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A contribution to geometric inequalities in Euclidean space forms

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Abstract

The aim of this article are Fenchel-Fáry-type inequalities, isoperimetric inequalities of Banchoff-Pohl-type for closed curves in euclidean space forms and Fenchel-Fáry-type inequalities for closed surfaces in euclidean spaces.

Keywords: Fenchel-Fáry inequality, isoperimetric inequality, euclidean space form, total absolute curvature, curves, surfaces.

MS Classification: 53C40, 53C65, 53A04, 53A05

1 Introduction

Differential geometric quantities of geometric objects, e.g. length, area, total absolut curvature, or topological quantities, e.g. betti numbers, are often related through geometric inequalities.
Here, we derive Fenchel-Fáry-type inequalities, isoperimetric inequalities of Banchoff-Pohl-type for closed curves in euclidean space forms, and Fenchel-Fáry-type inequalities for closed surfaces in euclidean spaces.
For geometric objects in euclidean spaces we use applications of classical techniques, i.e. Morse theory of linear level functions in connection with total absolute curvature, and integral geometric techniques in connection with Crofton formula and isoperimetric inequalities.
For geometric objects in euclidean space forms we use lifting through the universal riemannian covering into euclidean spaces. For euclidean space forms cf. [20].

Let c be a closed smooth curve in $n-$dimensional euclidean space E^n. Then there are the following inequalities for the total absolute curvature of c,

$$\operatorname{tac}(c) := \frac{1}{\pi} \int_c \mid \kappa \mid \; ds \; . \tag{1}$$

The Fenchel inequality

$$\operatorname{tac}(c) \geq 2 \; , \tag{2}$$

with equality holding if and only if c is a simple plane convex curve (cf. [10], [11], [3]).

The Fáry inequality, if c lies inside a ball $B(o,R) \subset E^n$ with center o and radius R,

$$\operatorname{tac}(c) \geq \frac{1}{\pi R} \cdot L(c) \; , \tag{3}$$

where $L(c)$ denotes the length of c. Equality holds if and only if c is a plane circle, traversed a number of times (cf. [5], [6], [9]).

And further the Banchoff-Pohl isoperimetric inequality

$$L(c)^2 \geq \frac{32\pi^3}{O_n \cdot O_{n-1}} \int\limits_{G^{n-2,n}} w_c(\xi)^2 \, d\xi \ , \tag{4}$$

where $w_c(\xi)$ denotes the winding number of c w.r.t. the $(n-2)-$dimensional plane $\xi \subset E^n$. Equality holds if and only if c is a circle, traversed in the same direction a number of times (cf. [2]). ($G^{n-2,n}$ = Grassmann manifold of all $(n-2)-$dimensional planes $\xi \subset E^n$; $d\xi$ = invariant volume density in $G^{n-2,n}$ (cf. [16]); O_n = surface area of the $n-$dimensional unit sphere $S^n \subset \mathbf{R}^{n+1}$.)

Remark: Concerning the total absolute curvature of surfaces in euclidean space forms, cf. [14].

2 Fenchel-Fáry inequalities in euclidean spaces

Proposition 2.1 *Let c be a closed smooth regular curve in E^n lying inside a ball $B(o,R)$ with center o and radius R. Then*

$$tac(c) \geq 2 + \frac{O_n}{2\pi R O_{n-1}} \cdot L(c) - \frac{2n}{R O_{n-1}} \cdot W_{n-1}(conv(c)) \ , \tag{5}$$

where $W_{n-1}(conv(c))$ denotes the $(n-1)-th$ quermassintegral of the convex hull $conv(c)$ of c (notation as in [16]). Equality holds if and only if c is a simple plane convex curve in case $n \geq 2$ or a circle with radius R, traversed a number of times, in case $n = 2$.

Proof: We consider the oriented hyperplane $\xi(u,p)$ in E^n with unit normal vector $u \in S^{n-1} = T^1_o E^n$ at distance p from $o \in E^n$. We take the orthogonal projection h_u of E^n onto the line $o + \mathbf{R} \cdot u$ (linear level function w.r.t. u) and its restriction $h_{u|c}$ along the curve c.
Through Morse theory each subarc of c induced by the intersection $\xi \cap c$ causes at least one critical point of $h_{u|c}$. Therefore, a.e. in $S^{n-1} \times \mathbf{R}$,

$$\nu(h_{u|c}) \geq 2 + n(\xi(u,p) \cap c) - 2 \cdot \chi(\xi(u,p) \cap \mathrm{conv}(c)) \ , \tag{6}$$

$\nu(h_{u|c})$ = number of critical points of $h_{u|c}$; $n(\xi(u,p) \cap c)$ = number of intersection points $\xi(u,p) \cap c$; $\chi(\xi(u,p) \cap \mathrm{conv}(c))$ = Euler characteristic of $\xi(u,p) \cap \mathrm{conv}(c)$. Note: $n(\xi \cap c) \neq 0$ transverse intersection points cause at least n critical points of $h_{u|c}$ and $\chi(\xi \cap \mathrm{conv}(c)) = 1$, $n(\xi \cap c) = 0$ implies $\chi(\xi \cap \mathrm{conv}(c)) = 0$ respectively.
Now, we integrate (6) over all oriented hyperplanes $\xi(u,p)$ intersecting $B(o,R)$, i.e. $u \in S^{n-1}$, $-R \leq p \leq R$, w.r.t. the invariant density $d\xi = du\, dp$ (cf. [16]). And we take into account the relation between total absolute curvature and Morse theory,

$$\mathrm{tac}(c) = \frac{1}{O_{n-1}} \int\limits_{S^{n-1}} \nu(h_{u|c}) \, du \tag{7}$$

(cf. [7], [12]),

the Crofton formula (cf. [16] (14.70))

$$\int_{\xi\cap B(o,R)\neq\emptyset} n(\xi\cap c)\, d\xi = \frac{O_n}{\pi}\cdot L(c)\ , \tag{8}$$

and the definition of quermassintegrals (cf. [16] (14.1))

$$\int_{\xi\cap B(o,R)\neq\emptyset} \chi(\xi\cap \mathrm{conv}(c))\, d\xi = \int_{\xi\cap \mathrm{conv}(c)\neq\emptyset} d\xi = 2n\cdot W_{n-1}(\mathrm{conv}(c)) \tag{9}$$

(Remark: Up to constant factors $W_{n-1}(\mathrm{conv}(c))$ equals the mean breadth of $\mathrm{conv}(c)$, or the integral of the $(n-2)$th mean curvature of $\partial\,\mathrm{conv}(c)$ respectively; especially for $n=2$: $2\,W_1(\mathrm{conv}(c)) = L(\partial\,\mathrm{conv}(c))$).

This yields (5).

To the equality case: Equality in (5) implies equality in (6).

This implies $c\subset \partial\,\mathrm{conv}(c)$. Because otherwise there exists $h_{u|c}$ with a non-degenerate critical point in the interior of $\mathrm{conv}(c)$. Take the corresponding critical level plane $\xi(u,p)$, fix u and vary p. This causes in (6) a jump exactly at the term $n(\xi\cap c)$ and hence a contradiction to equality in (6).

Moreover c is a plane curve. Because otherwise there exists $h_{u|c}$ with at least 3 non-degenerate critical points. Take the middle one of the corresponding critical level planes (w.l.o.g. these planes are pairwise distinct planes), fix u and vary p. As above, this causes a jump exactly at the term $n(\xi\cap c)$ in (6) and hence a contradiction.

Finally, if the convex plane curve c is a circle of radius R in case $n=2$ then $\chi(\xi\cap\mathrm{conv}(c)) = 1$ for all $\xi\cap B(o,R)\neq\emptyset$. Hence equality in (6) gives $\nu(h_{u|c}) = n(\xi\cap c)$ and c may be traversed a number of times. In all other cases there exists some ξ with $\xi\cap B(o,R)\neq\emptyset$ such that $\xi\cap c=\emptyset$ and $\xi\cap\mathrm{conv}(c)=\emptyset$. Hence equality in (6) gives $\nu(h_{u|c}) = 2$ and c is traversed once.

Remarks:

1) (5) implies the Fenchel inequality (2) (note: $n(\xi\cap c) - 2\cdot\chi(\xi\cap\mathrm{conv}(c))\geq 0$ in (6)).

(5) implies the sharp Fáry inequality (3) (cf. [5]) in case $n=2$ resp. a weaker version (cf. [9]) in case $n\geq 3$ (note: $2-2\cdot\chi(\xi\cap\mathrm{conv}(c))\geq 0$ in (6)).

2) The ball $B(o,R)$ may be the minimal ball circumscribed c. Then by the Jung inequality (cf. e.g. [4] 11.1)

$$R\leq \frac{n}{n+1}\cdot d\ , \tag{10}$$

$d=$ diameter of $c\subset E^n$, and we may rewrite (5) in terms of d.

3) Through the Favard inequalities relating the quermassintegrals W_{n-1}, W_{n-2} and W_{n-3} (cf. [4] 20.2), especially in dimension $n=3$,

$$W_2\leq \frac{1}{12\pi}\cdot\frac{\mathrm{area}(\partial\,\mathrm{conv}(c))^2}{\mathrm{vol}(\mathrm{conv}(c))}\ , \tag{11}$$

we may rewrite (5) in terms of surface area and volume of $\mathrm{conv}(c)$.

For knots $c\subset E^3$ there are improvements of the Fenchel inequality, e.g.

$$\mathrm{tac}(c)\geq 2\cdot b(c)\ , \tag{12}$$

where $b(c)$ denotes the bridge index of c (cf. e.g. [13]).

Proposition 2.2 *Let c be a smooth knot in E^3 lying inside a ball $B(o,R)$ with radius R. Then*

$$tac(c) \geq 2b(c) + \frac{1}{4R}L(c) - 2b(c)\frac{3}{4\pi R}W_2(conv(c)) + \frac{1}{8\pi R}\sum_{k=1}^{b(c)-1}(2b(c)-2k)\, m_{2k}(c)\ , \qquad (13)$$

where $m_{2k}(c)$ denotes the total measure in $G^{2,3}$ of those oriented planes ξ having exactly $2k$ intersection points $\xi \cap c$.

Proof: We start at

$$\begin{aligned}\nu(h_{u|c}) &\geq 2b(c) + \max(n(\xi(u,p)\cap c) - 2b(c), 0) = \\ &= 2b(c) + n(\xi \cap c) - 2b(c)\cdot\chi(\xi\cap \mathrm{conv}(c)) + \\ &\quad + \sum_{k=1}^{b(c)-1}(2b(c)-2k)\cdot \mathbf{1}_{2k} \end{aligned} \qquad (14)$$

($\mathbf{1}_{2k}$ = characteristic function of the subset in $G^{2,3}$ of those oriented planes ξ which have exactly $2k$ intersection points $\xi \cap c$).
Then, as in the proof of Proposition 2.1., we integrate over all oriented planes ξ intersecting $B(o,R)$ in order to get (13).

Remark: Formulas writing $m_{2k}(c)$ in terms of the curve c are given in [18] (for plane curves, cf. [1], [17]).

Let us now consider a closed smooth regular oriented surface $M \subset E^3$.
The Chern-Lashof inequality generalizes the Fenchel inequality, i.e.

$$\mathrm{tac}(M) \geq \beta(M) \geq 2\ , \qquad (15)$$

where $\mathrm{tac}(M) = \frac{1}{2\pi}\int_M \mid K(x) \mid\, dx$ denotes the total absolute Gauss curvature of M and $\beta(M)$ is the sum of the betti numbers of M, cf. [7], [12]. Equality holds if and only if M lies tight in E^3. Moreover, if M lies inside a ball $B(o,R)$ with radius R, then the Fáry inequalities (cf. [9], [15]) say

$$\mathrm{tac}(M) \geq \frac{2\pi}{R^{2-i}}\cdot \sigma_i(M) \quad (i=0,1)\ , \qquad (16)$$

$\sigma_i(M)$ = integral of the i th mean curvature of M (normalisations as in [16]). Equality holds if an only if M is a sphere of radius R.

Proposition 2.3 *Let M be a closed smooth regular surface embedded in E^3 bounding a compact domain D. Suppose that M lies inside a ball $B(o,R)$ with center o and radius R. Then*

$$tac(M) \geq 2 + \frac{1}{2\pi R}\mid \sigma_1(M) \mid - \frac{3}{2\pi R}\cdot W_2(conv(M))\,. \qquad (17)$$

Equality holds if and only if M is convex.

Proof: Through Morse theory each component of $\xi(u,p)\cap M$ causes at least 2 critical points of $h_{u|M}$. Therefore a.e.

$$\nu(h_{u|M}) \geq 2 + 2\cdot n_1(\xi(u,p)\cap M) - 2\cdot\chi(\xi(u,p)\cap \mathrm{conv}(M))\ , \qquad (18)$$

$n_1(\xi(u,p) \cap M) =$ number of components of the intersection $\xi(u,p) \cap M$.
Now, we integrate over all oriented planes $\xi(u,p)$ intersecting $B(o,R)$ w.r.t. the invariant density $d\xi = du\,dp$. We take into account

$$n_1(\xi \cap M) \geq | \chi(\xi \cap D) | \,, \tag{19}$$

and we use the relation between total absolute curvature and Morse theory (cf. [12])

$$\mathrm{tac}(M) = \frac{1}{O_{n-1}} \int\limits_{S^{n-1}} \nu(h_{u|M})\, du \,, \tag{20}$$

the kinematic formula (cf. [16] (14.79))

$$\int\limits_{\xi \cap B(o,R) \neq \emptyset} \chi(\xi \cap D)\, d\xi = 2 \cdot \sigma_1(M) \,, \tag{21}$$

and the definition of quermassintegrals (cf. [16]), in order to get (17).

To the equality case: Equality in (17) implies equality in (18). As in the proof of Proposition 2.1 this implies $M \subset \partial\,\mathrm{conv}(M)$. M is closed and regular, hence M is of sphere-type and $M = \partial\,\mathrm{conv}(M)$.

Remark: (17) implies the Fenchel inequality (note: $| \chi(\xi \cap D) | - \chi(\xi \cap \mathrm{conv}(M)) \geq 0$ in (18)). (17) implies the Fáry inequality for $i = 1$ (note: $1 - \chi(\xi \cap \mathrm{conv}(M)) \geq 0$ in (18)).

Proposition 2.4 *Let M be a closed smooth regular surface embedded in E^3 bounding a compact domain D. Suppose that M lies inside a ball $B(o,R)$ with center o and radius R. Then*

$$\begin{aligned} tac(M) \;\geq\; & 2\beta(M) + \frac{1}{2\pi R}|\sigma_1(M)| - 2\beta(M)\frac{3}{4\pi R} W_2(conv(M)) + \\ & + \frac{1}{8\pi R} \sum_{k=1}^{\beta(M)/2} (\beta(M) - 2k)\, m_k(M) \,, \end{aligned} \tag{22}$$

where $m_k(M)$ is the total measure in $G^{2,3}$ of those oriented planes ξ having exactly k intersection components $\xi \cap M$, $1 \leq k \leq \beta(M)/2$.

Proof: Starting at

$$\begin{aligned} \nu(h_{u|M}) \;\geq\; & \beta(M) + \max(2 \cdot n_1(\xi \cap M) - \beta(M), 0) \\ =\; & \beta(M) + 2n_1(\xi \cap M) - \beta(M)\,\chi(\xi \cap \mathrm{conv}(M)) + \\ & + \sum_{k=1}^{\beta(M)/2} (\beta(M) - 2k)\, \mathbf{1_k} \end{aligned} \tag{23}$$

($\mathbf{1_k}$ = characteristic function of the subset in $G^{2,3}$ of those oriented planes ξ having exactly k intersection components $\xi \cap M$), and proceeding as in Proposition 2.3, we get (22).

Remarks:

1) (22) implies the Chern-Lashof inequality
(note: $2n_1(\xi \cap M) + \sum_{k=1}^{\beta(M)/2} (\beta(M) - 2k)\, 1_k \geq \beta(M)\chi(\xi \cap \mathrm{conv}(M))$ in (23)).

2) Formulas writing $m_k(M)$ in terms of the surface M are given in [18].

3 Fenchel-Fáry inequalities in euclidean space forms

Proposition 3.1 *Let c be a closed smooth regular curve in a n-dimensional euclidean space form X. Let $x \in c$. Then*

$$tac(c) \geq \frac{\sigma_{x,a}+\sigma_{x,b}}{\pi} + \frac{n+1}{d(c)\, n\, O_{n-1}} \left(\frac{O_n}{2\pi} \left(L(c)+l(c,x)\right) - 2n\, W_{n-1}(conv(\tilde{c}_x)) \right) , \tag{24}$$

where $l(c,x)$ denotes the minimal length of loops through x homotopic to c, $d(c)$ denotes the diameter of c in X, see (25), $\tilde{c}_x$ is a universal lift of c_x (c_x = curve c cut up at x), and $\sigma_{x,a}$, $\sigma_{x,b}$ are the angles at x between the oriented subarc c_x and the oriented chord s_{xx} at start and finish (s_{xx} = geodesic segment from x to x homotopic to c_x) ($0 \leq \sigma_x \leq \pi$).
Equality holds if and only if c is a closed geodesic or a closed curve with 2 nd Frenet curvature (torsion) $\kappa_2 = 0$, 1 st Frenet curvature (curvature) $\kappa_1 \geq 0$ (w.r.t. suitable orientation) and rotation index $turn(c) = (2\pi)^{-1}(\sigma_{x,a}+\sigma_{x,b})$.

Proof: Let c be oriented. The subarc c_x is c cut up at x. We consider a lift $\tilde{c}_x$ of c_x w.r.t. the universal riemannian covering $\pi : E^n \to X$. $a, b \in E^n$ are start and finish of $\tilde{c}_x$. The lift $\tilde{s}_{xx}$ of the chord s_{xx} is the line segment in E^n between a and b, and $L(\tilde{s}_{xx}) = l(c,x)$.
Now we take the closed curve $\tilde{c}_x \cup \tilde{s}_{xx}$ in E^n with vertices at a resp. b and angles $\sigma_{x,a}$ resp. $\sigma_{x,b}$ there. We apply Proposition 2.1. We take into account $\mathrm{tac}(\tilde{c}_x) = \mathrm{tac}(c)$, $\mathrm{tac}(\tilde{s}_{xx}) = 0$, and that the vertices a and b produce total absolute curvature contribution $\frac{1}{\pi}(\pi - \sigma_{x,a} + \pi - \sigma_{x,b})$.
And we use the Jung inequality (10) to estimate the circumscribed radius of $\mathrm{conv}(\tilde{c}_x) \subset E^n$ through the diameter $d(c)$ of c in X, where

$$d(c) := \max_{x \in c} \left(\max_{y \in c_x} L(s_{xy}) \right) , \tag{25}$$

s_{xy} = chord from x to y = geodesic segment from x to y homotopic to the subarc $c_{|[x,y]}$.
This proves (24).

To the equality case:
Equality in (24) implies $\tilde{c}_x \cup \tilde{s}_{xx}$ simple plane convex. (Note: We used (5) and the Jung inequality (10), hence eventual circles in the equality case are traversed only once.) Therefore either $\tilde{c}_x = \tilde{s}_{xx}$ and hence c is a closed geodesic, or $\tilde{c}_x$ with $\kappa_2 = 0$, $\kappa_1 \geq 0$ and $\mathrm{turn}(\tilde{c}_x) = (2\pi)^{-1}(\sigma_{x,a}+\sigma_{x,b})$.
(Remark: The rotation index of a curve c with $\kappa_2 = 0$ is defined as the total variation of the angle between c and a fixed direction in the osculating tangent planes (fixed direction means parallel along c), normalized by $(2\pi)^{-1}$

Remarks:

1) For c nullhomotopic $\tilde{c}_x$ is closed, and we have $\sigma_{x,a} = \sigma_{x,b} = \pi$ and $l(c,x) = 0$.

2) If the chord s_{xx} is a closed geodesic then $\sigma_{x,a} = \sigma_{x,b}$.

3) Let X be a homogeneous euclidean space form, i.e. $X = T^{n-i} \times \mathbf{R}^i$, $0 \leq i \leq n$, T^{n-i} = $(n-i)$-dimensional flat torus (cf. [20] 2.7.1). Then all deck transformations are translations. And therefore $l(c,x) = l(c)$, s_{xx} is a closed geodesic, and $\sigma_{x,a} = \sigma_{x,b}$. In case of equality this implies $\sigma_x = 0$ (c closed geodesic) or $\sigma_x = \pi$ (c closed curve with $\kappa_2 = 0$, $\kappa_1 \geq 0$ and $\mathrm{turn}(c) = 1$).

4) If $x \in c$ produces the maximal right-hand side appearing in (24), then start or finish of $\tilde{c}_x$ lies on the boundary of $\mathrm{conv}(\tilde{c}_x)$.

(Because otherwise variation of x along c leaves $\mathrm{conv}(\tilde{c}_x)$ unchanged but in general changes $\sigma_{x,a}$ and $\sigma_{x,b}$ proportional to the curvature κ of c at x. And therefore, variation of x along c in the suitable direction in general increases the right-hand side in (24).)

5) The result in [8] applied to the arc $\tilde{c}_x$ in E^2 yields

$$\mathrm{tac}(c) \geq \frac{4}{\pi} \arccos \frac{l(c)}{L(c)} \ . \tag{26}$$

Example: Let us give a more detailed description in 2-dimensional euclidean space forms.

The case X orientable, i.e. $X =$ flat plane, cylinder or torus:
X is homogeneous, hence see remark 3.

The case X non-orientable, i.e. $X =$ flat Moebius band or Klein bottle:
If c is orientation-preserving, then s_{xx} is a closed geodesic and $\sigma_{x,a} = \sigma_{x,b}$.
If c is orientation-reversing, then in general s_{xx} is a broken geodesic and $\sigma_{x,b} = \sigma_{x,a} - \delta(c)$.
Considering the lifts of c_x, s_{xx} and of the generators of the 1 st fundamental group $\pi_1(X)$ of X, considering the deck transformations in E^2, and using some trigonometry, one can compute $\delta(c)$.
$X =$ flat Moebius band: $c \simeq \gamma^\alpha$
($\gamma =$ minimal closed geodesic which generates $\pi_1(X)$; α a suitable integer). Then

$$\delta(c) = \arccos \frac{\alpha\, L(\gamma)}{l(c,x)} \tag{27}$$

in case $\alpha \equiv 1 \bmod(2)$.
In case $\alpha \equiv 0 \bmod(2)$, c is orientation-preserving and $\delta(c) = 0$.
In both cases

$$l(c,x) = \sqrt{(\alpha\, L(\gamma))^2 + (2\, \bar{\alpha}\, d(x,\gamma))^2} \tag{28}$$

($d(x,\gamma) =$ distance beween x and γ; $\bar{\alpha} = \alpha \bmod(2)$).
$X =$ flat Klein bottle: $c \simeq \gamma_1^{\alpha_1}\, \gamma_2^{\alpha_2}$
(γ_1, $\gamma_2 =$ minimal closed geodesics which generate $\pi_1(X)$ under the relation $\gamma_2^{-1}\, \gamma_1\, \gamma_2 \gamma_1 = 1$; γ_1, γ_2 cut each other orthogonally; α_1, α_2 suitable integers). Then

$$\delta(c) = \arccos \frac{\alpha_2\, L(\gamma_2)}{l(c,x)} \tag{29}$$

in case $\alpha_2 \equiv 1 \bmod(2)$.
In case $\alpha_2 \equiv 0 \bmod(2)$, c is orientation-preserving and hence $\delta(c) = 0$.
In both cases

$$l(c,x) = \sqrt{(\alpha_1\, L(\gamma_1) + 2\, \bar{\alpha_2}\, d(x,\gamma_2))^2 + (\alpha_2\, L(\gamma_2))^2} \ . \tag{30}$$

Remark on non-homogeneous euclidean space forms X with $\dim(X) \geq 3$: In general the relation between $\sigma_{x,a}$ and $\sigma_{x,b}$ depends not only on the homotopy class of c or c_x respectivly, but also on the position of the tangent vector of c at x w.r.t. the action of the deck transformation from a to b.

4 Banchoff-Pohl inequalities in 2-dimensional euclidean space forms

Proposition 4.1 *Let c be a closed smooth regular curve in a 2-dimensional orientable euclidean space form X (i.e. X = flat plane, cylinder or torus). Then*

$$L(c)^2 \geq l(c)^2 + 2 \int_{T^1 X} W_c(p,u)\, dp\, du\ , \tag{31}$$

where $l(c)$ denotes the minimal length of closed curves in the homotopy class of c, and $W_c(p,u)$ denotes the generalized winding number of c w.r.t. $(p,u) \in T^1X$, see (34).
For c nullhomotopic equality holds if and only if c is a circle, i.e. c has constant curvature.
For c not nullhomotopic equality holds if and only if c is a closed geodesic.

Proof: We follow the line in [19].
The secant space S_c parametrizes the oriented secants of c together with their limiting positions. Let c be oriented and given by an immersion $f : S^1 \to X$; w.l.o.g. c has only transversal self-intersections. The interior of S_c is just $(S^1 \times S^1)\setminus(D \cup D_3)$ with $D := \{(x,y) \in S^1 \times S^1 \mid x = y\}$ and $D_3 := \{(x,y) \in S^1 \times S^1 \mid x \neq y\ ,\ f(x) = f(y)\ ,\ \text{and}\ c_{|[x,y]}\ \text{nullhomotopic in}\ X\}$. For $(x,y) \in \text{int}S_c$ the corresponding oriented secant is determined by the geodesic segment s_{xy} from $f(x)$ to $f(y)$ (chord) homotopic to the subarc $c_{|[x,y]}$. Equivalently, if we take the universal riemannian covering $\pi : E^2 \to X$, this means that the lifts of the chord s_{xy} from $f(x)$ to $f(y)$ or of the subarc $c_{|[x,y]}$ respectivly have the same final points if they start together. The boundary of S_c: The diagonal D of $S^1 \times S^1$ is replaced by two copies of D as boundary components, i.e. by the set D_1 of tangent geodesics of c at $f(x)$ oriented through the orientation of c, and by the set D_2 of oriented geodesics induced by the geodesic segments form $f(x)$ to $f(x)$ homotopic to c_x = curve c cut up at x. The finite set D_3 at each of its points is replaced by $T^1_{(x,y)}(S^1 \times S^1)$ parametrizing the oriented geodesics in X through $f(x)$.
For $(x,y) \in S_c$ let $r(x,y)$ be the length of the corresponding chord from $f(x)$ to $f(y)$. Let $\Phi : S_c \times [0, r(x,y)] \to T^1X$ be the smooth map with $\Phi(x,y,\rho) := (p,u)$, where u is the unit tangent vector of the oriented chord from $f(x)$ to $f(y)$ at the point p at distance ρ along the chord from $f(x)$.
Then (cf. [19] proof of Lemma 1)

$$\Phi^*(dp\, du) = \pi_1^*(\omega_2 \wedge \omega_{12}) \wedge d\rho \tag{32}$$

w.r.t. the orthonormal 2-frame $f(x)\, e_1\, e_2$ defined on S_c with e_1 = unit tangent vector of the chord from $f(x)$ to $f(y)$ (ω_i , ω_{ij} = associated Maurer-Cartan forms; π_1 = projection of $S_c \times [0,r]$ onto S_c).
Application of the coarea formula to Φ yields

$$\int_{S_c \times [0,r]} \Phi^*(dp\, du) = -\int_{T^1X} W_c(p,u)\, dp\, du \tag{33}$$

with the degree of Φ at $(p,u) \in T^1X$

$$-W_c(p,u) := \sum_{\substack{x\in S^1\\ f(x)\in g^-}} i_x \left(\sum_{\substack{y\in S^1\,,\, f(y)\in g^+\\ c_{|[x,y]}\simeq g_{|[f(x),f(y)]}}} i_y \right) , \tag{34}$$

where (p,u) defines the oriented geodesic g through p in direction u and p induces the geodesic subrays g^+, g^- of g emanating from p (g^+ this one from p in direction u; g^+ has the same, g^- the opposite orientation as g), and i_x and i_y are the algebraic intersection numbers of c with g at $f(x)$ and $f(y)$. (Note: The sum in (34) is finite because c is compact.)
Now we consider (w.l.o.g. c arc-length parametrized)

$$\int_{S_c} (dx \wedge dy + 2r\,\omega_2 \wedge \omega_{12}) + \int_{\partial S_c} r\omega_1 \,. \tag{35}$$

Using Stokes in S_c, taking into account the structure equations for the frame field, we rewrite (35)

$$\int_{S_c} (dx \wedge dy + dr \wedge \omega_1 + r\,\omega_2 \wedge \omega_{12}) \,. \tag{36}$$

(36) is always grater or equal to zero (cf. [19] proof of theorem, [2]), and equality holds if an only if $\sigma_x = -\sigma_y$ or $\sigma_x = \sigma_y = \pi$ or $\sigma_x = \sigma_y = -\pi$ (σ_x and σ_y = signed angles between the chord from $f(x)$ to $f(y)$ and the curve c at $f(x)$ and $f(y)$).
To the second integral in (35): $r_{|D_1} = 0$, $r_{|D_3} = 0$, $r_{|D_2} = \text{const.} = l(c)$ and therefore $\int_{\partial S_c} r\,\omega_1 = l(c) \int_{D_2} \omega_1 = -l(c)^2$ (note: the deck transformations are translations, hence the lift $\tilde{c}_x$ of c_x has parallel tangent vectors at start and finish; note: e_1 parallel along D_2).
This proves (31).

To the equality cases:
For c nullhomotopic, the lift $\tilde{c}$ of c is a closed curve in E^2. In case of equality we have $\sigma_x = -\sigma_y$ along S_c. This implies that $\tilde{c}$ is a circle in E^2. Hence c has constant curvature.
For c not nullhomotopic, the lifts $\tilde{c}_x$ are not closed. In case of equality we have $\sigma_x = -\sigma_y$ along S_c. Moreover we have $\sigma_x = \sigma_y$ along D_2 (note: $\tilde{c}_x$ has parallel tangent vectors at start and finish). Therefore $\sigma_x = \sigma_y = 0$ along D_2. This implies that $\tilde{c}_x$ is a line segment in E^2. Hence c is a closed geodesic in X.

Remarks:

1) For c nullhomotopic, the generalized winding number W_c of c is related to the classical winding number $w_{\tilde{c}}$ of the lift $\tilde{c}$ in E^2, namely

$$W_c(p,u) = \sum_{x\in\pi^{-1}(p)} w_{\tilde{c}}(x)^2 \,. \tag{37}$$

2) To closed curves c in non-orientable 2-dimensional euclidean space forms X (i.e. X = flat Moebius band or Klein bottle):
For c orientation-preserving, the proof runs as above and we get (31).
For c orientation-reversing, in general we have $r_{|D_2} \neq$ constant, and the boundary term raises

difficulties. But if we traverse c twice we get a new curve which is orientation-preserving, and we can proceed as above.

3) For simple closed curves c bounding a compact domain $D \subset X$ there are isoperimetric inequalities on surfaces, e.g.

$$L(c)^2 \geq 4\pi\, \chi(D)\, \mathrm{area}(D) \tag{38}$$

(cf. [4] 2.2 (8)).
For c simple nullhomotopic (31) and (38) coincide. In general for simple curves, (31) looks sharper than (38).

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