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Almost sure Cesàro and Euler summability of sequences of dependent random variables

By

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Abstract. For a sequence of real random variables C_α-summability is shown under conditions on the variances of weighted sums, comprehending and sharpening strong laws of large numbers (SLLN) of Rademacher-Menchoff and Cramér-Leadbetter, respectively. Further an analogue of Kolmogorov's criterion for the SLNN is established for E_α-summability under moment and multiplicativity conditions of 4th order, which allows to weaken Chow's independence assumption for identically distributed square integrable random variables. The simple tool is a composition of Cesàro-type and of Euler summability methods, respectively.

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1. Introduction. The classical Rademacher-Menchoff theorem ([20], [18]) states that for square integrable pairwise uncorrelated real random variables X_n satisfying $\sum Var(X_n)(\log n)^2/n^2 < \infty$, almost sure (a.s.) convergence of $\sum(X_n - EX_n)/n$ holds and thus, by the Kronecker lemma, the strong law of large numbers

$$\text{a.s.} \ \frac{1}{n+1}\sum_{k=0}^{n}(X_k - EX_k) \to 0,$$

i.e., a.s. (Cesàro) C_1-summability of $(X_n - EX_n)$ to 0. The classical proof uses a maximal inequality on partial sums obtained by a combinatorial argument and has been extended to more general forms of dependence of the $X_n\,'s$ (see [22], [23] among others). It allows to weaken the Cramér-Leadbetter condition [7] for the strong law of large numbers. In this paper, by an elementary argument, more generally a.s. C_α-summability of the sequence $(X_n - EX_n)$ to 0, $\alpha > 0$, i.e.,

$$\text{a.s.} \ \frac{1}{\binom{n+\alpha}{n}}\sum_{k=0}^{n}\binom{n-k+\alpha-1}{n-k}(X_k - EX_k) \to 0$$

(notation a.s. C_α-$\lim(X_n - EX_n) = 0$), is shown under an easily verifiable condition on variances of suitably weighted partial sums (Theorem 2.1). The proof is inspired by the fact that the composition of the $C_{\alpha/2}$-transform of a sequence with itself leads to the same convergence behavior as the C_α-transform. In the case $\alpha = 1$ the result leads to a further weakening of known conditions, especially of the Cramér-Leadbetter condition by a logarithmic factor (Corollaries 2.2 and 2.3).

As is well known, for independent square integrable real random variables the squared logarithmic factor in the Rademacher-Menchoff theorem may be omitted, which leads to Kolmogorov's criterion for the strong law of large numbers and also, by truncation, to Kolmogorov's strong law of large numbers $(X_0 + \ldots + X_n)/(n+1) \to 0$ a.s. for independent and identically distributed real random variables X_n with existence of $EX_n = 0$ (where the latter in this context is also necessary), see, e.g., [17], section 17. There exists a vast literature on weakening the independence assumptions there. For references see, e.g., [23], [4], [24]. Chow [6] showed that for independent and identically distributed X_n 's square integrability with $EX_n = 0$ is necessary and sufficient for a.s. (Euler) E_α- summability of (X_n) to 0, i.e.,

$$\text{a.s.} \sum_{k=0}^{n} \binom{n}{k} \alpha^k (1-\alpha)^{n-k} X_k \to 0$$

(notation a.s. E_α-$\lim X_n = 0$) with arbitrary fixed $\alpha \in (0,1)$. In this equivalence context the E_α-summability method may be replaced by the generally stronger Borel summability method [6], like in the context of Kolmogorov's strong law of large numbers the C_1-summability method may be replaced by the generally stronger Abel summability method [16]. In [6], for the sufficiency part complicated probabilistic tools (delayed averages, Hsu-Robbins-Erdös theorem) for independent identically distributed random variables were used. The results were generalized to power series methods of summability in [2] and [14]. A connection between $E|Y|^p < \infty$ and corresponding Riesz summability (fixed $p \geq 1$) was established in [3]. In this paper we establish a criterion for a.s. E_α-summability of sequences of random variables with finite fourth moment under a weakened independence assumption (strong multiplicativity of 4th order or m-dependence) and thus, by truncation, a.s. E_α-summability in the case of fourwise independent or m-dependent identically distributed square integrable real random variables (Theorems 2.2 and 2.3 with Remarks 2.7 and 2.8). The simple proof uses the fact that the composition of the $E_{\sqrt{\alpha}}$-transform of a sequence with itself is the E_α-transform.

2. Results. The following theorem yields a simple sufficient condition for C_α-summability of a sequence of random variables.

Theorem 2.1. Let $(X_n)_{n\in\mathbb{N}_0}$ be a sequence of square integrable real random variables. If

$$0 < \alpha < 1 \quad \text{and} \quad \sum_{n=1}^{\infty} \frac{1}{n^{2\alpha}} Var\left(\sum_{k=0}^{n} \binom{n-k+\frac{\alpha}{2}-1}{n-k} X_k\right) < \infty$$

or if

$$\alpha = 1 \quad \text{and} \quad \sum_{n=2}^{\infty} \frac{\log n}{n^2} Var\left(\sum_{k=0}^{n} \binom{n-k-\frac{1}{2}}{n-k} X_k\right) < \infty$$

or if

$$\alpha > 1 \quad \text{and} \quad \sum_{n=1}^{\infty} \frac{1}{n^{1+\alpha}} Var\left(\sum_{k=0}^{n} \binom{n-k+\frac{\alpha}{2}-1}{n-k} X_k\right) < \infty,$$

then

$$\text{a.s.} \quad C_\alpha\text{-}\lim(X_n - EX_n) = 0.$$

Remark 2.1. Theorem 2.1 for $\alpha = 2$ means that $\sum n^{-3} Var(X_1 + \ldots + X_n) < \infty$ implies a.s. C_2-$\lim(X_n - EX_n) = 0$. If additionally $(X_n - EX_n)$ is bounded from below, e.g., if $X_n \geq 0, \quad EX_n = O(1)$, then a classical elementary Tauberian theorem ([25], pp. 113, 117) immediately yields a.s. $C_1\text{-}\lim(X_n - EX_n) = 0$. This approach also leads to an elementary proof of Kolmogorov's strong law of large numbers for (pairwise) independent identically distributed integrable real random variables. See [24] with further references, especially [9] and [10] for an alternative elementary proof.

Remark 2.2. For the case $\alpha = 1$, in an analogous manner for a stochastic process $\{X_t;\ t \in \mathbb{R}_+\}$ in $\mathbb{R}$ with $EX_t^2 < \infty,\ t \in \mathbb{R}_+$, and continuity in squared mean ([17], section 37), one can show: If

$$\int_1^\infty \frac{\log t}{t^2} Var\left(\int_0^t \frac{1}{\sqrt{t-s}} X_s ds\right) dt < \infty,$$

then

$$\text{a.s.} \qquad \frac{1}{t}\int_0^t (X_s - EX_s)\, ds \to 0.$$

Corollary 2.1. Let $(X_n)_{n\in\mathbb{N}_0}$ be a sequence of square integrable pairwise uncorrelated random variables. If

$$0 < \alpha < 1 \quad \text{and} \quad \sum_{n=1}^\infty \frac{1}{n^{2\alpha}} Var(X_n) < \infty$$

or if

$$\alpha = 1 \quad \text{and} \quad \sum_{n=2}^\infty \frac{(\log n)^2}{n^2} Var(X_n) < \infty$$

or if

$$\alpha > 1 \quad \text{and} \quad \sum_{n=1}^\infty \frac{1}{n^2} Var(X_n) < \infty,$$

then

$$\text{a.s.} \quad C_\alpha\text{-}\lim(X_n - EX_n) = 0.$$

Remark 2.3. Corollary 2.1 for $\alpha > 1/2$ extends Theorem 9 in [8], where $\sup_n Var(X_n) < \infty$ is assumed. Corollary 2.1 for $\alpha = 1$ is a well-known consequence of the Rademacher-Menchoff theorem (compare Remark 2.4).

Corollary 2.2. Let $(X_n)_{n\in\mathbb{N}_0}$ be a sequence of square integrable real random variables centered at expectations. If

$$\sum_{n=2}^\infty \frac{\log n}{n^{3/2}} \sum_{i=2}^n \frac{1}{i^{1/2}} \left(\log \frac{n}{n+1-i}\right) (EX_iX_n)^+ < \infty, \tag{2.1}$$

then

$$\text{a.s.} \quad C_1\text{-}\lim X_n = 0.$$

Remark 2.4. Let as before the real random variables X_n be square integrable and centered at expectations. The Rademacher-Menchoff theorem ([20], [18]; see Révész ([21], 3.2, and, in a generalization, [17], Section 36, and [23], Theorem 3.7.2) states a.s. convergence of $\sum X_n/n$ and thus, via the Kronecker lemma, a.s. C_1-$\lim X_n = 0$, if the X_n 's are pairwise uncorrelated and $\sum(\log n)^2 EX_n^2/n^2 < \infty$. The two latter conditions can be weakened to

$$\sum \frac{(\log n)^2}{n^2} \sum_{i=1}^{n} (EX_iX_n)^+ < \infty \tag{2.2}$$

as can be obtained according to the proof of Theorem 2.1 in [22], compare also [24], Remark 5. Condition (2.1) in Corollary 2.2 is weaker than condition (2.2) as follows from the inequality $\log(1/1-x)) \leq x[1+\log(1/(1-x))],\ 0 \leq x < 1$. It further leads to an improvement of a condition of Cramér-Leadbetter type for C_1-summability ([7], p. 94, see also [19] and [24], p. 333) by a logarithmic factor (see the following corollary).

Corollary 2.3. Let $(X_n)_{n\in\mathbb{N}_0}$ be a sequence of square integrable real random variables centered at expectations and satisfying

$$(EX_iX_j)^+ \leq \quad c\frac{j^\beta}{[1+(j-i)^\beta][\log(2+j-i)]^\gamma},\ i \leq j$$

with $c \in \mathbb{R}_+$ and $\beta = 0,\ \gamma > 1$ or $0 < \beta < 1,\ \gamma > 2$ or $\beta = 1,\ \gamma > 4$. Then

$$\text{a.s.} \quad C_1\text{-}\lim X_n = 0.$$

Remark 2.5. Let $\beta_n > 0$ such that $\beta_n \uparrow \infty$. If in Theorem 2.1 for $\alpha = 1$, in Corollary 2.1 for $\alpha = 1$ and in Corollary 2.2 the condition is modified replacing the fractions $(\log n)/n^2$, $(\log n)^2/n^2$, $(\log n)/n^{3/2}$ by $(\log n)/\beta_n^2$, $(\log n)^2/\beta_n^2$, $(\log n)n^{1/2}/\beta_n^2$, respectively, then the assertion has to be modified replacing a.s. C_1-summability to 0 by

$$\text{a.s.} \ \frac{1}{\beta_n}\sum_{k=1}^{n}(X_k - EX_k) \to 0.$$

The proof is analogous. Thus by introducing the logarithmic factor in the mentioned conditions, one can avoid the additional conditions $X_n \geq 0$ and

$$\sup_n \ \beta_n^{-1}\sum_{k=1}^{n} EX_k < \infty \text{ or } \sup EX_n < \infty$$

in [4], [5], Theorem 1, and in [24], Remark 7. The a.s. convergence assertion can be interpreted as an assertion on weighted means of $Y_k - EY_k$, where $(\beta_k - \beta_{k-1})Y_k = X_k$.

A sequence (X_n) of real random variables with $E|X_n|^4 < \infty$ shall be called strongly multiplicative of 4th order, if

$$EX_iX_jX_kX_l = EX_iEX_jEX_kEX_l,$$

$$EX_i^2 X_j X_k = EX_i^2 EX_j EX_k,\ EX_i^3 X_j = EX_i^3 EX_j,\ EX_i^2 X_j^2 = EX_i^2 EX_j^2$$

for pairwise different indices (compare Alexits [1] and Révész [21]). For such sequences the following theorem yields a sufficient condition for E_α-summability ($0 < \alpha < 1$) which corresponds to Kolmogorov's condition for the strong law of large numbers.

Theorem 2.2. Let the sequence $(X_n)_{n\in\mathbb{N}_0}$ of real random variables with finite 4th moment be strongly multiplicative of 4th order. Assume

$$\sum_{n=1}^{\infty} \frac{E(X_n - EX_n)^4}{n^2} < \infty, \quad \sum_{n=1}^{\infty} \frac{[E(X_n - EX_n)^2]^2}{n^{3/2}} < \infty.$$

Then for each $\alpha \in (0,1)$.

$$\text{a.s.}\quad E_\alpha\text{-}\lim(X_n - EX_n) = 0.$$

Remark 2.6. The assumption $\sum n^{-3/2}[E(X_n - EX_n)^2]^2 < \infty$ in Theorem 2.2 is fulfilled, if $E(X_n - EX_n)^2 \uparrow$ and $\sum n^{-5/4} E(X_n - EX_n)^2 < \infty$. For these conditions imply $n^{-1/4}E(X_n - EX_n)^2 \to 0$ according to the proof of Olivier's theorem (see [15]).

Remark 2.7. One can transfer the proof of Theorem 2.2 (in section 3) to the case that the assumption of strong multiplicativity of 4th order is replaced by the assumption of m-dependence (with arbitrary fixed $m \in \mathbb{N}$), i.e., independence of the pair $(\mathcal{F}_0^j, \mathcal{F}_{j+m}^\infty) := (\mathcal{F}(X_0, \ldots, X_j), \mathcal{F}(X_{j+m}, X_{j+m+1}, \ldots))$ for each j. According to [13], Theorem 17.3.2, in the special case that (X_n) is a Gaussian sequence, m-dependence for m sufficiently large and ϕ-mixing, i.e.,

$$\phi_k := \sup_n \sup_{A\in\mathcal{F}_0^n, P(A)>0, B\in\mathcal{F}_{n+k}^\infty} |P(B|A) - P(B)| \to 0 \quad (k \to \infty),$$

are equivalent.

As a consequence of Theorem 2.2, by truncation we obtain the sufficiency part of Chow's [6] theorem on E_α-summability under a weakened independence assumption.

Theorem 2.3. Let the sequence $(X_n)_{n\in\mathbb{N}_0}$ of identically distributed square integrable real random variables be fourwise independent (independence of the quadrupel (X_i, X_j, X_k, X_l) for pairwise different indices). Then for each $\alpha \in (0,1)$

$$\text{a.s.}\quad E_\alpha\text{-}\lim(X_n - E_n) = 0.$$

Remark 2.8. In Theorem 2.3 the assumption of fourwise independence can be replaced by the assumption of m-dependence. The proof is the same except for use of Remark 2.7 instead of Theorem 2.2 itself.

3. Proofs. $c_1, c_2, \ldots$ will be suitable constants.

Proof of Theorem 2.1. The proof is inspired by the fact that for a sequence (s_n) in $\mathbb{R}$ the $C_{\alpha/2}$-transform of (s_n) is $C_{\alpha/2}$-summable if and only if (s_n) is C_α-summable, see, e.g., [12], p. 118, or [25], 54 III. Assume $EX_k = 0$, $k \in \mathbb{N}_0$, without loss of generality. The well-known relation

$$\binom{n+\alpha}{n} = (1 + \frac{\alpha}{1}) \ldots (1 + \frac{\alpha}{n}) \sim n^\alpha \tag{3.1}$$

(quotient of the left-hand side and the right-hand side is bounded away from 0 and ∞, which is obvious by taking log) will be used. Set

$$U_j := \sum_{k=0}^{j} \binom{j-k+\frac{\alpha}{2}-1}{j-k} X_k \quad (j=0,1,\ldots).$$

$\left(\binom{n+\frac{\alpha}{2}}{n}^{-1} U_n\right)$ is the $C_{\alpha/2}$-transform of (X_n). With $d_n = n^{-2\alpha}$ if $\alpha < 1$, $d_n = (\log n)/n^2$ if $\alpha = 1$, $d_n = n^{-(1+\alpha)}$ if $\alpha > 1$, the assumption $\sum d_n E U_n^2 < \infty$ implies a.s. $\sum d_n U_n^2 < \infty$ and thus

$$\text{a.s. } d_n \sum_{k=0}^{n} U_k^2 \to 0 \tag{3.2}$$

by the Kronecker lemma. From

$$(1-s)^{-\frac{\alpha}{2}} = \sum_{l=0}^{\infty} \binom{l+\frac{\alpha}{2}-1}{l} s^l, \ |s|<1,$$

one obtains, by taking squares,

$$\binom{m+\alpha-1}{m} = \sum_{l=0}^{m} \binom{m-l+\frac{\alpha}{2}-1}{m-l} \binom{l+\frac{\alpha}{2}-1}{l}, \qquad m=0,1,\ldots, \tag{3.3}$$

and therefore

$$\binom{n-k+\alpha-1}{n-k} = \sum_{j=k}^{n} \binom{n-j+\frac{\alpha}{2}-1}{n-j} \binom{j-k+\frac{\alpha}{2}-1}{j-k}, \qquad 0 \le k \le n,$$

thus

$$W_n := \frac{1}{\binom{n+\alpha}{n}} \sum_{j=0}^{n} \binom{n-j+\frac{\alpha}{2}-1}{n-j} U_j = \frac{1}{\binom{n+\alpha}{n}} \sum_{k=0}^{n} \binom{n-k+\alpha-1}{n-k} X_k.$$

(W_n) is the C_α-transform of (X_n). By the Cauchy-Schwarz inequality and (3.1)

$$\begin{aligned} |W_n|^2 &\le \frac{1}{\binom{n+\alpha}{n}^2} \sum_{j=0}^{n} \binom{n-j+\frac{\alpha}{2}-1}{n-j}^2 \sum_{j=0}^{n} |U_j|^2 \\ &\le c_1 \frac{1}{n^{2\alpha}} \sum_{k=2}^{n} k^{\alpha-2} \sum_{j=0}^{n} |U_j|^2 \le c_2 d_n \sum_{j=0}^{n} |U_j|^2 \qquad (n \ge 2). \end{aligned}$$

Now the assertion a.s. $W_n \to 0$ follows from (3.2). □

Proof of Remark 2.2. The proof follows the same line, using

$$\int_0^t \frac{1}{\sqrt{t-s}} \frac{1}{\sqrt{s}} ds = \int_0^1 \frac{du}{\sqrt{1-u}\sqrt{u}} = \pi, \quad t > 0,$$

instead of (3.3) for $\alpha = 1$. □

Proof of Corollary 2.1. Define (U_n) as in the proof of Theorem 2.1. By uncorrelatedness and by (3.1)

$$Var(U_n) \;\leq\; c_1 \sum_{k=0}^{n} (n-k+1)^{\alpha-2} Var(X_k).$$

If $0 < \alpha < 1$, then

$$\begin{aligned}\sum_{n=1}^{\infty} \frac{1}{n^{2\alpha}} Var(U_n) \;&\leq\; c_1 \sum_{k=0}^{\infty} \left(\sum_{n=\max\{1,k\}}^{\infty} \frac{1}{n^{2\alpha}} (n-k+1)^{\alpha-2} \right) Var(X_k) \\ &\leq\; c_2 \sum_{k=0}^{\infty} \frac{1}{(k+1)^{2\alpha}} Var(X_k).\end{aligned}$$

If $\alpha = 1$, then

$$\begin{aligned}\sum_{n=1}^{\infty} \frac{\log n}{n^2} Var(U_n) \;&\leq\; c_1 \sum_{k=0}^{\infty} \sum_{n=\max\{1,k\}}^{\infty} \frac{\log n}{n^2(n-k+1)} Var(X_k) \\ &\leq\; c_3 \sum_{k=0}^{\infty} \frac{(\log(k+2))^2}{(k+1)^2} Var(X_k),\end{aligned}$$

noticing

$$\int_k^{\infty} \frac{1}{x(x-k+1)} dx \;=\; O\left(\frac{\log k}{k}\right).$$

If $\alpha > 1$, then

$$\begin{aligned}\sum_{n=1}^{\infty} \frac{1}{n^{1+\alpha}} Var(U_n) \;&\leq\; c_1 \sum_{k=0}^{\infty} \left(\sum_{n=\max\{1,k\}}^{\infty} \frac{1}{n^{1+\alpha}} \frac{1}{(n-k+1)^{2-\alpha}} \right) Var(X_k) \\ &\leq\; c_4 \sum_{k=0}^{\infty} \frac{1}{(k+1)^2} Var(X_k),\end{aligned}$$

which for $\alpha \geq 2$ is obvious and which in the case $1 < \alpha < 2$ follows from

$$\frac{1}{k} \sum_{n=k+1}^{\infty} \frac{1}{(\frac{n}{k})^{1+\alpha}(\frac{n}{k}-1)^{2-\alpha}} \to \int_1^{\infty} \frac{dx}{x^{1+\alpha}(x-1)^{2-\alpha}} \;<\; \infty \text{ for } k \to \infty.$$

Now Theorem 2.1 together with the assumptions yields the assertion. □

Proof of Corollary 2.2. We apply Theorem 2.1 for $\alpha = 1$. We obtain

$$\begin{aligned}
&\sum_{n=2}^{\infty} \frac{\log n}{n^2} E\left(\sum_{k=0}^{n} \binom{n-k-\frac{1}{2}}{n-k} X_k\right)^2 \\
\leq\ & c_1 \left[\sum_{n=2}^{\infty} \frac{\log n}{n^2} \sum_{k=0}^{n} \frac{1}{n+1-k} EX_k^2 + 2\sum_{n=2}^{\infty} \frac{\log n}{n^2} \sum_{j=1}^{n} \sum_{i=0}^{j-1} \frac{1}{\sqrt{n+1-j}} \frac{1}{\sqrt{n+1-i}} E(X_iX_j)^+\right] \\
=:\ & c_1[A+2B],
\end{aligned}$$

then

$$A \leq \sum_{n=2}^{\infty} \frac{\log n}{n^3}(EX_0^2 + EX_1^2) + \sum_{k=2}^{\infty} \frac{\log k}{k^2} EX_k^2 + \sum_{k=2}^{\infty}\left(\sum_{n=k+1}^{\infty} \frac{\log n}{n^2(n+1-k)}\right) EX_k^2,$$

where for $k \geq 2$

$$\sum_{n=k+1}^{\infty} \frac{\log n}{n^2(n+1-k)} \leq 2\frac{\log k}{k} \int_k^{\infty} \frac{dx}{x(x+1-k)} \leq 4\frac{(\log k)^2}{k^2}.$$

As to B, we avoid to bound $1/\sqrt{n+1-i}$ by $1/\sqrt{n+1-j}$ for $i = 1, \ldots, j-1$, which would yield (2.2) as a sufficient condition, but write

$$\begin{aligned}
B &= \sum_{j=1}^{\infty} \sum_{i=0}^{j-1} \left(\sum_{n=j}^{\infty} \frac{\log n}{n^2} \frac{1}{\sqrt{n+1-j}} \frac{1}{\sqrt{n+1-i}}\right)(EX_iX_j)^+ \\
&\leq \sum_{n=2}^{\infty} \frac{\log n}{n^3} E(X_0X_1)^+ + 2\sum_{j=2}^{\infty} \sum_{i=0}^{j-1} \frac{\log j}{j} \left(\sum_{n=j}^{\infty} \frac{1}{(n+1)\sqrt{n+1-j}\sqrt{n+\frac{1}{2}-i}}\right)(EX_iX_j)^+,
\end{aligned}$$

where for $j \geq 2,\ 0 \leq i < j$

$$\begin{aligned}
\sum_{n=j}^{\infty} \frac{1}{(n+1)\sqrt{n+1-j}\sqrt{n+\frac{1}{2}-i}} &\leq \int_j^{\infty} \frac{dx}{x\sqrt{x-j}\sqrt{x-(i+\frac{1}{2})}} \\
&= \frac{1}{\sqrt{(i+\frac{1}{2})j}} \log \frac{\left(\sqrt{j} + \sqrt{i+\frac{1}{2}}\right)^2}{j-(i+\frac{1}{2})} \leq c_2 \frac{1}{(i+1)^{1/2} j^{1/2}} \log \frac{j}{j+1-i}.
\end{aligned}$$

Thus $A + 2B < \infty$ by (2.1), and the assertion is obtained from Theorem 2.1 with $\alpha = 1$.□

Proof of Corollary 2.3. First let $0 < \beta < 1,\ \ \gamma > 2$. We use Corollary 2.2. Noticing

$$\frac{\log n}{\log(2+n-i)} \leq 2\left(1 + \log \frac{n}{n+1-i}\right),\ \ i \leq n,$$

we obtain

$$\begin{aligned}
&\sum_{n=2}^{\infty}\frac{\log n}{n^{3/2}}\sum_{i=2}^{n}\frac{1}{i^{1/2}}\left(\log\frac{n}{n+1-i}\right)(EX_iX_n)^+\\
\le\ & c_1\sum_{n=2}^{\infty}\frac{(\log n)^{1-\gamma}}{n}\frac{1}{n}\sum_{i=2}^{n}\frac{1}{(\frac{i}{n})^{1/2}}\left(1+\log\frac{1}{\frac{1}{n}+1-\frac{i}{n}}\right)^{\gamma+1}\frac{1}{\frac{1}{n^\beta}+(1-\frac{i}{n})^\beta}\\
\le\ & c_2\sum_{n=2}^{\infty}\frac{(\log n)^{1-\gamma}}{n}\int_0^1\frac{1}{x^{1/2}}(1+\log\frac{1}{1-x})^{\gamma+1}\frac{1}{(1-x)^\beta}dx<\infty.
\end{aligned}$$

By this argument also the case $\beta=0,\gamma>2$ can be treated. As to the more general case $\beta=0,\gamma>1$ we refer to [24], Theorem 3. The case $\beta=1,\ \ \gamma>4$ is treated by verifying (2.2) (see [24], Remark 5). □

For the proof of Theorem 2.2 we need the following lemmas.

Lemma 3.1 (see [11], ch. VII, (5.11)). For each $p\in(0,1)$ a constant c^* exists such that for all $n\in\mathbb{N}$ and $k\in\{0,1,\dots,n\}$

$$\binom{n}{k}p^k(1-p)^{n-k}\le\frac{c^*}{\sqrt{n}}.$$

Lemma 3.2. Let $p\in(0,1)$. Then

$$\sum_{n=k}^{\infty}\frac{1}{n}\binom{n}{k}(1-p)^{n-k}p^k=\frac{1}{k}\quad(k\in\mathbb{N}),\tag{3.4}$$

$$\sum_{n=k}^{\infty}\frac{1}{n(n-1)}\binom{n}{k}(1-p)^{n-k}p^k=\frac{p}{k(k-1)}\qquad(k\in\{2,3,\dots,\}).\tag{3.5}$$

Proof of Lemma 3.2. (3.4) is equivalent to

$$\sum_{n=k}^{\infty}\binom{n-1}{k-1}(1-p)^{n-k}p^k=1\quad(k\in\mathbb{N}).\tag{3.6}$$

(3.5) is equivalent to

$$\sum_{n=k}^{\infty}\binom{n-2}{k-2}(1-p)^{(n-1)-(k-1)}p^{k-1}=1\quad(k\in\{2,,3,\dots,\}),$$

which is equivalent to (3.6). But (3.6) follows, with $q:=1-p$, from

$$(1-q)^{-k}\ =\ \sum_{j=0}^{\infty}\binom{-k}{j}(-q)^j=\sum_{n=k}^{\infty}\binom{n-1}{k-1}q^{n-k}\quad(k\in\mathbb{N}).$$

□

Lemma 3.3 (see [25], 64 II, and [12], Theorem 119). Let $\alpha, \beta \in (0,1)$. Then

$$\sum_{l=0}^{n} \binom{n}{l} \alpha^l (1-\alpha)^{n-l} \binom{l}{k} \beta^k (1-\beta)^{n-k} = \binom{n}{k} (\alpha\beta)^k (1-\alpha\beta)^{n-k}$$

for $n \in \mathbb{N}$, $k \in \{0,1,\ldots,n\}$, i.e., $E_{\alpha\beta}$ is the composition of E_α and E_β.

To make the paper more self-contained, in view of Lemma 3.3 we mention that, with $V = (v_{nk})_{n,l\in\mathbb{N}_0}$ defined by $v_{nk} = (-1)^k \binom{n}{k}$, one has $V = V^{-1}$ and $E_\alpha = V\ diag\ \{\alpha^n\}\ V$ $(0 < \alpha < 1)$ (E_α is a so-called Hausdorff matrix; see [12], ch. XI, and [25], section 72), thus $E_\alpha E_\beta = E_{\alpha\beta}$ $(0 < \alpha < 1,\ 0 < \beta < 1)$. This relation is also obtained by a probabilistic argument. Consider a branching process at times $0,1,2$ with size $Y_0 = n$ of the zero generation, for which each particle of the zero generation (of the first generation) independently of the other particles creates 1 new particle with probability α $(\beta) \in (0,1)$ and no new particle with probability $1-\alpha$ (1-β). Then the sizes Y_n of the generations with numbers $n = 0,1,2$ form a Markov chain (homogeneous in the case $\alpha = \beta$) with matrices E_α and E_β of transition probabilities, where obviously Y_1 and Y_2 have binomial distribution $b(n,\alpha)$ and $b(n,\alpha\beta)$, respectively, thus $E_\alpha E_\beta = E_{\alpha\beta}$.

Proof of Theorem 2.2. Let $\alpha \in (0,1)$ be fixed. Assume $EX_n = 0$ without loss of generality. Set

$$T_n := \sum_{k=0}^{n} \binom{n}{k} \sqrt{\alpha}^k (1-\sqrt{\alpha})^{n-k} X_k.$$

(T_n) is the $E_{\sqrt{\alpha}}$-transform of (X_n). First we show

$$\sum_{n=2}^{\infty} \frac{1}{(n+1)^{1/2}}\ ET_n^4 \ <\ \infty. \tag{3.7}$$

The left-hand side is bounded by

$$\begin{aligned}
&\sum_{n=2}^{\infty} \frac{1}{(n+1)^{1/2}} \sum_{k=0}^{n} \left[\binom{n}{k} \sqrt{\alpha}^k (1-\sqrt{\alpha})^{n-k}\right]^4 EX_k^4 \\
&+\sum_{n=2}^{\infty} \frac{1}{(n+1)^{1/2}} \sum_{j\neq k\in\{0,\ldots,n\}} \left[\binom{n}{j} \sqrt{\alpha}^j (1-\sqrt{\alpha})^{n-j} \binom{n}{k} \sqrt{\alpha}^k (1-\sqrt{\alpha})^{n-k}\right]^2 EX_j^2 EX_k^2 \\
\leq\ & c_1 \sum_{n=2}^{\infty} \frac{1}{(n+1)^2} \sum_{k=0}^{n} \binom{n}{k} \sqrt{\alpha}^k (1-\sqrt{\alpha})^{n-k} EX_k^4 \\
&+c_1 \sum_{n=2}^{\infty} \frac{1}{(n+1)^{3/2}} \left[\sum_{k=0}^{n} \binom{n}{k} \sqrt{\alpha}^k (1-\sqrt{\alpha})^{n-k} EX_k^2\right]^2 \\
&\text{(by Lemma 3.1)} \\
\leq\ & c_2 + c_3 \sum_{n=2}^{\infty} \frac{1}{(n+1)^2} \sum_{k=2}^{n} \binom{n}{k} \sqrt{\alpha}^k (1-\sqrt{\alpha})^{n-k} EX_k^4
\end{aligned}$$

$$+ \ c_3 \sum_{n=2}^{\infty} \frac{1}{(n+1)^{3/2}} \sum_{k=2}^{n} \binom{n}{k} \sqrt{\alpha}^k (1-\sqrt{\alpha})^{n-k} (EX_k^2)^2$$

(by the Cauchy-Schwarz inequality)

$$= \ c_2 + c_3 \sum_{k=2}^{\infty} \left(\sum_{n=k}^{\infty} \frac{1}{(n+1)^2} \binom{n}{k} \sqrt{\alpha}^k (1-\sqrt{\alpha})^{n-k} \right) EX_k^4$$

$$+ c_3 \sum_{k=2}^{\infty} \left(\sum_{n=k}^{\infty} \frac{1}{(n+1)^{3/2}} \binom{n}{k} \sqrt{\alpha}^k (1-\sqrt{\alpha})^{n-k} \right) (EX_k^2)^2$$

$$\leq \ c_2 + c_4 \sum_{k=2}^{\infty} \frac{1}{k(k-1)} EX_k^4 + c_4 \sum_{k=2}^{\infty} \frac{1}{k\sqrt{k-1}} (EX_k^2)^2$$

by Lemma 3.2 using first (3.5) and secondly (3.4) and (3.5) together with the Cauchy-Schwarz inequality. Now (3.7) follows from the assumptions. (3.7) implies a.s. $\sum (n+1)^{-1/2} T_n^4 < \infty$ and thus

$$\text{(3.8)} \qquad \text{a.s.} \quad \frac{1}{(n+1)^{1/2}} \sum_{k=0}^{n} T_k^4 \ \to \ 0$$

by the Kronecker lemma. Set

$$W_n := \sum_{k=0}^{n} \binom{n}{k} \sqrt{\alpha}^k (1-\sqrt{\alpha})^{n-k} T_k.$$

(W_n) is the $E_{\sqrt{\alpha}}$-transform of (T_n), i.e., by Lemma 3.3, the E_α-transform of (X_n). One obtains

$$|W_n|^4 \ \leq \ \sum_{k=0}^{n} \binom{n}{k} \sqrt{\alpha}^k (1-\sqrt{\alpha})^{n-k} T_k^4 \leq c_5 n^{-1/2} \sum_{k=0}^{n} T_k^4 \to 0 \quad \text{a.s.},$$

by Jensen's inequality, Lemma 3.1 and (3.8). Thus the assertion is obtained. □

The following lemma is well-known in the context of the proof of Kolmogorov's strong law of large numbers and will be used in the proof of Theorem 2.3. I denotes an indicator function.

Lemma 3.4 (see [11], ch. VII, p. 240, or [17], section 17). Let X be an integrable nonnegative random variable. Then $\sum n^{-2} E(XI_{[X \leq n]})^2 < \infty$.

Proof of Theorem 2.3. Assume $X_n \geq 0$ without loss of generality. The argument of the first step is well known from the proof of the classical Kolmogorov strong law of large numbers. Set $X_n^* := X_n I_{[X_n \leq \sqrt{n}]}$. Because of

$$\sum_{n=0}^{\infty} P\left[X_n \neq X_n^*\right] = \sum_{n=0}^{\infty} P\left[X_1^2 > n\right] \leq EX_1^2 < \infty,$$

a.s. $X_n = X_n^*$ from some index on (by the Borel-Cantelli lemma). Therefore and because of $EX_n^* = EX_1 I_{[X_1 \leq \sqrt{n}]} \to EX_1$, it suffices to show

$$\text{(3.9)} \qquad \text{a.s.} \quad E_\alpha\text{-}\lim(X_n^* - EX_n^*) = 0.$$

Lemma 3.4 yields $\sum n^{-2} E X_n^{*4} < \infty$, which together with $\sum n^{-3/2}[E(X_n^* - EX_n^*)^2]^2 \leq \sum n^{-3/2}(EX_1^2)^2 < \infty$ yields (3.9) by Theorem 2.2. □

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