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Abstract

In this paper we consider embedded eigenvalues of a Schrödinger Hamiltonian in a waveguide induced by a symmetric perturbation. It is shown that these eigenvalues become unstable and turn into resonances after twisting of the waveguide. The perturbative expansion of the resonance width is calculated for weakly twisted waveguides and the influence of the twist on resonances in a concrete model is discussed in detail.

1 Introduction

Quantum waveguides have been studied ever since the pioneering works by Lewin (see [20] and also [16, 21, 23]) appeared, even if only recently the problem of quantum transmission in waveguides has been considered. In this framework the spectral analysis of differential operators in tubular domains has become a research field of certain interest (see, e.g., [4, 5, 14, 15]). Moreover, with the introduction of nano-devices such as nanotubes, new open problems in quantum transmission for such structures appeared [6].

We consider a waveguide type domain $\Omega = \mathbb{R} \times \omega$ (see Figure 1, on the left), where the cross section ω of the waveguide is a bounded subset of $\mathbb{R}^2$. We impose Dirichlet boundary conditions at the boundary of Ω. The spectrum of the free operator $-\Delta$ on Ω is absolutely continuous and covers the half-line $[E_1, \infty)$, where E_1 is the lowest eigenvalue of the Dirichlet Laplacian on ω. It is a well known fact that the threshold of such spectrum is unstable against perturbations; indeed, a negative perturbation of $-\Delta$ will induce at least one bound state below E_1. The perturbation can be either of a potential type or of a geometric type, see [2, 3, 8, 14, 15] and references there. These new bound states correspond to the particles (electrons) which do not propagate along Ω, but remain localized in a bounded region of Ω.

Recently it has been shown,[13, 12], that the presence of bound states in Ω can be, up to certain extent, suppressed by another geometrical perturbation: the so called *twisting* (see Figure 1, on the right), see Section 2 for details. More exactly, the result of [13] shows that if the cross section ω is not rotationally symmetric and the tube Ω is twisted, even only locally, then the bound states for the perturbed Hamiltonian $-\Delta + V$ do not appear for any negative potential $V(x)$, but only if V is strong enough. In other words, one could say that a twisting of a tubular domain Ω stabilizes the transport of charged particles in Ω in the sense that it protects the particles to get trapped by weak perturbations. Similar result was obtained for two-dimensional waveguide with combined boundary conditions, where the twist is represented by the change of the boundary conditions at one point, [18].

However, the geometrical perturbations of the waveguide generically induce also the existence of *resonances*, i.e. metastable states with very long lifetimes, see [9, 10, 11]. These states correspond to the particles which remain localized in a bounded region for a very long time before they finally propagate to infinity.

It is the aim of the present paper to describe the influence of twisting on the resonances in the waveguides. More precisely, we start form the situation, in which the free Laplacian is perturbed by an attractive potential V, which decays at infinity along the waveguide direction. The point spectrum of the perturbed Hamiltonian $-\Delta + V(x)$, where $x \in \mathbb{R}$ represents the coordinate along the waveguide direction, consists, in addition to the bound states below E_1, of infinitely many eigenvalues embedded in the continuum $[E_1, \infty)$ (see Figure 2). It was shown in [9], for two-dimensional waveguides, that these embedded eigenvalues generically turn into resonances in the presence of a constant magnetic field. Following the method of [9] we show that this happens also when the magnetic field is replaced by the twisting, provided the cross section ω is not rotationally symmetric, see Theorem 1. For weak twisting we also give the perturbative expansion of the corresponding resonance width.

In order to obtain a precise estimate of the imaginary part of the resonances and, in particular, to prove that it is strictly negative we consider in Section 5 a concrete model in which the potential

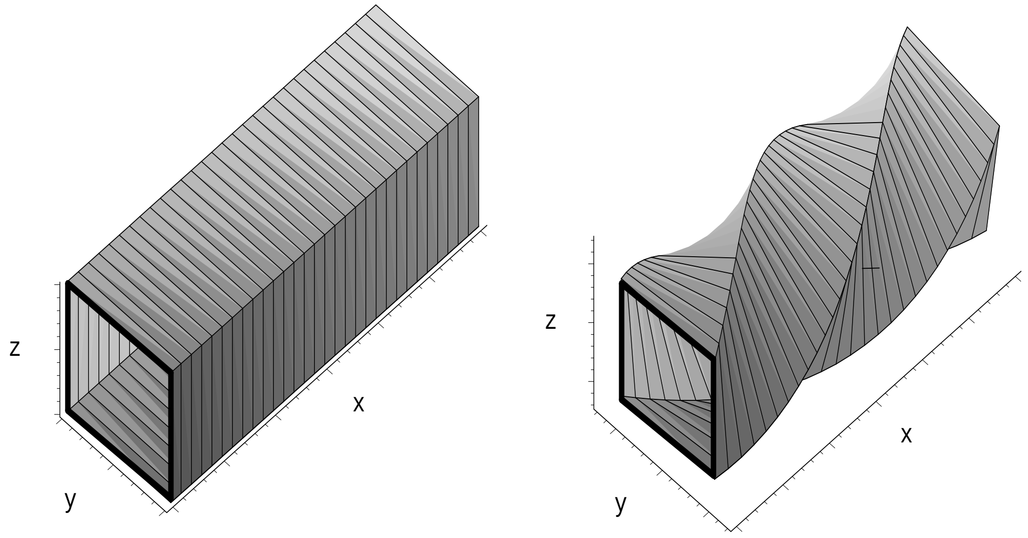

Figure 1: On the left, the plot of the surface of a rectangular waveguide without twisting; in the right, plot of the surface of the twisted rectangular waveguide. Bold line represents the boundary of ω.

V approximates a one-dimensional point interaction and an additional perturbation W is introduced. For such model we calculate the leading term of the imaginary part of a chosen resonance explicitly, see Proposition 1. In particular, we show that the twisting can also decrease the lifetime (i.e. increase the imaginary part) of an already existing resonance (induced by the perturbation W), provided W satisfies certain symmetry condition. We may thus say that twisting not only suppresses the creation of bound states, but also improves the particle transport in waveguides in the sense that it does not allow the existence of embedded eigenvalues or (in certain situations) it even might shorten the lifetimes of already existing resonances.

2 Preliminaries

Let ω be an open bounded and connected set in $\mathbb{R}^2$ and let α be a differentiable function from $\mathbb{R}$ to $\mathbb{R}$. For a given $x \in \mathbb{R}$ and $s := (y, z) \in \omega$ we define the mapping $f_\epsilon : \mathbb{R} \times \omega \to \mathbb{R}^3$ by

$$f_\varepsilon(x, s) = (x,\, y\cos(\varepsilon\,\alpha(x)) + z\sin(\varepsilon\,\alpha(x)),\, z\cos(\varepsilon\,\alpha(x)) - y\sin(\varepsilon\,\alpha(x))), \tag{1}$$

where $\varepsilon > 0$ is a real parameter. Furthermore, we introduce

$$\Omega_0 = \mathbb{R} \times \omega \quad \text{and} \quad \Omega_\varepsilon := f_\varepsilon(\Omega_0).$$

Clearly, Ω_ε is a tube which is twisted unless the function α is constant (e.g., in Figure 1 we plot, respectively, a rectangular tube without and with twisting).

For a real-valued measurable bounded function $V(x)$ on $\mathbb{R}$ we formally define the Hamiltonians

$$H_\varepsilon^0 = -\Delta \quad \text{and} \quad H_\varepsilon^V = -\Delta + V(x) \;\text{ in }\; L^2(\Omega_\varepsilon)$$

with Dirichlet boundary conditions at $\partial\Omega_\varepsilon$. The operator H_ε^V is associated with the closed quadratic form

$$Q_\varepsilon^V[\psi] := \int_{\Omega_\varepsilon} \left[|\nabla\psi|^2 + V(x)\,|\psi|^2\right] \mathrm{d}x\,\mathrm{d}s\,, \tag{2}$$

with the form domain $D(Q_\varepsilon^V) = \mathcal{H}_0^1(\Omega_\varepsilon)$. Given a test function $\psi \in C_0^\infty(\mathbb{R}\times\omega)$ it is useful to introduce the following shorthand,

$$\partial_\tau \psi := y\partial_z\psi - z\partial_y\psi. \tag{3}$$

As usual in such situations, in order to analyze the operator H_ε^V we pass from the twisted tube Ω_ε to the untwisted tube Ω_0 by means of a simple substitution of variables. This gives

$$Q_\varepsilon^V[\psi] = \int_{\Omega_0} \left(|\nabla_s\psi|^2 + |\partial_x\psi + \epsilon\dot\alpha(x)\partial_\tau\psi|^2 + V(x)\,|\psi|^2\right) \mathrm{d}x\,\mathrm{d}s\,,$$

where

$$\nabla_s\psi := (\partial_y\psi, \partial_z\psi)\,.$$

In other words, the operator H_ε^V acts on its domain in $L^2(\Omega_0)$ as

$$H_\varepsilon^V = -\partial_y^2 - \partial_z^2 - [\partial_x + \varepsilon\dot\alpha(x)\,\partial_\tau]^2 + V(x) = H_0^V + U_\epsilon^V\,,$$

where

$$H_0^V = -\partial_x^2 - \partial_y^2 - \partial_z^2 + V(x)$$

and

$$\begin{aligned} U_\varepsilon^V &= -[\partial_x + \varepsilon\dot\alpha(x)\,\partial_\tau]^2 + \partial_x^2 \\ &= -2\varepsilon\,\dot\alpha(x)\partial_x\,\partial_\tau - \varepsilon\,\ddot\alpha(x)\,\partial_\tau - \varepsilon^2\dot\alpha^2(x)\,\partial_\tau^2\,. \end{aligned}$$

Remark 1. The term U_ε^V is a symmetric operator on $L^2(\Omega_0)$ with Dirichlet boundary conditions on $\partial\Omega_0$.

In order to show that the embedded eigenvalues of H_0^V turn into the resonances once the waveguide is twisted, we employ the method of the exterior complex scaling in combination with the regular perturbation theory. We start by locating the spectrum of the untwisted model.

3 Spectrum of H_0^V

We will suppose that V satisfies the following

Assumption A. The function $V(x)$ is not identically equal to zero and

$$\int_{\mathbb{R}} (1+x^2)\,|V(x)|\,\mathrm{d}x < \infty \quad \text{and} \quad \int_{\mathbb{R}} V(x)\,\mathrm{d}x \le 0\,. \tag{4}$$

It then follows from [22] (see, e.g. Theorem XIII.110 in and its Notes) that the operator

$$h := -\partial_x^2 + V(x) \quad \text{in} \quad L^2(\mathbb{R})$$

possesses finitely many negative eigenvalues $\{\mu_j\}_{j=1}^N$, $N \ge 1$, each of multiplicity one. We denote by $\varphi_j(x)$ the corresponding normalized eigenfunctions. The essential spectrum of h covers the positive half-line $[0,\infty)$. On the other hand, it is well known that the operator $-\Delta_D^\omega$, i.e. the Dirichlet Laplacian on ω, is positive definite and has purely discrete spectrum. Let $\{E_n\}_{n=1}^\infty$ be the non-decreasing sequence of its eigenvalues and let $\chi_n(s)$ denote the associated normalized eigenfunctions. The set of such eigenfunctions is an orthonormal basis of $L^2(\omega)$ with Dirichlet boundary conditions on $\partial\omega$. We denote by

$$\Sigma = \{E = \mu_j + E_n,\ j = 1,\dots,N,\ n \ge 1\}$$

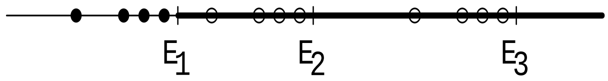

Figure 2: The discrete spectrum of H_0^V consists of finitely many simple eigenvalue below E_1 (denoted by full circle); the essential spectrum is given by the half-line $[E_1, +\infty)$. Furthermore, a non empty set of simple eigenvalues (denoted by empty circle) embedded in the essential spectrum occurs.

the set of eigenvalues of H_0^V with associated normalized eigenvectors

$$\psi_{n,j}(x,s) = \varphi_j(x)\chi_n(s),$$

and

$$\Sigma_+ = \Sigma \cap [E_1, +\infty),\ \Sigma_- = \Sigma \cap (-\infty, E_1)$$

where Σ_- is not empty since $\mu_j < 0$ for any j. Then, by the standard arguments, [22], the spectrum of

$$H_0^V = -\Delta + V(x)\,, \quad \text{in } L^2(\mathbb{R} \times \omega)$$

is given by $\sigma(H_0^V) = \sigma_d(H_0^V) \cup \sigma_{ess}(H_0^V)$, where

$$\sigma_d(H_0^V) = \Sigma_- \ \text{ and } \ \sigma_{ess}(H_0^V) = [E_1, \infty).$$

In addition, H_0^V possesses point spectrum embedded into the continuum given by Σ_+ (see Figure 2) .

We expect that when ε becomes non-zero then these embedded eigenvalues generically turn into resonances, which are the main object of our study.

Remark 2. Since the operator H_0^V commutes with complex conjugation then its eigenfunctions ψ can be assumed to be real-valued.

4 Complex scaling

Henceforth, we would like to employ the method of exterior complex scaling to the operator H_ε^V. In order to do so, we will need some assumptions on the functions V and α:

Assumption B. V extends to analytic function with respect to x in some sector

$$M_\beta := \{\zeta \in \mathbb{C} : |\arg \zeta| \le \beta\}, \quad \text{with} \quad \beta > 0.$$

Moreover, V is uniformly bounded in M_β.

Assumption C. α extends to analytic function with respect to x in

$$\mathcal{M}_\beta = M_\beta \cup \{\zeta \in \mathbb{C} \ : \ |\Im \zeta| \le \beta\}\,, \quad \text{with} \quad \beta > 0.$$

and $\dot\alpha$ is uniformly bounded in $\mathcal{M}_\beta$. In addition $\dot\alpha(x) > 0,\ \forall x \in \mathbb{R}$.

Remark 3. Since $\dot\alpha$ is uniformly bounded in $\mathcal{M}_\beta$ then from the Cauchy theorem it follows that $\ddot\alpha$ is uniformly bounded in $\mathcal{M}_{\beta'}$ for any $0 < \beta' < \beta$.

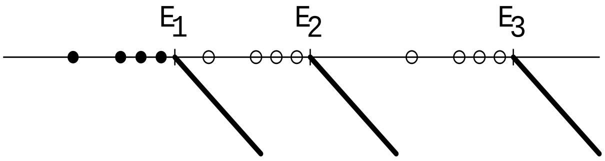

Figure 3: The discrete spectrum of $H_0^V(\theta)$ consists of a sequence of real and simple eigenvalues (denoted by circle); the essential spectrum is given by the half-lines $E_n + e^{-2i\Im\theta}\mathbb{R}^+$.

In analogy with [9] we introduce the mapping S_θ, which acts as a complex dilation in the longitudinal variable x:

$$(S_\theta\psi)(x,s) = e^{\theta/2}\psi(e^\theta x, s)\,, \quad \theta \in \mathbb{C}\,.$$

The transformed operator then takes the form

$$H_\varepsilon^V(\theta) = S_\theta H_\varepsilon^V S_\theta^{-1} = H_0^V(\theta) + U_\varepsilon^V(\theta)\,,$$

where

$$H_0^V(\theta) = S_\theta H_0^V S_\theta^{-1} = -e^{-2\theta}\,\partial_x^2 - \partial_y^2 - \partial_z^2 + V(e^\theta x)$$

and

$$U_\varepsilon^V(\theta) = S_\theta U_\varepsilon^V S_\theta^{-1} = -2\varepsilon\, e^{-\theta}\,\dot\alpha(e^\theta x)\partial_x\,\partial_\tau - \varepsilon\, e^{-\theta}\,\ddot\alpha(e^\theta x)\,\partial_\tau - \varepsilon^2\dot\alpha^2(e^\theta x)\,\partial_\tau^2\,. \tag{5}$$

Lemma 1. *Let V satisfy assumptions A and B, then $H_0^V(\theta)$ is an analytic family of type* A *with respect to θ. Furthermore, the spectrum of $H_0^V(\theta)$ has the form (see Figure 3)*

$$\sigma\left(H_0^V(\theta)\right) = \bigcup_n \left[E_n + e^{-2i\Im\theta}\mathbb{R}^+\right].$$

More precisely, the essential spectrum of $H_0^V(\theta)$ consists of the sequence of the half-lines $E_n + e^{-2i\Im\theta}\mathbb{R}^+$, $n = 1, 2, \ldots$, and the discrete spectrum of $H_0^V(\theta)$ consists of the set of eigenvalues $\mu_j + E_n$ with associated eigenvectors

$$\left[\psi_{n,j}(\theta)\right](x,s) = \left[S_\theta\psi_{n,j}\right](x,s) = e^{\theta/2}\varphi_j(e^\theta x)\chi_n(s). \tag{6}$$

Proof. It follows from Assumption B that the family of operators $H_0^V(\theta)$ in analytic of type A with respect to θ, see [17, Chap.7]. For what concerns its spectrum it is enough to remark that the operator

$$h(\theta) = S_\theta h S_\theta^{-1} = -e^{-2\theta}\,\partial_x^2 + V(e^\theta x)$$

in $L^2(\mathbb{R})$ has the spectrum given by

$$\sigma(h(\theta)) = \{\mu_1,\ \ldots,\ \mu_N\} \cup e^{-2i\Im\theta}\mathbb{R}^+\,.$$

□

Lemma 2. *Let V satisfy assumptions A and B and let α satisfies assumption C, then the operator $U_\varepsilon^V(\theta)$ is a relatively bounded perturbation of $H_0^V(\theta)$. Moreover, the family of operators $H_\varepsilon^V(\theta)$ is analytic of type A for all θ such that $|\theta| < R_\varepsilon$, where $R_\varepsilon \to \infty$ as $\varepsilon \to 0$.*

Proof. In order to prove this Lemma we introduce $R_\zeta = (H_0^V(\theta) - \zeta)^{-1}$, where ζ is a point from the resolvent set of $H_0^V(\theta)$. Note that both

$$\partial_x^2 \, R_\zeta \quad \text{and} \quad \partial_\tau^2 \, R_\zeta$$

are bounded operators (ω is a bounded domain). This follows from the fact that R_ζ maps $L^2(\Omega_0)$ into the domain of $H_0^V(\theta)$

$$D(H_0^V(\theta)) = \mathcal{H}^2(\Omega) \cap \mathcal{H}_0^1(\Omega),$$

which is contained in the domain of $I \otimes \partial_\tau^2$ as well as in the domain of $\partial_x^2 \otimes I$. For any $\psi \in C_0^\infty(\Omega)$ we have the estimate

$$\|\partial_x \partial_\tau \psi\| = (\partial_x \partial_\tau \psi, \partial_x \partial_\tau \psi)^{1/2} \leq \|\partial_\tau^2 \psi\|^{1/2} \, \|\partial_x^2 \psi\|^{1/2} \leq \|\partial_\tau^2 \psi\| + \|\partial_x^2 \psi\| \,,$$

where $\| \cdot \|$ denotes the norm in $L^2(\mathbb{R} \times \omega)$. This implies that

$$\|(\partial_x \partial_\tau + \partial_\tau^2) \, R_\zeta\| \leq C_\zeta \tag{7}$$

for some constant C_ζ. On the other hand, using the boundedness of $V(e^\theta x)$ we can estimate the first order term in (5) as follows

$$\begin{aligned} \|\partial_\tau \psi\|^2 &= (\psi, \partial_\tau^2 \psi) \leq (\psi, H_0^V(\theta) \, \psi) + c_1 \, \|\psi\|^2 \\ &= (R_\zeta \, (H_0^V(\theta) - \zeta) \, \psi, (H_0^V(\theta) - \zeta) \, \psi) + (c_1 + \zeta)\|\psi\|^2 \\ &\leq \|R_\zeta\| \, \|(H_0^V(\theta) - \zeta) \, \psi\|^2 + (c_1 + \zeta)\|\psi\|^2 \,, \end{aligned} \tag{8}$$

where c_1 is a positive constant. We can thus conclude that there exists a constant $c_2 > 0$ such that

$$\|U_\varepsilon^V(\theta) \, R_\zeta \, \psi\| \leq c_2(\varepsilon + \varepsilon^2) \, \|\psi\|$$

holds true for all $\psi \in C_0^\infty(\Omega_0)$.

To prove the second statement of the Lemma we first notice that by assumption B we have $D(H_0^V(\theta)) = D(H_0^V(0))$. By assumption C and [17, Sec. 7.2] it thus suffices to show that both $\partial_x \partial_\tau$ and ∂_τ are relatively bounded with respect to $H_0^V(\theta)$. However, this follows immediately from (7) and (8). □

Lemma 2 tells us that the eigenvalues of $H_\varepsilon^V(\theta)$ are analytic functions of θ. By a standard argument, [7], it turns out, that they are in fact independent of θ. The non-real eigenvalues of $H_\varepsilon^V(\theta)$ are identified with the resonances of H_ε^V, [7].

Remark 4. As a result of the previous proof it follows that $U_\varepsilon^V(\theta)$ is a regular perturbation of the operator $H_0^V(\theta)$. This enables us to apply the analytic perturbation theory to the eigenvalues of the operator $H_0^V(\theta)$.

Theorem 1. *Let $E = E_n + \mu_j \in \Sigma_+$ be a simple eigenvalue of $H_0^V(\theta)$. For any ball B centered in E there exists $\varepsilon^\star > 0$ such that for any ε with $|\varepsilon| < \varepsilon^\star$, there is an eigenvalue $E(\varepsilon)$ of $H_\varepsilon^V(\theta)$ belonging to B and with the imaginary part given by*

$$\Im E(\varepsilon) = -\varepsilon^2 a + O(\varepsilon^3) \tag{9}$$

where a is a constant independent of ε and equal to

$$a = \sum_{k \leq k^\star} \left| \langle \partial_\tau \chi_n, \chi_k \rangle_{L^2(\omega)} \right|^2 \langle v_j, \Im \hat{r}(E - E_k) v_j \rangle_{L^2(\mathbb{R})} \,. \tag{10}$$

Here

$$v_j = (-2\dot{\alpha}\partial_x + \ddot{\alpha})\varphi_j, \;\; k^\star = \max_k \{E_k - E < 0\}$$

and $\Im \hat{r}$ stands for the imaginary part of the reduced resolvent of $h = -\partial_x^2 + V$ with respect to the eigenvalue μ_j.

Proof. Let $\psi(\theta) = \psi_{n,j}(\theta)$ be the associated normalized eigenvector (6) belonging to E. We apply the regular perturbation theory saying that for some fixed $r > 0$ small enough and for any ε with modulus small enough, in the given ball $B_r(E)$ exists only one eigenvalue $E(\varepsilon)$ of $H_\varepsilon^V(\theta)$ with associated eigenvector

$$\psi^\varepsilon(\theta) = \frac{1}{2\pi i}\oint_{\partial B_r}\left[\zeta - H_\varepsilon^V(\theta)\right]^{-1}\psi(\theta)\mathrm{d}\zeta\,.$$

Furthermore, the regular perturbation theory also yields that this eigenvalues is given by means of the convergent perturbative series

$$E(\varepsilon) = \frac{\left\langle\bar\psi(\theta), H_\varepsilon^V(\theta)\psi^\varepsilon(\theta)\right\rangle_{L^2(\mathbb{R}\times\omega)}}{\left\langle\bar\psi(\theta), \psi^\varepsilon(\theta)\right\rangle_{L^2(\mathbb{R}\times\omega)}} = \sum_{m=0}^\infty e_m(\varepsilon),\;\; e_m = O(\varepsilon^m)$$

where, as usual,

$$e_0 = E \;\;\text{and}\;\; e_1 = \frac{\langle\bar\psi(\theta), U_\varepsilon^V(\theta)\psi(\theta)\rangle_{L^2(\mathbb{R}\times\omega)}}{\langle\bar\psi(\theta), \psi(\theta)\rangle_{L^2(\mathbb{R}\times\omega)}} = \frac{\langle\psi, U_\varepsilon^V\psi\rangle_{L^2(\mathbb{R}\times\omega)}}{\langle\psi,\psi\rangle_{L^2(\mathbb{R}\times\omega)}}$$

are constant with respect to θ. The constants e_0 and e_1 are real since U_ε^V symmetric (see Remark 1) and ψ is real-valued (see Remark 2). If we prove that $\Im e_2 = -\varepsilon^2 a + O(\varepsilon^3)$ for some $a > 0$ independent of ε then the stated result follows. To this end we recall that (see [22, §XII.6], [9])

$$\Im e_2 = \Im a_2\left(1 + O(\varepsilon)\right)\,,$$

where

$$\begin{aligned} a_2 &= -\frac{1}{2\pi i}\oint_{\partial B_r}\left\langle\bar\psi(\theta), U_\varepsilon^V(\theta)\left[\zeta - H_0^V(\theta)\right]^{-1}U_\varepsilon^V(\theta)\psi(\theta)\right\rangle_{L^2(\mathbb{R}\times\omega)}\frac{\mathrm{d}\zeta}{\zeta - E} \\ &= \lim_{\rho\to 0^+} f(\theta, E + i\rho) = \lim_{\rho\to 0^+} f(\theta = 0, E + i\rho) \end{aligned}$$

and

$$\begin{aligned} f(\theta,\zeta) &= -\left\langle\bar\psi(\theta), U_\varepsilon^V(\theta)\left[\zeta - H_\varepsilon^V(\theta)\right]^{-1}U_\varepsilon^V(\theta)\psi(\theta)\right\rangle_{L^2(\mathbb{R}\times\omega)} + \\ &\quad + \left|\left\langle\bar\psi(\theta), U_\varepsilon^V(\theta)\psi(\theta)\right\rangle_{L^2(\mathbb{R}\times\omega)}\right|^2(\zeta - E)^{-1}. \end{aligned}$$

Hence

$$a_2 = -\langle\psi, U_\varepsilon^V\hat R(E + i0)U_\varepsilon^V\psi\rangle_{L^2(\mathbb{R}\times\omega)} \tag{11}$$

where $\hat R(\zeta) = \left[\widehat{\zeta - H_0^V}\right]^{-1}$ is the reduced resolvent of H_0^V with respect to the eigenvalue E, see [17]. Recalling that ψ has the form

$$\psi(x,s) = \psi_{n,j}(x,s) = \varphi_j(x)\chi_n(s)$$

for some n and j and that $\{\chi_k(s)\}$ is a basis of $L^2(\omega)$ with Dirichlet boundary conditions, we obtain

$$U_\varepsilon^V\psi = \sum_k d_k(x)\chi_k, \;\;\text{where}\;\; d_k(x) = \langle\chi_k, U_\varepsilon^V\psi\rangle_{L^2(\omega)}\,.$$

We can thus write

$$a_2 = -\sum_k\left\langle d_k, [h - E \widehat{+ E_k} - i0]^{-1}d_k\right\rangle_{L^2(\mathbb{R})}\,. \tag{12}$$

Concerning the imaginary part of a_2 we emphasize that only finitely many terms on the r.h.s. of (12) have a non zero imaginary part. The latter follows from the fact that $E_k - E$ belongs to the resolvent set of h for any k large enough, more precisely for $k > k^\star$, where

$$k^\star = \max_k \{E_k - E < 0\} \,. \tag{13}$$

From

$$d_k(x) = \varepsilon v_j(x) \langle \partial_\tau \chi_n, \chi_k \rangle_{L^2(\omega)} [1 + O(\varepsilon)] \,, \quad v_j = (-2\dot{\alpha}\partial_x + \ddot{\alpha})\varphi_j$$

we can then conclude that

$$a_2 = -\varepsilon^2 A \, [1 + O(\varepsilon)] \,.$$

The equation (9) now follows because

$$A = A_{n,j} = \sum_{k \le k^\star} \left| \langle \partial_\tau \chi_n, \chi_k \rangle_{L^2(\omega)} \right|^2 \left\langle v_j, [h - \widehat{E + E_k} - i0]^{-1} v_j \right\rangle_{L^2(\mathbb{R})}$$

is independent of ε. Finally, introducing

$$\Im \hat{r}(\zeta) = \frac{1}{2} \left([h - \widehat{\zeta - i}0]^{-1} - [h - \widehat{\zeta + i}0]^{-1} \right)$$

we arrive at (10) since

$$a = \Im A = \sum_{k \le k^\star} \left| \langle \partial_\tau \chi_n, \chi_k \rangle_{L^2(\omega)} \right|^2 \langle v_j, \Im \hat{r}(E - E_k) v_j \rangle_{L^2(\mathbb{R})} \,.$$

□

Remark 5. Notice that if ω is rotationally symmetric, then $a = 0$.

Remark 6. We point out that $\Im r(\zeta)$ is a symmetric and positive operator for ζ real (see, e.g., [9]). We can thus *generically* expect that for any $\mathcal{E} > E_1$ fixed there exists $\varepsilon^\star > 0$ small enough such that $H_\varepsilon^V(\theta)$ does not have discrete spectrum in the interval $[E_1, \mathcal{E}]$ for any $0 < |\varepsilon| \le \varepsilon^\star$; more precisely, for any $\delta > 0$ the set

$$\sigma_d(H_\varepsilon^V(\theta)) \cap \{[E_1, \mathcal{E}] \times i[-\delta, +\delta]\}$$

is empty or it consists of finitely many points with imaginary part strictly negative (see Fig. 4). As a result it follows that any embedded eigenvalue $E \in \Sigma_+$ of the untwisted model turns into be a resonance for the twisted model once the twisting is applied. In particular, the twisted model does not admit embedded eigenvalues in the interval $[E_1, +\infty)$.

Remark 7. Finally, let us mention that there is one important difference between our result and that of [13]. Contrary to the effect of the twist on the ground state, the effect on the embedded eigenvalues occurs also when the Dirichlet boundary conditions at $\partial\Omega$ are replaced by the *Neumann conditions*. However, since all the calculations with the corresponding Neumann operators are completely analogous, we skip them and work only with the Dirichlet operators.

5 One concrete model

In the previous section we have seen, that the embedded eigenvalues under the influence of twisting generically turn into resonances.

In this section, we introduce an addition potential perturbation $W(x, s)$ and consider the operator

$$H_{\varepsilon,\kappa}^V = -\Delta + V(x) + \kappa \, W(x, s) \quad \text{in} \quad L^2(\Omega_\varepsilon) \,, \tag{14}$$

where κ is a small parameter. The potential function W is supposed to satisfy

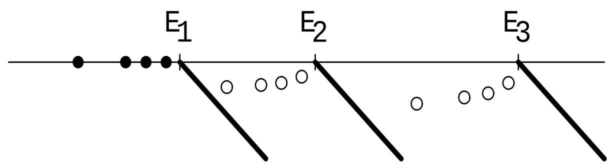

Figure 4: The discrete spectrum of $H^V_\varepsilon(\theta)$, for $\varepsilon \neq 0$ small enough, consists of two parts; the first part is given by the real and simple eigenvalues (full circle) below E_1, the second one is given by simple eigenvalues with real part larger that E_1 and with *imaginary part strictly negative* (empty circle).

Assumption D. For each fixed s the function $W(\cdot,\, s)$ satisfies (B).

For simplicity we put $\kappa = \varepsilon$. In the same way as in the previous section we can thus define the translated operator

$$\tilde H^V_\varepsilon(\theta) = S_\theta H^V_\varepsilon S_\theta^{-1} = H^V_0(\theta) + U^V_\varepsilon(\theta)\,, \tag{15}$$

where

$$H^V_0(\theta) = S_\theta H^V_0 S_\theta^{-1} = -e^{-2\theta}\,\partial_x^2 - \partial_y^2 - \partial_z^2 + V(e^\theta x)$$

and

$$U^V_\varepsilon(\theta) = -2\varepsilon\, e^{-\theta}\,\dot\alpha(e^\theta x)\partial_x\,\partial_\tau - \varepsilon\, e^{-\theta}\,\ddot\alpha(e^\theta x)\,\partial_\tau - \varepsilon^2\dot\alpha^2(e^\theta x)\,\partial_\tau^2 + \varepsilon\, W(e^\theta x, s)\,.$$

If $\dot\alpha = 0$, then the waveguide is straight, without twisting, and the embedded eigenvalues of H^V_ε in general turn into resonances due to the presence of the additional $W(x,s)$ provided W is not constant in s, see [9].

Our goal is to find out, how the presence of twisting changes the width of these resonances in the leading order of the perturbation series. To make this problem simpler we would like to consider a concrete model, in which V acts as a Dirac delta potential. However, as the Dirac delta potential is obviously not dilation analytic, see Assumption B, we will approximate it by the sequence

$$V_\nu(x) = -\frac{\nu}{2\cosh^2(\nu x)}, \qquad \nu > 0, \tag{16}$$

which converges to the delta function at zero as $\nu \to \infty$ in the sense of distributions. Moreover, to be able to give some quantitave results we assume that

$$\alpha(x) = x$$

and that ω satisfies the

Assumption E. The cross section ω is such that the two lowest eigenvalues E_1, E_2 of $-\Delta^D_\omega$ are simple and

$$E_2 - E_1 > \frac{1}{4}\,, \qquad C_1 = \langle \chi_1, \partial_\tau\, \chi_2\rangle_{L^2(\omega)} \neq 0\,. \tag{17}$$

Under assumptions (A — E) we then have

Proposition 1. *Let $\alpha(x) = x$ and consider the embedded eigenvalue $E = E_2 + \mu_1$ of the operator $H^{V_\nu}_0$, where V_ν is given by (16). Then in the vicinity of E there is an eigenvalue $E(\varepsilon)$ of $\tilde H^{V_\nu}_\varepsilon(\theta)$ with the imaginary part given by*

$$\Im E(\varepsilon) = -\varepsilon^2 a + O(\varepsilon^3)\,, \quad a > 0\,. \tag{18}$$

Moreover, if

$$W(x,s) = W(|x|,s), \tag{19}$$

then

$$\lim_{\nu\to\infty} a = C_1^2\,\frac{\sqrt{E_2-E_1-\frac{1}{4}}}{(E_2-E_1)^2} + \langle W\rangle\,, \tag{20}$$

where $\langle W\rangle \geq 0$*, see* (39)*,* (40)*.*

Remark 8. Note that

$$\Im E(\varepsilon) > \langle W\rangle\,,$$

which means that the twisting pushes the eigenvalue $E(\varepsilon)$ down in the complex plane, making thus the lifetime of the corresponding resonance shorter.

Remark 9. The assumption (E) on the cross section is quite weak. As an example one could take the rectangle $\omega = [0,1]\times[0,2]$. In this case one has

$$\sigma(-\Delta_\omega^D) = \left\{E_1 = \frac{5}{4}\pi^2,\ E_2 = 2\pi^2,\ E_3 = \frac{13}{4}\pi^2,\ E_4 = \frac{17}{4}\pi^2, \dots\right\}$$

and

$$\chi_1(y,z) = \sqrt{2}\,\sin(\pi\,y)\sin\left(\frac{\pi}{2}\,z\right) \quad,\quad \chi_2(y,z) = \sqrt{2}\,\sin(\pi\,y)\sin(\pi\,z)\ .$$

An explicit calculation then shows that

$$C_1 = \langle \chi_1, \partial_\tau\,\chi_2\rangle_{L^2(\omega)} = -\frac{2}{3} \neq 0\,.$$

On the other hand, it is clear that $C_1 = 0$ whenever ω is rotationally symmetric. Of course, in such a situation the twisting has no influence on $E(\varepsilon)$.

Remark 10. In a similar way one could calculate the imaginary parts of the eigenvalues of $\tilde{H}_\varepsilon^{V_\nu}(\theta)$ coming from the higher threshold energies $E_k\,, k \geq 3$. To avoid cumbersome computations we skip it.

5.1 Proof of Proposition 1

Equation (18) follows directly from Theorem 1. The rest of the proof will be given in two steps.

Spectrum of $h_\nu = -\partial_x^2 + V_\nu$

Following [19, §23] we set

$$s = \frac{1}{2}\left[-1+\sqrt{1+\frac{2}{\nu}}\right]\,. \tag{21}$$

the eigenvalue problem $h_\nu\tilde{\varphi}_j = \mu_j\tilde{\varphi}_j$ admits solutions

$$\mu_j = -\frac{\nu^2}{4}\left[-(2j-1)+\sqrt{1+\frac{2}{\nu}}\right]^2,\ 1\leq j < s+1 \tag{22}$$

with associated eigenfunctions

$$\tilde{\varphi}_j(x) = (1-\xi^2)^{e_j/2} F\left[e_j - s, e_j+s+1, e_j+1, \frac{1}{2}(1-\xi)\right], \tag{23}$$

where

$$\xi = \tanh(\nu x),\ e_j = \frac{\sqrt{-\mu_j}}{\nu} = \frac{1}{2}\left[-(2j-1) + \sqrt{1+\frac{2}{\nu}}\right], \tag{24}$$

F denote the hypergeometric function and $e_j - s = j - 1$. In particular, when $\nu \gg 1$ then $s \sim \frac{1}{2\nu} \ll 1$ and the spectrum of h consists of only one eigenvalue

$$\mu_1 = -\frac{\nu^2}{4}\left[-1 + \sqrt{1+\frac{2}{\nu}}\right]^2 \sim -\frac{1}{4} + O(\nu^{-1}) \tag{25}$$

with the associated normalized eigenvector

$$\varphi_1(x) = \frac{\tilde{\varphi}_1(x)}{\|\tilde{\varphi}_1(x)\|_{L^2(\mathbb{R})}},\ \tilde{\varphi}_1(x) = \left[1 - \tanh^2(\nu x)\right]^{e_1/2}. \tag{26}$$

We emphasize that $h_\nu \to h_\infty = -\partial_x^2 - \delta$ as $\nu \to +\infty$ in the norm resolvent (see, e.g., [1]), where δ denotes the Dirac's delta at $x = 0$.

Computation of the coefficient a

For $\nu \to \infty$ we have

$$\sigma_d(h_\nu) = \left\{\mu_1 = -\frac{1}{4} + O(\nu^{-1})\right\}. \tag{27}$$

We take ν large enough so that for the set $\Sigma = \{E = E_{j,n} = \mu_j + E_n\}$ of eigenvalues of $H_0^{V_\nu}$ holds

$$E_{1,1} = E_1 - \frac{1}{4} + O(\nu^{-1}) < E_1 < E_{1,2} = E_2 - \frac{1}{4} + O(\nu^{-1}) < E_2. \tag{28}$$

We then apply then the perturbative theory to the embedded eigenvalue

$$E = E_{1,2} = E_2 + \mu_1 \tag{29}$$

with the associated eigenvector

$$\psi(x,y,z) = \varphi_1(x)\chi_2(y,z). \tag{30}$$

In such a case $k^\star = 2$ and

$$\Im a_2 = \tag{31}$$
$$-\lim_{\rho\to 0^+} \Im\left\{\sum_{k=1,2} \langle d_k, [h_\nu - E + E_k - i\rho]^{-1} d_k\rangle_{L^2(\mathbb{R})} + \left|\langle \psi, U_\varepsilon^{V_\nu}\psi\rangle_{L^2(\mathbb{R}\times\omega}\right|^2 (i\rho)^{-1}\right\}$$

where

$$U_\varepsilon^{V_\nu} = -2\varepsilon\partial_x\partial_\tau - \varepsilon^2\partial_\tau^2 + \varepsilon W \quad \text{and} \quad d_k(x) = \langle \chi_k, U_\varepsilon^{V_\nu}\psi\rangle_{L^2(\omega)} \tag{32}$$

An integration by parts shows that $\langle \chi_2, \partial_\tau\chi_2\rangle_{L^2(\omega)} = 0$. We thus get

$$\begin{aligned}\langle \psi, U_\varepsilon^{V_\nu}\psi\rangle_{L^2(\mathbb{R}\times\omega)} &= -2\varepsilon\langle\varphi_1, \partial_x\varphi_1\rangle_{L^2(\mathbb{R})}\langle\chi_2, \partial_\tau\chi_2\rangle_{L^2(\omega)} \\ &\quad -\varepsilon^2\langle\varphi_1,\varphi_1\rangle_{L^2(\mathbb{R})}\langle\chi_2, \partial_\tau^2\chi_2\rangle_{L^2(\omega)} + \varepsilon\,\langle\varphi_1, W_{22}\varphi_1\rangle_{L^2(\mathbb{R})} \\ &= -C_0\varepsilon^2 + \varepsilon\,\langle\varphi_1, W_{22}\,\varphi_1\rangle_{L^2(\mathbb{R})},\end{aligned}$$

where

$$C_0 = \langle \chi_2, \partial_\tau^2 \chi_2 \rangle_{L^2(\omega)}, \quad W_{22}(x) = \langle \chi_2, W\chi_2 \rangle_{L^2(\omega)} .$$

Furthermore, from (32) it follows that

$$d_1(x) = -2\varepsilon C_1\, \partial_x \varphi_1 - \varepsilon^2 C_2\, \varphi_1 + \varepsilon\, W_{12}\, \varphi_1, \quad d_2(x) = -\varepsilon^2 C_0\, \varphi_1 + \varepsilon\, W_{22}\, \varphi_1$$

where

$$C_2 = \langle \chi_1, \partial_\tau^2 \chi_2 \rangle_{L^2(\omega)}, \quad W_{12}(x) = \langle \chi_2, W\chi_1 \rangle_{L^2(\omega)} .$$

Collecting all these facts and keeping in mind that $E = E_2 + \mu_1$ and φ_1 is the eigenfuction of h_ν with eigenvalue μ_1 we get after some tedious, but straightforward calculations, that

$$\lim_{\rho\to 0+} \Im \left\{ \langle d_2, [h_\nu - E + E_2 - i\rho]^{-1}\, d_2 \rangle_{L^2(\mathbb{R})} + \left| \langle \psi, U_\varepsilon^{V_\nu} \psi \rangle \right|^2_{L^2(\Omega_0)} (i\rho)^{-1} \right\} = 0 . \tag{33}$$

This implies

$$\begin{aligned} \Im a_2 &= -\lim_{\rho\to 0^+} \Im \langle d_1, [h_\nu - E + E_1 - i\rho]^{-1}\, d_1 \rangle_{L^2(\mathbb{R})} \\ &= -\lim_{\rho\to 0^+} \Im \left[4\varepsilon^2 C_1^2 \langle \partial_x \varphi_1, [h_\nu - E + E_1 - i\rho]^{-1}\, \partial_x \varphi_1 \rangle \right] \\ &\quad - \lim_{\rho\to 0^+} \Im \left[\varepsilon^2\, \langle W_{12}\, \varphi_1, [h_\nu - E + E_1 - i\rho]^{-1}\, W_{12}\, \varphi_1 \rangle \right] \\ &\quad - \lim_{\rho\to 0^+} 2\varepsilon^2\, C_1\, \Im \Big[\langle \partial_x \varphi_1, [h_\nu - E + E_1 - i\rho]^{-1}\, W_{12}\, \varphi_1 \rangle \\ &\qquad + \langle W_{12}\, \varphi_1, [h_\nu - E + E_1 - i\rho]^{-1}\, \partial_x \varphi_1 \rangle \Big] + O(\epsilon^3) . \end{aligned} \tag{34}$$

We now pass to the limit $\nu \to \infty$ which implies

$$\mu_1 \to -\frac{1}{4}, \quad \varphi_1 \to \phi = \sqrt{\frac{1}{2}}\, e^{-|x|/2}, \quad h_\nu \to h_\infty , \tag{35}$$

where the last limit is reached in the norm resolvent sense. We recall also that the resolvent $[h_\infty - \zeta]^{-1}$ of $h_\infty = -\partial_x^2 + \delta$ has the kernel given by

$$\mathcal{K}_\zeta(x,x') = \mathcal{K}_\zeta^0(x,x') + \mathcal{K}_\zeta^1(x,x') ,$$

where

$$\mathcal{K}_\zeta^0(x,x') = \frac{i}{2k}\, e^{ik|x-x'|}, \quad \mathcal{K}_\zeta^1(x,x') = \frac{1}{2k}\frac{1}{2k+i}\, e^{ik[|x|+|x'|]}, \quad \zeta = k^2,\ \Im k > 0 ,$$

see [1]. In our case

$$\zeta = k^2 = E - E_1 + i\rho = E_2 - E_1 + \mu_1 + i\rho . \tag{36}$$

We will denote by $\mathcal{K}_\zeta^0$ and $\mathcal{K}_\zeta^1$ the integral operators with the kernels $\mathcal{K}_\zeta^0(x,x')$ and $\mathcal{K}_\zeta^1(x,x')$ respectively. Since $\partial_x \varphi_1(x)$ is an odd function, it follows that $\mathcal{K}_\zeta^1 \partial_x \varphi_1 \equiv 0$ and

$$[\langle \partial_x \varphi_1, \left(\mathcal{K}_\zeta^0 + \mathcal{K}_\zeta^1\right) W_{12}\, \varphi_1 \rangle + \langle W_{12}\, \varphi_1, \left(\mathcal{K}_\zeta^0 + \mathcal{K}_\zeta^1\right) \partial_x \varphi_1 \rangle \Big] = 0 . \tag{37}$$

This means that in the limit $\nu \to \infty$ we have to replace $[h_\nu - E + E_1 - i\rho]^{-1}$ on the r.h.s. of (34) by $\mathcal{K}_\zeta^0$. An explicit computation then gives

$$\lim_{\rho\to 0^+} \Im \langle \partial_x \varphi_1, \mathcal{K}_\zeta^0\, \partial_x \varphi_1 \rangle = \frac{4\sqrt{E_2 - E_1 + \mu_1}}{\left[1 + 4(E_2 - E_1 + \mu_1)\right]^2} . \tag{38}$$

Finally, the contribution from W equals

$$\lim_{\rho\to 0^+} \Im\langle W_{12}\,\varphi_1, (\mathcal{K}_\zeta^0 + \mathcal{K}_\zeta^1) W_{12}\,\varphi_1\rangle = \frac{2b^2+1}{4b^2+1}\left(\int_{\mathbb{R}} W_{12}(x)\varphi_1(x)\cos(bx)\,dx\right)^2 \tag{39}$$

where

$$b = \sqrt{E_2 - E_1 + \mu_1}\,.$$

We denote by

$$\langle W\rangle := \frac{E_2 - E_1 + \frac{1}{4}}{2(E_2 - E_1)}\left(\int_{\mathbb{R}} W_{12}(x)\phi(x)\cos(bx)\,dx\right)^2 \tag{40}$$

its limit as $\nu \to \infty$. In view of (35) and (38) we get

$$\lim_{\nu\to\infty} \Im a_2 = -\varepsilon^2 \left(C_1^2\,\frac{\sqrt{E_2 - E_1 - \frac{1}{4}}}{(E_2 - E_1)^2} + \langle W\rangle \right).$$

The proof is complete.

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