

$$B_t^{GOU} := X_t = C_0 \exp(-\alpha t) + \sigma \int_0^t \exp(-\alpha(t-s))\, dB_s^v,$$

where C_0 is the initial random variable of the process B_t^{GOU}. By corollary 3.10 the expectation value of B_t^{GOU} is $S(B^{GOU})(0) = E(C_0)\exp(-\alpha t)$, and its covariance function is equal to

$$\begin{aligned}
&E\left((B_t^{GOU} - E(C_0)\exp(-\alpha t))(B_s^{GOU} - E(C_0)\exp(-\alpha s))\right)\\
&\quad= \exp(-\alpha(t+s))(E(C_0^2) - E(C_0)^2) + E(C_0\exp(-\alpha t)\sigma \int_0^s \exp(-\alpha(s-u))\, dB_s^v)\\
&\qquad+E(C_0\exp(-\alpha s)\sigma \int_0^t \exp(-\alpha(t-u))\, dB_s^v)\\
&\qquad+E(\sigma^2 \int_0^t \exp(-\alpha(t-u))\, dB_s^v \int_0^s \exp(-\alpha(s-u))\, dB_s^v).
\end{aligned}$$

Finally we present a process that can be considered as an extension of the Brownian Bridge.

Example 4.4 [Brownian bridge with B_t^v] An ordinary Brownian bridge is given for $0 < a < b$ and $t \in [0,T]$ by

$$B_t^{a\to b} := a\left(1 - \frac{t}{T}\right) + b\frac{t}{T} + \left(B_t - \frac{t}{T}B_T\right).$$

This definition motivates the extension to the generalized Brownian Bridge given by

$$B_t^{v,a\to b} := a\left(1 - \frac{t}{T}\right) + b\frac{t}{T} + \left(B_t^v - \frac{t}{T}B_T^v\right).$$

The expectation value is $E(B_t^{v,a\to b}) = E(B_t^{a\to b}) = a + (b-a)t/T$. The covariance function of $B_t^{v,a\to b}$ is equal to

$$E\left(\left(B_s^v - \frac{s}{T}B_t^v\right)\left(B_t^v - \frac{t}{T}B_T^v\right)\right) = v(s,t) - \frac{t}{T}v(s,T) - \frac{s}{T}v(s,T) + \frac{st}{T}v(T,T).$$

Example 4.5 [Application of short range dependency] Let $S(t)$ be the value of a brand. Suppose that the value of the brand is influenced by the investment rate of the current value of the brand $\mu(t)$. It seems reasonable that, if $S(t)$ is growing very fast, the increments of the value in the next few time periods change depending on what had happened just recently. This is a realistic short range influence as people begin to talk about the brand until they forget the advertisement. But on the other side one may imagine a long range dependency of $S(t)$ which stands for the long memory of the people. So, if we formulate this in a stochastic differential equation, we may get

$$S(t) = S_0 + \int_0^t \mu S(s)\,ds + \int_0^t \sigma_1 S(s)\,dB_s^H + \int_0^t \sigma_2 S(s)\,dB_s^s, \quad t > 0$$

with solution

$$S(t) = C \exp^{\diamond}(\mu t + \sigma_1 B_t^H + \sigma_2 B_t^s).$$

One can also construct such examples for stock prices and other processes, where the memory of the people is crucial.

5 Notes

Remark 5.1 If we use the Wick product than the stochastic integral's expectation turns out to be zero. There were other approachs to define the stochastic integral driven by fractional Brownian motion with the ordinary product. But the expectation of these integrals can have values nonequal to zero. Another problem with the Wick product is that the it is not closed as an operation in (L^2). Hence it can happen that for a stochastic process in (L^2) the integral with a Gaussian process with dependent increments is a stochastic distribution. But one can restrict the class of admissible integrands such that the integral is a random variable.

Remark 5.2 In the definition of Gaussian processes with dependent increments we started with the function $m(u,t)$, then the covariance function $v(s,t)$ followed. For applications it may be desirable to suppose a covariance function of the stochastic process, because the covariance function can be estimated more easily, and then to compute the function $m(u,t)$. The authors skipped this discussion in the present paper.

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