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TRAPPED MODES FOR AN ELASTIC PLATE WITH A PERTURBATION OF YOUNG'S MODULUS

CLEMENS FÖRSTER

ABSTRACT. We consider a linear elastic plate with stress-free boundary conditions in the limit of vanishing Poisson coefficient. We prove that under a local change of Young's modulus infinitely many eigenvalues arise in the essential spectrum which accumulate at a positive threshold. We give estimates on the accumulation rate and on the asymptotical behaviour of the eigenvalues.

1. INTRODUCTION

We consider a homogeneous and isotropic linear elastic medium on the domain $G = \mathbb{R}^2 \times J$ with $J = (-\frac{\pi}{2}, \frac{\pi}{2})$. For functions $u \in H^1(G; \mathbb{C}^3)$ we set

$$\epsilon(u) = \frac{1}{2}(\nabla u + (\nabla u)^T)$$

and[1]

$$(1.1) \qquad a_0[u] = 2 \int_G \langle \epsilon(u), \epsilon(u) \rangle_{\mathbb{C}^{3\times 3}} \, dx.$$

This is the quadratic form of the elasticity operator

$$(1.2) \qquad A_0 = -(\Delta + \operatorname{grad} \operatorname{div})$$

in $L^2(G; \mathbb{C}^3)$ for zero Poisson coefficient with stress-free boundary conditions. Here we have chosen a suitable set of units such that for Young's modulus E it holds $E = 2$. Let $f \in L^\infty(\mathbb{R}^2; [0,1])$ be a function of compact support which is extended to G by $f(x) = f(x_1, x_2)$ for $x = (x_1, x_2, x_3) \in G = \mathbb{R}^2 \times J$. The function f describes a local perturbation consisting in an area of reduced Young's modulus. For $\alpha \in (0, 1)$ we consider the perturbed operator A_α corresponding to the quadratic form

$$(1.3) \qquad a_\alpha[u] = 2 \int_G (1 - \alpha f)\langle \epsilon(u), \epsilon(u) \rangle_{\mathbb{C}^{3\times 3}} \, dx, \quad u \in H^1(G; \mathbb{C}^3).$$

The perturbation gives rise to local oscillations of the plate which are situated around the perturbation. These oscillations are called trapped modes and correspond to eigenvalues of A_α which are embedded into its essential spectrum. We prove the existence of infinitely many eigenvalues $\varkappa_k(\alpha), k \in \mathbb{N}$, of A_α for $\alpha \in (0,1)$ which accumulate to a certain threshold $\Lambda > 0$. Furthermore, we give two sorts of estimates which characterise the asymptotical behaviour of the eigenvalues. On the one hand we prove the small coupling asymptotics

$$(1.4) \qquad \varkappa_k(\alpha) = \Lambda - \alpha^2 (\Lambda \pi \lambda_k(K))^2 + o(\alpha^2) \quad \text{as} \quad \alpha \to 0.$$

Here $\lambda_k(K)$ are the eigenvalues of a compact model operator K specified in (5.20). On the other hand we prove the accumulation rate asymptotics

$$(1.5) \qquad \ln(\Lambda - \varkappa_k(\alpha)) = -2k \ln k + o(k \ln k) \quad \text{as} \quad k \to \infty.$$

The topic of this paper is related to a series of works on trapped modes in perturbed quantum, acoustic and elastic waveguides, see among others [1, 2, 3, 4, 5, 6, 7] and references therein. While trapped modes for quantum and acoustic waveguides correspond to eigenvalues of the scalar-valued Laplacian, the investigation of trapped modes in elastic waveguides leads to the spectral analysis

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[1]For two matrices $A, B \in \mathbb{C}^{3\times 3}$ we set $\langle A, B \rangle_{\mathbb{C}^{3\times 3}} = \sum_{i,j=1}^3 A_{ij} \overline{B}_{ji}$.

of the elasticity operator which acts on vector-valued functions. Because of the more involved structure of this operator, new and in some degree curious spectral effects can be observed.

These effects depend on the specific structure of the symbol which is associated with the elasticity operator. In [7] this dependence was exploited for the elasticity operator on the strip $\Gamma = \mathbb{R} \times J$. Performing a Fourier transform in the unbounded direction on sees that the spectrum arises from an infinite series of band functions depending on the Fourier parameter. Considering the operator only for specific spatial and internal symmetry cases, the lowest band function attains its minimum at two non-zero points $\xi = \pm\varkappa$ of the Fourier coordinate ξ. This yields two trapped modes when a suitable perturbation is applied.

Within this article we consider the plate $G = \mathbb{R}^2 \times J$ which is obtained by rotating the strip Γ around the axis which is along the bounded dimension. As we will see, the lowest band function in this case is obtained by rotating the corresponding band function of the strip case around the axis $\xi = 0$. Its minimum is now attained on the whole circle $\{\xi \in \mathbb{R}^2 : |\xi| = \varkappa\}$. This corresponds to infinitely many standing waves with minimal kinetic energy. Therefore, indeed *infinitely* many eigenvalues arise, when a perturbation is applied. This kind of "strongly degenerated" spectral minimum was treated in [8] for rather general operators. We will make use of the concepts which were developed there. But in contrast to the applications presented in [8], we have to deal with super-polynomial accumulation rates of the eigenvalues. Such accumulation rates were recently observed in different situations, see for example [9, 10, 11]. In general, they rest on the combination of compactly supported perturbations applied to strongly degenerated operators.

The structure of the paper is as follows. In Section 2 we state the problem in detail. In Section 3 we describe the spatial and internal symmetries of the problem which will help us to distinguish the embedded eigenvalues from the surrounding essential spectrum. We give a description of the spectral minimum mentioned above which is mainly based on [7].

In Section 4 we prove the existence of infinitely many eigenvalues. Further auxiliary material is provided in Section 5. We introduce general classes of operators which allow us to handle super-polynomial eigenvalue asymptotics beyond the well-known Σ_p-classes for compact operators. Then we state a special type of Birman-Schwinger principle appropriate to our problem. Finally we introduce the method of [8] and describe the behaviour of the Birman-Schwinger operator along the spectral minimum of the unperturbed operator.

The main results about the asymptotical behaviour of the embedded eigenvalues are stated in Section 6, followed by the main proofs in Section 7. The idea of the proofs is to obtain properties from the spectrum of the unbounded elasticity operator by investigating the compact Birman-Schwinger operator. This is done by extracting that part of the Birman-Schwinger operator which is responsible for the leading term in the eigenvalue asymptotics, calculating its properties and estimating the remainder terms. The latter two problems are treated separately in Sections 8 and 9.

1.2. **Notations.** Throughout this article, $n(s, T)$ denotes the number of singular values of a compact operator T above $s > 0$ including multiplicities. For general selfadjoint operators T, $n_\pm(s, T)$ denotes the number of eigenvalues above and below $s \in \mathbb{R}$, respectively.

2. STATEMENT OF THE PROBLEM

We denote the stress tensor by $\sigma(u) = \lambda(\mathrm{Tr}\,\epsilon(u))\mathrm{I} + 2\mu\epsilon(u)$ where $\mu = \frac{E}{2(1+\nu)}$ and $\lambda = \frac{E\nu}{(1+\nu)(1-2\nu)}$ are the Lamé constants, ν is the Poisson ratio and E is Young's modulus. Then the quadratic form corresponding to the operator of linear elasticity subject to stress-free boundary conditions

is given by

$$(2.1) \qquad a_0[u] = \int_G \langle \sigma(u), \epsilon(u) \rangle_{\mathbb{C}^{3\times3}} \, dx,$$

which is well-defined for all functions $u \in H^1(G; \mathbb{C}^3)$.

In this paper we stress on the special case of zero Poisson coefficient. Choosing a suitable set of units such that $E = 2$, (2.1) turns into

$$(2.2) \qquad a_0[u] = 2 \int_G \langle \epsilon(u), \epsilon(u) \rangle_{\mathbb{C}^{3\times3}} \, dx.$$

This form is associated with the positive self-adjoint operator

$$(2.3) \qquad A_0 = -(\Delta + \operatorname{grad} \operatorname{div})$$

on the domain

$$D(A_0) = \left\{ u \in H^2(G; \mathbb{C}^3) : \partial_j u_3 + \partial_3 u_j = 0, \ j = 1, 2, 3 \quad \text{on} \quad \mathbb{R}^2 \times \{\pm\tfrac{\pi}{2}\} \right\}.$$

The inequality

$$(2.4) \qquad a_0[u] \le 2\|u\|^2_{H^1(G;\mathbb{C}^3)}, \qquad u \in H^1(G; \mathbb{C}^3)$$

is obvious. Moreover, the reverse estimate

$$(2.5) \qquad a_0[u] + \|u\|^2 \ge c(G)\|u\|^2_{H^1(G;\mathbb{C}^3)}, \qquad u \in H^1(G; \mathbb{C}^3), \quad c(G) > 0$$

holds, where $\|\cdot\|$ denotes the norm in $L^2(G; \mathbb{C}^3)$. This is the well-known Korn inequality [12]. Hence, the class of functions $u \in L^2(G; \mathbb{C}^3)$, for which the integral (2.2) is well-defined and finite, coincides with $H^1(G; \mathbb{C}^3)$. Therefore, the form a_0 is closed on the domain $D[a_0] = H^1(G; \mathbb{C}^3)$.

For $f \in L^\infty(\mathbb{R}^2; [0,1])$ with compact support, extended to G by $f(x) = f(x_1, x_2)$ for $x \in G = \mathbb{R}^2 \times J$, we define the form

$$(2.6) \qquad v[u] := 2 \int_G f \langle \epsilon(u), \epsilon(u) \rangle_{\mathbb{C}^{3\times3}} \, dx, \qquad u \in D[a_0].$$

Considering the form

$$(2.7) \qquad a_\alpha[u] := a_0[u] - \alpha v[u]$$

for $\alpha \in (0,1)$, which corresponds to (1.3), we see that this form is also closed on the domain $D[a_\alpha] = D[a_0] = H^1(G; \mathbb{C}^3)$. Therefore it induces a positive self-adjoint operator A_α in $L^2(G; \mathbb{C}^3)$.

The spectrum of the operator A_0 is purely absolutely continuous and coincides with $[0, \infty)$. It is well-known [13], that a local change of the boundary conditions or a local change of the quadratic form will not change the essential spectrum. Therefore the essential part of the spectrum of A_α fills the non-negative semi-axis, too. In this paper we shall discuss the existence of positive eigenvalues of the operator A_α which are embedded into its continuous spectrum.

3. Auxiliary material I

3.1. **Spatial and internal symmetries.** In analogy to [7] we define the subspaces H_j, $j = 1, 2$ of $H := L^2(G; \mathbb{C}^3)$ as

$$H_j := \big\{ u \in H : u_l(\cdot, -x_3) = (-1)^{j-1} u_l(\cdot, x_3), \ l = 1, 2,$$
$$u_3(\cdot, -x_3) = (-1)^j u_3(\cdot, x_3) \big\}.$$

Then $H = H_1 \oplus H_2$. Further let

$$H_3 := \{ u \in H_1 : \ u = (u_1(x_1, x_2), u_2(x_1, x_2), 0) \}.$$

It forms a subspace in H_1. The orthogonal complement H_4 to H_3 in H_1 consists of all functions $w \in H_1$ for which

$$\int_J w_i(\cdot, x_3) \, dx_3 = 0 \quad \text{in} \quad L^2(\mathbb{R}^2; \mathbb{C}) \quad \text{for} \quad i = 1, 2.$$

Let P_j be the orthogonal projections onto H_j, $j = 1, \ldots, 4$. Then $P_j P_1 = P_1 P_j = P_j$ for $j = 3, 4$. A simple calculation shows, that

$$D[a_\alpha^{(j)}] := P_j D[a_\alpha] \subset D[a_\alpha] , \quad j = 1, \ldots, 4$$

and

$$a_\alpha[u, w] = 0 \quad \text{for all} \quad u \in D[a_\alpha^{(l)}], \ w \in D[a_\alpha^{(j)}]$$

if $l, j = 2, 3, 4$ and $l \neq j$. Hence, these subspaces are reducing for the operator A_α and

$$(3.1) \qquad A_\alpha = A_\alpha^{(3)} \oplus A_\alpha^{(4)} \oplus A_\alpha^{(2)} \quad \text{on} \quad H = H_3 \oplus H_4 \oplus H_2 ,$$

where the operators $A_\alpha^{(j)}$ are the restrictions of A_α to $D(A_\alpha^{(j)}) = D(A_\alpha) \cap H_j$ and correspond to the closed forms $a_\alpha^{(j)}$, given by the differential expression (1.3) on $D[a_\alpha^{(j)}]$, $j = 2, 3, 4$. Put

$$(3.2) \qquad A_\alpha^{(1)} = A_\alpha^{(3)} \oplus A_\alpha^{(4)} \quad \text{on} \quad H_1 = H_3 \oplus H_4 ,$$

being the restriction of A_α to $D(A_\alpha) \cap H_1$. Then it holds

$$(3.3) \qquad A_\alpha = A_\alpha^{(1)} \oplus A_\alpha^{(2)} \quad \text{on} \quad H = H_1 \oplus H_2 .$$

Decomposition (3.3) reflects the spatial symmetry of the operator A_α, while (3.2) exploits the specific internal structure of A_α. These symmetries depend on the above-mentioned restriction to perturbation functions f which are constant in x_3-direction. We also note that (3.2) fails for elasticity operators with non-zero Poisson coefficients.

The hierarchy of symmetry spaces H_j for the vector-valued displacement functions carries over to the matrix-valued stress functions. We set

$$H^\times = \{w \in L^2(G; \mathbb{C}^{3 \times 3}) : w = w^T\}$$

and define the subspaces $H_j^\times$ of $H^\times$ by

$$H_1^\times := \{w \in H^\times : w_{ij}(\cdot, -x_3) = \delta w_{ij}(\cdot, x_3), \text{where}$$
$$\delta = 1 \text{ for } i, j \in \{1, 2\} \text{ or } i = j = 3 \text{ and } \delta = -1 \text{ otherwise}\},$$
$$H_4^\times := \{w \in H_1^\times : \int_J w_{ij}(\cdot, x_3) \, \mathrm{d}x_3 = 0 \text{ in } L^2(\mathbb{R}^2; \mathbb{C}) \text{ for } i, j \in \{1, 2\}\},$$
$$H_2^\times := H^\times \ominus H_1^\times, \qquad H_3^\times := H_1^\times \ominus H_4^\times.$$

Then

$$H^\times = H_3^\times \oplus H_4^\times \oplus H_2^\times$$

and

$$\epsilon(u) \in H_j^\times \quad \text{for} \quad u \in D[a_0] \cap H_j.$$

For functions $u \in D(Q) := D[a_0^{(4)}]$ we define the operator $Qu = \sqrt{2}\epsilon(u)$. It holds that $A_0^{(4)} = Q^*Q$. In Section 5.2 we will need the operator

$$(3.4) \qquad U := Q \left(A_0^{(4)}\right)^{-\frac{1}{2}}, \qquad U : H_4 \to H_4^\times$$

which is an isometry between H_4 and the range of U,

$$R(U) = H_4^\times \ominus (\ker Q^*).$$

This follows from

$$\|Uv\|_\times^2 = \|Q(A_0^{(4)})^{-\frac{1}{2}}v\|_\times^2 = \|\sqrt{2}\epsilon((A_0^{(4)})^{-\frac{1}{2}}v)\|_\times^2 = a_0^{(4)}[(A_0^{(4)})^{-\frac{1}{2}}v] = \|v\|^2$$

for all $v \in H_4$ where $\| \cdot \|_\times$ denotes the norm in $L^2(G; \mathbb{C}^{3 \times 3})$.

3.2. **Separation of variables for A_0.** Applying the unitary Fourier transform Φ in (x_1, x_2)-direction and its inverse Φ^*, one finds that $\Phi A_0 \Phi^*$ permits the orthogonal decomposition

$$\Phi A_0 \Phi^* = \int_{\mathbb{R}^2}^{\oplus} A(\xi)\, \mathrm{d}\xi \qquad \text{on} \qquad H = \int_{\mathbb{R}^2}^{\oplus} h\, \mathrm{d}\xi, \quad h = L^2(J; \mathbb{C}^3).$$

The self-adjoint operators $A(\xi)$ are given by the differential expressions

$$(3.5) \qquad A(\xi) = \begin{pmatrix} -\partial_3^2 + |\xi|^2 + \xi_1^2 & \xi_1 \xi_2 & -i\xi_1 \partial_3 \\ \xi_1 \xi_2 & -\partial_3^2 + |\xi|^2 + \xi_2^2 & -i\xi_2 \partial_3 \\ -i\xi_1 \partial_3 & -i\xi_2 \partial_3 & -2\partial_3^2 + |\xi|^2 \end{pmatrix}$$

on the domains

$$(3.6) \qquad \begin{aligned} D(A(\xi)) = \{u \in H^2(J; \mathbb{C}^3) : \partial_3 u_3|_{t=\pm\pi/2} = (\partial_3 u_1 + i\xi_1 u_3)|_{t=\pm\pi/2} = 0, \\ (\partial_3 u_2 + i\xi_2 u_3)|_{t=\pm\pi/2} = 0\}. \end{aligned}$$

The symmetry (3.1) extends to the operators $A(\xi)$. Indeed, put

$$h_j := \{u \in h : u_{1/2}(t) = (-1)^{j-1} u_{1/2}(-t), u_3(t) = (-1)^j u_3(-t)\}, \ j = 1, 2,$$

$$h_3 := \mathrm{span}\{(1,0,0)^T, (0,1,0)^T\}, \quad h_4 := \{u \in h_1 : \int_J u_i(t)\, \mathrm{d}t = 0, \ i = 1, 2\}.$$

Then we have

$$(3.7) \qquad H_j = \int_{\mathbb{R}^2}^{\oplus} h_j\, \mathrm{d}\xi \qquad \text{and} \qquad \Phi A_0^{(j)} \Phi^* = \int_{\mathbb{R}^2}^{\oplus} A^{(j)}(\xi)\, \mathrm{d}\xi, \qquad j = 1, \dots, 4,$$

where the operators $A^{(j)}(\xi)$ are the restrictions of $A(\xi)$ to $D(A^{(j)}(\xi)) = D(A(\xi)) \cap h_j$. Moreover, it holds

$$(3.8) \qquad \begin{aligned} A(\xi) = A^{(1)}(\xi) \oplus A^{(2)}(\xi) \quad \text{on} \quad h = h_1 \oplus h_2\ , \\ A(\xi) = A^{(3)}(\xi) \oplus A^{(4)}(\xi) \oplus A^{(2)}(\xi) \quad \text{on} \quad h = h_3 \oplus h_4 \oplus h_2\ . \end{aligned}$$

The operators $A^{(j)}(\xi)$ correspond to quadratic forms $a^{(j)}(\xi)$ being closed on the domains $D[a^{(j)}(\xi)] = H^1(J; \mathbb{C}^3) \cap h_j$, $j = 1, \dots, 4$.

3.3. **The spectral analysis of the operator $A_0^{(4)}$.** During this paper the spectral decomposition of the operator $A_0^{(4)}$ shall be of particular interest. Because of the decomposition (3.7) we have in fact to carry out the spectral analysis of the operators $A^{(4)}(\xi)$. Being the restrictions of the non-negative second order Sturm-Liouville systems (3.5) to $D(A(\xi)) \cap h_4$, the operators $A^{(4)}(\xi)$ have a non-negative discrete spectrum, which accumulates to infinity only. The eigenvalues of $A^{(4)}(\xi)$ are invariant under rotations of ξ. If λ is an eigenvalue of $A^{(4)}(\xi)$ with eigenfunction $u(\xi) = (u_1(\xi), u_2(\xi), u_3(\xi))^T$, then, for any unitary matrix $M \in \mathbb{R}^{2 \times 2}$, λ is also an eigenvalue of $A^{(4)}(M\xi)$ with the eigenfunction $u(M\xi)$ fulfilling

$$\begin{pmatrix} u_1(M\xi) \\ u_2(M\xi) \end{pmatrix} = M \begin{pmatrix} u_1(\xi) \\ u_2(\xi) \end{pmatrix}, \qquad u_3(M\xi) = u_3(\xi).$$

Therefore, we can restrict ourselves to the special case $\xi = \binom{0}{r}, r \geq 0$ where

$$A^{(4)}(\xi) = A^{(4)}(r) = \begin{pmatrix} -\partial_3^2 + r^2 & 0 & 0 \\ 0 & -\partial_3^2 + 2r^2 & -ir\partial_3 \\ 0 & -ir\partial_3 & -2\partial_3^2 + r^2 \end{pmatrix}.$$

The operator $A^{(4)}(r)$ can be written as $A^{(4)}(r) = \hat{A}^{(4)}(r) \oplus \check{A}^{(4)}(r)$ where the latter two operators are the restrictions of $A^{(4)}(r)$ to

$$D(\hat{A}^{(4)}(r)) = \{u \in D(A^{(4)}(r)) : u_1 = 0\},$$

$$D(\check{A}^{(4)}(r)) = \{u \in D(A^{(4)}(r)) : u_2 = u_3 = 0\}.$$

For the eigenvalues $\check{\lambda}_k(r)$ of $\check{A}^{(4)}(r)$ it holds

$$(3.9) \qquad \check{\lambda}_k(r) = r^2 + 4k^2, \quad k \in \mathbb{N}.$$

The eigenvalues $\hat{\lambda}_k(r)$ of the operator $\hat{A}^{(4)}(r)$ however exhibit a nontrivial structure which was analysed in [7]. The eigenvalues are simple for any fixed r and the functions $r \mapsto \hat{\lambda}_k(r)$ are real analytic. The spectral minimum

$$\tag{3.10} \Lambda = \inf \bigcup_{r \geq 0} \sigma(\hat{A}^{(4)}(r)) = \inf_{r \geq 0} \hat{\lambda}_1(r)$$

is achieved for exactly one non-trivial value $r = \varkappa > 0$. Moreover it holds

$$\tag{3.11} \hat{\lambda}_1(\varkappa + \varepsilon) = \Lambda + q^2 \varepsilon^2 + O(\varepsilon^3) \quad \text{as} \quad \varepsilon \to 0$$

for a certain $q > 0$. Applying the function $u(t) = (0, 1 - \frac{\pi}{2} \cos(t), i \frac{\pi}{2} \sin(t))^T$ to the quadratic form $\hat{a}^{(4)}(r)$ associated with $\hat{A}^{(4)}(r)$ yields for $r = 1$

$$\frac{\hat{a}^{(4)}(1)[u]}{\|u\|^2_{L^2(J;\mathbb{C}^3)}} = 2$$

and therefore $\Lambda \leq 2$. Comparing this with (3.9) we achieve for the ground state $\lambda_1(r)$ of $A^{(4)}(r)$

$$\Lambda = \inf \bigcup_{r \geq 0} \sigma(A^{(4)}(r)) = \inf_{r \geq 0} \lambda_1(r)$$

where the infimum is attained exactly at $r = \varkappa$. As stated in [7], it holds

$$\tag{3.12} \lambda_1(\varkappa + \varepsilon) = \Lambda + q^2 \varepsilon^2 + O(\varepsilon^3) \quad \text{as} \quad \varepsilon \to 0$$

for some $q > 0$. A numerical evaluation gives

$$\tag{3.13} \begin{aligned} \varkappa &= 0.632138 \pm 10^{-6}\,, \\ \Lambda &= 1.887837 \pm 10^{-6}\,, \\ q &= 0.849748 \pm 10^{-6}. \end{aligned}$$

Also from [7] we obtain that the eigenfunction corresponding to $\lambda_1(r)$ can be given by $\psi_1 = \tilde{\psi}_1/\|\tilde{\psi}_1\|_{L^2(J;\mathbb{C}^3)}$ where

$$\tilde{\psi}_1(r, x_3) := \big(0, ir\, d_1(r, x_3), d_2(r, x_3)\big)^T$$

and

$$\tag{3.14} \begin{aligned} d_1(r, t) &:= r\beta \cos\left(\tfrac{\pi}{2}\beta\right) \cos(\gamma t) + \tfrac{\gamma^2 \beta}{r} \cos\left(\tfrac{\pi}{2}\gamma\right) \cos(\beta t), \\ d_2(r, t) &:= -r\beta\gamma \cos\left(\tfrac{\pi}{2}\beta\right) \sin(\gamma t) + r\gamma^2 \cos\left(\tfrac{\pi}{2}\gamma\right) \sin(\beta t) \end{aligned}$$

with $\beta = \sqrt{\lambda_1(r) - r^2}$ and $\gamma = \sqrt{\lambda_1(r)/2 - r^2}$. Note that this is only valid for $\gamma \neq 0$ which is fulfilled in a neighbourhood of $r = \varkappa$.

For general $\xi \in \mathbb{R}^2$ in a neighbourhood of $|\xi| = \varkappa$ we denote the eigenfunction corresponding to $\lambda_1(\xi)$ again by $\psi_1 = \tilde{\psi}_1/\|\tilde{\psi}_1\|_{L^2(J;\mathbb{C}^3)}$ where now

$$\tag{3.15} \tilde{\psi}_1(\xi, x_3) = \big(i\xi_1\, d_1(|\xi|, x_3), i\xi_2\, d_1(|\xi|, x_3), d_2(|\xi|, x_3)\big)^T.$$

Moreover, the spectral minimum of $A^{(4)}(\xi)$ is attained for $|\xi| = \varkappa$ where $\lambda_1(\xi) = \Lambda$. For such ξ the functions

$$\tag{3.16} w_\xi(x) = \psi_1(\xi, x_3) e^{i\xi \cdot \binom{x_1}{x_2}}$$

fulfil

$$\tag{3.17} -(\Delta + \operatorname{grad} \operatorname{div}) w_\xi(x) = \Lambda w_\xi(x) \quad \text{for} \quad x \in G.$$

4. Existence of eigenvalues

Theorem 4.1. *If $f \not\equiv 0$ in L^2-sense, then $A_\alpha^{(4)}$ has infinitely many eigenvalues in $[0, \Lambda)$ for all $\alpha \in (0, 1)$.*

The idea of the proof is to construct suitable test spaces from the functions w_ξ in (3.16) such that the quadratic form of $A_\alpha^{(4)}$ is strictly lower than Λ for all normed functions of these spaces. Then the proof follows from an application of the variational principle.

4.1. **Proof of Theorem 4.1.** First we define suitable cut-off functions for w_ξ. Let $\tilde{\zeta} \in \mathrm{C}^\infty(\mathbb{R})$ with $\tilde{\zeta}(t) = 1$ for $t < 1$, $\tilde{\zeta}(t) = 0$ for $t > 2$ and $0 \le \tilde{\zeta}(t) \le 1$ for $t \in (1, 2)$ and set

$$\zeta_\varepsilon(x) := \tilde{\zeta}(\varepsilon \ln |x|) \quad \text{for} \quad x \in \mathbb{R}^2, \, \varepsilon > 0.$$

It holds $\zeta_\varepsilon \in \mathrm{C}_0^\infty(\mathbb{R}^2)$ and

$$(4.1) \qquad \int_{\mathbb{R}^2} |\nabla \zeta_\varepsilon|^2 \, \mathrm{d}x \to 0 \quad \text{for} \quad \varepsilon \to 0.$$

Next, for arbitrary $m \in \mathbb{N}$ we choose $\xi^k \in \mathbb{R}^2, k = 1, \ldots, m$ with $|\xi^k| = \varkappa$ such that $\xi_1^k \neq \xi_1^l$ for $k \neq l$. We set

$$u_k^{(\varepsilon)}(x) = \zeta_\varepsilon(x_1, x_2) w_{\xi^k}(x) \quad \text{for} \quad x \in G$$

and define the test space

$$\mathcal{E}_m^{(\varepsilon)} = \mathrm{span}\{u_k^{(\varepsilon)} : k = 1, \ldots, m\}.$$

As we show below,

$$(4.2) \qquad \dim \mathcal{E}_m^{(\varepsilon)} = m \quad \text{for all} \quad \varepsilon > 0.$$

Therefore every element of $\mathcal{E}_m^{(\varepsilon)}$ can be represented in a unique way as

$$u_\eta = \sum_{k=1}^m \eta_k u_k^{(\varepsilon)} \quad \text{for} \quad \eta \in \mathbb{C}^m.$$

We recall that $a_\alpha[u] = a_0[u] - \alpha v[u]$ for $u \in D[a_0]$ where the quadratic form v was defined in (2.6). As we will prove below it holds

$$(4.3) \qquad \inf_{0 \neq u_\eta \in \mathcal{E}_m^{(\varepsilon)}} \frac{v[u_\eta]}{|\eta|^2} \ge c_0$$

for some $c_0 > 0$ independent of sufficiently small $\varepsilon > 0$ and

$$(4.4) \qquad \sup_{0 \neq u_\eta \in \mathcal{E}_m^{(\varepsilon)}} \frac{a_0^{(4)}[u_\eta] - \Lambda \|u_\eta\|^2}{|\eta|^2} \to 0 \quad \text{as} \quad \varepsilon \to 0.$$

Then for sufficiently small $\varepsilon > 0$ there exists $\delta(\varepsilon) \in (0, \alpha c_0)$ such that

$$\begin{aligned} a_\alpha^{(4)}[u_\eta] = a_0^{(4)}[u_\eta] - \alpha v[u_\eta] &= a_0^{(4)}[u_\eta] - \Lambda \|u_\eta\|^2 - \alpha v[u_\eta] + \Lambda \|u_\eta\|^2 \\ &< \delta(\varepsilon)|\eta|^2 - \alpha c_0 |\eta|^2 + \Lambda \|u_\eta\|^2 \\ &< \Lambda \|u_\eta\|^2 \end{aligned}$$

for all $u_\eta \in \mathcal{E}_m^{(\varepsilon)}$. From Glazman's Lemma [14, 10.2.2] it follows the existence of at least m eigenvalues (including multiplicities) below Λ. Because m was arbitrary, this proves the theorem.

4.2. **Proof of (4.2).** Choose an interval $I \subset \mathbb{R}$ and $x_2 \in \mathbb{R}$ such that $\zeta_\varepsilon|_{I \times \{x_2\}} = 1$. Recall the definition of w_ξ in (3.16) and of ψ_1 in (3.15). Choose $x_3 \in J$ such that $d_2(\varkappa, x_3) \neq 0$. Then for arbitrary $\eta \in \mathbb{C}^m$ the condition

$$\sum_{k=1}^m \eta_k u_k^{(\varepsilon)}(x) = 0, \quad x \in \mathbb{R}^2 \times J$$

evaluated for $x \in I \times \{x_2\} \times \{x_3\}$ implies

$$\sum_{k=1}^m \eta_k \mathrm{e}^{i\xi_1^k x_1} = 0, \quad x_1 \in I.$$

Because $\xi_1^k \neq \xi_1^l$ for $k \neq l$ the functions $x_1 \mapsto \mathrm{e}^{i\xi_k x_1}, k = 1, \ldots, m$ are linearly independent. Hence, $\eta = 0$. This proves the linear independence of the functions $u_k^{(\varepsilon)}$.

4.3. **Proof of** (4.3)**.** Because $f \neq 0$ in L^2-sense there exists $\Omega \subset \mathbb{R}^2$ open, nonempty and $c > 0$ such that for χ_Ω being the characteristic function of Ω it holds $c\chi_\Omega(x) \leq f(x)$ for almost every $x \in \mathbb{R}^2$. Hence,

$$\inf_{|\eta|=1} v[u_\eta] = \inf_{|\eta|=1} 2 \int_G f|\epsilon(u_\eta)|^2 \, \mathrm{d}x \geq \inf_{|\eta|=1} 2c \int_{\Omega \times J} |\epsilon(u_\eta)|^2 \, \mathrm{d}x =: c_0.$$

We consider only sufficiently small $\varepsilon > 0$ such that $\zeta_\varepsilon|_\Omega = 1$. Then c_0 does not depend on ε. If $c_0 = 0$ there must be some function $v_\eta \in \mathcal{E}_m^{(\varepsilon)}, |\eta| = 1$ such that $\int_{\Omega \times J} |\epsilon(v_\eta)|^2 \, \mathrm{d}x = 0$. Particularly this means

$$0 = \partial_3 u_{\eta,3}(x) = \sum_{k=1}^m \eta_k \partial_3 u_{k,3}^{(\varepsilon)}(x) \quad \text{for} \quad x \in \Omega \times J.$$

In analogy to Section 4.2 we obtain $\eta = 0$ which is obviously contrary to the requirement $|\eta| = 1$. This proves $c_0 > 0$.

4.4. **Proof of** (4.4)**.** Note that $\|u\|_a = \left(a_0^{(4)}[u] - \Lambda\|u\|^2\right)^{\frac{1}{2}}$ for $u \in D[a_0^{(4)}]$ defines a norm in $D[a_0^{(4)}]$. Therefore it holds that

$$(4.5) \qquad \|u_\eta\|_a^2 \leq \left(\sum_{k=1}^m |\eta_k| \, \|u_k^{(\varepsilon)}\|_a\right)^2 \leq |\eta|^2 \sum_{k=1}^m \|u_k^{(\varepsilon)}\|_a^2.$$

A direct calculation using the definitions of w_ξ in (3.16) and ψ_1 in (3.15) yields

$$\|u_k^{(\varepsilon)}\|_a^2 = a_0^{(4)}[u_k^{(\varepsilon)}] - \Lambda\|u_k^{(\varepsilon)}\|^2$$

$$= \int_G (2|\psi_{1,1}|^2 + |\psi_{1,3}|^2)|\partial_1\zeta_\varepsilon|^2 + (2|\psi_{1,2}|^2 + |\psi_{1,3}|^2)|\partial_2\zeta_\varepsilon|^2 +$$

$$+ |\psi_{1,2}\,\partial_1\zeta_\varepsilon + \psi_{1,1}\,\partial_2\zeta_\varepsilon|^2 \, \mathrm{d}x.$$

Hence,

$$\|u_k^{(\varepsilon)}\|_a^2 \leq C \int_{\mathbb{R}^2} |\nabla\zeta_\varepsilon|^2 \, \mathrm{d}(x_1, x_2)$$

for $C > 0$ independent of ε. The result follows now from (4.1) and (4.5).

5. Auxiliary material II

5.1. **The classes Σ_π and $\mathcal{S}^\pi$.** In order to describe the exponential accumulation rate of the eigenvalues of $A_\alpha^{(4)}$ at Λ we need to generalise the well-known weak Neumann-Schatten classes Σ_p, see [14, 11.6], and the classes $\mathcal{S}^p$ introduced by Safronov [15, 8]. They are both designed to measure polynomial accumulation rates only. We develop these generalisations only as far as it is needed here. Particularly we abandon to postulate conditions ensuring basic properties like linearity or ideal structure for the generalised classes. For a systematic treatment of such extensions see for example [16, 17].

Let $\pi : \mathbb{R}_+ \to \mathbb{R}_+$ be a function with $\lim_{s\to 0} \pi(s) = 0$. Then for suitable compact operators T we can define the functionals

$$\Delta_\pi(T) = \limsup_{s\to 0} \pi(s)n(s, T), \quad \delta_\pi(T) = \liminf_{s\to 0} \pi(s)n(s, T).$$

We denote by Σ_π the class of compact operators T which fulfil $\Delta_\pi(T) < \infty$. Analogously, for suitable operator functions $\mathcal{T} : \mathbb{R}_+ \to \mathfrak{S}_\infty$ we define the functionals

$$\Delta_\pi(\mathcal{T}) = \limsup_{s\to 0} \pi(s)n(1, \mathcal{T}(s)), \quad \delta_\pi(\mathcal{T}) = \liminf_{s\to 0} \pi(s)n(1, \mathcal{T}(s)).$$

We denote by $\mathcal{S}^\pi$ the class of operator functions $\mathcal{T}$ which fulfil $\Delta_\pi(\beta\mathcal{T}) < \infty$ for all $\beta > 0$. Furthermore, $\mathcal{S}_0^\pi$ is the class of all $\mathcal{T} \in \mathcal{S}^\pi$ fulfilling $\Delta_\pi(\beta\mathcal{T}) = 0$ for all $\beta > 0$. We will need the following two lemmata which were also stated in [8] for the polynomial case.

Lemma 5.1. *For $\mathcal{T} \in \mathcal{S}^\pi$ and $\mathcal{T}_0 \in \mathcal{S}_0^\pi$*

(5.1) $$\lim_{\beta \to 1} \Delta_\pi(\beta \mathcal{T}) = \Delta_\pi(\mathcal{T}) \quad implies \quad \Delta_\pi(\mathcal{T} + \mathcal{T}_0) = \Delta_\pi(\mathcal{T}) \quad and$$

(5.2) $$\lim_{\beta \to 1} \delta_\pi(\beta \mathcal{T}) = \delta_\pi(\mathcal{T}) \quad implies \quad \delta_\pi(\mathcal{T} + \mathcal{T}_0) = \delta_\pi(\mathcal{T}).$$

Proof. From Ky-Fan's inequality [14, 11.1.3] it follows

$$n(1, \mathcal{T}(s) + \mathcal{T}_0(s)) \le n(1 - \varepsilon, \mathcal{T}(s)) + n(\varepsilon, \mathcal{T}_0(s)) \quad \text{for} \quad \varepsilon, s > 0$$

which leads to

$$\Delta_\pi(\mathcal{T} + \mathcal{T}_0) \le \limsup_{s \to 0} \big(\pi(s) n(1 - \varepsilon, \mathcal{T}(s)) + \pi(s) n(\varepsilon, \mathcal{T}_0(s)) \big) \le \Delta_\pi(\tfrac{1}{1-\varepsilon} \mathcal{T}).$$

Using the condition from (5.1) we obtain

$$\Delta_\pi(\tfrac{1}{1-\varepsilon} \mathcal{T}) \to \Delta_\pi(\mathcal{T}) \quad \text{as} \quad \varepsilon \to 0$$

and therefore

$$\Delta_\pi(\mathcal{T} + \mathcal{T}_0) \le \Delta_\pi(\mathcal{T}).$$

Analogously it holds

(5.3) $$\Delta_\pi(\mathcal{T}) \le \Delta_\pi(\tfrac{1}{1-\varepsilon}(\mathcal{T} + \mathcal{T}_0)) + \Delta_\pi(-\tfrac{1}{\varepsilon} \mathcal{T}_0) \to \Delta_\pi(\mathcal{T} + \mathcal{T}_0)$$

which proves the first implication. The second one is established in the same way. $\square$

Lemma 5.2. *Let $\mathcal{T} : \mathbb{R}_+ \to \mathfrak{S}_\infty$ fulfil $\mathcal{T}(s) \to T$ in $\mathfrak{S}_\infty$ as $s \to 0$. Then for all $\pi : \mathbb{R}_+ \to \mathbb{R}_+$ with $\lim_{s \to 0} \pi(s) = 0$ it holds $\mathcal{T} \in \mathcal{S}_0^\pi$.*

Proof. Let $\mathcal{U}(s) = T$ for $s > 0$. Obviously, it holds $\mathcal{U} \in \mathcal{S}_0^\pi$. Furthermore,

$$\Delta_\pi(\beta(\mathcal{T} - \mathcal{U})) = \limsup_{s \to 0} \pi(s) n(\beta^{-1}, \mathcal{T}(s) - T) = 0 \quad \text{for} \quad \beta > 0$$

because $n(\beta^{-1}, \mathcal{T}(s) - T)$ is bounded as $\|\mathcal{T}(s) - T\| \to 0$ for $s \to 0$. Therefore, $\mathcal{T} - \mathcal{U} \in \mathcal{S}_0^\pi$. Since $\mathcal{U}$ fulfils the condition in (5.1), Lemma 5.1 yields $\mathcal{T} \in \mathcal{S}_0^\pi$. $\square$

5.2. **A modified Birman-Schwinger principle.** Using the results from Section 3.1 we define a modified Birman-Schwinger type operator

(5.4) $$\mathcal{Y}_\alpha(\tau) := \left(\frac{\Lambda - \tau}{A_0^{(4)} - \Lambda + \tau} \right)^{\frac{1}{2}} V_\alpha \left(\frac{\Lambda - \tau}{A_0^{(4)} - \Lambda + \tau} \right)^{\frac{1}{2}}$$

in H_4 for $\tau \in (0, \Lambda)$ and $\alpha \in (0, 1)$ where

(5.5) $$V_\alpha = U^* \sqrt{\alpha f} (I - \alpha \sqrt{f} \, U U^* \sqrt{f})^{-1} \sqrt{\alpha f} \, U$$

and U is the isometry defined in (3.4). Since $f \in L^\infty(\mathbb{R}^2; [0, 1])$, the operator V_α is bounded. The following version of the Birman-Schwinger principle [18, 19] is crucial for our considerations.

Lemma 5.3. *For $\tau \in (0, \Lambda)$ and $\alpha \in (0, 1)$ it holds*

(5.6) $$n_- \left(\Lambda - \tau, A_\alpha^{(4)} \right) = n_+ (1, \mathcal{Y}_\alpha(\tau)).$$

Proof. For convenience, we put $A := A_0^{(4)}$ and $g_\alpha := \sqrt{\alpha f}$. Furthermore, the expression $L \prec H$ means that L is a subspace of H. Using Glazman's lemma [14, 10.2.2] we obtain

$$n_-(\delta, A_\alpha^{(4)}) = \sup_{L \prec D[a_0^{(4)}]} \{ \dim L : a_0^{(4)}[u] - \alpha v[u] < \delta \|u\|^2, \quad 0 \ne u \in L \}$$

$$= \sup_{L \prec D[a_0^{(4)}]} \{ \dim L : \|(A - \delta)^{\frac{1}{2}} u\|^2 < \alpha v[u], \quad 0 \ne u \in L \}.$$

Applying the substitution $v := (A - \delta)^{\frac{1}{2}} u$ and Glazman's lemma we obtain

$$n_-(\delta, A_\alpha^{(4)}) = \sup_{L \prec H_4} \{ \dim L : \|v\|^2 < \|g_\alpha Q(A - \delta)^{-\frac{1}{2}} v\|_\times^2, \quad 0 \ne v \in L \}$$

$$= n_+(1, (A - \delta)^{-\frac{1}{2}} Q^* g_\alpha^2 Q(A - \delta)^{-\frac{1}{2}})$$

where $\|\cdot\|_\times$ denotes the norm in $L^2(G; \mathbb{C}^{3\times3})$. Next, we define a bounded operator $\mathcal{W}: H_4 \to H_4^\times$ by

$$\mathcal{W} = g_\alpha Q(A-\delta)^{-\frac{1}{2}} = g_\alpha U A^{\frac{1}{2}}(A-\delta)^{-\frac{1}{2}}.$$

As is well-known [14, 3.10], the discrete and essential spectrum of $\mathcal{W}^*\mathcal{W}$ and $\mathcal{W}\mathcal{W}^*$ coincide, except for the point zero. Because

$$\mathcal{W}\mathcal{W}^* = g_\alpha UU^* g_\alpha + \delta g_\alpha U(A-\delta)^{-1}U^* g_\alpha,$$

we obtain

$$n_-(\delta, A_\alpha^{(4)}) = n_+(1, g_\alpha UU^* g_\alpha + \delta g_\alpha U(A-\delta)^{-1}U^* g_\alpha)$$

$$= \sup_{L \prec H_4^\times} \{\dim L : \|\tilde{V}_\alpha^{-\frac{1}{2}} w\|_\times^2 < \|\mathcal{T}_\alpha w\|^2, \ 0 \neq w \in L\}$$

where

$$\tilde{V}_\alpha = (I - g_\alpha UU^* g_\alpha)^{-1} \quad \text{and} \quad \mathcal{T}_\alpha = \sqrt{\delta}(A-\delta)^{-\frac{1}{2}}U^* g_\alpha.$$

Applying the substitution $\tilde{w} := \tilde{V}_\alpha^{-\frac{1}{2}} w$ we get

$$n_-(\delta, A_\alpha^{(4)}) = \sup_{L \prec H_4^\times} \{\dim L : \|\tilde{w}\|_\times^2 < \|\mathcal{T}_\alpha \tilde{V}_\alpha^{\frac{1}{2}} \tilde{w}\|^2, \ 0 \neq \tilde{w} \in L\}$$

$$= n_+(1, \tilde{V}_\alpha^{\frac{1}{2}} \mathcal{T}_\alpha^* \mathcal{T}_\alpha \tilde{V}_\alpha^{\frac{1}{2}})$$

Since the spectrum of $\tilde{V}_\alpha^{\frac{1}{2}} \mathcal{T}_\alpha^* \mathcal{T}_\alpha \tilde{V}_\alpha^{\frac{1}{2}}$ and $\mathcal{T}_\alpha \tilde{V}_\alpha \mathcal{T}_\alpha^*$ coincides, except in the point zero, we obtain

$$n_-(\delta, A_\alpha^{(4)}) = n_+(1, \mathcal{T}_\alpha \tilde{V}_\alpha \mathcal{T}_\alpha^*)$$

which is (5.6) for $\delta = \Lambda - \tau$. $\square$

We recall that U, defined in (3.4), is an isometry between H_4 and $R(U)$. Therefore the operator UU^* is an orthogonal projection onto $R(U)$ fulfilling

$$0 \leq UU^* \leq I.$$

Here 0 and I are the zero and identity operator in $H_4^\times$. Hence, from the definition of V_α in (5.5) we get

$$(5.7) \qquad\qquad U^* \alpha f U \leq V_\alpha \leq U^* \frac{\alpha}{1-\alpha} f U.$$

Defining

$$(5.8) \qquad\qquad \mathcal{Y}(\tau) := \left(\frac{\Lambda - \tau}{A_0^{(4)} - \Lambda + \tau}\right)^{\frac{1}{2}} U^* f U \left(\frac{\Lambda - \tau}{A_0^{(4)} - \Lambda + \tau}\right)^{\frac{1}{2}}$$

we obtain

$$(5.9) \qquad\qquad \alpha \mathcal{Y}(\tau) \leq \mathcal{Y}_\alpha(\tau) \leq \frac{\alpha}{1-\alpha} \mathcal{Y}(\tau)$$

for $\tau \in (0, \Lambda)$ and $\alpha \in (0, 1)$. This allows us to work with the simplified Birman-Schwinger type operator $\mathcal{Y}(\tau)$ instead of $\mathcal{Y}_\alpha(\tau)$.

Note that the operator $\mathcal{Y}(\tau)$ is compact. This can be obtained by considering the operator

$$(5.10) \qquad\qquad (\Lambda - \tau) \sqrt{f} U \left(A_0^{(4)} - \Lambda + \tau\right)^{-1} U^* \sqrt{f}$$

in $H_4^\times$ which has the same spectrum beyond zero as $\mathcal{Y}(\tau)$. Since

$$U \left(A_0^{(4)} - \Lambda + \tau\right)^{-1} U^* = Q \left(A_0^{(4)}(A_0^{(4)} - \Lambda + \tau)\right)^{-1} Q^*$$

is an operator of order minus two and f has compact support, the compactness of (5.10) is a consequence of the Rellich-Kondrachov Theorem [20].

5.3. **Reduction to the spectral minimum of $A_0^{(4)}$.** We develop a slightly modified version of [8]. As in Section 3.3, $\lambda_1(\xi)$ denotes the lowest branch of eigenvalues of $A^{(4)}(\xi)$. The corresponding normed eigenfunction in $L^2(J; \mathbb{C}^3)$ is $\psi_1(\xi, \cdot)$. We set

$$(5.11) \qquad \lambda_0(\xi) := \lambda_1(\xi) - \Lambda \quad \text{for} \quad \xi \in \Xi$$

where $\Xi := \lambda_1^{-1}([\Lambda, \Lambda + \delta))$ for suitable $\delta > 0$. The spectral projection of $A_0^{(4)}$ onto $[\Lambda, \Lambda + \delta)$ is given by

$$(\Pi_c u)(x) := \Phi^*_{\xi \to \binom{x_1}{x_2}} \left(\chi_\Xi(\xi) \, \langle (\Phi u)(\xi, \cdot), \psi_1(\xi, \cdot) \rangle_{L^2(J;\mathbb{C}^3)} \, \psi_1(\xi, x_3) \right), \ u \in H_4.$$

Then the unitary operator

$$\Pi_0 : \Pi_c H_4 \to L^2(\Xi), \qquad (\Pi_0 u)(\xi) := \langle (\Phi u)(\xi, \cdot), \psi_1(\xi, \cdot) \rangle_{L^2(J;\mathbb{C}^3)}$$

maps $\Pi_c H_4$ into the Fourier transformed scalar-valued regime. Next, we define

$$P(\xi) := \begin{cases} +\sqrt{\lambda_0(\xi)} & \text{for} \quad |\xi| \geq \varkappa, \\ -\sqrt{\lambda_0(\xi)} & \text{for} \quad |\xi| \leq \varkappa. \end{cases}$$

From the properties of $\lambda_1(\xi)$ which were collected in Section 3.3 we obtain $P \in \mathrm{C}^1(\Xi)$. Furthermore, P depends only on $|\xi|$. Using (3.12) and (5.11) we obtain

$$|\nabla P(\xi)| = q \quad \text{for} \quad |\xi| = \varkappa$$

and therefore $|\nabla P(\xi)| > 0$ for $\xi \in \Xi$ if we choose $\delta > 0$ sufficiently small. Now we are able to describe $A_0^{(4)}$ by a suitable direct integral. Let

$$M_\lambda = \{\xi \in \mathbb{R}^2 : P(\xi) = \lambda\} \quad \text{for} \quad \lambda \in \Theta := (-\sqrt{\delta}, \sqrt{\delta}).$$

In particular, $M_0 = \{\xi \in \mathbb{R}^2 : |\xi| = \varkappa\}$ is the set where $\lambda_1(\cdot)$ attains its minimum. We denote by $\mathrm{d}M_\lambda$ the measure on M_λ which is induced by $\mathrm{d}\xi$ on $\mathbb{R}^2$. Furthermore, we set $\mathrm{d}\mu_\lambda := |\nabla P|^{-1}\mathrm{d}M_\lambda$ and

$$G(\lambda) := L^2(M_\lambda, \mathrm{d}\mu_\lambda) \quad \text{for} \quad \lambda \in \Theta.$$

The measures $\mathrm{d}\xi$ and $\mathrm{d}\mu_\lambda \mathrm{d}\lambda$ coincide on Ξ. With these definitions we obtain

$$\lambda_0 = \int_\Theta^\oplus \lambda^2 \, \mathrm{d}\lambda \quad \text{on} \quad L^2(\Xi) = \int_\Theta^\oplus G(\lambda) \, \mathrm{d}\lambda$$

where λ^2 has to be interpreted as the operator of multiplication by λ^2 in $G(\lambda)$. The spaces $G(\lambda)$ are unitarily equivalent. We denote by $U_\lambda : G(\lambda) \to G(0)$ the corresponding unitary operator. It follows that the space $L^2(\Xi)$ is unitarily equivalent to

$$G_\Theta := L^2(\Theta) \otimes G(0)$$

where the corresponding unitary operator $\Upsilon : L^2(\Xi) \to G_\Theta$ is given by

$$(\Upsilon u)(\lambda) = U_\lambda(u|_{M_\lambda}) \quad \text{for} \quad u \in L^2(\Xi) \quad \text{and almost every} \quad \lambda \in \Theta.$$

Finally, we define the isometry

$$(5.12) \qquad F := \Upsilon \Pi_0$$

between $\Pi_c H_4$ and G_Θ which reduces the operator $A_0^{(4)} - \Lambda$ to its essential behaviour near the spectral minimum. Namely, if we define $J_\lambda u := (Fu)(\lambda)$ for $u \in \Pi_c H_4$ and almost every $\lambda \in \Theta$ it holds

$$(5.13) \qquad J_\lambda(A_0^{(4)} - \Lambda)u = \lambda^2 J_\lambda u.$$

5.4. **Reduction of $\mathcal{Y}(\tau)$ to the spectral minimum.** For $u \in H_4^\times$ it holds

$$J_\lambda \Pi_c U^* \sqrt{f} u = U_\lambda \left[\Pi_0 \Pi_c (Q(A_0^{(4)})^{-\frac{1}{2}})^* \sqrt{f} u \right]_{M_\lambda}$$

$$= U_\lambda \left[\langle \Phi \sqrt{f} u, \varphi_1 \rangle_{L^2(J;\mathbb{C}^{3 \times 3})} \right]_{M_\lambda}$$

where

$$
\begin{aligned}
\varphi_1(\xi, x_3) &= \frac{1}{\sqrt{\lambda_1(\xi)}} (\Phi Q \Phi^* \psi_1)(\xi, x_3) \\
&= \frac{1}{\sqrt{2\lambda_1(\xi)}} \left(\left(\begin{smallmatrix} i\xi \\ \partial_3 \end{smallmatrix} \right) \psi_1(\xi, x_3) + \left(\left(\begin{smallmatrix} i\xi \\ \partial_3 \end{smallmatrix} \right) \psi_1(\xi, x_3) \right)^T \right)
\end{aligned}
$$
(5.14)

for $\xi \in \Xi$ and $x_3 \in J$. Here $\lambda_1(\xi)$ denotes the lowest eigenvalue of $A^{(4)}(\xi)$ corresponding to the normed eigenfunction $\psi_1(\xi, \cdot)$ as stated in Section 3.3. Because of the compact support of f and the smoothness of φ_1 in ξ, the function $\langle \Phi \sqrt{f} u, \varphi_1 \rangle_{L^2(J;\mathbb{C}^{3 \times 3})}$ is also smooth in ξ for every $u \in H_4^\times$. Therefore, the operator $X : H_4^\times \to G(0)$ with

(5.15) $$Xu := \left[\langle \Phi \sqrt{f} u, \varphi_1 \rangle_{L^2(J;\mathbb{C}^{3 \times 3})} \right]_{M_0} \quad \text{for} \quad u \in H_4^\times$$

is well-defined. Moreover, there exists a constant $C > 0$ such that

(5.16) $$\| Xu - J_\lambda \Pi_c U^* \sqrt{f} u \|_{G(0)} \leq C\lambda \|u\|_\times \quad \text{for} \quad u \in H_4^\times, \ \lambda \in \Theta.$$

The essential spectral properties of the Birman-Schwinger operator (5.8) can be found in the operator

(5.17) $$\mathcal{L}(\tau) := F^* \mathcal{K}(\tau) \mathcal{K}^*(\tau) F \oplus \mathbb{O} \quad \text{on} \quad \Pi_c H_4 \oplus (I - \Pi_c) H_4,$$

where $\mathcal{K}(\tau)$ is given by

(5.18) $$\mathcal{K}(\tau) \ : \ H_4^\times \to G_\Theta \ : \ u \mapsto \eta_\tau \otimes Xu.$$

The function

(5.19) $$\eta_\tau(\lambda) = \left(\frac{\Lambda - \tau}{\lambda^2 + \tau} \right)^{\frac{1}{2}} \quad \text{for} \quad \lambda \in \Theta$$

corresponds to the outer terms of the simplified Birman-Schwinger operator (5.8). We will see that the spectrum of $\mathcal{L}(\tau)$ is basically determined by the spectrum of the operator

(5.20) $$K = XX^*$$

which is a compact integral operator in $L^2(M_0, \mathrm{d}\mu_0)$ with kernel

(5.21) $$k(\eta, \xi) = \frac{1}{(2\pi)^2} \int_G f(x) \, \mathrm{e}^{i(\eta - \xi) \cdot \left(\begin{smallmatrix} x_1 \\ x_2 \end{smallmatrix} \right)} \langle \varphi_1(\eta, x_3), \varphi_1(\xi, x_3) \rangle_{\mathbb{C}^{3 \times 3}} \, \mathrm{d}x, \quad \eta, \xi \in M_0.$$

6. STATEMENT OF THE MAIN RESULTS

We assume $f \not\equiv 0$ in L^2-sense. Let $(\varkappa_l(\alpha))_l$ be the eigenvalues of $A_\alpha^{(4)}$ below Λ in nondecreasing order including multiplicities. These eigenvalues are embedded eigenvalues for the complete operator A_α. Let furthermore $(\lambda_l(K))_l$ be the eigenvalues of K in non-increasing order including multiplicities.

Theorem 6.1. *For all $l \in \mathbb{N}$ it holds that*

(6.1) $$\varkappa_l(\alpha) = \Lambda - \alpha^2 (\Lambda \pi \lambda_l(K))^2 + o(\alpha^2) \quad \text{as} \quad \alpha \to 0.$$

For the calculation of the eigenvalues $\lambda_l(K)$ in the case of rotationally symmetric perturbations see Section 8. There we express $\lambda_l(K)$ in terms of a double series and provide estimates on $\lambda_l(K)$ for sufficiently big l.

For our second result we define the function

(6.2) $$w(t) := t^{-2t}.$$

Note that the inverse function w^{-1} exists if we consider $w : (t_0, \infty) \to (0, \tau_0)$ for sufficiently big $t_0 > 0$ which corresponds to small $\tau_0 > 0$. Now we can state the estimate on the counting function for eigenvalues of $A_\alpha^{(4)}$.

Theorem 6.2. *It holds that*

$$(6.3) \qquad \lim_{\tau \to 0} \frac{n_-(\Lambda - \tau, A_\alpha^{(4)})}{w^{-1}(\tau)} = 1.$$

From Theorem 6.2 we can derive the following two estimates on the eigenvalues of $A_\alpha^{(4)}$.

Corollary 6.3. *For every $\varepsilon > 0$ there exists $N_\varepsilon > 0$ such that*

$$(6.4) \qquad w((1 + \varepsilon)k) \leq \Lambda - \varkappa_k(\alpha) \leq w((1 - \varepsilon)k) \quad for \quad k > N_\varepsilon.$$

Corollary 6.4. *It holds that*

$$(6.5) \qquad \ln(\Lambda - \varkappa_k(\alpha)) = -2k \ln k + o(k \ln k) \quad as \quad k \to \infty.$$

Remark 6.5. It is interesting to note that the estimates in Theorem 6.2, Corollary 6.3 and 6.4 do not depend on the perturbation. This is a consequence of the super-polynomial type of eigenvalue accumulation, as can be seen in Section 8.3.

Remark 6.6. The result (6.5) is very similar to asymptotics obtained in [9, 10]. There it was shown that the negative eigenvalues Λ_n of the three-dimensional Pauli operator with constant magnetic field perturbed by a compactly supported potential fulfil

$$(6.6) \qquad \ln(-\Lambda_k) = -2k \ln k + O(k) \quad as \quad k \to \infty.$$

Although the Pauli operator is quite different to the elasticity operator, there is a common property which may explain the similarity of the accumulation rate asymptotics: in both cases an operator with strongly degenerated spectral edge is perturbed by a compactly supported perturbation.

We suppose that differences between the two problems become visible by a refined analysis of the asymptotic formulas. In [11] a detailed investigation of the Pauli operator was accomplished yielding also the second term of the asymptotic expansion for (6.6). It is an interesting open problem whether the methods from [11] can be used to obtain also a refined formula for the elastic problem.

7. Proofs of the main estimates

In the following we denote by $\lambda_l(\cdot)$ the eigenvalues of the corresponding non-negative compact operator in non-increasing order including multiplicities.

Proof of Theorem 6.1. From the definition of $\mathcal{L}(\tau)$ in (5.17) and from the minimax principle [14, 9.2.4] it follows that

$$(7.1) \qquad \lambda_l(\mathcal{L}(\tau)) = \|\eta_\tau\|_{L^2(\Theta)}^2 \lambda_l(K) = \frac{2(\Lambda - \tau)\arctan(\sqrt{\delta/\tau})}{\sqrt{\tau}} \lambda_l(K)$$

and therefore

$$\lambda_l(\sqrt{\tau}\mathcal{L}(\tau)) = \Lambda \pi \lambda_l(K) + o(1) \quad as \quad \tau \to 0, \quad l \in \mathbb{N}.$$

Lemma 9.2 which we will prove below gives us

$$\|\sqrt{\tau}\mathcal{Y}(\tau) - \sqrt{\tau}\mathcal{L}(\tau)\| \leq C(\sqrt{\tau} + \sqrt[4]{\tau}) \to 0 \quad as \quad \tau \to 0$$

which leads to

$$(7.2) \qquad \lambda_l(\sqrt{\tau}\mathcal{Y}(\tau)) \to \Lambda \pi \lambda_l(K) \quad as \quad \tau \to 0.$$

From the Birman-Schwinger principle (5.6) it follows that

$$\lambda_l(\mathcal{Y}_\alpha(\Lambda - \varkappa_l(\alpha))) = 1.$$

With (5.9) this yields

$$\lambda_l(\alpha\mathcal{Y}(\Lambda - \varkappa_l(\alpha))) \leq 1 \leq \tfrac{1}{1-\alpha}\lambda_l(\alpha\mathcal{Y}(\Lambda - \varkappa_l(\alpha)))$$

and therefore

(7.3) $$\lambda_l(\alpha \mathcal{Y}(\Lambda - \varkappa_l(\alpha))) \to 1 \quad \text{as} \quad \alpha \to 0.$$

Since $A_\alpha^{(4)} \geq (1 - \alpha)A_0^{(4)}$ it holds that $\inf \sigma(A_\alpha^{(4)}) \geq (1 - \alpha)\Lambda$. Hence,

$$\Lambda - \varkappa_l(\alpha) \to 0 \quad \text{as} \quad \alpha \to 0.$$

If we replace now τ by $\Lambda - \varkappa_l(\alpha)$ in (7.2) and compare it with (7.3), we obtain

$$\alpha^{-1}\sqrt{\Lambda - \varkappa_l(\alpha)} \to \Lambda \pi \lambda_l(K) \quad \text{as} \quad \alpha \to 0.$$

This concludes the proof. $\square$

Proof of Corollary 6.4. From Corollary 6.3 it follows that we can find $N \in \mathbb{N}$ and $\varepsilon_k > 0$ for $k \in \mathbb{N}$ with $\varepsilon_k \to 0$ as $k \to \infty$ such that

$$\ln w((1 + \varepsilon_k)k) \leq \ln(\Lambda - \varkappa_k(\alpha)) \leq \ln w((1 - \varepsilon_k)k) \quad \text{for} \quad k > N.$$

From the definition of w in (6.2) we obtain

$$\ln w((1 \pm \varepsilon_k)k) = -2k \ln k + o(k \ln k) \quad \text{as} \quad k \to \infty.$$

$\square$

Proof of Corollary 6.3. From Theorem 6.2 it follows that for every $\varepsilon > 0$ there exists an $N_\varepsilon \in \mathbb{N}$ such that

$$\frac{k}{w^{-1}(\Lambda - \varkappa_k(\alpha))} = \frac{n_-(\varkappa_k(\alpha) + 0, A_\alpha^{(4)})}{w^{-1}(\Lambda - \varkappa_k(\alpha))} \leq \frac{1}{1 - \varepsilon} \quad \text{for} \quad k > N_\varepsilon.$$

Therefore, the upper estimate in (6.4) follows. The lower estimate is obtained analogously. $\square$

For the proof of Theorem 6.2 we need a small auxiliary result which states a very special behaviour of the function $w(t) = t^{-2t}$.

Lemma 7.1. *For all $c > 0$ it holds that*

(7.4) $$\lim_{\tau \to 0} \frac{w^{-1}(c\tau)}{w^{-1}(\tau)} = 1.$$

Proof. By the mean value theorem we obtain that for sufficiently small $\tau > 0$ and certain $\xi \in (\min(\tau, c\tau), \max(\tau, c\tau))$

$$\left| \ln \frac{w^{-1}(c\tau)}{w^{-1}(\tau)} \right| = |\ln w^{-1}(c\tau) - \ln w^{-1}(\tau)| = \frac{|1 - c|\tau}{w^{-1}(\xi)|w'(w^{-1}(\xi))|}$$

$$\leq \frac{|1 - c^2|\xi}{c\, w^{-1}(\xi)|w'(w^{-1}(\xi))|}.$$

Because $x := w^{-1}(\xi) \to \infty$ as $\tau \to 0$ we obtain

$$0 \leq \lim_{\tau \to 0} \left| \ln \frac{w^{-1}(c\tau)}{w^{-1}(\tau)} \right| \leq \lim_{x \to \infty} \frac{|1 - c^2|}{c} \frac{w(x)}{x|w'(x)|} = 0.$$

This proves (7.4). $\square$

In the following we use the notations

(7.5) $$\varrho(\tau) := \frac{1}{w^{-1}(\tau)}, \quad \pi(\tau) := \frac{1}{v^{-1}(\tau)}, \quad w(t) = t^{-2t}, \quad v(t) = t^{-t}.$$

Note that $\pi(\tau) = \varrho(\tau^2)$.

Proof of Theorem 6.2. As we will prove in Section 8 below, the operator K defined in (5.20) fulfils

(7.6) $$\Delta_\pi(K) = 1, \qquad \delta_\pi(K) = 1.$$

The functionals Δ and δ were introduced in Section 5.1. From the definition of $\mathcal{L}(\tau)$ in (5.17) and from the minimax principle [14, 9.2.4] it follows that

$$\lambda_l(\mathcal{L}(\tau)) = \|\eta_\tau\|_{L^2(\Theta)}^2 \lambda_l(K) = \frac{2(\Lambda - \tau)\arctan(\sqrt{\delta/\tau})}{\sqrt{\tau}} \lambda_l(K).$$

Therefore, if we set $\zeta_\tau := 2(\Lambda - \tau)\arctan(\sqrt{\delta/\tau})$, we obtain for arbitrary $\beta > 0$

$$n(1, \beta\mathcal{L}(\tau)) = n((\beta\zeta_\tau)^{-1}\sqrt{\tau}, K).$$

Note that $\zeta_\tau \to \pi\Lambda$ as $\tau \to 0$. Using Lemma 7.1 together with (7.5) and (7.6) we obtain

$$\begin{aligned}
\Delta_\varrho(\beta\mathcal{L}) &= \limsup_{\tau \to 0} \varrho(\tau) n(1, \beta\mathcal{L}(\tau)) \\
&= \limsup_{\tau \to 0} \frac{\varrho(\tau)}{\varrho((\beta\zeta_\tau)^{-2}\tau)} \varrho([(\beta\zeta_\tau)^{-1}\sqrt{\tau}]^2) n((\beta\zeta_\tau)^{-1}\sqrt{\tau}, K) \\
&= \Delta_\pi(K) = 1.
\end{aligned}$$

In the same way we achieve the result for $\delta_\varrho(\beta\mathcal{L})$. From the definition of $\mathcal{L}$ in (5.17) it follows that $n(1, \beta\mathcal{L}(\tau)) = n(1, \sqrt{\beta}F^*\mathcal{K}(\tau))$ and therefore

(7.7) $$\Delta_\varrho(\beta F^*\mathcal{K}) = 1, \qquad \delta_\varrho(\beta F^*\mathcal{K}) = 1 \quad \text{for all} \quad \beta > 0.$$

Let us denote by

$$\mathcal{Z}(\tau) := \sqrt{f}U\left(\frac{\Lambda - \tau}{A_0^{(4)} - \Lambda + \tau}\right)^{\frac{1}{2}}$$

the right part of the Birman-Schwinger operator (5.8). In Lemma 9.1 below we will show that $\mathcal{Z}^*(\tau) - F^*\mathcal{K}(\tau)$ converges in $\mathfrak{S}_\infty$ to a compact operator as $\tau \to 0$. Therefore, we can apply Lemma 5.2 for this operator family and obtain

$$\beta(\mathcal{Z}^*(\tau) - F^*\mathcal{K}(\tau)) \in \mathcal{S}_0^\varrho \quad \text{for} \quad \beta > 0.$$

By (7.7) we have proven that $\beta F^*\mathcal{K} \in \mathcal{S}^\varrho$ and $\Delta_\varrho(\beta F^*\mathcal{K}) = 1$ for all $\beta > 0$. Hence, Lemma 5.1 is applicable which yields

(7.8) $$\Delta_\varrho(\beta\mathcal{Y}) = \Delta_\varrho(\sqrt{\beta}\mathcal{Z}^*) = \Delta_\varrho(\sqrt{\beta}F^*\mathcal{K}) = 1 \quad \text{for} \quad \beta > 0.$$

Using (5.9) we achieve

(7.9) $$\Delta_\varrho(\mathcal{Y}) = \Delta_\varrho(\alpha\mathcal{Y}) \le \Delta_\varrho(\mathcal{Y}_\alpha) \le \Delta_\varrho(\tfrac{\alpha}{1-\alpha}\mathcal{Y}) = \Delta_\varrho(\mathcal{Y}).$$

This and the same procedure for $\delta_\varrho(\mathcal{Y}_\alpha)$ yield

$$\Delta_\varrho(\mathcal{Y}_\alpha) = 1, \qquad \delta_\varrho(\mathcal{Y}_\alpha) = 1.$$

The proof is completed by applying the Birman-Schwinger principle (5.6). $\qquad\square$

8. TREATMENT OF THE OPERATOR K

8.1. **Calculation of the eigenvalues.** As mentioned in Section 5.4, the operator K is an integral operator in $L^2(M_0, \mathrm{d}\mu_0)$ with kernel

(8.1) $$k(\eta, \xi) = \frac{1}{(2\pi)^2} \int_G f(x)\, e^{i(\eta - \xi)\cdot\binom{x_1}{x_2}} \langle\varphi_1(\eta, x_3), \varphi_1(\xi, x_3)\rangle_{\mathbb{C}^{3\times 3}}\, \mathrm{d}x, \quad \eta, \xi \in M_0.$$

If we remember the definition of φ_1 in (5.14) and of ψ_1 in (3.15) and set

$$p := \varkappa^2\|d_1\|_J^2 + \|d_2\|_J^2, \quad p_0 = 2\|d_2'\|_J^2, \quad p_1 = \|d_1' + d_2\|_J^2, \quad p_2 = 2\|d_1\|_J^2,$$

where $\|\cdot\|_J$ is the norm in $L^2(J)$ and $d_{1/2}(t) := d_{1/2}(\varkappa, t)$, we obtain

(8.2) $$k(\eta, \xi) = \frac{p_2(\eta\cdot\xi)^2 + p_1(\eta\cdot\xi) + p_0}{\Lambda p(2\pi)^2} \int_{\mathbb{R}^2} f(x)e^{ix\cdot(\eta-\xi)}\, \mathrm{d}x.$$

From now on we restrict ourselves to the case of functions f which are rotationally symmetric. Setting $a := \sup\{|x| : x \in \operatorname{ess\,supp} f\}$ we obtain for the integral on the right hand side

$$\int_{\mathbb{R}^2} f(x) e^{ix \cdot (\eta - \xi)} \, dx = \int_0^a \int_0^{2\pi} f(r) r e^{ir|\eta - \xi| \cos \varphi} \, d\varphi \, dr$$

$$= 2\pi a^2 \sum_{k=0}^{\infty} \frac{(-1)^k}{(k!)^2} \left(\frac{a|\eta - \xi|}{2} \right)^{2k} \int_0^1 f(ar) r^{2k+1} \, dr.$$

With the substitutions

$$\eta = \varkappa \begin{pmatrix} \cos s \\ \sin s \end{pmatrix}, \quad \xi = \varkappa \begin{pmatrix} \cos t \\ \sin t \end{pmatrix}, \quad d\mu_0 = \varkappa |[\nabla P]_{M_0}|^{-1} \, dt = \frac{\varkappa}{q} \, dt$$

we can consider the operator K also as operator in $L^2((0, 2\pi))$. Using

$$\eta \cdot \xi = \varkappa^2 \cos(s - t), \quad |\eta - \xi|^2 = 2\varkappa^2 (1 - \cos(s - t))$$

and setting

$$\tilde{f}_k := \int_0^1 f(ar) r^{2k+1} \, dr,$$

the kernel of K can be rewritten as

$$(8.3) \qquad k(s, t) = \frac{a^2 \varkappa}{2\pi \Lambda pq} \sum_{m=0}^2 p_m (\varkappa^2 \cos(s - t))^m \sum_{k=0}^{\infty} \frac{(a^2 \varkappa^2 (\cos(s - t) - 1))^k}{2^k (k!)^2} \tilde{f}_k.$$

Obviously, $k(s, t)$ is a power series in $\cos(s - t)$. Therefore, $k(s, t)$ can also be written as

$$(8.4) \qquad k(s, t) = \sum_{l=-\infty}^{\infty} \mu_{|l|} \frac{e^{il(s-t)}}{2\pi}$$

which means that K has exactly the eigenfunctions $s \mapsto e^{\pm ins}$ corresponding to the eigenvalues μ_n for $n \in \mathbb{N}_0$. The eigenvalue μ_n is given by

$$\mu_n = \mu_n e^{in \cdot 0} = \int_0^{2\pi} k(0, t) e^{int} \, dt$$

$$(8.5) \qquad = \frac{a^2 \varkappa}{\Lambda pq} \sum_{m=0}^2 p_m \varkappa^{2m} \sum_{k=0}^{\infty} \frac{(a\varkappa)^{2k} \tilde{f}_k}{2^k (k!)^2} \cdot \frac{1}{2\pi} \int_0^{2\pi} (\cos(t))^m (\cos(t) - 1)^k e^{int} \, dt$$

$$= \frac{a^2 \varkappa}{\Lambda pq} \sum_{m=0}^2 p_m \varkappa^{2m} \sum_{k=|n-m|_+}^{\infty} \frac{(a\varkappa)^{2k} \tilde{f}_k}{2^k (k!)^2} \sum_{l=|\lceil \frac{m-n}{2} \rceil|_+}^{\lfloor \frac{k-n+m}{2} \rfloor} \frac{(-1)^{k+n+m}}{2^{2l+n}} \binom{k}{2l+n-m} \binom{2l+n}{l}.$$

Here $|j|_+ = \frac{1}{2}(|j| + j)$ for $j \in \mathbb{N}$.

8.2. **Estimates for the eigenvalues.** From a detailed analysis of the latter expression which will be given in Appendix A it follows that we can find $c > 0$ and $N \in \mathbb{N}$ such that

$$(8.6) \qquad c^{-1} (2n)^3 f_{2n} \left(\frac{a\varkappa e}{2n} \right)^{2n} < \mu_n < c(2n)^3 f_{2n} \left(\frac{a\varkappa e}{2n} \right)^{2n} \quad \text{for all} \quad n > N$$

where

$$(8.7) \qquad f_k := \int_0^1 f(ar) \, r^{k-3} \, dr.$$

From (8.4) we see that the eigenspaces of the eigenvalues μ_n have dimension two for $n \geq 1$ and dimension one for $n = 0$. If we denote by $\lambda_n(K)$ for $n \in \mathbb{N}$ the eigenvalues of K in non-increasing order including multiplicities, we obtain for a certain constant $c > 0$ and all sufficiently big $n \in 2\mathbb{N}$

$$(8.8) \qquad c^{-1} n^3 f_n \left(\frac{a\varkappa e}{n} \right)^n < \lambda_n(K) = \lambda_{n+1}(K) < cn^3 f_n \left(\frac{a\varkappa e}{n} \right)^n.$$

8.3. **Estimates for the number of eigenvalues.** Note that every non-trivial function $f \in L^\infty(\mathbb{R}^2; [0,1])$ with compact support can be estimated from above and from below in L^2-sense by some characteristic function of a ball in $\mathbb{R}^2$ multiplied by a constant $C \in (0,1]$.

If we replace f by its lower and upper estimate in the definition of K, we obtain a lower and an upper estimate for the operator K. Since the spectrum of K is invariant under translations of f, we can actually restrict ourselves to the case of balls with centre at zero, i.e.

$$f = C\chi_{\{x \in \mathbb{R}^2 : |x| < a\}}$$

for $C \in (0,1]$ and $a > 0$. In this case, $f_n = C(n-2)^{-1}$ for $n > 2$, and the estimate (8.8) can be simplified to

$$(8.9) \qquad \left(\frac{1}{cn}\right)^n < \lambda_n(K) < \left(\frac{c}{n}\right)^n, \quad n \in \mathbb{N}$$

for a certain constant $c > 0$. Obviously, (8.9) holds true for K with arbitrary perturbing function f, if c is choosen sufficiently large depending on f.

Let $v(t) := t^{-t}$ as in (7.5) and $v_h(t) := (ht)^{-t}$ for $h > 0$. The inverse functions of v and v_h exist, if we consider these functions for sufficiently large t only. We need the following limit result.

Lemma 8.1. *For every $h > 0$ it holds that*

$$(8.10) \qquad \lim_{\tau \to 0} \frac{v^{-1}(\tau)}{v_h^{-1}(\tau)} = 1.$$

Proof. Since $v(t) = \exp(-t \ln t)$ and $v_h(t) = \exp(-t \ln ht)$ it holds that

$$v^{-1}(\tau) = \tilde{v}^{-1}(-\ln \tau), \quad v_h^{-1}(\tau) = \tilde{v}_h^{-1}(-\ln \tau), \qquad \tilde{v}(t) = t \ln t, \quad \tilde{v}_h(t) = t \ln ht.$$

Therefore, (8.10) is proven by showing that

$$(8.11) \qquad \lim_{\tau \to \infty} \frac{\tilde{v}^{-1}(\tau)}{\tilde{v}_h^{-1}(\tau)} = 1.$$

Assume $0 < h \leq 1$. In this case $\tilde{v}_h(t) \leq \tilde{v}(t)$ and $\tilde{v}^{-1}(\tau) \leq \tilde{v}_h^{-1}(\tau)$ for all sufficiently large $t, \tau > 0$. Furthermore, for any $\varepsilon \in (0,1)$ we find $t_\varepsilon > 0$ such that

$$\tilde{v}((1-\varepsilon)t) \leq (1-\varepsilon)\tilde{v}(t) \leq \tilde{v}_h(t)$$

for all $t > t_\varepsilon$. Therefore, $\tilde{v}_h^{-1}(\tau) \leq (1-\varepsilon)^{-1}\tilde{v}^{-1}(\tau)$ for all sufficiently large τ and hence

$$1 - \varepsilon \leq \liminf_{\tau \to \infty} \frac{\tilde{v}^{-1}(\tau)}{\tilde{v}_h^{-1}(\tau)} \leq \limsup_{\tau \to \infty} \frac{\tilde{v}^{-1}(\tau)}{\tilde{v}_h^{-1}(\tau)} \leq 1$$

for all $\varepsilon \in (0,1)$. This proves (8.11). The proof for the case $h > 1$ is obtained analogously. $\square$

For $\tau \in [\lambda_{k+1}(K), \lambda_k(K))$, k sufficiently large, we obtain from (8.9) that

$$\frac{n(\tau, K)}{v_{\frac{1}{c}}^{-1}(\tau)} \leq \frac{k}{v_{\frac{1}{c}}^{-1}(\lambda_k(K))} \leq 1$$

and therefore, using (8.10) and the definitions from (7.5),

$$(8.12) \qquad \Delta_\pi(K) = \limsup_{\tau \to 0} \pi(\tau) n(\tau, K) = \limsup_{\tau \to 0} \frac{v^{-1}(\tau)}{v_{\frac{1}{c}}^{-1}(\tau)} \frac{n(\tau, K)}{v^{-1}(\tau)} \leq 1.$$

On the other hand, for $\tau \in [\lambda_{k+1}(K), \lambda_k(K))$, k sufficiently large, we have

$$\frac{n(\tau, K)}{v_{\tilde{c}}^{-1}(\tau)} \geq \frac{k}{v_{\tilde{c}}^{-1}(\lambda_{k+1}(K))} \geq 1$$

with some constant $\tilde{c} > c$ independent of τ and k, which is again a consequence of (8.9). Therefore,

$$\delta_\pi(K) = \liminf_{\tau \to 0} \pi(\tau) n(\tau, K) = \liminf_{\tau \to 0} \frac{v^{-1}(\tau)}{v_{\tilde{c}}^{-1}(\tau)} \frac{n(\tau, K)}{v^{-1}(\tau)} \geq 1.$$

Hence, we obtain $1 \leq \delta_\pi(K) \leq \Delta_\pi(K) \leq 1$ which proves (7.6).

9. Estimates for the remainder terms

Lemma 9.1. *The operator $\mathcal{Z}^*(\tau) - F^*\mathcal{K}(\tau)$ converges in $\mathfrak{S}_\infty$ as $\tau \to 0$.*

Proof. Let us note that $\mathcal{Z}^*(\tau)$ and $F^*\mathcal{K}(\tau)$ are compact operators for all $\tau > 0$ since $\mathcal{Y}(\tau)$ and $\mathcal{L}(\tau)$ are compact. We can split $\mathcal{Z}(\tau)$ into

$$\mathcal{Z}(\tau) = \mathcal{Z}(\tau)\Pi_c + \mathcal{Z}(\tau)(I - \Pi_c).$$

From the boundedness of $(A_0^{(4)} - \Lambda)^{-\frac{1}{2}}(I - \Pi_c)$ it follows immediately that

$$\mathcal{Z}(\tau)(I - \Pi_c) = \sqrt{f}\, U \left(\frac{\Lambda - \tau}{A_0^{(4)} - \Lambda + \tau} \right)^{\frac{1}{2}} (I - \Pi_c)$$

converges in $\mathfrak{S}_\infty$ as $\tau \to 0$. The proof is accomplished if we can show that also

$$\mathcal{Q}(\tau) := F\Pi_c \mathcal{Z}^*(\tau) - \mathcal{K}(\tau)$$

converges in $\mathfrak{S}_\infty$ as $\tau \to 0$. This is done by a detailed analysis of the spectral structure developed in Sections 5.3 and 5.4. We recall that due to (5.13) and the definition of η_τ in (5.19) it holds

$$\Pi_c \mathcal{Z}^*(\tau) = \Pi_c \left(\frac{\Lambda - \tau}{A_0^{(4)} - \Lambda + \tau} \right)^{\frac{1}{2}} U^* \sqrt{f} = F^* \eta_\tau F\Pi_c U^* \sqrt{f}.$$

Therefore,

$$\mathcal{Q}(\tau)u = \eta_\tau F\Pi_c U^* \sqrt{f}\, u - \eta_\tau \otimes Xu = \eta_\tau Ru, \quad \text{for} \quad u \in H_4^\times,$$

where the operator $R : H_4^\times \to G_\Theta$ is given by

$$R = F\Pi_c U^* \sqrt{f} - I \otimes X.$$

For $\tau_1 > \tau_2 > 0$ and $u \in H_4^\times$ we obtain that

$$\|(\mathcal{Q}(\tau_1) - \mathcal{Q}(\tau_2))u\|_{G_\Theta}^2 = \int_\Theta \|(\eta_{\tau_1}(\lambda) - \eta_{\tau_2}(\lambda))(Ru)(\lambda)\|_{G(0)}^2 \, \mathrm{d}\lambda$$

$$= \int_\Theta |\eta_{\tau_1}(\lambda) - \eta_{\tau_2}(\lambda)|^2 \, \|J_\lambda \Pi_c U^* \sqrt{f}\, u - Xu\|_{G(0)}^2 \, \mathrm{d}\lambda$$

$$\leq C\|u\|_\times^2 \int_\Theta \lambda^2 |\eta_{\tau_1}(\lambda) - \eta_{\tau_2}(\lambda)|^2 \, \mathrm{d}\lambda$$

for a certain $C > 0$. This is a direct consequence of (5.16). The integral on the right hand side can be estimated by $C_1\sqrt{\tau}$ for a certain constant $C_1 > 0$. Therefore, we achieve

$$\|(\mathcal{Q}(\tau_1) - \mathcal{Q}(\tau_2))u\|_{G_\Theta}^2 \leq CC_1\sqrt{\tau}\|u\|_\times^2$$

which proves the convergence of $\mathcal{Q}(\tau)$ as $\tau \to 0$. $\qquad\square$

Lemma 9.2. *There exists a constant $C > 0$ such that for all sufficiently small $\tau > 0$*

$$(9.1) \qquad\qquad \|\mathcal{Y}(\tau) - \mathcal{L}(\tau)\| \leq C(1 + 1/\sqrt[4]{\tau}).$$

Proof. On $(I - \Pi_c)H_4$ the operator $\mathcal{Y}(\tau)$ is bounded as $\tau \to 0$ whereas the operator $\mathcal{L}(\tau)$ is identically zero. On the other hand, for $u \in \Pi_c H_4$ it holds

$$|\langle(\mathcal{Y}(\tau) - \mathcal{L}(\tau))u, u\rangle| = |\langle \mathcal{Z}(\tau)u, \mathcal{Z}(\tau)u\rangle - \langle \mathcal{K}^*(\tau)Fu, \mathcal{K}^*(\tau)Fu\rangle|$$

$$(9.2) \qquad \leq \|(\mathcal{Z}(\tau) - \mathcal{K}^*(\tau)F)u\|^2 + 2|\mathrm{Re}\langle(\mathcal{Z}(\tau) - \mathcal{K}^*(\tau)F)u, \mathcal{K}^*(\tau)Fu\rangle|$$

$$\leq \|(\mathcal{Z}(\tau) - \mathcal{K}^*(\tau)F)u\|^2 + 2\|(\mathcal{Z}(\tau) - \mathcal{K}^*(\tau)F)u\|\, \|\mathcal{K}^*(\tau)Fu\|.$$

From Lemma 9.1 it follows that $\mathcal{Z}(\tau) - \mathcal{K}^*(\tau)F$ is bounded as $\tau \to 0$, and from (7.1) in the proof of Theorem 6.1 it follows that

$$\|\mathcal{K}^*(\tau)F\| = \sqrt{\lambda_1(\mathcal{L}(\tau))} \leq \frac{c}{\sqrt[4]{\tau}}$$

for a certain $c > 0$. Inserting this into (9.2) we obtain (9.1). $\qquad\square$

APPENDIX A

We want to derive estimate (8.6). Setting $C_1 := a^2 \varkappa (\Lambda pq)^{-1}$ we obtain from (8.5) for $n \geq 2$ that

$$(A.1) \qquad \mu_n = C_1 \sum_{m=0}^{2} p_m \varkappa^{2m} \sum_{k=0}^{\infty} \frac{(-1)^k (a\varkappa)^{2(n+k-m)} \tilde{f}_{n+k-m}}{2^{n+k-m}((n+k-m)!)^2} \sum_{l=0}^{\lfloor \frac{k}{2} \rfloor} \frac{1}{2^{n+2l}} \binom{k+n-m}{2l+n-m} \binom{2l+n}{l}.$$

With the estimate

$$(A.2) \qquad \sum_{l=0}^{\lfloor \frac{k}{2} \rfloor} \frac{1}{2^{n+2l}} \binom{k+n-m}{2l+n-m} \binom{2l+n}{l} \leq \frac{(k+n-m)! \, (k+n)(k+n-1)}{2^n \, n!} \sum_{l=0}^{\lfloor \frac{k}{2} \rfloor} \frac{1}{4^l l!}$$

$$\leq \frac{(k+n-m)! \, n^2 (k+1)^2}{2^n \, n!} e^{\frac{1}{4}}$$

and with the definition $C_2 := \sum_{m=0}^{2} p_m 2^m a^{-2m}$ we obtain

$$\mu_n \leq C_1 C_2 e^{\frac{1}{4}} \left(\frac{a\varkappa}{2} \right)^{2n} \frac{n^2 \tilde{f}_{n-2}}{n!} \sum_{k=0}^{\infty} \frac{(a\varkappa)^{2k}(k+1)^2}{2^k(n+k-2)!}.$$

Using $(n+k-2)! \geq (n-2)! \, k!$ and Stirling's formula

$$\sqrt{2\pi n} \left(\frac{n}{e} \right)^n \leq n! \leq e^{\frac{1}{12n}} \sqrt{2\pi n} \left(\frac{n}{e} \right)^n$$

we achieve

$$\mu_n \leq C_1 C_2 \, 4 e^{\frac{1}{4}+(a\varkappa)^2} \left(\frac{a\varkappa}{2} \right)^{2n} \frac{n^4 \tilde{f}_{n-2}}{(n!)^2}$$

$$\leq C_1 C_2 \frac{e^{\frac{1}{4}+(a\varkappa)^2}}{4\pi} (2n)^3 \tilde{f}_{n-2} \left(\frac{a\varkappa e}{2n} \right)^{2n}.$$

This proves the upper estimate in (8.6). In order to prove the lower estimate, we start again with formula (A.1) and estimate it from below. If we define $C_3 := p_2 \varkappa^4 (a\varkappa)^{-4}$ and use (A.2), we obtain

$$\mu_n \geq C_1 C_3 \left(\frac{a\varkappa}{2} \right)^{2n} \frac{n^4 \tilde{f}_{n-2}}{(n!)^2} - C_1 C_2 e^{\frac{1}{4}} \left(\frac{a\varkappa}{2} \right)^{2n} \frac{n^2 \tilde{f}_{n-2}}{n!} \sum_{k=1}^{\infty} \frac{(a\varkappa)^{2k}(k+1)^2}{2^k(n+k-2)!}.$$

Estimating the sum in the second term and using Stirling's formula, we achieve

$$\mu_n \geq \left(C_1 C_3 - \frac{1}{n} C_1 C_2 \, 4(a\varkappa)^2 e^{\frac{1}{4}+(a\varkappa)^2} \right) \frac{n^4 \tilde{f}_{n-2}}{(n!)^2} \left(\frac{a\varkappa}{2} \right)^{2n}$$

$$\geq \frac{C_1 C_3 - \frac{1}{n} C_1 C_2 \, 4(a\varkappa)^2 e^{\frac{1}{4}+(a\varkappa)^2}}{16\pi e^{\frac{1}{6n}}} (2n)^3 \tilde{f}_{n-2} \left(\frac{a\varkappa e}{2n} \right)^{2n}.$$

This proves the lower bound in (8.6) for sufficiently large $n \in \mathbb{N}$.

REFERENCES

[1] D. V. Evans, M. Levitin, D. Vassiliev: Existence theorems for trapped modes. *J. Fluid Mech.* **261** 21–31 (1994)

[2] W. Bulla, F. Gesztesy, W. Renger and B. Simon: Weakly coupled bound states in quantum waveguides. *Proc. Amer. Math. Soc.* **125** 1487–1495 (1997)

[3] I. Roitberg, D. Vassiliev and T. Weidl: Edge resonance in an elastic semi-strip. *Quart. J. Mech. Appl. Math.* **51** 1–14 (1998)

[4] D. Gridin, R. V. Craster and A. T. I. Adamou: Trapped modes in curved elastic plates. *Proc. R. Soc. A* **461** 1181–1197 (2005)

[5] T. Ekholm, H. Kovařík and D. Krejčiřík: A Hardy inequality in twisted waveguides. *Preprint: arxiv: math-ph/0512050* (2005)

[6] V. Zernov, A. V. Pichugin and J. Kaplunov: Eigenvalue of a semi-infinite elastic plate. *Proc. R. Soc. A* **462** 1255–1270 (2006)

[7] C. Förster and T. Weidl: Trapped modes for an elastic strip with perturbation of the material properties. *Q. Jl. Mech. Appl. Math.* **59** 399–418 (2005)

[8] A. Laptev, O. Safronov, and T. Weidl: Bound states asymptotics for elliptic operators with strongly degenerating symbols. *Nonlinear problems in mathematical physics and related topics, I, In Honor of Professor O. A. Ladyzhenskaya.*, Int. Math. Ser. 1, Kluwer/Plenum, New York, 233–246, (2002)

[9] M. Melgaard and G. Rozenblum: Eigenvalue asymptotics for weakly perturbed Dirac and Schrödinger operators with constant magnetic fields of full rank. *Comm. Part. Diff. Eq.* **28** 697–736 (2003)

[10] G. D. Raikov and S. Warzel: Quasi-classical versus non-classical spectral asymptotics for magnetic Schrödinger operators with decreasing electric potentials. *Rev. Math. Phys.* **14** 1051–1072 (2002)

[11] N. Filonov and A. Pushnitski: Spectral asymptotics of Pauli operators and orthogonal polynomials in complex domains. *Comm. Math. Phys.* (to appear)

[12] J. Gobert: Une inégalité fondamentale de la théorie de l'élasticité. (French) *Bull. Soc. Roy. Sci. Liége* **31** 182–191 (1962)

[13] M. S. Birman: Perturbations of the continuous spectrum of a singular elliptic operator by varying the boundary and the boundary conditions. (Russian) *Vestnik Leningrad. Univ.* **17** 22–55 (1962)

[14] M. S. Birman and M. Z. Solomjak: *Spectral Theory of Self-Adjoint Operators in Hilbert Space.* D. Reidel Publishing Company, 1987 (original: Leningrad University Press, 1980)

[15] O. Safronov: Discrete spectrum in a gap of continuous spectrum for variable sign perturbations with large coupling constant. *St. Petersburg Math. J.* **8** 307–331 (1997)

[16] I. C. Gohberg and M. G. Kreĭn: *Introduction to the Theory of Linear Nonselfadjoint Operators.* American Mathematical Society, Providence, 1969 (translation)

[17] T. Weidl: General operator ideals of weak type. *St. Petersburg Math. J.* **4** 503–525 (1993)

[18] M. S. Birman: On the spectrum of singular boundary-value problems. (Russian) *Mat. Sb. (N.S.)* **55** (1961) 125–174, translated in *Amer. Math. Soc. Trans. (2)* **53** 23–80 (1966)

[19] Y. Schwinger: On the bound states of a given potential. *Proc. Nat. Acad. Sci. USA* **47** 122–129 (1961)

[20] R. A. Adams and J. J. F. Fournier: *Sobolev Spaces.* Elsevier, 2003

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