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A universal strong law of large numbers for conditional expectations via nearest neighbors

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Abstract
For k_n-nearest neighbor estimates of a regression Y on X (d-dimensional random vector X, integrable real random variable Y) based on observed independent copies of (X, Y), strong universal pointwise consistency is shown, i.e., strong consistency P_X-almost everywhere for general distribution of (X, Y). With tie-breaking by indices, this means validity of a universal strong law of large numbers for conditional expectations $E(Y|X = x)$.
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1 Introduction

The estimation sequence (m_n) is called strongly universally pointwise consistent, if

$$\text{almost surely } m_n(x) \to m(x) \quad \text{mod } \mu \tag{1}$$

for all distributions of (X, Y) with $E|Y| < \infty$. If, in the case that μ is concentrated in a single point $x^* \in \mathbb{R}^d$, (1) immediately yields Kolmogorov's strong law of large numbers (SLLN) $(Y_1 + \ldots + Y_n)/n \to EY_1 = m(x^*)$ almost surely, then the strong universal consistency result can be considered as a universal strong law of large numbers for conditional expectations. In literature one finds several results on strong universal pointwise consistency which concern modifications of averaging estimates: kernel estimates with truncated Y_i's in Kozek *et al.* [9], modified recursive partitioning estimate in Algoet [1], (modified) truncated kernel estimate, modified recursive kernel estimate and modified truncated nearest neighbor estimate in Algoet and Györfi [2], (semi-)recursive partitioning and (semi-)recursive kernel estimate in Walk [14]. Strong pointwise consistency of the classical Nadaraya-Watson kernel estimate was established under boundedness or moment conditions (stronger than $E|Y| < \infty$) by Devroye [3], Greblicki *et al.* [7], Zhao and Fang [16], Stute [13] and Kozek *et al.* [9] or under regularity conditions in Mukerjee [10] and Kozek *et al.* [9]. Györfi *et* *al.* [8] mention strong universal pointwise consistency of the classical kernel estimate and the classical nearest neighbor estimate as open problems. This paper gives an affirmative answer for the latter estimate, i.e., it states (1) for k_n-nearest neighbor estimates for suitable (k_n) under the only condition $E|Y| < \infty$ (Theorem 1). This result comprehends a universal strong law of large numbers for conditional expectations. Tools in the proof are Etemadi's [6] device to prove classical strong laws of large numbers, a variant of the generalized Lebesgue density theorem concerning $Em_n(x) \to m(x) \mod \mu$ (Lemma 1), the Efron-Stein inequality for variances in Steele's [11] version (Lemma 2), a sharpened covering lemma for nearest neighbors (Lemma 3) with corollaries (Lemmas 4 - 8).

2 Result

For the definition $m_n(x)$ of the k_n-nearest neighbor estimate, $x \in \mathbb{R}^d$ fixed, the data $(X_1, Y_1), \ldots, (X_n, Y_n)$ are reordered according to increasing values of $||X_i - x||$ (euclidean norm) where the reordered data sequence is denoted by

$$(X_{1,n}(x), Y_{1,n}(x)), \ldots, (X_{n,n}(x), Y_{n,n}(x))$$

with $X_{k,n}(x)$ as the so-called kth nearest neighbor (k-NN) of x in $\{X_1, \ldots, X_n\}$. The k_n-NN regression function estimate is defined by

$$\begin{aligned} m_n(x) &:= \frac{1}{k_n} \sum_{i=1}^{k_n} Y_{i,n}(x) \\ &= \frac{1}{k_n} \sum_{i=1}^{n} Y_i I_{[X_i \text{ is among the } k_n \text{ NNs of } x \text{ in } \{X_1, \ldots, X_n\}]} \end{aligned} \tag{2}$$

with $k_n \in \{1, \ldots, n-1\}$, $n \geq 2$, where I denotes an indicator function.

We use two rules for breaking a so-called tie $||x_{i_1} - x|| = \ldots = ||x_{i_j} - x||$. As to the first rule (called purely random tie-breaking), let (X, V) be a random vector, where V is independent of (X, Y) and uniformly distributed on $[0, 1]$. We also artificially enlarge the random data set by introducing real random variables $V_1, V_2, \ldots$ such that the $(d+2)$-dimensional random vectors $(X, V, Y), (X_1, V_1, Y_1), (X_2, V_2, Y_2), \ldots$ are independent and identically distributed. Especially the $V_i's$ have uniform distribution on $[0, 1]$, and each (X_i, V_i) is distributed as (X, V). Ties, now in context with $||(x_i, V_i) - (x, V)||$ instead of $||x_i - x||$, appear only with probability zero. In contrast to the global rule described in Györfi *et al* [8], pp. 86, 87, we use this enlargement only in the above context, i.e., only in the realized tie situation. The second rule for breaking the tie consists in declaring $x_{i_{l'}}$ to be "closer" than $x_{i_{l''}}$ if $i_{l'} < i_{l''}$ (tie-breaking by indices). The formulations in this paper concern both rules, except one of the rules is expressly mentioned.

The following theorem states strong universal consistency of the k_n-nearest neighbor estimates.

Theorem 1. Assume $E|Y| < \infty$, and let $k_n \in \{1, \ldots, n-1\}$, $n \geq 2$, with $k_n \uparrow, k_n/n^\beta \to c \in (0, \infty)$ with $0 < \beta < 1$. Let the k_n-nearest neighbor estimation be defined by (2) with purely random tie-breaking or tie-breaking by indices. Then (1) holds.
Remark 1. If in Theorem 1 especially μ is concentrated on $\{x^*\}$ for some $x^* \in \mathbb{R}^d$, then almost surely $m_n(x^*) \to m(x^*) = EY$. In the case of tie-breaking by indices, this means

$$\text{almost surely } \frac{Y_1 + \ldots + Y_{k_n}}{k_n} \to EY,$$

thus, because $\{k_n; n \geq 2\} \subset \{n_0, n_0 + 1, \ldots\}$ for some n_0,

$$\text{almost surely } \frac{Y_1 + \ldots + Y_n}{n} \to EY.$$

Therefore Theorem 1 is a universal strong law of large numbers for conditional expectations.

3 Proofs

First we give some tools (Lemmas 1 - 8) and then prove Theorem 1. Let the assumptions of Theorem 1 be fulfilled.
Lemma 1.

$$Em_n(x) \to m(x) \mod \mu.$$

Proof. We shall use $k_n/n \to 0$. We notice $E(Y_i|X_i) = m(X_i)$ and thus

$$Em_n(x) = \frac{1}{k_n} \sum_{i=1}^{n} Em(X_i) I_{[X_i \text{ is among the } k_n \text{ NNs of } x \text{ in } \{X_1, \ldots, X_n\}]}.$$

Because the random vectors (X_1, Y_i) are independent and identically distributed, under both rules of tie breaking the left-hand side has the same value. Therefore we may restrict to the case of purely random tie-breakig and obtain

$$Em_n(x) = \frac{n}{k_n} Em(X_1) I_{[X_1 \text{ is among the } k_n \text{ NNs of } x \text{ in } \{X_1, \ldots, X_n\}]}.$$

We shall use the argument in the proof of the generalized pointwise Lebesgue density theorem (see, e.g., Wheeden and Zygmund ([15], chapter 10), and Györfi *et al.* ([8], section 24.2)) and of a further generalization due to Greblicki *et al.* ([7] (see also Györfi *et al.* ([8], Lemma 24.8)).

In the first step, for an arbitrary μ-integrable $f : \mathbb{R}^d \to \mathbb{R}$ we show existence of a constant c depending on d such that

$$\mu \left\{ x \in \mathbb{R}^d : \sup_n \frac{n}{k_n} E|f(X_1)| I_{[X_1 \text{ is among the } k_n \text{ NNs of } x \text{ in } \{X_1, \ldots, X_n\}]} > \alpha \right\} \leq \frac{c}{\alpha} \int |f(y)| \mu(dy)$$

for any $\alpha > 0$. Set

$$A_t^{(n)} := \{y \in \mathbb{R}^d : P\,[y \text{ is among the } k_n \text{ NNs of } x \text{ in } \{y, X_2, \ldots, X_n\}\ ] > t\},\ t \in (0, 1),$$

which is $\emptyset$ or a ball in $\mathbb{R}^d$ centered at x. Then

$$\begin{aligned}
&\frac{n}{k_n} E|f(X_1)| I_{[X_1 \text{ is among the } k_n \text{ NNs of } x \text{ in } \{X_1,\dots,X_n\}]} \\
= \ & \frac{n}{k_n} \int |f(y)| P\left[y \text{ is among the } k_n \text{ NN's of } x \text{ in } \{y, X_2, \dots, X_n\}\right] \mu(dy) \\
= \ & \frac{n}{k_n} \int |f(y)| \int_0^1 I_{A_t^{(n)}}(y) dt \mu(dy) \\
= \ & \frac{n}{k_n} \int_0^1 \left[\int_{A_t^{(n)}} |f(y)| \mu(dy) \right] dt \\
= \ & \int_0^1 \left[\int_{A_t^{(n)}} |f(y)| \mu(dy) \right] dt \Big/ \int_0^1 \mu(A_t^{(n)}) dt \\
\leq \ & \sup_{\text{rational } t>0} \int_{A_t^{(n)}} |f(y)| \mu(dy) / \mu(A_t^{(n)}), \\
\leq \ & \sup_{h \in H} \int_{S_{x,h}} |f(y)| \mu(dy) / \mu(S_{x,h})
\end{aligned}$$

for a suitable countable set $H \subset (0, \infty)$. This together with the well-known fact that

$$\mu(\{x \in \mathbb{R}^d : \sup_{h \in H} \int_{S_{x,h}} |f(y)| \mu(dy) / \mu(S_{x,h}) > \alpha\}) \leq \frac{c}{\alpha} \int |f(y)| \mu(dy)$$

for any $\alpha > 0$ with some constant c depending on d (see, e.g., Wheeden and Zygmund ([15], Lemma 10.47), and Györfi *et al.* ([8], Lemma 24.4)), yields the desired auxiliary result.

In the second step, for an arbitrary fixed $\varepsilon > 0$ we choose a continuous function $g : \mathbb{R}^d \to \mathbb{R}$ with compact support such that $\int |m(y) - g(y)| \mu(dy) < \frac{\varepsilon^2}{2(c+1)}$ with constant c from the first step. For each $x \in \text{support}(\mu)$, because of $k_n/n \to 0$, one has

$$\text{almost surely } ||X_{(k_n,n)}(x) - x|| \to 0,$$

which is a consequence of the strong law of large numbers (see Györfi *et al.* ([8], Lemma 6.1)), thus

$$\text{almost surely} \quad \frac{1}{k_n} \sum_{i=1}^{n} |g(X_i) - g(x)| I_{[X_i \text{ is among the } k_n \text{ NNs of } x \text{ in } \{X_1,\dots,X_n\}]} \to 0$$

and

$$d_n(x) := \frac{n}{k_n} E|g(X_1) - g(x)| I_{[X_1 \text{ is among the } k_n \text{ NN's of } x \text{ in } \{X_1,\dots,X_n\}]} \to 0.$$

One has, for $x \in \mathrm{support}(\mu)$,

$$\begin{aligned}
& |Em_n(x) - m(x)| \\
\leq\ & \frac{n}{k_n} E|m(X_1) - m(x)| I_{[X_1 \text{ is among the } k_n \text{ NN's of } x \text{ in } \{X_1,\ldots,X_n\}]} \\
\leq\ & \frac{n}{k_n} E|m(X_1) - g(X_1)| I_{[X_1 \text{ is among the } k_n \text{ NN's of } x \text{ in } \{X_1,\ldots,X_n\}]} + |m(x) - g(x)| + d_n(x) \\
=:\ & p_n(x) + |m(x) - g(x)| + d_n(x) \\
=\ & p_n(x) + |m(x) - g(x)| + o(1).
\end{aligned}$$

Define the set

$$T_\varepsilon := \{x \in \mathbb{R}^d : \sup_n p_n(x) + |m(x) - g(x)| > \varepsilon\}.$$

By the first step and the Markov inequality

$$\begin{aligned}
& \mu(T_\varepsilon) \\
\leq\ & \mu(\{x \in \mathbb{R}^d : \sup_n p_n(x) > \varepsilon/2\}) + \mu\left(\{x \in \mathbb{R}^d : |m(x) - g(x)| > \varepsilon/2\}\right) \\
\leq\ & \frac{2c+2}{\varepsilon} \int |m(x) - g(x)| \mu(dx) \ \leq\ \varepsilon.
\end{aligned}$$

Now $\varepsilon \to 0$ yields the assertion. □

The Efron-Stein [5] inequality on variances in Steele's [11] version (see Györfi *et al.* [8] for further references) will be formulated for the independent identically distributed random vectors $Z_1 = (X_1, Y_1), \ldots, Z_n = (X_n, Y_n)$, $\tilde{Z}_1 = (\tilde{X}_1, \tilde{Y}_1), \ldots, \tilde{Z}_n = (\tilde{X}_n, \tilde{Y}_n)$.

Lemma 2. Let $f : \mathbb{R}^{(d+1)n} \to \mathbb{R}$ be measurable with square integrability of $f(Z_1, \ldots, Z_n)$. Then

$$Var(f(Z_1, \ldots, Z_n)) \leq \frac{1}{2} \sum_{j=1}^{n} E|f(Z_1, \ldots, Z_j, \ldots, Z_n) - f(Z_1, \ldots, \tilde{Z}_j, \ldots, Z_n)|^2.$$

Let γ_d be the minimal number of closed cones $C_1, \ldots, C_{\gamma_d}$ of angle $\pi/4$ which are centered at 0 with different central directions such that their union covers $\mathbb{R}^d$. The following lemma sharpens Corollary 6.1 in Györfi *et al.* [8], which deals with $P[\ x$ is among the k NNs of X in $\{x, X_2, \ldots, X_n\}]$.

Lemma 3. Let $x \in \mathbb{R}^d$, $1 \leq k < n$. With purely random tie-breaking,

$$P\,[x \text{ is } k\text{-}NN \text{ of } X \text{ in } \{x, X_2, \ldots, X_n\}] \ \leq \frac{\gamma_d}{n}.$$

Proof. We use ideas from the proof of Corollary 6.1 in Györfi *et al.* [8]. Let $k \geq 2$. The treatment of the case $k = 1$ is analogous, but simpler. For $i \in \{1, \ldots, n\}$ let $X_{J_1(i)}, \ldots, X_{J_{k-1}(i)}$ with random indices $J_1(i) < \ldots < J_{k-1}(i)$ in $\{1, \ldots, i-1, i+1, \ldots, n\}$ be the $k-1$ NNs of X_i in $\{X_1, \ldots, X_{i-1}, X_{i+1}, \ldots, X_n\}$. Then,

$$P\,[x \text{ is } k\text{-NN of } X \text{ in } \{x, X_2, \ldots, X_n\}]$$

$$
\begin{aligned}
&= \frac{1}{n}\sum_{i=1}^{n} P\left[x \text{ is } k\text{-NN of } X_i \text{ in } \{x, X_1, \dots, X_{i-1}, X_{i+1}, \dots, X_n\}\right] \\
&\quad \text{(by symmetry)} \\
&\le \frac{1}{n}\sum_{i=1}^{n} P\left[x \text{ is 1st NN of } X_i \text{ in } \{x, X_1, \dots, X_{i+1}, X_{i+1}, \dots, X_n\}\right. \\
&\qquad\qquad \left.\backslash\{X_{J_1(i)}, \dots, X_{J_{k-1}(i)}\}\right] \\
&= \frac{1}{n} E \sum_{i=1}^{n} I_{\left[x \text{ is 1st NN of } X_i \text{ in } \{x, X_1, \dots, X_{i+1}, X_{i+1}, \dots, X_n\}\right.} \\
&\qquad\qquad {}_{\left.\backslash\{X_{J_1(i)}, \dots, X_{J_{k-1}(i)}\}\right]}\,.
\end{aligned}
$$

Because for $u, u' \in x+C_j$ ($j \in \{1, \dots, \gamma_d\}$) with $u \ne x$ the inequality $||u-x|| \le ||u'-x||$ implies $||u-u'|| < ||u'-x||$ and thus $||u-u'|| \ge ||u'-x||$ implies $||u-x|| > ||u'-x||$, we can notice: if x is the 1st NN of some X_i in $x+C_j$ ($i = 1, \dots, n$) in the set $A_{i,j} \cup \{x\}$ with $A_{i,j}$ consisting of those X_l ($l \in \{1, \dots, i-1, i+1, \dots, n\}\backslash\{J_1(i), \dots, J_{k-1}(i)\}$) falling into $x+(C_j\backslash\{0\})$, then X_i (in $x+C_j$) is the unique 1st NN of x in $A_{i,j}$. Thus the number of such X_i's is at most γ_d and the expected sum above is bounded by γ_d. This yields the assertion. □

Lemma 4. Let $1 \le k < n$ and $f : \mathbb{R}^d \to \mathbb{R}_+$ be measurable. Then

$$\sum_{j=1}^{n} E f(X_j) I_{[X_j \text{ is } k\text{-NN of } X \text{ in } \{X_1, \dots, X_n\}]} \le \gamma_d E f(X).$$

Proof. Because the random vectors (X_i, Y_i) are independent and identically distributed, under both rules of tie-breaking the left-hand side has the same value. Therefore we may restrict to the case of purely random tie-breaking and obtain

$$
\begin{aligned}
&\sum_{j=1}^{n} E f(X_j) I_{[X_j \text{ is } k\text{-NN of } X \text{ in } \{X_1, \dots, X_n\}]} \\
&= \sum_{j=1}^{n} \int f(x) P\left[x \text{ is } k\text{-NN of } X \text{ in } \{X_1, \dots, X_{j-1}, x, X_{j+1}, \dots, X_n\}\right] \mu(dx) \\
&\le \gamma_d \int f(x)\mu(dx) \\
&\quad \text{(by Lemma 3)} \\
&= \gamma_d E f(X).
\end{aligned}
$$

□

Lemma 5. Let $1 \le k < n$ and Y_j be square integrable.

a) $\sum_{j=1}^{n} E Y_j^2 I_{[X_j \text{ is } k\text{-NN of } X \text{ in } \{X_1, \dots, X_n\}]} \le \gamma_d E Y^2$.

b) $EY_{k,n}(X)^2 \leq \gamma_d EY^2$.

Proof.

a) The inequality is obtained by Lemma 4 with $f(X_j) = E(Y_j^2|X_j)$.

b) We obtain

$$\begin{aligned} EY_{k,n}(X)^2 &= \sum_{j=1}^{n} EY_{k,n}(X)^2 I_{[X_j \text{ is } k\text{-NN of } X \text{ in } \{X_1,\ldots,X_n\}]} \\ &= \sum_{j=1}^{n} EY_j^2 I_{[X_j \text{ is } k\text{-NN of } X \text{ in } \{X_1,\ldots,X_n\}]} \\ &\leq \gamma_d EY^2 \end{aligned}$$

by part a) □

Lemma 6. Let $q > 0,\ 1 \leq k < n$.

a)

$$P[Y_n > q,\ X_n \text{ is } k\text{-NN of } X \text{ in } \{X_1,\ldots,X_n\}] \leq \frac{\gamma_d}{n} P[Y > q].$$

b)

$$P[Y_n > q,\ X_n \text{ is among the } k \text{ NNs of } X \text{ in } \{X_1,\ldots,X_n\}] \leq \gamma_d \frac{k}{n} P[Y > q].$$

Proof.

a) The left-hand side below concerns tie-breaking by indices as well as purely random tie-breaking, with differing probability values. We obtain

$$\begin{aligned} & P\left[Y_n > q,\ X_n \text{ is } k\text{-NN of } X \text{ in } \{X_1,\ldots,X_n\}\right] \\ \leq & \int P\left[Y_n > q | X_n = x\right] P\left[x \text{ is } k\text{-NN of } X \text{ in } \{X_1,\ldots,X_{n-1},x\}\right. \\ & \qquad \left.\text{under purely random tie-breaking}\right] \mu(dx) \\ \leq & \frac{\gamma_d}{n} \int P[Y_n > q | X_n = x] \mu(dx) \\ & \text{(by Lemma 3)} \\ = & \frac{\gamma_d}{n} P[Y > q]. \end{aligned}$$

b) Immediately by part a). □

Lemma 7. Let $Y_j \geq 0$ be square integrable. Let $1 \leq k < M \leq N$. Then

$$\begin{aligned} & \int Var\left(\sum_{j=1}^{M} Y_j I_{[X_j \text{ is among the } k \text{ NNs of } x \text{ in } \{X_1,\dots,X_N\}]}\right)\mu(dx) \\ \leq \;& 2\gamma_d k E Y^2. \end{aligned}$$

Proof. Let $(\tilde{X}_1, \tilde{Y}_1), \dots, (\tilde{X}_N, \tilde{Y}_N)$ be $(d+1)$-dimensional random vectors such that $(X_1, Y_1), \dots, (X_N, Y_N), (\tilde{X}_1, \tilde{Y}_1), \dots, (\tilde{X}_n, \tilde{Y}_N)$ are independent and identically distributed. With

$$\begin{aligned} F_{N,j}(x) &:= [X_j \text{ is among the } k \text{ NNs of } x \text{ in } \{X_1,\dots,X_N\} \\ G_{N,j}(x) &:= \left[\tilde{X}_j \text{ is among the } k \text{ NNs of } x \text{ in } \{X_1,\dots,X_{j-1},\tilde{X}_j,X_{j+1},\dots,X_N\}\right], \end{aligned}$$

by Lemma 2 we obtain

$$\begin{aligned} & Var\left(\sum_{j=1}^{M} Y_j I_{F_{N,j}(x)}\right) \\ \leq \;& \frac{1}{2}\sum_{j=1}^{M} E\left(Y_j - \tilde{Y}_j\right)^2 I_{F_{N,j}(x)\cap G_{N,j}(x)} + \frac{1}{2}\sum_{j=1}^{M} E\left(Y_j^2 + Y_{k+1,N}(x)^2\right) I_{F_{N,j}(x)\cap G_{N,j}(x)^{\complement}} \\ + \;& \frac{1}{2}\sum_{j=1}^{M} E(\tilde{Y}_j^2 + Y_{k,N}(x)^2) I_{F_{N,j}(x)^{\complement}\cap G_{N,j}(x)} + \frac{1}{2}\sum_{j=M+1}^{N} EY_{k+1,N}(x)^2 I_{F_{N,j}(x)\cap G_{N,j}(x)^{\complement}} \\ + \;& \frac{1}{2}\sum_{j=M+1}^{N} EY_{k,N}(x)^2 I_{F_{N,j}(x)^{\complement}\cap G_{N,j}(x)}, \end{aligned}$$

where on $F_{N,j}(x)^{\complement} \cap G_{N,j}(x)$ $X_{k,N}(x)$ is only the $(k+1)$-NN of x in $\{X_1,\dots,X_{j-1},\tilde{X}_j, X_{j+1},\dots,X_N\}$. Thus, by symmetry,

$$\begin{aligned} & Var\left(\sum_{j=1}^{M} Y_j I_{F_{N,j}(x)}\right) \\ \leq \;& \sum_{j=1}^{M} EY_j^2 I_{F_{N,j}(x)} + \sum_{j=1}^{M} EY_{k+1,N}(x)^2 I_{F_{N,j}(x)} + \sum_{j=M+1}^{N} EY_{k+1,N}(x)^2 I_{F_{N,j}(x)} \\ \leq \;& \sum_{j=1}^{M} EY_j^2 I_{F_{M,j}(x)} + kEY_{k+1,N}(x)^2. \end{aligned}$$

Now

$$\int Var\left(\sum_{j=1}^{M} Y_j I_{F_{N,j}(x)}\right)\mu(dx)$$

$$
\begin{aligned}
&\leq \sum_{l=1}^{k}\sum_{j=1}^{M} EY_j^2 I_{[X_j \text{ is } l\text{-NN of } X \text{ in } \{X_1,\ldots,X_N\}]} + kEY_{k+1,N}(X)^2 \\
&\leq \gamma_d \sum_{l=1}^{k} EY^2 + \gamma_d k EY^2 \\
&\quad \text{(by Lemma 5a,b)} \\
&\leq 2\gamma_d k EY^2.
\end{aligned}
$$

□

Lemma 8. Let $Y_j \geq 0$ be square integrable. Let $1 \leq k < M < N \leq (1+\rho)M$, with $\rho > 0$. Then

$$
\int Var\left(\sum_{j=M+1}^{N} Y_j I_{[X_j \text{ is among the } k \text{ NNs of } x \text{ in } \{X_1,\ldots,X_M,X_j\}]}\right)\mu(dx)
$$
$$
\leq 4\rho(1+\rho)\gamma_d k EY^2.
$$

Proof. Let $(\tilde{X}_1,\tilde{Y}_1),\ldots,(\tilde{X}_N,\tilde{Y}_N)$ be $(d+1)$-dimensional random vectors such that $(X_1,Y_1),\ldots,(X_N,Y_N),(\tilde{X}_1,\tilde{Y}_1),\ldots,(\tilde{X}_N,\tilde{Y}_N)$ are independent and identically distributed. By Lemma 2 we have

$$
\begin{aligned}
&\mathrm{V}ar\left(\sum_{j=M+1}^{N} Y_j I_{[X_j \text{ is among the } k \text{ NNs of } x \text{ in } \{X_1,\ldots,X_M,X_j\}\ ]}\right) \\
&\leq \frac{1}{2}\sum_{l=1}^{M} E\Big|\sum_{j=M+1}^{N} Y_j\Big(I_{[X_j \text{ is among the } k \text{ NNs of } x \text{ in } \{X_1,\ldots,X_M,X_j\}]} \\
&\qquad - I_{[X_j \text{ is among the } k \text{ NNs of } x \text{ in } \{X_1,\ldots,X_{l-1},\tilde{X}_l,X_{l+1},\ldots,X_M,X_j\}\ ]}\Big)\Big|^2 \\
&\quad +\frac{1}{2}\sum_{l=M+1}^{N} E|Y_l I_{[X_l \text{ is among the } k \text{ NNs of } x \text{ in } \{X_1,\ldots,X_M,X_l\}\ ]} \\
&\qquad -\tilde{Y}_l I_{[\ \tilde{X}_l \text{ is among the } k \text{ NNs of } x \text{ in } \{X_1,\ldots,X_M,\tilde{X}_l\}]}|^2 \\
&=: V_1(x) + V_2(x).
\end{aligned}
$$

Then, similarly to the proof of Lemma 7, by symmetry

$$
\begin{aligned}
V_1(x) &\leq 2\sum_{l=1}^{M} E\Big(\sum_{j=M+1}^{N} Y_j I_{[X_l \text{ is among the } k \text{ NNs of } x \text{ in } \{X_1,\ldots,X_M,X_j\}} \\
&\qquad I_{[X_j \text{ is } (k+1)\text{-NN of } x \text{ in } \{X_1,\ldots,X_M,X_j\}]}\Big)^2 \\
&\leq 2E\left[\left(\sum_{j=M+1}^{N} Y_j I_{[X_j \text{ is } (k+1)\text{-NN of } x \text{ in } \{X_1,\ldots,X_M,X_j\}]}\right)^2\right.
\end{aligned}
$$

$$\sum_{l=1}^{M} I_{[X_l \text{ is among the } k \text{ NNs of } x \text{ in } \{X_1,\ldots,X_M\}]}\Bigg]$$

$$\begin{aligned}
&= 2kE\left(\sum_{j=M+1}^{N} Y_j I_{[X_j \text{ is } (k+1)\text{-NN of } x \text{ in } \{X_1,\ldots,X_M,X_j\}]}\right)^2 \\
&= 2k\sum_{j=M+1}^{N} EY_j^2 I_{[X_j \text{ is } (k+1)\text{-NN of } x \text{ in } \{X_1,\ldots,X_M,X_j\}]} \\
&\quad +2k \sum_{\substack{i,j\in\{M+1,\ldots,N\}\\ i\neq j}} EY_iY_j I_{[X_i \text{ is } (k+1)\text{-NN of } x \text{ in } \{X_1,\ldots,X_M,X_i\}]} \\
&\qquad\qquad I_{[X_j \text{ is } (k+1)\text{-NN of } x \text{ in } \{X_1,\ldots,X_M,X_j\}]} \\
&=: 2kW_1(x) + 2kW_2(x).
\end{aligned}$$

Now

$$\begin{aligned}
&\int W_1(x)\mu(dx) \\
&= \sum_{j=M+1}^{N} EY_j^2 I_{[X_j \text{ is } (k+1)\text{-NN of } X \text{ in } \{X_1,\ldots,X_M,X_j\}]} \\
&= \sum_{j=M+1}^{N} \int E(Y_j^2|X_j=x)P[x \text{ is } (k+1)\text{-NN of } X \text{ in } \{X_1,\ldots,X_M,x\}]\mu(dx) \\
&\quad \text{(with label } j \text{ for } x \text{ in case of tie-breaking by indices)} \\
&\leq (N-M)\int E(Y^2|X=x)P\,[x \text{ is } (k+1)\text{-NN of } X \text{ in } \{X_1,\ldots,X_M,x\} \\
&\qquad\qquad \text{under purely random tie-breaking}]\,\mu(dx) \\
&\quad \text{(as in the proof of Lemma 6a)} \\
&\leq \frac{N-M}{M+1}\gamma_d \int E(Y^2|X=x)\mu(dx) \\
&\quad \text{(by Lemma 3)} \\
&\leq \rho\gamma_d EY^2.
\end{aligned}$$

Further

$$\begin{aligned}
&\int W_2(x)\mu(dx) \\
&= 2E \sum_{\substack{i,j\in\{M+1,\ldots,N\}\\ i\neq j}} Y_iY_j I_{[X_i \text{ is } (k+1)\text{-NN of } X \text{ in } \{X_1,\ldots,X_M,X_i,X_j\},]} \\
&\qquad\qquad I_{[X_j \text{ is } (k+2)\text{-NN of } X \text{ in } \{X_1,\ldots,X_M,X_i,X_j\}]} \\
&\quad \text{(by symmetry)}
\end{aligned}$$

$$\leq \; 2\int\int \sum_{\substack{i,j\in\{M+1,\dots,N\}\\ i\neq j}} E(Y_i|X_i=x)E(Y_j|X_j=\tilde{x})P[x \text{ is } (k+1)\text{-NN of } X$$

$$\text{in } \{X_1,\dots,X_M,x,\tilde{x}\}, \tilde{x} \text{ is } (k+2)\text{-NN of } X \text{ in } \{X_1,\dots,X_M,x,\tilde{x}\}$$

$$\text{under purely random tie-breaking}]\ \mu(dx)\mu(d\tilde{x})$$

(as before)

$$= \; 2(N-M)(N-M-1)E\left(Y_{M+1}Y_{M+2}\right.$$
$$\left. I_{[X_{M+1} \text{ is } (k+1)\text{-NN of } X \text{ in } \{X_1,\dots,X_{M+2}\},\ X_{M+2} \text{ is } (k+2)\text{-NN of } X \text{ in } \{X_1,\dots,X_{M+2}\}]}\right)$$
$$\leq \; \frac{(N-M)^2}{M^2}E\Bigg[(Y_{k+1,M+2}(X)^2+Y_{k+2,M+2}(X)^2)$$
$$\sum_{j=1}^{M+2} I_{[X_j \text{ is } (k+1)\text{-NN of } X \text{ in } \{X_1,\dots,X_{M+2}\}]} \sum_{j=1}^{M+2} I_{[X_j \text{ is } (k+2)\text{-NN of } X \text{ in } \{X_1,\dots,X_{M+2}\}}\Bigg]$$
$$= \; \frac{(N-M)^2}{M^2}\left[EY_{k+1,M+2}(X)^2+EY_{k+2,M+2}(X)^2\right]$$

(each time under purely random tie-breaking)

$$\leq \; 2\rho^2\gamma_d EY^2$$

by Lemma 5b. Finally

$$V_2(x) \leq 2\sum_{l=M+1}^{N} EY_l^2 I_{[X_l \text{ is among the k NNs of } x \text{ in } \{X_1,\dots,X_M,X_l\}]},$$

thus

$$\int V_2(x)\mu(dx)$$
$$\leq \; 2\sum_{l=M+1}^{N}\int E(Y_l^2|X_l=x)P\left[x \text{ is among the } k \text{ NNs of } X \text{ in } \{X_1,\dots,X_M,x\}\right]\mu(dx)$$

(with label l for x in case of tie-breaking by indices)

$$\leq \; 2\int E(Y^2|X=x)\sum_{l=M+1}^{N} P\left[x \text{ is among the } k \text{ NNs of } X \text{ in } \{X_1,\dots,X_M,x\}\right]$$

$$\text{under purely random tie-breaking}]\ \mu(dx)$$

(as before)

$$\leq \; 2\frac{N-M}{M+1}\gamma_d k\int E(Y^2|X=x)\mu(dx)$$

(by Lemma 3)

$$\leq \; 2\rho\gamma_d kEY^2.$$

Thus the assertion is obtained. □

Proof of Theorem 1. We use Etemadi's [6] device to prove strong laws of large numbers. Without loss of generality assume $Y_i \geq 0$. For $c > 0$ set $Y_i^{[c]} := Y_i I_{[Y_i \leq c]}$. Further set

$$m_n^{(n)}(x) := \frac{\sum_{i=1}^{n} Y_i^{[k_n]} I_{[X_i \text{ is among the } k_n \text{ NNs of } x \ \text{ in } \{X_1,\dots,X_n\}]}}{k_n}, \ x \in \mathbb{R}^d.$$

In the first step we show that almost surely for μ-almost all $x \in \mathbb{R}^d$ the event

$$B_i(x) := [\text{for some } n \geq i : Y_i > k_n, \ X_i \text{ is among the } k_n \ NNs \text{ of } x \text{ in } \{X_1,\dots,X_n\}]$$

occurs for only finitely many $i \in \mathbb{N}$. Thus

$$\text{almost surely } m_n^{(n)}(x) - m_n(x) \to 0 \quad \text{mod } \mu. \tag{3}$$

Let $r_l := \min\{j \in \mathbb{N}; k_j = l\}, \ l \in \mathbb{N}$. For $i \in \mathbb{N}$ we notice

$$\begin{aligned}
& \int P(B_i(x))\mu(dx) \\
= \ & P\,[\text{ for some } n \geq i : Y_i > k_n, \ X_i \text{ is among the } k_n \text{ NNs of } X \text{ in } \{X_1,\dots,X_n\}] \\
= \ & P\Big([Y_i > k_i, \ X_i \ \text{ is among the } k_i \ \text{ NNs of } X \ \text{ in } \{X_1,\dots,X_i\}] \\
& \quad \cup \Big(\bigcup_{l > k_i} [Y_i > l, \ X_i \text{ is among the } l \text{ NNs of } X \text{ in } \{X_1,\dots,X_{r_l}\}\ ] \Big) \\
= \ & P\Big([Y_i > k_i, \ X_i \ \text{ is among the } k_i \ \text{ NNs of } X \ \text{ in } \{X_1,\dots,X_i\}] \\
& \quad \cup \Big(\bigcup_{l > k_i} [Y_i > l, X_i \text{ is } \ l\text{-NN of } X \ \text{ in } \ \{X_1,\dots,X_{r_l}\}] \Big) \\
& \text{(with pairwise disjoint events)}
\end{aligned}$$

$$\begin{aligned}
\leq \ & \gamma_d \frac{k_i}{i} P\,[Y > k_i] \\
& + \sum_{l=k_i+1}^{\infty} P\,[Y_i > l, \ X_i \ \text{ is } l\text{-NN of } X \ \text{ in } \{X_1,\dots,X_{r_l}\}] \\
& \text{(by Lemma 6b)} \\
=: \ & A_i + D_i.
\end{aligned}$$

Further

$$\sum_{i=1}^{\infty} A_i = \gamma_d \sum_{i=1}^{\infty} \frac{k_i}{i} P\,[Y > k_i]$$

$$
\begin{aligned}
&\leq c_1 \sum_{i=1}^{\infty} i^{\beta-1} P\left[Y > \frac{1}{c_1} i^{\beta}\right] \\
&\leq \frac{c_1^2}{\beta} \int_0^{\infty} P\left[Y > t\right] dt \\
&= \frac{c_1^2}{\beta} EY < \infty
\end{aligned}
$$

with some constant $c_1 \in (0, \infty)$, and

$$
\begin{aligned}
\sum_{i=1}^{\infty} D_i &= \sum_{l=1}^{\infty} \sum_{i=1}^{r_l - 1} P\left[Y_i > l,\ X_i \text{ is } l\text{-NN of } X \text{ in } \{X_1, \ldots, X_{r_l}\}\right] \\
&\leq \gamma_d \sum_{l=1}^{\infty} P\left[Y > l\right] \\
&\quad \text{(by Lemma 4 with } f(X_i) = E(I_{[Y_i > l]} | X_i)) \\
&\leq \gamma_d EY < \infty.
\end{aligned}
$$

Thus

$$
\sum_{i=1}^{\infty} \int P(B_i(x)) \mu(dx) < \infty.
$$

Now the Borel-Cantelli lemma yields the desired result.

In the second step we show

$$
\text{almost surely } \ m_N^N(x) \to m(x) \quad \text{mod } \mu. \tag{4}
$$

Set $l_n := \lfloor a^n \rfloor$ for fixed $a > 1$. For N, n so large that $k_{l_{n+1}} \leq l_n < N \leq l_{n+1}$, we have

$$
\begin{aligned}
& m_n^*(x) \\
:= & \frac{1}{k_{l_{n+1}}} \sum_{i=1}^{l_n} Y_i^{[k_{l_n}]} I_{[X_i \text{ is among the } k_{l_n} \text{ NNs of } x \text{ in } \{X_1, \ldots, X_{l_{n+1}}\}]} \\
\leq & \ m_N^{(N)}(x) \\
\leq & \frac{1}{k_{l_n}} \sum_{i=1}^{l_n} Y_i^{[k_{l_{n+1}}]} I_{[X_i \text{ is among the } k_{l_{n+1}} \text{ NNs of } x \text{ in } \{X_1, \ldots, X_{l_n}\}]} \\
& + \frac{1}{k_{l_n}} \sum_{i=l_n+1}^{l_{n+1}} Y_i^{[k_{l_{n+1}}]} I_{[X_i \text{ is among the } k_{l_{n+1}} \text{ NNs of } x \text{ in } \{X_1, \ldots, X_{l_n}, X_i\}]} \\
=: & \ m_n'(x) + m_n''(x).
\end{aligned} \tag{5}
$$

First we show

$$
\text{almost surely } \ m_n^*(x) - Em_n^*(x) \to 0 \quad \text{mod } \mu, \tag{6}
$$

$$\text{almost surely } \; m'_n(x) - Em'_n(x) \to 0 \quad \text{mod } \mu, \tag{7}$$
$$\text{almost surely } \; m''_n(x) - Em''_n(x) \to 0 \quad \text{mod } \mu. \tag{8}$$

It suffices to show

$$\int \sum Var(m^*_n(x))\mu(dx) < \infty, \tag{9}$$

$$\int \sum Var(m'_n(x))\mu(dx) < \infty, \tag{10}$$

$$\int \sum Var(m''_n(x))\mu(dx) < \infty. \tag{11}$$

By Lemma 7 we obtain

$$\begin{aligned}
& \sum \int Var(m^*_n(x))\mu(dx) \\
\leq \;& 2\gamma_d \sum \frac{1}{k^2_{l_{n+1}}} k_{l_n} E\left(Y^{[k_{l_n}]}\right)^2 \\
\leq \;& c_2 \sum \frac{1}{a^{(n+1)\beta}} \int_0^{c'a^{n\beta}} t^2 P_Y(dt) \\
\leq \;& c_2 \int_0^\infty \frac{1}{a^{s\beta}} \int_0^{c'a^{s\beta}} t^2 P_Y(dt) ds \\
\leq \;& c_2 \int_0^\infty \left(\int_{\frac{ln(t/c')}{\beta \ln a}}^\infty a^{-s\beta} ds \right) t^2 P_Y(dt) \\
= \;& \frac{c_2 c'}{\beta \ln a} \int t P_Y(dt) \\
= \;& \frac{c_2 c'}{\beta \ln a} EY < \infty
\end{aligned}$$

with suitable constants $c', c_2 \in (0, \infty)$, thus (9). Analogously, by Lemmas 7 and 8, we obtain (10) and (11), respectively. Now for $\delta > 0$ choose $k'_n \in \{1, \ldots, n-1\}$ such that $k'_n = \lceil (1+\delta)a^\beta k_n \rceil$ for large n. By Lemma 1

$$\begin{aligned}
& \limsup Em'_n(x) \\
\leq \;& \lim \frac{k'_{l_n}}{k_{l_n}} \frac{1}{k'_{l_n}} E \sum_{i=1}^{l_n} Y_i I_{[X_i \text{ is among the } k'_{l_n} \text{ NNs of } x \text{ in } \{X_1, \ldots, X_{l_n}\}]} \\
= \;& (1+\delta)a^\beta m(x) \quad \text{mod } \mu\,.
\end{aligned}$$

Further

$$
\begin{aligned}
& \limsup Em''_n(x) \\
\leq \; & \limsup \frac{1}{k_{l_n}} \sum_{i=l_n+1}^{l_{n+1}} EY_i I_{[X_i \text{ is among the } k_{l_{n+1}} \text{ NNs of } x \text{ in } \{X_1,\dots,X_{l_n},X_i\}]} \\
\leq \; & \limsup \frac{l_{n+1}-l_n}{k_{l_n}} EY I_{[X \text{ is among the } k_{l_{n+1}} \text{ NNs of } x \text{ in } \{X_1,\dots,X_{l_n},X\}]} \\
& \text{(the latter expectation under purely random tie-breaking)} \\
= \; & \lim \frac{l_{n+1}-l_n}{k_{l_n}} \frac{k_{l_{n+1}}}{l_n+1} \frac{1}{k_{l_{n+1}}} E \sum_{i=1}^{l_n+1} Y_i I_{[X_i \text{ is among the } k_{l_{n+1}} \text{ NNs of } x \text{ in } \{X_1,\dots,X_{l_n+1}\}]} \\
& \text{(the expectation under purely random tie-breaking)} \\
= \; & a^\beta (a-1) m(x) \mod \mu
\end{aligned}
$$

by Lemma 1. We notice that for arbitrary $C > 0$ one has $k_{l_n} > C$ for n sufficiently large, further

$$
\begin{aligned}
& \frac{1}{l_{n+1}} E \sum_{i=1}^{l_{n+1}} Y_i^{[C]} I_{[X_i \text{ is among the } k_{l_n} \text{ NNs of } x \text{ in } \{X_1,\dots,X_{l_{n+1}}\}]} \\
\leq \; & \frac{1}{l_n} E \sum_{i=1}^{l_n} Y_i^{[C]} I_{[X_i \text{ is among the } k_{l_n} \text{ NNs of } x \text{ in } \{X_1,\dots,X_{l_{n+1}}\}]}
\end{aligned}
$$

(with equality in the case of purely random tie-breaking). Once more by Lemma 1 together with (5),(6),(7),(8), we then obtain

$$
\begin{aligned}
\text{almost surely} \quad & \frac{1}{a}\frac{1}{a^\beta} E\left(Y^{[C]} | X = x\right) \;\leq\; \liminf E\, m^*_n(x) \\
= \; & \liminf m^*_n(x) \;\leq\; \liminf m^{(N)}_N(x) \\
\leq \; & \limsup m^{(N)}_N(x) \;\leq\; \limsup m'_n(x) + \limsup m''_n(x) \\
= \; & \limsup Em'_n(x) + \limsup Em''_n(x) \\
\leq \; & [(1+\delta) + (a-1)] a^\beta m(x) \mod \mu.
\end{aligned}
$$

Letting $\delta \downarrow 0$, $a \downarrow 1$ and $C \uparrow \infty$ we obtain (4).

Now (3) and (4) yield the assertion. □

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