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Option Pricing in a Black-Scholes Market with Memory *

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Abstract

We price contingent claims by a replicating portfolio in a Black-Scholes market with memory. The Black-Scholes market is given by a Gaussian process with dependent increments. The wealth process of the replicating portfolio will be formulated by the use of the Wick product, which has its interpretation by the existence of an equal classical portfolio. Further a chain rule for stochastic distribution processes is proved.

1 Introduction

In the last five years several approaches to price options in markets with stock prices have been driven by a fractional Brownian motion has been published (see e.g. [HuOk], [HuOkSa]). The most interesting feature of these markets is that the stock price process has dependent increments. There the construction of a stochastic integral and a replicating portfolio uses a product for Hida distributions called Wick product. Several problems of the interpretation of these models especially of the Wick product are discussed in [BjHu] and [Be]. These arguments would normally force to deny these economic models.

In [DiSc] an integral calculus for a large class of Gaussian processes with dependent increments is formulated, which contains the fractional Brownian motion as a special case. In order to define the stochastic integral and the replicating portfolio the Wick product is employed, too. In the present paper we suggest how to interpretate this Wick portfolio. In the second and third section of this paper results on the white noise calculus and the stochastic integral developped in [DiSc] are sketched. In the fourth section a chain rule for stochastic distribution processes is proved, and it is shown that in special cases it is identical with Itô's rule. In the fifth section we formulate the Black-Scholes market with memory and price a path-independent contingent claim in a Black-Scholes market with memory. It turns out that the value of such an options is the solution of a partial differential equation, namely a weighted heat equation with boundary condition given by the pay-off profile of the option.

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2 Gaussian processes and white noise calculus

2.1 The construction of the Schwartz space and its dual

Let $|\cdot|_0$ be the norm of $L^2(\mathbb{R})$. We sketch the construction of the Schwartz space $S(\mathbb{R})$ with the locally convex topology and its dual $S'(\mathbb{R})$ with the weak topology. Let $\langle \omega, \eta \rangle$ denote the bilinear pairing with $\omega \in S'(\mathbb{R})$ and $\eta \in S(\mathbb{R})$. It follows that $\langle \cdot, \cdot \rangle$ is the inner product of $L^2(\mathbb{R})$ if $\omega, \eta \in L^2(\mathbb{R})$. The following construction of the Schwartz space and its dual is presented, for instance, in [Ku], Chapter 3.2. Let $A := -\frac{d^2}{dx^2} + x^2 + 1$, so A is densely defined on $L^2(\mathbb{R})$. With the Hermite polynomial of degree n

$$H_n(x) := (-1)^n e^{x^2} \left(\frac{d}{dx} \right)^n e^{-x^2}$$

we define

$$e_n(x) := \frac{1}{\sqrt{\sqrt{\pi} 2^n n!}} H_n(x) e^{\frac{-x^2}{2}}.$$

The functions $e_n(x)$ are eigenfunctions of A and the corresponding eigenvalue is $2n+2$, $n \in \mathbb{N}_0$. The operator A^{-1} is bounded on $L^2(\mathbb{R})$, especially A^{-p} is a Hilbert-Schmidt operator for any $p > \frac{1}{2}$. Let for each $p \geq 0$, $|f|_p := |A^p f|_0$. The norm is given by the eigenvalues as

$$|f|_p = \left(\sum_{n=0}^{\infty} (2n+2)^{2p} \langle f, e_n \rangle^2 \right)^{1/2}.$$

We define

$$S_p(\mathbb{R}) := \{ f; f \in L^2(\mathbb{R}), |f|_p < \infty \}$$

and with these spaces the Schwartz space $S(\mathbb{R})$ can be represented by $S(\mathbb{R}) = \cap_{p \geq 0} S_p(\mathbb{R})$. This construction leads to the Gel'fand triple $S(\mathbb{R}) \subset L^2(\mathbb{R}) \subset S'(\mathbb{R})$. Furthermore, we get the following continuous inclusion maps

$$S(\mathbb{R}) \subset S_p(\mathbb{R}) \subset L^2(\mathbb{R}) \subset S'_p(\mathbb{R}) \subset S'(\mathbb{R}).$$

Let $\mathcal{B}$ denote the Borel σ-algebra on $S'(\mathbb{R})$, i.e., the σ-algebra generated by the weak topology. One can show by the use of the Bochner-Minlos theorem that there is a unique Gaussian measure μ on $(S'(\mathbb{R}), \mathcal{B})$. The space $(S'(\mathbb{R}), \mathcal{B}, \mu)$ is called *white noise*, and the space (L^2) denotes $L^2(S'(\mathbb{R}), \mathcal{B}, \mu)$. The bilinear form $\langle \omega, f \rangle$ with $f \in L^2(\mathbb{R})$ and $\omega \in S'(\mathbb{R})$ is declared by

$$\lim_{k \to \infty} \langle \omega, \eta_k \rangle = \langle \omega, f \rangle$$

with $\eta_k \to f$ and $\{\eta_k\} \subset S(\mathbb{R})$. It is possible to show that $\langle \cdot, f \rangle = \int_{\mathbb{R}} f(s)\, dB_s$ is a random variable in (L^2) for all $f \in L^2(\mathbb{R})$. The random variable $\langle \cdot, f \rangle$ has expectation value zero and variance $|f|_0^2$.

2.2 A class of Gaussian processes with dependent increments

Suppose $m(\cdot,\cdot) : \mathbb{R}^2 \to \mathbb{R}$ such that for all $t \in \mathbb{R}$ the function $m(u,t) \in L^2(\mathbb{R})$. Define $v(s,t) = \int_{\mathbb{R}} m(u,t)m(u,s)\,du$. Then we have the stochastic process

$$B_t^v := \langle \cdot, m(u,t)\rangle = \int_{\mathbb{R}} m(u,t)\,dB_u, \quad t \in \mathbb{R},$$

with ordinary Brownian motion B_u. We choose $m(u,t)$ such that $\frac{d}{dt}m(u,t) \in S_p'(\mathbb{R})$ for all $t \in \mathbb{R}$ and for some $p \geq 0$. The process B_t^v is a Gaussian process. Its covariance function is given by

$$v(s,t) = \int_{S'(\mathbb{R})} B_s^v B_t^v \, d\mu = \int_{\mathbb{R}} m(u,s)m(u,t)\,du.$$

Now we show some properties of B_t^v which follows by assumed properties of $m(u,t)$. Let $m(u,0) \equiv 0$, hence $B_0^v = 0$, and B_t^v is a centered Gaussian process. The properties $v(t,t) \geq 0$ and $v(s,t) = v(t,s)$ are obvious. It is natural to request that B_t^v is pathwise continuous. This is ensured by the supposed continuity of the function $\langle \omega, m(u,\cdot)\rangle : \mathbb{R} \to \mathbb{R}$, $\omega \in S'(\mathbb{R})$. In Section 5 we will additionally require that the function $|m(\cdot,t)|_0^2$ is continuously differentiable and strictly monotone increasing in t with $t > 0$ and that for $T > 0$

$$\lim_{t \to T} \frac{(T-t)}{\sqrt{|m(u,T)|_0^2 - |m(u,t)|_0^2}} < \infty.$$

Further we suppose that there is a $\eta \in S(\mathbb{R})$ such that for all $t \in [0,T]$, $T > 0$

$$\langle \frac{dm(u,t)}{dt}, \eta(u)\rangle = 1([0,T])(t) \tag{1}$$

holds, where $1([0,T])$ is the indicator function of the intervall $[0,T]$. In the following we give some special instances of B_t^v.

Example 2.1 [Ordinary Brownian motion] The stochastic process $B_t^v = B_t$ is the ordinary Brownian motion if $m(u,t) = 1([0,t])(u)$. Its covariance function is $v(s,t) = \min(t,s)$. This example is further discussed in [Ku], Chapter 3.1.

Example 2.2 [A Gaussian process with short range dependency] Let B_t^s, $t \in \mathbb{R}$, be a centered Gaussian process with covariance function $v(s,t)$, such that $v(s,\cdot)$ has a global maximum and $\lim_{t\to\infty} v(s,t) = 0$ for all fixed $s \in \mathbb{R}$. Then we call B_t^s a short range Brownian motion. As an example consider $m(u,t) = t^2 \exp(-(u-t)^2)$, hence $v(s,t) = kt^2s^2\exp(-(t-s)^2/2)$ with a constant k.

The fractional Brownian motion B_t^H can be formulated in this framework, too (see [DiSc] or [Be]).

2.3 The construction of the Hida test and distribution space

In this section the construction of the Hida test and Hida distribution spaces (S) and $(S)^*$ are outlined. These spaces are used to derivate the Gaussian process B_t^v and to define a stochastic integral driven by these Gaussian processes. A more detailed description can be found e.g. in [Ku], Chapter 3.3, and in [Be], Chapter 5.3. Together with (L^2) they form also a Gel'fand triple $(S) \subset (L^2) \subset (S)^*$. Let $\langle\langle \Phi, \zeta \rangle\rangle$ denote the bilinearform of $\Phi \in (S)^*$ and $\zeta \in (S)$. For $f \in L^2(\mathbb{R}^n)$ the multiple Wiener integral with respect to ordinary Brownian motion is defined by

$$I_n(f) := n! \int_{\mathbb{R}^n} f(t_1, t_2, t_3, \ldots, t_n)\, dB_{t_1} dB_{t_2} \ldots dB_{t_n}.$$

The following proposition is the chaos decompostion of (L^2) (see [Be], Theorem 1.4.8).

Proposition 2.3 *For all $F \in (L^2)$ there is a unique sequence $(f_n)_{n \in \mathbb{N}_0}$ such that $f_n \in L^2(\mathbb{R}^n)$ is symmetric and*

$$F = \sum_{n=0}^{\infty} I_n(f_n)$$

with convergence in (L^2).

For A as above the operator $\Gamma(A)$ on (L^2) is defined by

$$\Gamma(A)F = \sum_{n=0}^{\infty} I_n(A^{\otimes n} f_n)$$

with $F \in (L^2)$. Let $(S)_n = \Gamma(A)^n((L^2))$ with inner product

$$((F, G))_n := E[\Gamma(A)^{-n} F \cdot \Gamma(A)^{-n} G]$$

with $F, G \in (L^2)$. Now we set similarly as above $(S) := \cap_{n \in \mathbb{N}} (S)_n$. With $(S)'_n := (S)_{-n}$, we get $(S)^* := \cup_{n \in \mathbb{N}} (S)_{-n}$. The topologies of (S) and of $(S)^*$ are given by the projective limit topology and weak topology, respectively (see [Ku], Chapter 2.2). As before there are continuous inclusion maps

$$(S) \subset (S)_n \subset (L^2) \subset (S)_{-n} \subset (S)^*.$$

2.4 S-transform and Wick product

Now we introduce the S-transform from $(S)^*$ into the set of the functions from $S(\mathbb{R})$ to $\mathbb{R}$.

For $\eta \in S(\mathbb{R})$ set $I(\eta) := I_1(\eta) = \int_{\mathbb{R}} \eta(s)\, dB_s$.

Proposition 2.4 *For all $\eta \in S(\mathbb{R})$ the random variable $\exp(I(\eta) - 1/2|\eta|_0^2)$ is a Hida test function and the set*

$$\left\{ e^{I(\eta) - 1/2|\eta|_0^2} : \eta \in S(\mathbb{R}) \right\}$$

is total in (S).

The proof can be found in [Ku], Proposition 5.10.

Definition 2.5 *The S-transform of a Hida distribution $\Phi \in (S)^*$ is defined by*

$$(S\Phi)(\eta) := \langle\langle \Phi, e^{I(\eta)-1/2|\eta|_0^2} \rangle\rangle, \quad \eta \in S(\mathbb{R}).$$

Remark 2.6 The image of the S-transform is discussed in [PoSt]. There it is proved that the S-transform as a mapping from $S(\mathbb{R})$ to $\mathbb{R}$ has some analytic properties. The S-transform is an injective mapping.

This follows from the preceeding proposition (see [Ku], Proposition 5.10).

Proposition 2.7 *The S-transform of $\langle \cdot, f \rangle$ with $f \in S'(\mathbb{R})$ is given by*

$$S(\langle \cdot, f \rangle)(\eta) = \langle f, \eta \rangle$$

For further explanation about the notation and the properties of the term $\langle \cdot, f \rangle$ with $f \in S'(\mathbb{R})$ see [Ku], Chapter 3.4. The proof is given in [Ku], Proposition 5.9. This proposition will be used to define the stochastic integral driven by Gaussian processes with dependent increments. For further details of the S-transform see [Ku], Chapter 5, or [PoSt].

Example 2.8 The S-transform of the ordinary Brownian motion B_t is

$$\begin{aligned} S(B_t)(\eta) &= \langle\langle B_t, e^{I(\eta)-1/2|\eta|_0^2} \rangle\rangle \\ &= \int_{S'(\mathbb{R})} B_t e^{I(\eta)-1/2|\eta|_0^2}\, d\mu \\ &= \int_{\mathbb{R}} 1([0,t])(u)\eta(u)\, du = \int_0^t \eta(u)\, du, \end{aligned}$$

by the use of proposition 2.7.

Example 2.9 For the short range Brownian motion B_t^s we get

$$\begin{aligned} S(B_t^s)(\eta) &= \langle\langle B_t^s, e^{I(\eta)-1/2|\eta|_0^2} \rangle\rangle \\ &= \int_{S'(\mathbb{R})} B_t^s e^{I(\eta)-1/2|\eta|_0^2}\, d\mu \\ &= \int_{\mathbb{R}} \eta(u) t^2 \exp(-(u-t)^2)\, du, \end{aligned}$$

by the use of proposition 2.7.

Now one can define the Wick product as follows (see [Ku], Chapter 8.4, page 92):

Definition 2.10 *The Wick product of two Hida distributions Φ and Ψ in $(S)^*$, denoted by $\Phi \diamond \Psi$, is the unique Hida distribution in $(S)^*$ such that $S(\Phi \diamond \Psi)(\eta) = S(\Phi)(\eta)S(\Psi)(\eta)$ for all $\eta \in S(\mathbb{R})$.*

3 Stochastic calculus for Gaussian processes with dependent increments

3.1 The white noise $\mathbf{W_t^v}$

In this subsection we differentiate B_t^v. The following definition is taken from [Be], Chapters 5.3 and 5.4.

Definition 3.1 *Let I be an interval in $\mathbb{R}$. A mapping $X : I \to (S)^*$ is called a stochastic distribution process. A stochastic distribution process is called differentiable in the $(S)^*$-sense, if*

$$\lim_{h\to 0} \frac{X_{t+h} - X_t}{h}$$

exists in $(S)^*$.

Now we are prepared to compute dB_t^v/dt in the $(S)^*$-sense and to define the stochastic integral $\int_{\mathbb{R}} X_t \, dB_t^v$ as a white noise integral by use of the Wick product.

The next theorem presents a criterion for differentiability (see [Be], Theorem 5.3.9). Let I be as above.

Theorem 3.2 *Let $F : I \to S'(\mathbb{R})$ be differentiable in the sense of the weak topology of $S'(\mathbb{R})$. Then $\langle \cdot, F(t)\rangle$ is a differentiable stochastic distribution process and*

$$\frac{d}{dt}\langle \cdot, F(t)\rangle = \langle \cdot, \frac{d}{dt}F(t)\rangle.$$

So we obtain a theorem for the derivative of the Gaussian process B_t^v in the $(S)^*$-sense.

Theorem 3.3 (Derivative of B_t^v)

$$W_t^v := \frac{d}{dt}B_t^v = \langle \cdot, \frac{d}{dt}m(u,t)\rangle$$

The S-transform of W_t^v is

$$S(W_t^v)(\eta) = \langle \frac{d}{dt}m(u,t), \eta(u)\rangle.$$

Proof. The proof is given by Theorem 3.2. □

With δ_t denoting the Dirac distribution we get

Example 3.4 [Ordinary Brownian motion] For the derivative of the ordinary Brownian motion B_t, we use $\frac{d}{dt}1([0,t]) = \delta_t$ so $dB_t/dt =: W_t = \langle \cdot, \delta_t\rangle$, further $S(W_t)(\eta) = \eta(t)$ (see [Ku], Chapter 3.1).

Example 3.5 [Short range Brownian motion] The derivative of B_t^s is obviously given by

$$\langle \cdot, \frac{d}{dt}t^2 \exp(-(t-u)^2)\rangle = \langle \cdot, 2t \exp(-(t-u)^2) + t^2(-2(u-t)) \exp(-(u-t)^2)\rangle.$$

In this way one can also get the derivative of the fractional Brownian motion W_t^H (see [Be] or [DiSc]). The stochastic distribution process W_t^v is called white noise of B_t^v. With the white noise W_t^v we can define $\int_{\mathbb{R}} X_t \, dB_t^v$.

3.2 White noise integral and stochastic differential equations driven by $\mathbf{B_t^v}$

We start with the definition of the white noise integral (see [Ku], Chapter 13).

Definition 3.6 *The stochastic distribution process $X_. : \mathbb{R} \to (S)^*$ is white noise integrable, if there is a $\Psi \in (S)^*$ such that, for all $\eta \in S(\mathbb{R})$, $(SX_.)(\eta) \in L^1(\mathbb{R})$ and*

$$(S\Psi)(\eta) = \int_{\mathbb{R}} (SX_t)(\eta)\, dt.$$

This definition makes sense as the S-transform is injective.

Definition 3.7 *If $X_t \diamond W_t^v$ is white noise integrable, the stochastic integral of the stochastic distribution process X_t is given by*

$$\int_{\mathbb{R}} X_t\, dB_t^v := \int_{\mathbb{R}} X_t \diamond W_t^v\, dt.$$

This definition coincides in the case of the fractional Brownian motion with the definition of the fractional Itô integral (see Bender ([Be]), Øksendal and Hu [HuOk]). Further consider a continuously differentiable stochastic distribution process Z_t.

Definition 3.8 *If $X_t \diamond (dZ_t)/(dt)$ is white noise integrable, then the stochastic distribution process X_t has a stochastic integral $\int_{\mathbb{R}} X_t\, dZ_t := \int_{\mathbb{R}} X_t \diamond (dZ_t)/(dt) dt$.*

The following theorem is inspired by Bender's theorem for fractional Itô integrals (see [Be], Chapter 5) and proved in [DiSc].

Theorem 3.9 *Let $a, b \in \mathbb{R}$, $X_. : [a,b] \to (S)_{-p}$ be continuous for some $p \in \mathbb{N}$, and $W_.^v : \mathbb{R} \to (S)_{-q}$ be continuous for some $q \in \mathbb{N}$. Then $\int_a^b X_t\, dB_t^v$ exists. Further for any sequence of tagged partitions $\tau_n = (\pi_k^{(n)}, t_k^{(n)})$ of $[a,b]$ with $\lim_{n\to\infty} \max\{|\pi_k - \pi_{k-1}|; k = 1, .., n\} = 0$, we have*

$$\lim_{n\to\infty} \sum_{k=0}^{n} X_{t_{k-1}^{(n)}} \diamond \left(B^v_{\pi_k^{(n)}} - B^v_{\pi_{k-1}^{(n)}} \right) = \int_a^b X_t\, dB_t^v$$

with limit in $(S)^*$. Let $\int_0^t X_s\, dB_s^v$ exist and $\int_0^t X_s\, dB_s^v \in (L^2)$. Then $E(\int_0^t X_s\, dB_s^v) = 0$.

The S-transform of $\int_{\mathbb{R}} X_t\, dB_t^v$ is therefore given by

$$S\left(\int_{\mathbb{R}} X_t\, dB_t^v \right)(\eta) = \int_{\mathbb{R}} S(X_t \diamond W_t^v)(\eta)\, dt = \int_{\mathbb{R}} (SX_t)(\eta) \langle \frac{d}{dt} m(u,t), \eta(u) \rangle\, dt.$$

Example 3.10 We calculate

$$\int_0^t B_s^v \, dB_s^v = \int_0^t B_s^v \diamond W_s^v \, ds = \frac{1}{2} \, (B_t^v)^{\diamond 2} \, ,$$

where $(\cdot)^{\diamond 2}$ is the Wick square. By the Wick calculus presented in [Be], Chapter 1, or [Ku], Chapter 8, we get

$$\frac{1}{2} \, (B_t^v)^{\diamond 2} = \frac{1}{2} \, (B_t^v)^2 - \frac{1}{2} |m(u,t)|_0^2.$$

Note that this example coincides with the case of ordinary Brownian motion with the Itô-integral and with the case of fractional Brownian motion with the fractional Itô integral.

Example 3.11 Consider the stochastic differential equation

$$dX_t = \mu X_t \, dt + \sigma X_t \, dB_t^v,$$

with constants $\mu \in \mathbb{R}$ and $\sigma > 0$, which is declared by the integral equation

$$\begin{aligned} X_t &= \int_0^t \mu X_s \, ds + \int_0^t \sigma X_s \, dB_s^v \\ &= \int_0^t (\mu X_s + \sigma X_s \diamond W_s^v) \, ds \\ &= \int_0^t X_s \diamond (\mu + \sigma W_s^v) \, ds, \end{aligned}$$

Its solution is given by

$$X_t = \exp^{\diamond}(\mu t + \sigma B_t^v) = \exp(\mu t + \sigma B_t^v - \frac{1}{2} \sigma^2 |m(u,t)|_0^2),$$

where $\exp^{\diamond}(\langle \cdot, f \rangle) = \sum_{k=0}^{\infty} (\langle \cdot, f \rangle)^{\diamond k} / k!$, and some Wick calculus is used in the last equation, see e.g. [Be], Chapter 1.

4 A Chain Formula for stochastic distribution processes

In this section we deduce a chain formula for a certain class of stochastic distribution processes. It will be proved, that this chain formula includes Itô's rule in the case of the Brownian motion. For its proof we will need the following characterization of convergence in $(S)^*$ (see [Ku], Theorem 8.6, page 86, or [PoSt]).

Theorem 4.1 *Let (Φ_n) be a sequence in $(S)^*$ and $F_n(\eta) = S(\Phi_n)(\eta)$. Then (Φ_n) converges in $(S)^*$ if and only if the following conditions are satisfied:*

(i) $\lim_{n\to\infty} F_n(\eta)$ *exists for all* $\eta \in S(\mathbb{R})$.

(ii) There are positive constants K, a *and* p *such that for all* $n \in \mathbb{N}$ *and* $\eta \in S(\mathbb{R})$

$$|F_n(\eta)| \le K \exp(a|\eta|_p^2) \tag{2}$$

In [Ku], Chapter 5.3, an identity between Wick powers and powers with ordinary product is given, which will be used several times in the following proofs. For $f \in L^2(\mathbb{R})$ and $n \in \mathbb{N}_0$ it holds

$$(\langle \cdot, f\rangle)^n = \sum_{k=0}^{n/2} \binom{n}{2k} (2k-1)!!\, |f|_0^{2k} (\langle \cdot, f\rangle)^{\diamond(n-2k)},$$

where by convention $(2k-1)!! = (2k-1)(2k-3)\cdots 3\cdot 1$ and $(-1)!! = 1$. Further, the upper limit of the index is to be understood as the integer part of $n/2$.

Theorem 4.2 *Let* $f(\cdot,\cdot) : \mathbb{R} \times (S)^* \to (S)^*$, *such that* $f(t,X) = \sum_{k=0}^n a_k(t) X^{\diamond k}$ *with continuously differentiable functions* $a_k(t)$. *Then the chain formula*

$$\begin{aligned} f(b, B_b^v) - f(a, B_a^v) &= \int_a^b \left(\frac{\partial}{\partial t} f(t, B_t^v) + \frac{\partial}{\partial x} f(t, B_t^v) \diamond W_t^v \right) dt \\ &= \int_a^b \frac{\partial}{\partial t} f(t, B_t^v)\, dt + \int_a^b \frac{\partial}{\partial x} f(t, B_t^v)\, dB_t^v \end{aligned}$$

holds.

Proof. The S-transform of $f(t, B_t^v)$ is given by $g(t;\eta) := \sum_{k=0}^n a_k(t)(S(B_t^v)(\eta))^k$. It follows that the derivative of the function $g(t;\eta)$ equals

$$\frac{dg(t;\eta)}{dt} = \sum_{k=0}^n \left(\frac{da_k(t)}{dt} (S(B_t^v)(\eta))^k + a_k(t) k (S(B_t^v)(\eta))^{k-1} S(W_t^v)(\eta) \right),$$

by use of Theorems 3.2 and 3.3. After integrating both sides with respect to t, the S-transform of the assertion follows. □

Theorem 4.3 *Let* f *be as in Theorem 4.2. If* B_t^v *is the ordinary Brownian motion* B_t, *then the chain rule as given above and Itô's rule coincide.*

Proof. Due to the Wick calculus the function $f(t, B_t)$ has two representations, one with Wick product, and the other with ordinary product. The chain rule is declared with respect to the representation with the Wick product and Itô's rule ist declared with respect to the representation with the ordinary product. Both will be calculated and by comparing the summands it will be shown, that they are equal. As the chain rule and Itô's rule are linear mappings, it is sufficient to check the assertion for $a(t)(B_t)^n$ with any continuously differentiable $a(t)$ and any $n \in \mathbb{N}$. First we show the representation with the Wick product of $a(t)(B_t)^n$ by use of the Wick calculus and the chain rule

$$
\begin{aligned}
a(t)(B_t)^n &= a(t)\sum_{k=0}^{n/2}\binom{n}{2k}(2k-1)!!\, t^k (B_t)^{\diamond(n-2k)} \\
&= \int_0^t \frac{da(u)}{du}\sum_{k=0}^{n/2}\binom{n}{2k}(2k-1)!!\, u^k (B_u)^{\diamond(n-2k)}\, du \\
&\quad + \int_0^t a(u)\sum_{k=0}^{n/2}\binom{n}{2k}(2k-1)!!\, k u^{k-1} (B_u)^{\diamond(n-2k)}\, du \\
&\quad + \int_0^t a(u)\sum_{k=0}^{n/2}\binom{n}{2k}(2k-1)!!\, u^k (n-2k) (B_u)^{\diamond(n-2k-1)}\, dB_u \\
&=: I_1 + I_2 + I_3
\end{aligned}
$$

Applying Itô's rule to $a(t)(B_t)^n$ and then using the Wick representation it follows

$$
\begin{aligned}
a(t)(B_t)^n &= \int_0^t \frac{da(u)}{du}(B_u)^n\, du + \int_0^t a(u) n (B_u)^{n-1}\, dBu + \frac{1}{2}\int_0^t a(u)n(n-1)(B_u)^{n-2}\, du \\
&= \int_0^t \frac{da(u)}{du}\sum_{k=0}^{n/2}\binom{n}{2k}(2k-1)!!\, u^k (B_u)^{\diamond(n-2k)}\, du \\
&\quad + \int_0^t a(u) n \sum_{k=0}^{(n-1)/2}\binom{(n-1)}{2k}(2k-1)!!\, u^k (B_u)^{\diamond(n-2k-1)}\, dB_u \\
&\quad + \frac{1}{2}\int_0^t a(u)n(n-1)\sum_{k=0}^{(n-2)/2}\binom{(n-2)}{2k}(2k-1)!!\, u^k (B_u)^{\diamond(n-2k-2)}\, du \\
&=: I_4 + I_5 + I_6.
\end{aligned}
$$

The term I_1 is equal to I_4. The integral I_2 is equal to I_6, because with $k+1 =: m$ in I_6 we get

$$
\begin{aligned}
I_6 &= \frac{1}{2}\int_0^t a(u)n(n-1)\sum_{k=0}^{(n-2)/2}\binom{(n-2)}{2k}(2k-1)!!\, u^k (B_u)^{\diamond(n-2k-2)}\, du \\
&= \int_0^t a(u)\sum_{m=1}^{n/2}\frac{(n-2)!\, n\,(n-1)\, 2m\,(2m-1)}{2(n-2m)!(2m)!}(2m-3)!!\, u^{m-1}(B_u)^{\diamond(n-2m)}\, du \\
&= \int_0^t a(u)\sum_{m=1}^{n/2}\binom{n}{2m}(2m-1)!!\, m u^{m-1}(B_u)^{\diamond(n-2m)}\, du \\
&= I_2.
\end{aligned}
$$

The integral I_3 is equal to I_5, note that both sums have the same largest k if n is odd, and if n is even, then the last term in I_3 vanishes because of the factor $(n-2k)$, hence both sums end again at the same k. In order to show that both integrals are equal, we calculate

$$\begin{aligned}
I_5 &= \int_0^t a(u)n \sum_{k=0}^{(n-1)/2} \binom{(n-1)}{2k} (2k-1)!!\, u^k (B_u)^{\diamond(n-2k-1)}\, dB_u \\
&= \int_0^t a(u) \sum_{k=0}^{(n-1)/2} \frac{n!}{(n-1-2k)!(2k)!} (2k-1)!!\, u^k (B_u)^{\diamond(n-2k-1)}\, dB_u \\
&= \int_0^t a(u) \sum_{k=0}^{n/2} \binom{n}{2k} (2k-1)!!(n-2k)\, u^k (B_u)^{\diamond(n-2k-1)}\, dB_u \\
&= I_3.
\end{aligned}$$

□

In the next step the chain rule for $\lim_{n\to\infty} \sum_{k=0}^{n} a(t)(B_t^v)^{\diamond k}$ will be considered. The idea is to approximate functions by these Wick polynomials and to define a class of functions to which the chain rule applies. One of these functions is the already known Wick exponential $\exp^{\diamond}(\langle\cdot, f\rangle) = \sum_{k=0}^{\infty}(\langle\cdot, f\rangle)^{\diamond k}/k!$ with $f \in S'(\mathbb{R})$.

Definition 4.4 *Let D be a subset of $(S)^*$. A function $f : \mathbb{R} \times D \to (S)^*$ admits a Wick representation in D, if there exists a sequence of Wick polynomials $\{a_k(t)X^{\diamond k}\}_{k=0}^{\infty}$ with continuously differentiable $a_k(t)$, such that for all $X \in D$*

$$f(t, X) = \lim_{n\to\infty} \sum_{k=0}^{n} a_k(t) X^{\diamond k}$$

with convergence in $(S)^*$.

Now the question arises which functions f have this property. A sufficent condition for this is given in the following corollary.

Corollary 4.5 *Suppose that $g : \mathbb{R} \to \mathbb{R}$ has a power series representation with $g(x) = \sum_{k=0}^{\infty} g_k x^k$ for all $x \in \mathbb{R}$, and that there exist positive constants C, p and a such that for $X \in (S)^*$ and for all $\eta \in S(\mathbb{R})$*

$$|g(S(X)(\eta))| \leq C \exp(a|\eta|_p^2). \tag{3}$$

Then

$$g^{\diamond}(X) := \left(\sum_{k=0}^{\infty} g_k X^{\diamond k} \right) \in (S)^*.$$

Proof. We have to check the first condition of Theorem 4.1. The convergence of $F_n(\eta) := \sum_{k=0}^n g_k(S(X)(\eta))^k$ is follows from the convergence of $\sum_{k=0}^n g_k x^k$ for all $x \in \mathbb{R}$. □

In order to show where condition (3) may fail, we give the following example.

Example 4.6 Let $h \in L^2(\mathbb{R})$. Then $\exp^\diamond(\langle \cdot, h\rangle^{\diamond k}) \notin (S)^*$ for $k > 2$. We calculate the S-transform and get

$$S\left(\sum_{n=0}^{\infty} \frac{(\langle \cdot, h\rangle)^{\diamond(kn)}}{n!}\right)(\eta) = \sum_{n=0}^{\infty} \frac{(\langle h, \eta\rangle^k)^n}{n!} = \exp(\langle h, \eta\rangle^k).$$

Now choose $h = \eta$, so condition (3) fails.

Theorem 4.7 *Let $D := \{\langle \cdot, h\rangle,\ h \in S'(\mathbb{R})\}$. Suppose that $f : \mathbb{R} \times D \to (S)^*$ admits a Wick representation in D, then the chain rule for f holds with $b > a$*

$$f(b, B_b^v) - f(a, B_a^v) = \int_a^b \left(\frac{\partial f(t, B_t^v)}{\partial t} + \frac{\partial f(t, B_t^v)}{\partial x} \diamond W_t^v\right) dt.$$

Proof. Observe that $f(t, B_t^v) = \lim_{n\to\infty} \sum_{k=0}^n a_k(t)(B_t^v)^{\diamond k}$ and

$$\sum_{k=0}^n \left(a_k(b)(B_b^v)^{\diamond k} - a_k(a)(B_a^v)^{\diamond k}\right) = \int_a^b \sum_{k=0}^n \left(\frac{da_k(u)}{du}(B_u^v)^{\diamond k} + a_k(u)k(B_u^v)^{\diamond(k-1)} \diamond W_u^v\right) du$$

Taking limits $n \to \infty$ on both sides proves the assertion. □

As with Itô's chain rule it is helpful to have a chain rule which applies on functions of solutions of stochastic differential equations, too.

Theorem 4.8 *Suppose $f : \mathbb{R} \times D \to (S)^*$ admits a Wick representation, and let the stochastic distribution process $X_\cdot : \mathbb{R} \to D$ be given by*

$$X(t) - X(0) = \int_0^t (\mu(u) + \sigma(u) \diamond W_u^v)\, du$$

with two stochastic distribution processes μ and σ such that the integral exists. Then the chain rule for $f(t, X_t)$ holds with $b > a$

$$f(b, X_b) - f(a, X_a) = \int_a^b \left(\frac{\partial f(u, X_u)}{\partial t} + \frac{\partial f(u, X_u)}{\partial x} \diamond (\mu(u) + \sigma(u) \diamond W_u^v)\right) du.$$

Proof. The proof is obvious by taking the S-transform on both sides, and using the Wick representation of f and the integral representation of X_t. Fubini's theorem is applied to change the order of integration and S-transformation. □

Theorem 4.9 (Chain rule) *Suppose $f : \mathbb{R} \times D \to (S)^*$ admits a Wick representation and let $X. : \mathbb{R} \to D$ be a continuously differentiable stochastic distribution process. Then for $b > a$ it holds*

$$f(b, X_b) - f(a, X_a) = \int_a^b \frac{\partial f(u, X_u)}{\partial t} + \frac{\partial f(u, X_u)}{\partial x} \diamond \frac{dX_u}{du}\, du.$$

Proof. The proof is given by calculating the S-transform as above. □

Remark 4.10 The chain rule can be generalized without effort to vector-valued stochastic distribution processes $f : \mathbb{R} \times ((S)^*)^{\otimes n} \to (S)^*$, which admit a corresponding Wick representation.

5 Pricing in a Black-Scholes market with memory

The aim of this section is to derive a pricing formula for options. To begin with we discuss the properties of our market model. Its formulation is similar to [HuOk].

5.1 Market assumptions

Suppose the market offers two types of investment assets. Fix some $T > 0$. Firstly, a bond $A(t)$ with constant interest rate r follows

$$\frac{dA(t)}{dt} = r\, A(t) \tag{4}$$

with $t \in [0, T]$ and $A(0) = 1$. Secondly, consider a stock whose price process $S(t)$ is the solution of the stochastic differential equation

$$\frac{dS(t)}{dt} = \mu S(t) + \sigma S(t) \diamond W_t^v, \tag{5}$$

with the constants $\mu \in \mathbb{R}$ and $\sigma > 0$ and $t \in [0, T]$. Its unique solution is given by

$$S(t) = S(0) \exp(\mu t + \sigma B_t^v - 1/2\sigma^2 |m(u,t)|_0^2)$$

(for details see [DiSc]). A portfolio or a trading strategy is given by a two dimensional process $\theta(t, S(t)) = (g(t, S(t)), h(t, S(t)))$, where g and h are the quantities of bonds and stocks held at time t, respectively. In preceeding papers about fractional Brownian motion in finance at least two possibilities to model the value of a portfolio are suggested (see [BjHu]). One of them says that the wealth process of the *Wick portfolio* is

$$V^w(t, \theta) = g^w(t, S(t)) \diamond A(t) + h^w(t, S(t)) \diamond S(t) = g^w(t, S(t)) \cdot A(t) + h^w(t, S(t)) \diamond S(t).$$

Here the Wick product at $A(t)$ is changed to the ordinary product because $A(t)$ is deterministic, and the second possibility is the wealth process of the portfolio with ordinary product in both investment possibilities such that

$$V^o(t,\theta) = g^o(t,S(t)) \cdot A(t) + h^o(t,S(t)) \cdot S(t).$$

In [BjHu] there were several critical comments on the wealth process of the Wick portfolio. These arguments would normally force to deny the Wick portfolio. But we will show that under certain circumstances one can deduce from a given portfolio with ordinary products a Wick portfolio, such that both portfolios are equal in (L^2). Let

$$D := \{S(0)\exp(\mu t + \sigma B_t^v - 1/2\sigma^2 |m(u,t)|_0^2) : t \in [0,T]\}. \tag{6}$$

So here D is the set of random variables describing the stock prices, which can occur in $[0,T]$. Let $g^o, h^o : [0,T] \times \mathbb{R} \to \mathbb{R}$, such that $g^o(t,x) = \sum_{k=0}^\infty g_k^o(t)x^k$ and $h^o(t,x) = \sum_{k=0}^\infty h_k^o(t)x^k$ for all $x \in \mathbb{R}$ and continuously differentiable $g_k^o(t)$ and $h_k^o(t)$ and suppose $g^o(t,X), h^o(t,X) \in (L^2)$ for all $X \in D$ and for all $t \in [0,T]$.

Theorem 5.1 *Let D, g^o and h^o be as above. Then there exists a unique Wick portfolio wealth process $V^w(t,\theta)$ such that for all $t \in [0,T]$ it holds a.s. that*

$$\begin{aligned} V^o(t,\theta) &= g^o(t,S(t)) \cdot A(t) + h^o(t,S(t)) \cdot S(t) \\ = V^w(t,\theta) &= g^w(t,S(t)) \diamond A(t) + h^w(t,S(t)) \diamond S(t), \end{aligned}$$

where g^w, h^w admit a Wick representation in D. For a Wick portfolio wealth process $V^w(t,\theta)$, where g^w and h^w admit a Wick representation in D, there is a unique wealth process of a portfolio with ordinary products $V^o(t,\theta)$, such that $V^o(t,\theta) = V^w(t,\theta)$ a.s. and for all $t \in [0,T]$.

Proof. We calculate for $n \in \mathbb{N}_0$, where $X^{\diamond 0} := 1$ with $X \in D$,

$$\begin{aligned} (S(t))^{\diamond n} &= (S(0))^n (\exp^\diamond(\mu t + \sigma B_t^v))^{\diamond n} \\ &= (S(0))^n \exp(n\mu t)(\exp^\diamond(\langle \cdot, \sigma m(u,t)\rangle))^{\diamond n} \\ &= (S(0))^n \exp(n\mu t)\exp^\diamond(\langle \cdot, n\sigma m(u,t)\rangle) \\ &= (S(0))^n \exp(n\mu t + n\sigma B_t^v - \frac{1}{2}n^2\sigma^2 |m(u,t)|_0^2) \\ &= (S(0))^n \exp(n\mu t + n\sigma B_t^v - \frac{1}{2}n\sigma^2 |m(u,t)|_0^2)\exp(\frac{1}{2}(n-n^2)\sigma^2|m(u,t)|_0^2) \\ &= (S(0))^n (\exp(\mu t + \sigma B_t^v - \frac{1}{2}\sigma^2|m(u,t)|_0^2))^n \exp(\frac{1}{2}(n-n^2)\sigma^2|m(u,t)|_0^2) \\ &= (S(t))^n \exp(\frac{1}{2}(n-n^2)\sigma^2|m(u,t)|_0^2). \end{aligned}$$

Applying this identity to the portfolio with ordinary products

$$\begin{aligned}
V^o(t,\theta) &= g^o(t,S(t))\cdot A(t) + h^o(t,S(t))\cdot S(t)\\
&= \sum_{k=0}^{\infty} A(t) g_k^o(t)(S(t))^k + \sum_{k=0}^{\infty} h_k^o(t)(S(t))^{k+1}\\
&= \sum_{k=0}^{\infty} A(t) g_k^o(t)\exp(\frac{1}{2}(k^2-k)\sigma^2|m(u,t)|_0^2)(S(t))^{\diamond k}\\
&\qquad + \sum_{k=0}^{\infty} h_k^o(t)\exp(\frac{1}{2}((k+1)^2-(k+1))\sigma^2|m(u,t)|_0^2)(S(t))^{\diamond(k+1)}\\
&= \left(\sum_{k=0}^{\infty} g_k^w(t)(S(t))^{\diamond k}\right) A(t) + \left(\sum_{k=0}^{\infty} h_k^w(t)(S(t))^{\diamond k}\right)\diamond S(t)\\
&= g^w(t,S(t))\diamond A(t) + h^w(t,S(t))\diamond S(t) = V^w(t,\theta),
\end{aligned}$$

where

$$g_k^w(t) := g_k^o(t)\exp\left(\frac{1}{2}(k^2-k)\sigma^2|m(u,t)|_0^2\right)$$

and

$$h_k^w(t) := h_k^o(t)\exp\left(\frac{1}{2}((k+1)^2-(k+1))\sigma^2|m(u,t)|_0^2\right).$$

The uniqueness of the Wick representation follows by the fact that the Wick product of two Hida distributions is unique. □

This justifies the use of the Wick portfolio and we assume the wealth process $V(t,\theta) := V^w(t,\theta)$ to have the representation of the Wick portfolio, where g, h admit a Wick representation in D.

Definition 5.2 *A trading strategy is self-financing if*

$$V(t,\theta) - V(0,\theta) = \int_0^t g(t,S(t))\,dA(t) + \int_0^t h(t,S(t))\,dS(t)$$

where integrals are defined in the $(S)^*$*-sense.*

Definition 5.3 *The portfolio* θ *is called an arbitrage for the market given by (4) and (5), if it is self-financing and* $V(0,\theta)=0$, $V(T,\theta)\geq 0$ *and* $\mu(\{\omega; V(T,\theta)(\omega)>0\})>0$, *where* μ *is the probability measure introduced in Subsection 2.1. Let* Θ *be the class of self-financing trading strategies in this market.*

Theorem 5.4 *There is no arbitrage in the class of self-financing trading strategies* Θ.

Proof. The wealth process $V(t,\theta)$ of a self-financing trading strategy θ satisfies with $v(t,S(t)) = g(t,S(t))A(t) + h(t,S(t)) \diamond S(t)$

$$V(T,\theta) = V(0,\theta) + \int_0^T g(u,S(u))rA(u)\,du \tag{7}$$

$$+ \int_0^T \mu\, h(u,S(u)) \diamond S(u)\,du + \int_0^T \sigma\, h(u,S(u)) \diamond S(u)\,dB_u^v \tag{8}$$

$$= V(0,\theta) + \int_0^T v(u,S(u))r\,du \tag{9}$$

$$+ \int_0^T (\mu - r)h(u,S(u)) \diamond S(u)\,du + \int_0^T \sigma\, h(u,S(u)) \diamond S(u)\,dB_u^v. \tag{10}$$

According to (1) there exists a $\tilde{\eta} \in S(\mathbb{R})$ such that

$$\langle \frac{dm(u,t)}{dt}, \tilde{\eta}(u) \rangle = \frac{r-\mu}{\sigma} 1([0,T])(t).$$

Evaluating the S-transform of (7) with $\tilde{\eta}$ we get

$$S(V(T,\theta))(\tilde{\eta}) = V(0,\theta) + \int_0^T rS(V(u,\theta))(\tilde{\eta})\,du.$$

The solution of this integral equation is

$$S(V(T,\theta))(\tilde{\eta}) = V(0,\theta)\exp(rT).$$

Suppose that $V(0,\theta) = 0$, then $S(V(T,\theta))(\tilde{\eta}) = 0$. Because the measures μ and $\mu_\eta(\cdot) := S(1(\cdot))(\eta)$, the latter induced by the S-transform, are equivalent (for further details see [Be], Chapters 2 and 6), it follows that $V(T,\theta) = 0$ with respect to the measure μ. □

Now we specify the class of contingent claims which will be priced in the next subsection.

Definition 5.5 *The class of path independent contingent claims $\mathcal{X}$ with expiry date T consists of those contingent claims $X(T)$, whose payoff functions admit a representation $p(T,S(T)) = \sum_{k=0}^{\infty} p_k (S(T))^{\diamond k}$ and satisfy $p(T,S(T)) \in (L^2)$.*

Due to arguments above one can also deduce a power series representation of $p(T,S(T))$ in $S(T)$ with ordinary products. We call this market the Black-Scholes market with memory, where the memory is given by covariance function $v(s,t)$. The challenge of pricing these contingent claims at time 0 is to determine the value of the replicating portfolio at time 0, which we present in the next subsection.

5.2 Pricing of contingent claims in the Black-Scholes market with memory

We call a contingent claim in the market *attainable*, if there exists a self-financing trading strategy θ that replicates the contingent claim X.

Theorem 5.6 (Pricing of contingent claims) *The contingent claim $X \in \mathcal{X}$ is attainable in the Black-Scholes market with memory. The value process $v(t, S(t)) = V(t, \theta)$ of the portfolio θ replicating the contingent claim X satisfies the following stochastic partial differential equation*

$$rv_s(t, S(t)) \diamond S(t) - rv(t, S(t)) + v_t(t, S(t)) = 0 \tag{11}$$

with boundary condition $v(T, S(T)) = p(T, S(T))$.

Proof. The wealth process is defined by $v(t, S(t)) = g(t, S(t))A(t) + h(t, S(t)) \diamond S(t)$. We apply the chain rule from Theorem 4.9 with D as in (6) to the wealth process and get

$$v(t, S(t)) - v(0, S(0)) = \int_0^t v_t(u, S(u))du + \int_0^t v_s(u, S(u)) \diamond S(u) \diamond (\mu + \sigma W_u^v)du.$$

Because the replicating portfolio is self-financing it follows

$$v(t, S(t)) - v(0, S(0)) = \int_0^t g(u, S(u))rA(u)\,du + \int_0^t h(u, S(u)) \diamond S(u) \diamond (\mu + \sigma W_u^v)\,du.$$

Comparing both results we get

$$\begin{aligned}
&\int_0^t v_t(u, S(u))\,du + \int_0^t v_s(u, S(u)) \diamond S(u) \diamond (\mu + \sigma W_u^v)\,du \\
&= \int_0^t g(u, S(u))rA(u)\,du + \int_0^t h(u, S(u)) \diamond S(u) \diamond (\mu + \sigma W_u^v)\,du
\end{aligned}$$

and this becomes to

$$\begin{aligned}
&\int_0^t (v_t(u, S(u)) + \mu v_s(u, S(u)) \diamond S(u) - g(u, S(u))rA(u) - \mu h(u, S(u)) \diamond S(u))\,du \\
&+ \int_0^t \sigma(v_s(u, S(u)) \diamond S(u) - h(u, S(u)) \diamond S(u))\,dB_t^v = 0.
\end{aligned}$$

By regarding the S-transform one can deduce that a.s. and for each $u \in [0, T]$

$$v_s(u, S(u)) - h(u, S(u)) = 0 \tag{12}$$

and

$$v_t(u, S(u)) + \mu v_s(u, S(u)) \diamond S(u) - g(u, S(u))rA(u) - \mu h(u, S(u)) \diamond S(u) = 0.$$

By applying $-rv(t,S(t)) + rh(t,S(t)) \diamond S(t) = -rg(t,S(t))A(t)$ we get

$$v_t(u,S(u)) - rv(t,S(t)) + rv_s(u,S(u)) \diamond S(t) = 0,$$

thus the pricing formula holds. In order to show that the contingent claim is attainable we investigate the payoff function $p(T,S(T)) = \sum_{k=0}^{\infty} p_k(S(T))^{\diamond k}$. The wealth process $v(t,S(t))$ also admits a Wick representation $\sum_{k=0}^{\infty} v_k(t)(S(t))^{\diamond k}$. Therefore it follows

$$\sum_{k=0}^{\infty} v_k(T)(S(T))^{\diamond k} = \sum_{k=0}^{\infty} p_k(S(T))^{\diamond k},$$

hence $v_k(T) = p_k$. If we plug $\sum_{k=0}^{\infty} v_k(t)(S(t))^{\diamond k}$ in the pricing formula and compare the coefficients of $(S(t))^{\diamond k}$, the equation for $k \in \mathbb{N}_0$

$$\frac{dv_k(t)}{dt} = (r - rk)v_k(t)$$

follows. So $v_k(t) = c_k \exp((r-rk)t)$ and $p_k = c_k \exp((r-rk)T)$ with the constants c_k. This shows that the contingent claim can be replicated. The self-financing replicating portfolio is given by (12) and by

$$g(t,S(t)) = \frac{v(t,S(t)) - v_s(t,S(t)) \diamond S(t)}{A(t)}.$$

□

Due to considerations above this stochastic partial differential equation (11) admits a representation with ordinary products.

Theorem 5.7 (The pricing partial differential equation) *The stochastic partial differential equation with boundary condition in Theorem 5.6 can be transformed into the deterministic partial differential equation*

$$v_t^o(t,s) + rsv_s^o(t,s) + \frac{1}{2}\sigma^2 \frac{d|m(u,t)|_0^2}{dt} s^2 v_{ss}^o(t,s) = rv^o(t,s), \tag{13}$$

where $v^o(T,s) = p(T,s)$ and $v^o(t,S(t))$ denotes the representation of the wealth process with ordinary product.

Proof. The wealth process admits a Wick representation, so

$$v(t,S(t)) = \sum_{k=0}^{\infty} a_k(t)(S(t))^{\diamond k}.$$

Now we regard the partial derivatives in the stochastic partial differential equation involving the Wick representation

$$v_t(t,S(t)) = \sum_{k=0}^{\infty} \frac{da_k(t)}{dt}(S(t))^{\diamond k}$$
$$v_s(t,S(t)) = \sum_{k=0}^{\infty} a_k(t)k(S(t))^{\diamond(k-1)}.$$

The derivative with respect to t of the representation with ordinary product as in the proof of Theorem 5.1

$$v^o(t,S(t)) = \sum_{k=0}^{\infty} a_k(t)\exp(1/2(k-k^2)\sigma^2|m(u,t)|_0^2)(S(t))^k$$

is

$$\begin{aligned}
v_t^o(t,S(t)) &= \sum_{k=0}^{\infty} \frac{da_k(t)}{dt}\exp(1/2(k-k^2)\sigma^2|m(u,t)|_0^2)(S(t))^k \\
&\quad - \sum_{k=0}^{\infty} a_k(t)\frac{1}{2}(k-1)k\sigma^2\frac{d|m(u,t)|_0^2}{dt}\exp(1/2(k-k^2)\sigma^2|m(u,t)|_0^2)(S(t))^k \\
&= \sum_{k=0}^{\infty} \frac{da_k(t)}{dt}\exp(1/2(k-k^2)\sigma^2|m(u,t)|_0^2)(S(t))^k \\
&\quad - (S(t))^2\frac{\sigma^2}{2}\frac{d|m(u,t)|_0^2}{dt}\sum_{k=0}^{\infty} a_k(t)(k-1)k\exp(1/2(k-k^2)\sigma^2|m(u,t)|_0^2)(S(t))^{k-2} \\
&= v_t(t,S(t)) - (S(t))^2\frac{\sigma^2}{2}\frac{d|m(u,t)|_0^2}{dt}v_{ss}^o(t,S(t)).
\end{aligned}$$

The partial derivatives with respect to $S(t))$ are equal with the coefficients $S(t)\cdot v_s^o(t,S(t)) = S(t)\diamond v_s(t,S(t))$. By substituting all terms we get the partial differential equation with the same boundary condition. □

Remark 5.8 This pricing equation involves the classical case with ordinary Brownian motion, because

$$\frac{d|1([0,t])|_0^2}{dt} = \frac{d\,t}{dt} = 1.$$

Remark 5.9 Some contingent claims have a payoff function $p(T,S(T))$ which is not smooth with respect to the stock price $S(T)$. On first view this may be in conflict with smoothness requirements on the wealth process. However this problem can be overcome as in the classical case, where the partial differential equation is solved on $[0,T)$ having the solution $v(t,S(t))$, if $\lim_{t\to T} v(t,S(t)) = p(T,S(T))$, which is met in many cases, the boundary condition can be satisfied. So the payoff function $p(T,S(T))$ does not need to satisfy smoothness properties.

An European call option EC_K is a contingent claim, whose payoff function is $p(T,S(T)) = \max\{(S(T)-K),0\}$. Instead of trying to investigate whether EC_K belongs to the set of admissible contingent claims $\mathcal{X}$, one can price the European call option by solving the partial differential equation. But it is necessary that the wealth process $v^o_{EC_K}(t,S(t))$ admits a Wick representation in $S(t)$, so the stochastic partial differential equation (11) can be satisfied, too. We get the wealth process of the European call by solving the deterministic partial differential equation. Let $N(x) := (2\pi)^{-1/2}\int_{-\infty}^{x}\exp(-z^2/2)dz$.

Theorem 5.10 *The value process of the European call option in the Black-Scholes market with memory is*

$$v^o(t,s) = sN(d_1(t,s)) - K\exp(-r(T-t))N(d_2(t,s)),$$

where

$$d_1(t,S(t)) := \frac{\ln(S(t)/K) + r(T-t) + 1/2\sigma^2(|m(u,T)|_0^2 - |m(u,t)|_0^2)}{\sigma\sqrt{(|m(u,T)|_0^2 - |m(u,t)|_0^2)}}$$

and

$$d_2(t,S(t)) := \frac{\ln(S(t)/K) + r(T-t) - 1/2\sigma^2(|m(u,T)|_0^2 - |m(u,t)|_0^2)}{\sigma\sqrt{(|m(u,T)|_0^2 - |m(u,t)|_0^2)}}$$

Proof. The solution satisfies the partial differential equation with the boundary condition. Note that

$$rv^o(t,S(t)) = rsN(d_1(t,s)) - rK\exp(-r(T-t))N(d_2(t,s)) =: D_1 + D_2,$$

With $(dN(t))/(dt) =: \phi(t)$ and $\beta(t) := \sigma\sqrt{|m(u,T)|_0^2 - |m(u,t)|_0^2}$ the derivatives turn out to be

$$\begin{aligned}
&v^o_t(t,s)\\
&= s\phi(d_1(t,s))\frac{\partial d_1(t,s)}{\partial t} - rK\exp(-r(T-t))N(d_2(t,s)) - K\exp(-r(T-t))\phi(d_2(t,s))\frac{\partial d_2(t,s)}{\partial t}\\
&= s\phi(d_1(t,s))\left(\frac{-r - \frac{1}{2}\sigma^2\frac{d|m(u,t)|_0^2}{dt}}{\beta(t)} + \sigma^2\frac{(\ln(s/k) + r(T-t) + \frac{1}{2}(\beta(t))^2)\frac{1}{2}\frac{d|m(u,t)|_0^2}{dt}}{(\beta(t))^3}\right)\\
&\quad - rK\exp(-r(T-t))N(d_2(t,s))\\
&\quad - K\exp(-r(T-t))\phi(d_2(t,s))\cdot\\
&\qquad \cdot\left(\frac{-r + \frac{1}{2}\sigma^2\frac{d|m(u,t)|_0^2}{dt}}{\beta(t)} + \sigma^2\frac{(\ln(s/k) + r(T-t) - \frac{1}{2}(\beta(t))^2)\frac{1}{2}\frac{d|m(u,t)|_0^2}{dt}}{(\beta(t))^3}\right)\\
&=: D_3 + D_4 + D_5 + D_6 + D_7 + D_8 + D_9,
\end{aligned}$$

where $D_3 = s\phi(d_1(t,s))(-r)/(\beta(t))$ and the same way for D_7. The term in (13) including the partial derivative with respect to s can be computed as follows

$$\begin{aligned}
&rsv_s^o(t,s)\\
&= rsN(d_1(t,s)) + rs^2\phi(d_1(t,s))\frac{1}{s\beta(t)} - rsK\exp(-r(T-t))\phi(d_2(t,s))\frac{1}{s\beta(t)}\\
&=: D_{10} + D_{11} + D_{12}.
\end{aligned}$$

The term in (13) including the second partial derivative with respect to s equals

$$\begin{aligned}
&\frac{1}{2}\sigma^2 s^2 \frac{d|m(u,t)|_0^2}{dt} v_{ss}^o(t,s)\\
&\quad= \sigma^2 s^2 \frac{d|m(u,t)|_0^2}{dt}\phi(d_1(t,s))\frac{1}{s\beta(t)}\\
&\qquad - \frac{1}{2}\sigma^2 s^3 \frac{d|m(u,t)|_0^2}{dt}\phi(d_1(t,s))d_1(t,s)\frac{1}{s^2(\beta(t))^2}\\
&\qquad - \frac{1}{2}\sigma^2 s^3 \frac{d|m(u,t)|_0^2}{dt}\phi(d_1(t,s))\frac{1}{s^2\beta(t)}\\
&\qquad + \frac{1}{2}\sigma^2 s^2 \frac{d|m(u,t)|_0^2}{dt}K\exp(-r(T-t))\phi(d_2(t,s))d_2(t,s)\frac{1}{s^2(\beta(t))^2}\\
&\qquad + \frac{1}{2}\sigma^2 s^2 \frac{d|m(u,t)|_0^2}{dt}K\exp(-r(T-t))\phi(d_2(t,s))\frac{1}{s^2\beta(t)}\\
&\quad=: D_{13} + D_{14} + D_{15} + D_{16} + D_{17}.
\end{aligned}$$

Now we compare the summands and notice that $D_1 = D_{10}$, $D_2 = D_6$, $D_3 = D_{11}$, $D_4 + D_{15} = D_{13}$, $D_5 = D_{14}$, $D_7 = D_{12}$, $D_8 = D_{17}$ and $D_9 = D_{16}$.

The wealth process v^o admits a Wick representation, because it has a representation with power series in s with convergence radius equal to infinity with each $t \in [0,T)$ and the process satisfies $v(t, S(t)) \in (L^2)$, so the Wick representation of the wealth process solves the stochastic partial differential equation in Theorem 5.6. Therefore it is the wealth process of the replicating portfolio. □

Remark 5.11 We assumed that μ, σ and r are constants. But without effort one can formulate the results with deterministic and continuously differentiable $\mu(t), \sigma(t)$ and $r(t)$, where $\sigma(t) > c$ with a positiv constant c.

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