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Preprint 2006/013

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ISSN **1613-8309**

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Ulrich Brehm and Wolfgang Kühnel

Abstract. We give a classification of all polyhedral maps on the torus admitting a vertex transitive automorphism group. Equivalently, we classify all equivelar polyhedral maps on the torus. These are precisely the ones which are quotients of the regular tessellations $\{3,6\}$, $\{6,3\}$ or $\{4,4\}$ by a pure translation group. An explicit formula for the number of combinatorial types with n vertices is obtained in terms of arithmetic functions in elementary number theory, such as the number of integer divisors of n. The asymptotic behaviour for $n \to \infty$ is also discussed, and an example is given for n such that the number of distinct equivelar triangulations of the torus with n vertices is larger than n itself. The numbers of regular and chiral maps are determined separately, as well as the ones for all other kinds of symmetry. Furthermore, arithmetic properties of the integers of type p^2+pq+q^2 (or p^2+q^2, resp.) can be interpreted and visualized by the hierarchy of covering maps between regular and chiral equivelar maps or type $\{3,6\}$ (or $\{4,4\}$, resp.).

Key words: *regular tessellation, equivelar, triangulated tori, weakly regular, 6-connected graph, modular group, Dirichlet character, binary quadratic form*

MSC classifications: Primary 52B70; Secondary 05C10, 11A25.

1 Introduction

It is well known that there are many distinct triangulations of the torus where every vertex is contained in precisely 6 edges. Classical examples are the unique (and chiral) 7-vertex torus [6, Fig.8.4] (also called the *Möbius torus* after A.Möbius [19, p.553]) $\{3,6\}_{(2,1)}$ and the regular 12-vertex torus $\{3,6\}_{(2,2)}$, see Figure 1. The latter is not unique in the sense that there are other 12-vertex triangulations of the torus which look locally the same but which are globally not isomorphic (and which are not regular). The 7-vertex torus is distinguished by a number of properties in the class of abstract triangulations, in the class of twofold triple systems, and in the class of tight embeddings of the torus into Euclidean space, see [14], [15, 2.19]. On the other hand,

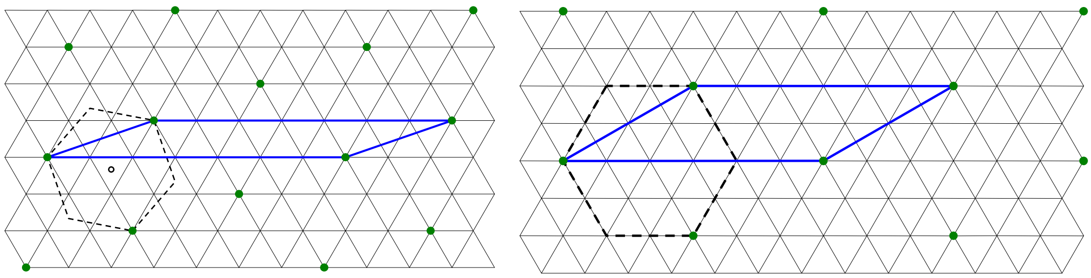

Figure 1: The unique 7-vertex torus and the regular 12-vertex torus

can you tell in a simple way how many combinatorially distinct types exist with, say, 24 vertices? In this case we expect to see a double covering (or possibly several ones) of the regular 12-vertex torus, but what are the others? Actually there are eleven distinct combinatorial types with 24 vertices, see below. In this note, we are going to investigate an explicit formula for the number of combinatorially distinct equivelar polyhedral maps with n vertices on the torus. The classical cases of regular and chiral maps on the torus were treated in [6, Ch.8]. Partial further results were previously obtained by Altshuler [1] and by Datta and Upadhyay [7] in terms of lists of orbits of triangles and by Negami [20] in terms of embeddings of 6-connected graphs. Furthermore, by a computer check Lutz [16] has determined all combinatorial types of triangulated surfaces with a small number of vertices and with a vertex transitive automophism group, and Knödler [13] found algebraic conditions in the case where n is a prime number. For a survey about polyhedral maps on surfaces in general see [4].

Definition A (polyhedral or non-polyhedral) map on a surface is called equivelar if there are numbers p and q such that every vertex is contained in precisely q edges and every facet contains exactly p vertices and edges, compare [18]. Consequently, an equivelar map admits an abstract Schläfli symbol $\{p, q\}$ which describes locally the nature of the map. However, the Schläfli symbol does not determine the map globally. This is true only if the surface is simply connected which leads to the classical regular maps on the sphere, the Euclidean plane and the hyperbolic plane, see [6], [17]. For an equivelar map on the torus the only possibilities for the Schläfli symbole are $\{3, 6\}$, $\{6, 3\}$ or $\{4, 4\}$. By duality between $\{3, 6\}$ and $\{6, 3\}$ we have to deal only with the two cases of equivelar triangulations (the case $\{3, 6\}$) and equivelar quadrangulations (the case $\{4, 4\}$). In the sequel $\mathbf{T}(n)$ denotes the number of (polyhedral) n-vertex equivelar triangulations, and $\mathbf{Q}(n)$ denotes the number of (polyhedral) n-vertex equivelar quadrangulations of the torus (up to combinatorial isomorphy).

2 Equivelar triangulations

Notations A triangulation is called polyhedral if the intersection of two triangules is either empty or a common vertex or a common edge. In order to fix notations, let $\{3, 6\}$ denote the regular tessellation of the plane by equilateral triangles. We may assume that the edge length is 1 and that the vector $(1, 0)$ points into the direction of an edge. Then in cartesian coordinates the vectors $\mathbf{a} = (1, 0)$ and $\mathbf{b} = \frac{1}{2}(1, \sqrt{3})$ generate a lattice Γ in the plane. Its automorphism group $G(\Gamma)$ is a Coxeter group. We can also think of Γ as the normal abelian subgroup of $G(\Gamma)$ which consists of all translations $z\mathbf{a} + z'\mathbf{b}$ with integers z, z'.

For any triangular map on the torus such that every vertex is in precisely six edges, the universal covering is the regular tessellation $\{3, 6\}$, and the torus itself is a quotient of it by a sublattice Γ' of Γ. Again Γ acts transitively on the quotient of $\{3, 6\}$ by Γ', and its automorphism group $A = G(\Gamma)/\Gamma'$ acts transitively on the set of vertices. This can be formulated in the following proposition (compare [7]).

Proposition 1 ([20])
Let T be a triangulated torus. Then the following conditions are equivalent:

1. *T is equivelar,*
2. *each vertex of T is 6-valent,*
3. *the edge graph of T is 6-connected,*
4. *T has a vertex transitive automorphism group,*
5. *T can be obtained as a quotient of the regular tessellation $\{3, 6\}$ by a sublattice (or a pure translation group) Γ'.*

From the graph-theoretical point of view such graphs are also called 6-regular. Minimal graphs of that type which triangulate a torus were studied in [8].

Additional symmetry The isotropy group of a triangulated torus T with a vertex transitive automorphism group is normally understood as the subgroup fixing one vertex. In our case this always contains a point reflection as a central element. If it contains in addition an element of order three (rotation by $2\pi/3$) then, by combination with the point reflection, it contains also an element of order six (rotation by $\pi/3$). Consequently, the isotropy group is either of order two (the general case) or of order four (with one line reflection) or cyclic of order six (with rotations only) or dihedral of order twelve (with rotations and a line reflection). In the literature, such a triangulated torus with a cyclic isotropy group C_6 is usually called chiral because a line reflection will produce two isomorphic versions (right-handed and left-handed), in case of the dihedral group D_6 it is called regular or reflexive because the automorphism groups act transitively on flags, see [6], [17]. The number of flags is $12n$ for an n-vertex torus.

Theorem 1 ([6])
Any chiral or regular torus is of the form $\{3,6\}_{p,q}$ *with integers* p,q*. This notation means that the sublattice* Γ' *is generated by the translation* $p\mathbf{a}+q\mathbf{b}$ *and all rotations by* $\pi/3$*. The number of vertices of the torus* $\{3,6\}_{p,q}$ *is* $n=p^2+pq+q^2$*.*
A line reflection transforms $\{3,6\}_{p,q}$ *into* $\{3,6\}_{q,p}$ *(these two are the right-handed and the left-handed version). Moreover we have* $\{3,6\}_{q,p}=\{3,6\}_{p+q,-q}$*. The triangulation is regular (with a* D_6 *as isotropy group) if and only if* $pq(p-q)=0$*. In this case we have* $\{3,6\}_{q,p}=\{3,6\}_{p,q}$*.*

As an example, the 7-vertex torus is the chiral $\{3,6\}_{2,1}$, the regular 12-vertex torus is $\{3,6\}_{2,2}$

Remark: Steiner triple systems and quadruple systems which are related with chiral maps are studied in [15]. One example is the union of two copies of a $\{3,6\}_{3,1}$ which are glued together along the 13 vertices on each side. It can be considered as a 2-fold branched covering of one copy of $\{3,6\}_{3,1}$.

Definition *Two triangulated tori* T,T' *are called* isomorphic *if there is a bijection* $\Phi: V\to V'$ *between the two sets* V,V' *of vertices such that* $\langle abc\rangle$ *is a triangle of* T *if and only if* $\langle\Phi(a)\Phi(b)\Phi(c)\rangle$ *is a triangle of* T'*.*

The associated matrix We represent a suitable fundamental domain of a weakly regular triangulated torus as a parallelogram, spanned by two vectors $\mathbf{x}$ and $\mathbf{y}$, in the euclidean plane which is tessellated as $\{3,6\}$. We may assume that $\mathbf{x}$ is parallel to $\mathbf{a}$, so it can be represented as $\mathbf{x}=a\cdot\mathbf{a}$. Furthermore we my assume that the second vector can be written as $\mathbf{y}=b\cdot\mathbf{a}+c\cdot\mathbf{b}$. In matrix notation, we represent the triangulated torus by the associated matrix

$$\begin{pmatrix} a & b\\ 0 & c\end{pmatrix}$$

where a,c are positive numbers and where b may be any integer. In the case of regular or chiral tori $\{3,6\}_{(p,q)}$ it is also convenient to represent the fundamental domain by the associated matrix

$$\begin{pmatrix} p & -q\\ q & p+q\end{pmatrix}.$$

Lemma 1 *The number* n *of vertices in the torus equals the determinant of the associated matrix.*

This follows from the simple observation that the determinant $\det(\mathbf{xy})$ equals the area of the parallelogram which in turn must be equal to the number of triangles times $\frac{1}{4}\sqrt{3}$ (the latter is the area of one equilateral triangle with edge length 1). Furthermore, we have $\det(\mathbf{xy}) = \det(\mathbf{ab})\cdot\det\begin{pmatrix} a & b\\ 0 & c\end{pmatrix} = \frac{1}{2}\sqrt{3}ac$. On the other hand, the number of triangles equals twice the number of vertices.

Examples The 7-vertex torus can be represented be the matrix $\begin{pmatrix} 7 & 2\\ 0 & 1\end{pmatrix}$ whereas the regular 12-vertex torus can be represented by $\begin{pmatrix} 6 & 2\\ 0 & 2\end{pmatrix}$. The chiral or regular version is the matrix $\begin{pmatrix} 2 & -1\\ 1 & 3\end{pmatrix}$ for the 7-vertex torus and $\begin{pmatrix} 2 & -2\\ 2 & 4\end{pmatrix}$ for the regular 12-vertex torus.

This matrix representation is unique for the fundamental domain but it is not for the triangulated tori themselves since any torus can be represented by several fundamental domains. For the 7-vertex torus we could also choose

$$\begin{pmatrix} 7 & 9\\ 0 & 1\end{pmatrix} = \begin{pmatrix} 7 & 2\\ 0 & 1\end{pmatrix}\cdot\begin{pmatrix} 1 & 1\\ 0 & 1\end{pmatrix}$$

instead of $\begin{pmatrix} 7 & 2 \\ 0 & 1 \end{pmatrix}$. Furthermore, the matrices $\begin{pmatrix} 3 & 0 \\ 0 & 4 \end{pmatrix}$ and

$$\begin{pmatrix} 3 & -3 \\ 0 & 4 \end{pmatrix} = \begin{pmatrix} 3 & 0 \\ 0 & 4 \end{pmatrix} \cdot \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}$$

represent the same torus with 12 vertices (up to isomorphy). The transition from one matrix to the other corresponds to a shearing of the fundamental domain. Furthermore, rotation and reflection does not change the triangulation of the torus but the associated matrix is multiplied by some matrix from the left. We formulate this in the following proposition.

Proposition 2 *Two matrices M_1 and M_2 represent isomorphic tori if and only if there exist matrices $R \in G_0$ and $S \in GL(2, \mathbb{Z})$ such that*

$$M_2 = RM_1S.$$

Here $G_0 \cong D_6$ denotes the group of rotations and reflections fixing the origin in the tessellation $\{3, 6\}$, that is, the group generated by the matrices

$$A = \begin{pmatrix} 0 & -1 \\ 1 & 1 \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} -1 & -1 \\ 0 & 1 \end{pmatrix}.$$

Furthermore, $GL(2, \mathbb{Z})$ denotes the group of invertible integer matrices, also known as the modular group.

Proof: Since A is a rotation by $\pi/3$ and B is a line reflection, it is obvious that multiplication with them from the left does not change the combinatorial type of the torus. The action of the modular group from the right corresponds to elementary operations with rows and columns which preserve the determinant of the matrix (up to sign). Geometrically, this corresponds to modifying the fundamental domain by a change of the basis with integer coefficients.

Vice versa, if two fundamental domains represent the same combinatorial type of a triangulated torus, then it must be possible to transform one into the other by exactly these operations: rotations, reflections and change of the basis. □

If we want to achieve a unique normal form of a matrix for a given triangulated torus, we have to take these modifications into account.

Proposition 3 1. *Any combinatorial type of an equivelar triangular maps on the torus with n vertices is obtained from an associated matrix of the following type*

$$\begin{pmatrix} a & b \\ 0 & c \end{pmatrix}$$

where $ac = n$, $a, c > 0$ and $-c \leq b < a - c$. The last condition means that b is unique modulo a. However, the same map may be represented by several matrices of this type. Negami [20] uses the symbol $T(a, b, c)$ for the triangulated torus with the corresponding matrix.

2. *For determining the number of combinatorial types of polyhedral equivelar triangulations we have to rule out the cases where loops or double edges occur. An example of such a case is any of the associated matrices $\begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix}$ or $\begin{pmatrix} a & 0 \\ 0 & 2 \end{pmatrix}$.*

Theorem 2 *Let n be any given natural number. We are interested in the number $\mathbf{T}(n)$ of all non-isomorphic equivelar (polyhedral) triangulations of the torus with n vertices. Let $\mathbf{T_i}(n)$ denote the number of non-isomorphic such triangulations having an isotropy group of order $2i$. Then $\mathbf{T_i}(n) = 0$ unless $i = 1, 2, 3, 6$, and the following formula holds*

$$6\mathbf{T_1}(n) + 3\mathbf{T_2}(n) + 2\mathbf{T_3}(n) + \mathbf{T_6}(n) = \begin{cases} \sigma(n) - 12 & \text{if } n \text{ is even} \\ \sigma(n) - 6 & \text{if } n \text{ is odd}, \end{cases}$$

where $\sigma(n)$ denotes the sum of all integer divisors of n (including n itself):

$$\sigma(n) = \sum_{k|n} k.$$

Proof. From Proposition 3 we obtain the number of possible associated matrices as follows: For any positive integer divisor a of n we get precisely a distinct associated matrices. If we add up these numbers we get the sum over all positive integer divisors of n which is nothing but the function $\sigma(n)$. However, in this counting certain cases are counted more than once. If the isotropy group is of order 2 we count it six times (by rotations), if the isotropy group is of order 4 we count it three times. This is caused by the fact that the choice of a basis $\mathbf{a}, \mathbf{b}$ is geometrically not unique because there are six possibilities, just by rotating the basis. In general: If the isotropy group is of order $2i$ we count it $6/i$ times. This leads to the formula in Theorem 2 if we take out those triangular maps which are not polyhedral. There are 6 cases such if n is odd and 12 cases if n is even. □

Remark: The following explicit formula for $\sigma(n)$ is well known [10, §16.7]: Let

$$n = \prod_{i=1}^{r} p_i^{a_i}$$

with distinct prime numbers p_i then we have

$$\sigma(n) = \prod_{i=1}^{r} \frac{p_i^{a_i+1} - 1}{p_i - 1}$$

or, alternatively,

$$\sigma(n) = \prod_{i=1}^{r} \sum_{j=0}^{a_i} p_i^j = \prod_{i=1}^{r} (1 + p_i + p_i^2 + ... + p_i^{a_i})$$

Furthermore we have [21, p.10]

$$\sum_{n=1}^{\infty} \sigma(n) n^{-s} = \zeta(s)\zeta(s-1)$$

where $\zeta(s) = 1 + 2^{-s} + 3^{-s} + 4^{-s} + \cdots$ is the Riemann ζ-function.

Example: For $n = 7$ we have the following possibilities for associated matrices in Proposition 3:

$$\begin{pmatrix} a & b \\ 0 & c \end{pmatrix} = \begin{pmatrix} 7 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 7 & 1 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 7 & 2 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 7 & 3 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 7 & 4 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 7 & 5 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 7 & 6 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & -7 \\ 0 & 7 \end{pmatrix}.$$

Among them, only the cases $\begin{pmatrix} 7 & 2 \\ 0 & 1 \end{pmatrix}$ and $\begin{pmatrix} 7 & 4 \\ 0 & 1 \end{pmatrix}$ are polyhedral. These are the two chiral versions $\{3,6\}_{2,1}$ and $\{3,6\}_{1,2}$ of the 7-vertex torus. The other six instances have either loops or double edges, compare [20, Table 1]. This verifies the equations $2\mathbf{T_3}(7) = \sigma(7) - 6 = 2$, $\mathbf{T_6}(7) = \mathbf{T_2}(7) = \mathbf{T_1}(7) = 0$. It gives a proof of the uniqueness of the polyhedral 7-vertex torus.

Theorem 3 (explicit number of combinatorial types)
Let $\mathbf{T_i}(n)$ denote the number of non-isomorphic weakly regular triangulations of the torus with n vertices having an isotropy group of order $2i$. Let $d(n)$ denote the number of integer divisors of n, Then the following hold:

1. *$\mathbf{T_6}(n)$ equals the number of possibilities to represent n as a sum p^2+pq+q^2 with $pq(p-q)=0$ and with $q \leq p$,*

2. *$\mathbf{T_3}(n)$ equals the number of possibilities to represent n as a sum p^2+pq+q^2 with $pq(p-q) \neq 0$ and with $q < p$,*

3. $\mathbf{T_2}(n) = \begin{cases} d(n) - 2 - \mathbf{T_6}(n) & \text{if } n \text{ is odd} \\ d(\frac{n}{2}) - 2 & \text{if } n \equiv 2 \bmod 4 \\ d(\frac{n}{4}) + d(\frac{n}{2}) - 4 - \mathbf{T_6}(n) & \text{if } n \equiv 0 \bmod 4 \end{cases}$

4. *$\mathbf{T_1}(n)$ is uniquely determined by Theorem 2.*

The numbers $\mathbf{T_6}(n)$ and $\mathbf{T_3}(n)$ are determined more explicitly as follows.

$$\mathbf{T_6}(n) = \begin{cases} 1 & \text{if } n \geq 9 \text{ is either a square or } 3 \text{ times a square} \\ 0 & \text{otherwise} \end{cases}$$

$$\mathbf{T_3}(n) = \begin{cases} \left[\frac{1}{2}\prod_{i=1}^{r}(a_i+1)\right] & \text{if all numbers } b_0, \ldots, b_s \text{ are even} \\ 0 & \text{otherwise} \end{cases}$$

where $n = \prod_{i=1}^{r} p_i^{a_i} \cdot 2^{b_0} \cdot \prod_{j=1}^{s} q_j^{b_i} \cdot 3^{c_0}$ with distinct prime numbers $p_i \equiv 1(6)$ and $q_i \equiv 5(6)$ and arbitrary c_0.

In particular, if $n = m^2$ is a square then $\mathbf{T_6}(n) = 1, \mathbf{T_3}(n) = 0$. If n is prime then $\mathbf{T_6}(n) = 0, \mathbf{T_3}(n) = 1$. As an example for $\mathbf{T_3}(n) > 1$, we have $\mathbf{T_3}(133) = 2$ because $11^2 + 11 + 1 = 133 = 9^2 + 9 \cdot 4 + 4^2$.

Recall the following expression of $d(n)$ in terms of the Riemann ζ-function [21, p.10]:

$$\sum_{n=1}^{\infty} d(n) n^{-s} = (\zeta(s))^2$$

Corollary 1 *For any n the number of non-isomorphic equivelar triangulations of the torus is*

$$\mathbf{T}(n) = \frac{1}{6}\sigma(n) + \frac{1}{2}\mathbf{T_2}(n) + \frac{2}{3}\mathbf{T_3}(n) + \frac{5}{6}\mathbf{T_6}(n) - \begin{cases} 1 \text{ if } n \text{ is odd} \\ 2 \text{ if } n \text{ is even} \end{cases}$$

where $\mathbf{T_2}, \mathbf{T_3}, \mathbf{T_6}$ are given in Theorem 3 above.

Corollary 2 *If n is a prime number then we have $\mathbf{T}(n) = \left[\frac{n-1}{6}\right]$. If $n = 3p$ with a prime number p then $\mathbf{T}(n) = \left[\frac{2p+4}{3}\right]$.*

Proof of Theorem 3.

Part 1 is fairly clear since $\mathbf{T_6}(n)$ equals the number of non-isomorphic regular maps on the torus with n vertices. By Theorem 1 there are only the cases $p = 0$ or $q = 0$ or $p = q$. Hence n is either a square or 3 times a square, and there is at most one map with n vertices in any case.

Part 2 follows from Lemma 3 below.

For the proof of Part 3 we go back to the associated matrix $\begin{pmatrix} a & b \\ 0 & c \end{pmatrix}$ in Proposition 3. We just have to determine the cases with an isotropy group of order four (which must be isomorphic with

$\mathbb{Z}_2 \oplus \mathbb{Z}_2$). This kind of symmetry occurs precisely for integer matrices (up to equivalence given in Proposition 2) of type $\begin{pmatrix} a & -c/2 \\ 0 & c \end{pmatrix}$, $\begin{pmatrix} a & (a-c)/2 \\ 0 & c \end{pmatrix}$. Note that we have only to consider common symmetries of the hexagonal lattice and the sublattice under consideration.

If n is odd then a and c are also odd, and we just count the number of integer divisors of n. However, this includes possible regular cases $\begin{pmatrix} 3c & c \\ 0 & c \end{pmatrix}$ or $\begin{pmatrix} c & 0 \\ 0 & c \end{pmatrix}$ (which are already considered by the computation of $\mathbf{T_6}$) and two non-polyhedral cases. The equation $\mathbf{T_2}(n) = d(n) - 2 - \mathbf{T_6}(n)$ follows.

If $n \equiv 2(4)$ then either a is even and c is odd or vice versa. If a is even then there is no possibility for b, so this case cannot occur. If c is even we are left with only the possibility $b = -c/2$ with $c \neq n$ and $c \neq 2$ (these cases are not polyhedral). So we count just the integer divisors of $\frac{n}{2}$ and have to omit two cases. None of theses cases is regular, so $\mathbf{T_6}(n) = 0$. The equation $\mathbf{T_2}(n) = d(\frac{n}{2}) - 2$ follows.

If $n \equiv 0(4)$ then for any matrix $\begin{pmatrix} a & -c/2 \\ 0 & c \end{pmatrix}$ c must be an even number, and for $\begin{pmatrix} a & (a-c)/2 \\ 0 & c \end{pmatrix}$ both a and c must be even. So in the former case we count the integer divisors of $n/2$, in the latter case we count the integer divisors of $n/4$. Four of the cases are not polyhedral ($a = 1, 2, c = 2$ in the first case, $a = 2$ in the second). Possible regular cases $\begin{pmatrix} 3c & c \\ 0 & c \end{pmatrix}$ have to be omitted here since they are counted by the $\mathbf{T_6}$. The equation $\mathbf{T_2}(n) = d(\frac{n}{2}) + d(\frac{n}{4}) - 4 - \mathbf{T_6}(n)$ follows.

The proof of part 4 follows directly from 1. - 3. in connection with Theorem 2. □

Lemma 2 *The set of integers of the form $n = p^2 + pq + q^2$ is closed under multiplication. Furthermore, any square is of this form, and every prime factor p_i of a square-free number $p^2 + pq + q^2 = n = p_1 \cdots p_\ell$ is again of this form.*

This is well known in number theory, see [21]. It follows by the primitive Dirichlet character χ_D associated with the discriminant $\beta^2 - 4\alpha\gamma = D = -3$ of the binary quadratic form $\alpha x^2 + \beta xy + \gamma y^2$. The number $\widetilde{r}(n)$ of possibilities to represent an integer n as a sum $p^2 + pq + q^2$ can be calculated in terms of this Dirichlet character. Since the character is multiplicative, it can never vanish for prime factors of a square-free product unless it vanishes for the product also. In more detail we have $\sum_n \widetilde{r}(n) n^{-s} = \zeta(s) L(s, \chi_D)$ where $L(s, \chi_D) = \sum_n \chi_D(n) n^{-s}$ denotes the Dirichlet L-series for χ_D.

Lemma 3 *For any given n the number $\widetilde{r}(n)$ of possible distinct representations as $n = p^2 + pq + q^2$ is the following: Let $n = \prod_{i=1}^r p_i^{a_i} \cdot 2^{b_0} \cdot \prod_{j=1}^s q_j^{b_i} \cdot 3^{c_0}$ with distinct prime numbers $p_i \equiv 1(6)$ and $q_i \equiv 5(6)$ then*

$$\widetilde{r}(n) = \begin{cases} \prod_{i=1}^r (a_i + 1) & \text{if all numbers } b_0, \ldots, b_s \text{ are even} \\ 0 & \text{otherwise} \end{cases}.$$

Consequently, $\mathbf{T_3}(n) = \left[\frac{1}{2}\widetilde{r}(n)\right]$. In particular, every prime number $n \equiv 1(6)$ admits a unique representation $n = p^2 + pq + q^2$.

This is well known in number theory, see [12, §12.4] in a slightly different form. One possibility to prove it is the following: This function $\widetilde{r}(n)$ is multiplicative. Therefore it suffices to determine its value for powers of prime numbers. Here the Dirichlet character χ_D can be used.

Corollary 3 *There are integers n such that $\mathbf{T}(n) > n$. In other words: The number of non-isomorphic and equivelar n-vertex triangulations can be larger than n.*

Proof: The following asymptotic formula is well known as Gronwall's theorem, see [9], [10, §18.3]:

$$\overline{\lim_{n\to\infty}} \frac{\sigma(n)}{n \log\log n} = e^{\gamma}$$

where $\gamma = 0.5772...$ denotes the Euler-Mascheroni constant. Consequently we have

$$\overline{\lim_{n\to\infty}} \frac{\mathbf{T}(n)}{n \log\log n} = \frac{1}{6} e^{\gamma}.$$

The assertion follows. An explicit example (probably the smallest one) is the number

$$n = 2^5 \cdot 3^3 \cdot 5^2 \cdot 7 \cdot 11 \cdot 13 \cdot 17 \cdot 19 \cdot 23 \cdot 29 \cdot 31 \qquad \text{with}$$

$$\sigma(n) = 63 \cdot 40 \cdot 31 \cdot 8 \cdot 12 \cdot 14 \cdot 18 \cdot 20 \cdot 24 \cdot 30 \cdot 32 > 6n.$$

By a more elementary argument given in [11, p.36] we have

$$\frac{\sigma(n!)}{n!} \geq 1 + \frac{1}{2} + \cdots + \frac{1}{n}$$

which also proves our assertion. □

Proposition 4 (normal form of the associated matrix)
The set of all combinatorial types of equivelar triangulations of the torus with n vertices with a unique shortest straight edge loop is in $(1-1)$-correspondence with the set of associated matrices of the following type

$$\begin{pmatrix} a & b \\ 0 & c \end{pmatrix}$$

where $ac = n$, $a, c > 0$ and $-\frac{c}{2} \leq b \leq \frac{a-c}{2}$ and in addition

$a \geq 3$

$c > \gcd(a, b)$ *and* $c > \gcd(a, b + c)$

$b \neq 1$ *and* $b \neq \frac{a-1}{2}$ *if* $c = 1$

$b \neq 1$ *if* $c = 2$.

The proof is not given here. It is based on a more detailed check of the various cases in Proposition 2 which are equivalent to each other under rotation or reflection. In the remaining case where no unique straight edge loop exists there is a similar normal form

For small values of n these numbers are listed in the following Table I, compare [20, Table 1]:

n	$\mathbf{T_1}(n)$	$\mathbf{T_2}(n)$	$\mathbf{T_3}(n)$	$\mathbf{T_6}(n)$	$\mathbf{T}$(n)
7	0	0	1	0	1
8	0	1	0	0	1
9	1	0	0	1	2
10	1	0	0	0	1
11	1	0	0	0	1
12	2	1	0	1	4
13	1	0	1	0	2
14	2	0	0	0	2
15	2	2	0	0	4
16	2	2	0	1	5
17	2	0	0	0	2
18	4	1	0	0	5
19	2	0	1	0	3
20	4	2	0	0	6
21	3	2	1	0	6
22	4	0	0	0	4
23	3	0	0	0	3
24	5	6	0	0	11
25	4	0	0	1	5
26	5	0	0	0	5
27	5	1	0	1	7
28	6	2	1	0	9
29	4	0	0	0	4
30	9	2	0	0	11

Table I

Remark: In particular cases the number $\mathbf{T}(n)$ was also calculated or estimated by Altshuler in [1]. In his terminology these are the cases of maps with a normal Hamiltonian circuit. In our terminology this concerns the cases $c = 1$ and $a|(c - b)$.

3 Coverings between triangulated tori

Any equivelar triangulated torus T induces a hierarchy of (orientation-preserving) covering maps $T' \to T$, including the universal covering $\widetilde{T} \to T$. On the other hand, not every such torus admits a covering map down to some other (still polyhedral) torus. An example is the minimal 7-vertex torus. In this section we analyze a few results on such coverings, in particular in the regular or chiral cases. Such coverings were also investigated by [3] including a discussion of the so-called chirality index.

The number of chiral and regular maps on the torus with n vertices equals the number of possibilities to represent n as $n = p^2 + pq + q^2$ with integers $p \geq q \geq 0$. If n is a prime number then the torus is chiral and unique. Furthermore such a triangulated torus is indecomposable in the following sense:

Definition *A triangulated torus $\{3,6\}_{p,q}$ is called* decomposable *if there is a k-sheeted simplicial covering map $\Phi : \{3,6\}_{p,q} \to \{3,6\}_{r,s}$ onto another torus for some k with $1 < k < n = p^2+pq+q^2$. Otherwise $\{3,6\}_{p,q}$ is called* indecomposable.

Obviously, if n is either a prime number or the square of a prime m where m is not of the form $p^2 + pq + q^2$ then there is no such k since the number of sheets nust divide the number of vertices and n/k must be of the form $p^2 + pq + q^2$. It is a less trivial question whether any given integer divisor k of n can be realized as the number of sheets of such a simplicial covering map.

Definition (product between chiral or regular maps)
For two given chiral or regular maps $\{3,6\}_{p,q}$ and $\{3,6\}_{r,s}$ we define a *-product

$$\{3,6\}_{(p,q)*(r,s)}$$

by $(p,q) * (r,s) := (pr - qs, qr + (p+q)s)$. The neutral element is $\{3,6\}_{1,0}$ (which, however, is not polyhedral). This commutative *-product is motivated by the matrix product

$$\begin{pmatrix} p & -q \\ q & p+q \end{pmatrix} \cdot \begin{pmatrix} r & -s \\ s & r+s \end{pmatrix} = \begin{pmatrix} pr - qs & -(ps + qr + qs) \\ ps + qr + qs & pr + qr + ps \end{pmatrix}$$

of the two associated matrices.

Proposition 5 *There are covering maps*

$$\{3,6\}_{(p,q)*(r,s)} \to \{3,6\}_{p,q} \quad \textit{and} \quad \{3,6\}_{(p,q)*(r,s)} \to \{3,6\}_{r.s}$$

for any choice of p, q, r, s.

Proof. We have to consider the lattices Γ of $\{3,6\}$ generated by $\mathbf{a}$ and $\mathbf{b}$ as above with sublattices Γ' generated by $p\mathbf{a} + q\mathbf{b}$ and $-q\mathbf{a} + p\mathbf{b}$. and Γ'' generated by $r(p\mathbf{a} + q\mathbf{b}) + s(-q\mathbf{a} + p\mathbf{b})$ and $-s(p\mathbf{a} + q\mathbf{b}) + r(-q\mathbf{a} + p\mathbf{b})$. So we have $\{3,6\}_{(p,q)*(r,s)} = \{3,6\}/\Gamma''$ and $\{3,6\}_{p,q} = \{3,6\}/\Gamma'$. Consequently we have a covering map $\{3,6\}/\Gamma'' \to \{3,6\}/\Gamma'$, just by dividing out Γ'/Γ'', regarded as an abelian group of decktransformations. By commutativity of the *-operation the same holds after pairwise interchanging p, q and r, s. □

Corollary 4 *A trisngulated torus $\{3,6\}_{p,q}$ is decomposable if and only if the pair (p,q) admits a nontrivial decomposition in the sense of the *-product above (nontrivial means that no factor is the neutral element). So in some sense the indecomposable tori are the prime elements with respect to the *-product.*

Corollary 5 *Whenever $p^2 + pq + q^2 = k(r^2 + rs + s^2)$ with nonnegative integers, p, q, r, s, k then there is a k-sheeted simplicial covering from $\{3,6\}_{p,q}$ to either $\{3,6\}_{r,s}$ or $\{3,6\}_{s,r}$. In particular, if $p = mr$ and $q = ms$ then there is an m^2-sheeted simplicial covering from $\{3,6\}_{p,q}$ to $\{3,6\}_{r,s}$. Consequently, up to such coverings, it suffices to consider the case where p and q have no common divisor.*

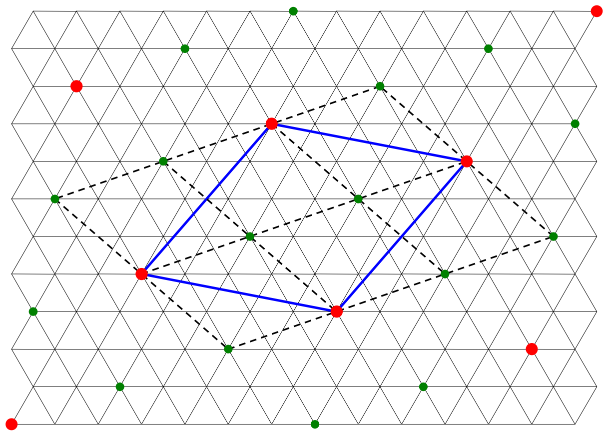

Figure 2: 3-sheeted covering from $\{3,6\}_{1,4}$ onto $\{3,6\}_{2,1}$

Examples: 1. There is a 3-sheeted covering map from $\{3,6\}_{1,4}$ with 21 vertices onto $\{3,6\}_{2,1}$ with 7 vertices. This is based on the *-product $(1,4) = (1,1) * (2,1)$. If we represent the fundamental domain of the former by a hexagon then this can be decomposed into three rhombic fundamental domains of the latter, see Figure 2.

2. There is a 7-sheeted covering map from $\{3,6\}_{11,1}$ onto $\{3,6\}_{3,2}$. If we represent the fundamental domain of the 7-vertex torus by a hexagon then it contains 7 copies of rhombi (each consisting of 2 triangles). Similarly, the hexagonal fundamental domain of $\{3,6\}_{11,1}$ with 133 vertices can be divided into 7 fundamental domains of $\{3,6\}_{3,2}$ with 19 vertices each. This defines a 7-sheeted covering. Similarly, we have a 19-sheeted covering from $\{3,6\}_{11,1}$ onto $\{3,6\}_{2,1}$. Notice that $\{3,6\}_{9,4}$ is another chiral map with the same number $n = 133$ of vertices which is combinatorially distinct from $\{3,6\}_{11,1}$. There is also a 7-sheeted covering from $\{3,6\}_{4,9}$ onto $\{3,6\}_{3,2}$.

3. There is a 7-sheeted covering from the regular $\{3,6\}_{0,7}$ onto the chiral $\{3,6\}_{2,1}$ and also a 7-sheeted covering from the chiral $\{3,6\}_{14,7}$ onto the regular $\{3,6\}_{7,0}$.

4 Equivelar quadrangulations

This section deals with tori which are decomposed into quadrilaterals. We call such a decomposition a quadrangulation. A quadrangulation is called polyhedral if the intersection of two quadrilaterals is either empty or a common vertex or a common edge. In particular this implies that any further decomposition into triangles by choosing a diagonal in each quadrilateral will be a polyhedral triangulation.

Proposition 6 *Let T be a quadrangulated torus. Then the following conditions are equivalent:*

1. *T is equivelar,*
2. *each vertex of T is 4-valent,*
3. *T has a vertex transitive automorphism group,*

4. *T can be obtained as a quotient of the regular tessellation $\{4,4\}$ by a sublattice (or a pure translation group) Γ'.*

Theorem 4 ([6])
Any chiral or regular torus is of the form $\{4,4\}_{p,q}$ with integers p, q. This notation means that the sublattice Γ' is generated by the cartesian vector (p,q) and all rotations by $\pi/2$. The number of vertices of the torus $\{4,4\}_{p,q}$ is $n = p^2 + q^2$.
A line reflection transforms $\{4,4\}_{p,q}$ into $\{4,4\}_{q,p}$ (these two are the right-handed and the left-handed version). Moreover we have $\{4,4\}_{q,p} = \{4,4\}_{p,-q}$. The quadrangulation is regular (with a D_4 as isotropy group) if and only if $pq(p-q) = 0$. In this case we have $\{4,4\}_{q,p} = \{4,4\}_{p,q}$.

The associated matrix We represent a suitable fundamental domain of a weakly regular quadrangulated torus as a parallelogram, spanned by two vectors $\mathbf{x}$ and $\mathbf{y}$, in the euclidean plane which is tessellated as $\{4,4\}$ where $\mathbf{a} = (1,0)$ and $\mathbf{b} = (0,1)$ are the standard basis vectors in a cartesian grid. We may assume that $\mathbf{x}$ is parallel to $\mathbf{a}$, so it can be represented as $\mathbf{x} = a \cdot \mathbf{a}$. Furthermore we my assume that the second vector can be written as $\mathbf{y} = b \cdot \mathbf{a} + c \cdot \mathbf{b}$. In matrix notation, we represent the quadrangulated torus by the associated matrix

$$\begin{pmatrix} a & b \\ 0 & c \end{pmatrix}$$

where a, c are positive numbers and where b may be any integer. In the case of regular or chiral tori $\{4,4\}_{(p,q)}$ it is also convenient to represent the fundamental domain by the associated matrix

$$\begin{pmatrix} p & -q \\ q & p \end{pmatrix}.$$

By analogy, Lemma 1 carries over to this case: *The number n of vertices in the torus equals the determinant of the associated matrix.*

Similarly, Proposition 2 remains valid if we replace the isotropy group D_6 of $\{3,6\}$ by the isotropy group D_4 of the tessellation $\{4,4\}$. Proposition 3 can be carried over

Theorem 5 *Let n be any given natural number. We are interested in the number $\mathbf{Q}(n)$ of all non-isomorphic equivelar (polyhedral) quadrangulations of the torus with n vertices. Let $\mathbf{Q_i}(n)$ denote the number of non-isomorphic types having an isotropy group of order $2i$. Then $\mathbf{Q_i}(n) = 0$ unless $i = 1,2,4$, and the following formula holds*

$$4\mathbf{Q_1}(n) + 2\mathbf{Q_2}(n) + \mathbf{Q_4}(n) = \begin{cases} \sigma(n) - 16 & \text{if } n \text{ is even} \\ \sigma(n) - 8 & \text{if } n \text{ is odd ,} \end{cases}$$

where $\sigma(n)$ is as above.

The proof follows by analogy with the proof of Theorem 2.

Theorem 6 (explicit number of combinatorial types)
Let $\mathbf{Q_i}(n)$ denote the number of non-isomorphic weakly regular quadrangulations of the torus with n vertices having an isotropy group of order $2i$. We have $\mathbf{Q_2}(n) = \mathbf{Q_2}^{ch}(n) + \widetilde{\mathbf{Q_2}}(n)$ where $\mathbf{Q_2}^{ch}(n)$ denotes the number of instances with a cyclic isotropy group (the chiral cases) and $\widetilde{\mathbf{Q_2}}(n)$ denotes the number of cases with a non-cyclic isotropy group. Let $d(n)$ denote the number of integer divisors of n, Then the following hold:

1. *$\mathbf{Q_4}(n)$ equals the number of possibilities to represent n as a sum $p^2 + q^2$ with $pq(p-q) = 0$ and with $q \leq p$. More precisely, $\mathbf{Q_4}(n) = 1$ if $n = m^2$ or if $n = 2m^2$ and $\mathbf{Q_4}(n) = 0$ otherwise.*

2. $\mathbf{Q_2}^{ch}(n)$ *equals the number of possibilities to represent* n *as a sum* p^2+q^2 *with* $pq(p-q)\neq 0$ *and with* $q<p$. *This number can be expressed in terms of the Riemann* ζ*-function and the Dirichlet L-function [21, p.14]:*

$$\sum_{n=1}^{\infty}\big(\mathbf{Q_4}(n)+2\mathbf{Q_2}^{ch}(n)\big)n^{-s}=\zeta(s)L(s)$$

where $L(s)=1-3^{-s}+5^{-s}-7^{-s}+\cdots$.

3. $\widetilde{\mathbf{Q_2}}(n)=\begin{cases} d(n)-2-\mathbf{Q_4}(n) & \text{if } n \text{ is odd}\\ \frac{1}{2}d(n)+d(\frac{n}{2})-4-\mathbf{Q_4}(n) & \text{if } n\equiv 2 \text{ mod } 4\\ \frac{1}{2}\Big(d(n)+d(\frac{n}{4}\Big)+d(\frac{n}{2})-6-\mathbf{Q_4}(n) & \text{if } n\equiv 0 \text{ mod } 4\end{cases}$

4. $\mathbf{Q_1}(n)$ *is uniquely determined by Theorem* 5.

Corollary 6 *For any* n *the number of non-isomorphic equivelar quadrangulations of the torus is*

$$\mathbf{Q}(n)=\frac{1}{4}\sigma(n)+\frac{1}{2}\mathbf{Q_2}(n)+\frac{3}{4}\mathbf{Q_4}(n)-\begin{cases}2 \text{ if } n \text{ is odd}\\ 4 \text{ if } n \text{ is even}\end{cases}$$

where $\mathbf{Q_2}=\mathbf{Q_2}^{ch}+\widetilde{\mathbf{Q_2}}$ *and* $\mathbf{Q_4}$ *are given in Theorem* 5 *above.*

Corollary 7 *If* n *is a prime number then we have* $\mathbf{Q}(n)=\left[\frac{n-5}{4}\right]$.

This follows from Lemma 5 below since for prime n we have $\mathbf{Q_2}(n)=1$ if $n\equiv 1(4)$ and $\mathbf{Q_2}(n)=0$ otherwise.

Proof of Theorem 6.

Part 1 is obvious. Part 2 follows from Lemma 5 below since $\mathbf{Q_4}(n)+2\mathbf{Q_2}^{ch}(n)=r(n)$. The normalization is made such that our $r(n)$ is $\frac{1}{4}$ times the $r(n)$ in [21].

Part 3: Here we have an isotropy group which is non-cyclic of order 4, hence isomorphic with $\mathbb{Z}_2\oplus\mathbb{Z}_2$. In this case a fundamental domain of the torus in the tessellated plane can be brought into exactly one of the following four distinct shapes which we have to deal with separately:

Case 1. The fundamental domain can be a non-square rectangle where the edges are parallel to the edges of the $\{4,4\}$-tessellation.
Case 2. It can be a non-square rhombus whose diagonals are parallel to the edges of the $\{4,4\}$-tessellation.
Case 3. It can be a non-square rectangle where the edges are parallel to the diagonals in the $\{4,4\}$-tessellation.
Case 4. It can be a non-square rhombus whose diagonals are parallel to the diagonals of the $\{4,4\}$-tessellation.

Accordingly we have associated matrices of the following types, given by the two edges of the rectangle or one edge and one diagonal of the rhombus, respectively:

1. The associated matrices of type $\begin{pmatrix}a & 0\\ 0 & c\end{pmatrix}$ with $2<c<a$ and $n=ac$ give rise to $[\frac{1}{2}d(n)]-1$ distinct polyhedral quadrangulations if n is odd and $[\frac{1}{2}d(n)]-2$ if n is even.

2. (for even n only) The associated matrices of type $\begin{pmatrix}a & a/2\\ 0 & c\end{pmatrix}$ with $2<2c<a$, a even and $n=ac$ give rise to $[\frac{1}{2}d(\frac{n}{2})]-1$ distinct polyhedral quadrangulations.

3. (for even n only) The associated matrices of type $\begin{pmatrix}a & -c\\ a & c\end{pmatrix}$ with $2<c<a$ and $n=2ac$ give rise to $[\frac{1}{2}d(\frac{n}{2})]-1$ distinct polyhedral quadrangulations if $n\equiv 2(4)$ and $[\frac{1}{2}d(\frac{n}{2})]-2$ if $n\equiv 0(4)$.

4. Finally, the associated matrices of type $\begin{pmatrix} a & (a-c)/2 \\ a & (a+c)/2 \end{pmatrix}$ with $2 < c < a$ and $n = ac$, $a+c$ even, give rise to $[\frac{1}{2}d(n)] - 1$ distinct polyhedral quadrangulations if n is odd and $[\frac{1}{2}d(\frac{n}{4})] - 1$ if $n \equiv 0(4)$ and 0 if $n \equiv 2(4)$. By summing up we obtain the formula for $\widetilde{\mathbf{Q_2}}(n)$ in Theorem 6. We have to use the face that $d(n)$ is odd if and only if n is a square.

Part 4 follows from 1.-3. in connection with Theorem 5. □

Lemma 4 *The set of integers of the form $n = p^2+q^2$ is closed under multiplication. Furthermore, any square is of this form, and every prime factor p_i of a square-free number $p^2+q^2 = n = p_1 \cdots p_\ell$ is again of this form.*

This follows as Lemma 2. In our case we have $D = -4$, and the Dirichlet L-series reduces to the ordinary Dirichlet function $L(s) = 1 - 3^{-s} + 5^{-s} - 7^{-s} + \cdots$ since in this case $\chi(n) = +1$ if $n \equiv 1(4)$, $\chi(n) = -1$ if $n \equiv -1(4)$, $\chi(n) = 0$ if n is even.

Lemma 5 *For any given n the number $r(n)$ of possible distinct representations as $n = p^2 + q^2$ is the following: Let $n = \prod_{i=1}^r p_i^{a_i} \cdot 2^{b_0} \cdot \prod_{j=1}^s q_j^{b_i}$ with distinct prime numbers $p_i \equiv 1(4)$ and $q_i \equiv 3(4)$ then*

$$r(n) = \begin{cases} \prod_{i=1}^r (a_i+1) & \text{if all numbers } b_0, \ldots, b_s \text{ are even} \\ 0 & \text{otherwise} \end{cases} .$$

Consequently, ${\mathbf{Q_2}}^{ch}(n) = \left[\frac{1}{2}r(n)\right]$. In particular, every prime number $n \equiv 1(4)$ admits a unique representation $n = p^2 + q^2$ (up to interchanging p and q).

This is well known in number theory, see [10, §16.9]. One possibility to prove it is the following: This function $r(n)$ is multiplicative. Therefore it suffices to determine its value for powers of prime numbers. Here the Dirichlet character χ (see above) can be used, compare [5, 1.8].

For small values of n these numbers are listed in the following Table II:

n	$\mathbf{Q_1}(n)$	$\widetilde{\mathbf{Q_2}}(n)$	${\mathbf{Q_2}}^{ch}(n)$	$\mathbf{Q_4}(n)$	$\mathbf{Q}$(n)
9	1	0	0	1	2
10	0	0	1	0	1
11	1	0	0	0	1
12	2	2	0	0	4
13	1	0	1	0	2
14	2	0	0	0	2
15	3	2	0	0	5
16	3	1	0	1	5
17	2	0	1	0	3
18	3	2	0	1	6
19	3	0	0	0	3
20	5	2	1	0	8
21	5	2	0	0	7
22	5	0	0	0	5
23	4	0	0	0	4
24	8	6	0	0	14
25	5	0	1	1	7
26	6	0	1	0	7
27	7	2	0	0	9
28	9	2	0	0	11
29	5	0	1	0	6
30	12	4	0	0	16

Table II

5 Coverings between quadrangulated tori

In this section we are going to briefly carry over some results from Section 3 to (orientation preserving) covering maps between quadrangulated tori of type $\{4,4\}$.

Definition *A quadrangulated torus* $\{4,4\}_{p,q}$ *is called* decomposable *if there is a* k*-sheeted simplicial covering map* $\Phi : \{4,4\}_{p,q} \to \{4,4\}_{r,s}$ *onto another torus for some* k *with* $1 < k < n = p^2+q^2$*. Otherwise* $\{4,4\}_{p,q}$ *is called* indecomposable.

Definition (product between chiral or regular maps)
For two given chiral or regular maps $\{4,4\}_{p,q}$ and $\{4,4\}_{r,s}$ we define a *-product

$$\{4,4\}_{(p,q)*(r,s)}$$

by $(p,q)*(r,s) := (pr-qs, qr+ps)$. The neutral element is $\{4,4\}_{1,0}$ (which, however, is not polyhedral). This commutative *-product is motivated by the matrix product

$$\begin{pmatrix} p & -q \\ q & p \end{pmatrix} \cdot \begin{pmatrix} r & -s \\ s & r \end{pmatrix} = \begin{pmatrix} pr-qs & -(qr+ps) \\ qr+ps & pr-qs \end{pmatrix}$$

of the associated matrices.

Proposition 7 *There are covering maps*

$$\{4,4\}_{(p,q)*(r,s)} \to \{4,4\}_{p,q} \quad \textit{and} \quad \{4,4\}_{(p,q)*(r,s)} \to \{4,4\}_{r.s}$$

for any choice of p,q,r,s*.*

The proof follows as the one of Proposition 5.

Corollary 8 *A quadrangulated torus* $\{4,4\}_{p,q}$ *is decomposable if and only if the pair* (p,q) *admits a nontrivial decomposition in the sense of the *-product above (nontrivial means that no factor is the neutral element). So the indecomposable tori are the prime elements with respect to the *-product.*

Corollary 9 *Whenever* $p^2+q^2 = k(r^2+s^2)$ *with nonnegative integers,* p,q,r,s,k *then there is a* k*-sheeted simplicial covering from* $\{4,4\}_{p,q}$ *to either* $\{4,4\}_{r,s}$ *or* $\{4,4\}_{s,r}$*.*

Final remark: There is a (1-1)-correspondence between the set of positive integer representations of type $n = p^2+q^2$ on the one hand and the set of (not necessarily polyhedral) regular or chiral maps of type $\{4,4\}_{p,q}$ on the other hand. By Corollary 7 this preserves the partial ordering given by factorization on the one hand and the existence of covering maps (or *-product factorization) on the other hand.. Similarly, from Section 3 we obtain the same (1-1)-correspondence between the set of positive integer representations of type $n = p^2+pq+q^2$ on the one hand and the set of (not necessarily polyhedral) regular or chiral maps of type $\{3,6\}_{p,q}$ on the other hand. Corollary 5 gives the same structure in this case. Prime numbers on the one hand correspond to indecomposable maps on the other hand. This can be regarded as a geometric interpretation of certain results in number theory. In fact, one can vivualize the various ways of representing an integer as p^2+q^2 or p^2+pq+q^2, resp., by the various regular or chiral maps and coverings between them.

Acknowledgment: We thank Rainer Dietmann for explaining to us some mysteries of binary quadratic forms in number theory and Karla Nestler for providing the figures.

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