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Homogenization with corrector for periodic differential operators. Approximation of solutions in the Sobolev class $H^1(\mathbb{R}^d)$

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Abstract

We continue to study a class of matrix periodic elliptic second order differential operators $\mathcal{A}_\varepsilon$ in $\mathbb{R}^d$ with rapidly oscillating coefficients (depending on $\mathbf{x}/\varepsilon$). This class was considered in [BSu1,2,4]. The *homogenization problem* in the small period limit is studied. We obtain approximation for the resolvent $(\mathcal{A}_\varepsilon + I)^{-1}$ in the operator norm from $L_2(\mathbb{R}^d)$ to $H^1(\mathbb{R}^d)$ with error of order ε. In this approximation, the *corrector* is taken into accout. Besides, the $(L_2 \to L_2)$-approximations of the so called fluxes are obtained.

Contents

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0 Introduction

0.1

The present paper is a continuation of a series of papers [BSu2–4] on homogenization theory for one class of (matrix) *differential operators* (DO's) acting in $L_2(\mathbb{R}^d)$. This class is rather wide, and it includes many applications. The operator-theoretic constructions (see [BSu2, Ch. 1], [BSu3] and Ch. 1 below) lie in the basis of our approach. The work [Su1,2] about homogenization of the Maxwell system is based on the same abstract material. However, in general case, the Maxwell operator can not be studied on the basis of the class of DO's considered in [BSu2,4] and in the present paper. Applications of the operator-theoretic material to DO's are based on using the Floquet-Bloch theory (precisely, on applying the Gelfand transform) in combination with an elementary scale transformation.

Formally, the present paper can be read independently of [BSu2–4]. The necessary notions and objects are defined again, the necessary results from [BSu2–4] are cited.

0.2. The class of operators

We consider elliptic positive second order matrix DO's in $L_2(\mathbb{R}^d;\mathbb{C}^n)$ admitting a representation (factorization) of the form

$$\mathcal{A} = \mathcal{A}(g,f) = f(\mathbf{x})^* b(\mathbf{D})^* g(\mathbf{x}) b(\mathbf{D}) f(\mathbf{x}). \tag{0.1}$$

Here $b(\mathbf{D})$ is a homogeneous matrix first order DO with constant coefficients. Its symbol $b(\boldsymbol{\xi})$ is an $(m \times n)$-matrix of rank n (it is assumed that $m \geq n$). The matrix-valued functions $f(\cdot)$ (of size $n \times n$) and $g(\cdot)$ (of size $m \times m$) are assumed to be *periodic* with respect to some *lattice* Γ in $\mathbb{R}^d$ and such that

$$g(\mathbf{x}) > 0; \quad g, g^{-1} \in L_\infty, \quad f, f^{-1} \in L_\infty.$$

More precise description of the operators (0.1) can be found below in Subsection 4.1.

It is useful to start with the study of a more narrow class of operators of the form

$$\widehat{\mathcal{A}} = \widehat{\mathcal{A}}(g) = b(\mathbf{D})^* g(\mathbf{x}) b(\mathbf{D}). \tag{0.2}$$

Correspondingly, we accept the „two-level“ order of exposition.

The operators (0.1) and (0.2) are considered as selfadjoint operators in $L_2(\mathbb{R}^d; \mathbb{C}^n)$. The bottom of the spectrum of $\mathcal{A}(g, f)$ is the point $\lambda = 0$:

$$\min \operatorname{spec} \mathcal{A}(g, f) = 0. \tag{0.3}$$

Along with the operators (0.1) and (0.2), we consider the following operators, whose coefficients are rapidly oscillating for small $\varepsilon > 0$:

$$\mathcal{A}_\varepsilon(g, f) = f(\varepsilon^{-1}\mathbf{x})^* b(\mathbf{D})^* g(\varepsilon^{-1}\mathbf{x}) b(\mathbf{D}) f(\varepsilon^{-1}\mathbf{x}), \quad \varepsilon > 0, \tag{0.4}$$

$$\widehat{\mathcal{A}}_\varepsilon(g) = b(\mathbf{D})^* g(\varepsilon^{-1}\mathbf{x}) b(\mathbf{D}), \quad \varepsilon > 0. \tag{0.5}$$

0.3

The *homogenization* problem can be treated as the problem of the asymptotic description of the behavior of the resolvent $(\mathcal{A}_\varepsilon(g, f) + I)^{-1}$ as $\varepsilon \to 0$. For simplicity, now we shall speak about operators of the form (0.5). In the classical homogenization theory, the following fact plays a crucial role: there exists an effective DO $\widehat{\mathcal{A}}^0 = \widehat{\mathcal{A}}(g^0)$ of the form (0.2) with the constant (*effective*) matrix g^0 such that the *resolvent* $(\widehat{\mathcal{A}}_\varepsilon(g) + I)^{-1}$ *converges* (in some sense) to the resolvent $(\widehat{\mathcal{A}}^0 + I)^{-1}$. Usually the strong (or even weak) convergence is considered. Further correction terms to $(\widehat{\mathcal{A}}^0 + I)^{-1}$ are constructed. The main of them is the term $\varepsilon K(\varepsilon)$ of order ε, where the operator $K(\varepsilon)$ is called a *corrector*. The error estimates are obtained under the smoothness conditions on the coefficients f and g and smoothness conditions on the right-hand sides of the corresponding equations.

The rule of constructing g^0 for a given matrix g is known. It is not quite elementary, but is visible. Under our conditions, this rule is described below in §5.

For the first time in the homogenization theory, the error estimates of order ε (sharp-order) in the L_2-norm uniform with respect to the right-hand sides were obtained in [BSu1,2]. In terms of the resolvents, these estimates have the following form:

$$\|(\widehat{\mathcal{A}}_\varepsilon + I)^{-1} - (\widehat{\mathcal{A}}^0 + I)^{-1}\|_{L_2(\mathbb{R}^d;\mathbb{C}^n) \to L_2(\mathbb{R}^d;\mathbb{C}^n)} \le C\varepsilon. \tag{0.6}$$

The constant C depends only on the lattice Γ, on the upper and lower bounds for the matrix-valued function g and on the constants in the inequality (4.2) (see below). The similar but somewhat more complicated approximations were obtained for operators of the form (0.4).

0.4. The method of investigation

We put

$$(T_\varepsilon u)(\mathbf{y}) = \varepsilon^{d/2} u(\varepsilon \mathbf{y}), \quad \mathbf{y} \in \mathbb{R}^d.$$

The *scaling operator* T_ε is unitary in $L_2(\mathbb{R}^d; \mathbb{C}^n)$. Obviously,

$$(\widehat{\mathcal{A}}_\varepsilon + I)^{-1} = \varepsilon^2 T_\varepsilon^* (\widehat{\mathcal{A}} + \varepsilon^2 I)^{-1} T_\varepsilon,$$
$$(\widehat{\mathcal{A}}^0 + I)^{-1} = \varepsilon^2 T_\varepsilon^* (\widehat{\mathcal{A}}^0 + \varepsilon^2 I)^{-1} T_\varepsilon.$$

These identities allow us to deduce (0.6) from the estimate

$$\|(\widehat{\mathcal{A}} + \varepsilon^2 I)^{-1} - (\widehat{\mathcal{A}}^0 + \varepsilon^2 I)^{-1}\|_{L_2(\mathbb{R}^d;\mathbb{C}^n) \to L_2(\mathbb{R}^d;\mathbb{C}^n)} \le C\varepsilon^{-1}. \qquad (0.7)$$

This way the inequality (0.6) has been obtained in [BSu1,2]. The operators T_ε, T_ε^* have no influence, when we „return“ from (0.7) to (0.6), since they are unitary.

The passage from (0.7) to (0.6) is quite trivial. At the same time, it clearly illustrates that the homogenization procedure is a manifestation of the spectral *threshold effect*: we have to know the behavior of the resolvent of a periodic operator $\widehat{\mathcal{A}} = \widehat{\mathcal{A}}(g)$ near the bottom $\lambda = 0$ (see (0.3)) of its spectrum. It is the latter problem that was solved in [BSu1,2].

In order to obtain estimate (0.7), we transform the operator $\widehat{\mathcal{A}}$ applying the (unitary) Gelfand transformation $\mathcal{V}$ (see definition below in Subsection 4.3). Let Ω be the cell of the lattice Γ, and let $\mathbf{k} \in \mathbb{R}^d$ be the parameter (the *quasimomentum*). Under the Gelfand transformation $\mathcal{V}$, the operator $\widehat{\mathcal{A}}$ turns into the operator of multiplication by an appropriate operator-valued function $\widehat{\mathcal{A}}(\mathbf{k})$. For each $\mathbf{k}$, the operators $\widehat{\mathcal{A}}(\mathbf{k})$ are selfadjoint operators in $L_2(\Omega; \mathbb{C}^n)$ with compact resolvent. The latter allows us to use the general spectral analytic perturbation theory (with respect to the one-dimensional parameter $t = |\mathbf{k}|$) and to obtain some analog of the inequality (0.7) for the operator $(\widehat{\mathcal{A}}(\mathbf{k}) + \varepsilon^2 I)^{-1}$ with the constant independent of $\mathbf{k}$. Then application of the inverse transformation $\mathcal{V}^*$ leads to (0.7).

The method described above may seem to be somewhat roundabout. But it is this method that openes the way to delicate calculations and sharp-order estimates. This method turned out to be effective also for obtaining more accurate approximation of the resolvent $(\widehat{\mathcal{A}}_\varepsilon + I)^{-1}$ in the $(L_2(\mathbb{R}^d; \mathbb{C}^n))$-operator norm with the error estimate of order ε^2 (see [BSu4]). For this, terms of order ε must be included in approximation, i. e., the *corrector* must be taken into account. Evidently, the corresponding constructions become more complicated.

We emphasize that, if, for instance, one studies convergence of $(\widehat{\mathcal{A}}_\varepsilon + I)^{-1}$ to $(\widehat{\mathcal{A}}^0 + I)^{-1}$ in the strong sense, then applying the scaling transformation gives nothing. Indeed, then on the „coming back“ step the factors T_ε, T_ε^* are the obstacles for the limit procedure as $\varepsilon \to 0$.

0.5. The corrector

In [BSu4], the following estimate has been obtained:

$$\|(\widehat{\mathcal{A}}_\varepsilon + I)^{-1} - (\widehat{\mathcal{A}}^0 + I)^{-1} - \varepsilon \widetilde{K}(\varepsilon)\|_{L_2(\mathbb{R}^d;\mathbb{C}^n) \to L_2(\mathbb{R}^d;\mathbb{C}^n)} \le C\varepsilon^2. \qquad (0.8)$$

As it was shown in [BSu4], the bounded operator $\widetilde{K}(\varepsilon)$ (the *corrector*), must contain three terms:

$$\widetilde{K}(\varepsilon) = K_1(\varepsilon) + K_2(\varepsilon) + K_3. \tag{0.9}$$

Note that $K_2(\varepsilon) = K_1(\varepsilon)^*$ and that K_3 does not depend on ε. The expression for $\widetilde{K}(\varepsilon)$ is given below in (10.9). Clearly, $\widetilde{K}(\varepsilon)$ is not defined uniquely, since we can add some terms of order $O(\varepsilon)$ to it.

In the traditional homogenization theory, the corrector contains only one term, which differs from our term $K_1(\varepsilon)$ by the absence of the additional *smoothing operator*. Our smoothing operator Π_ε is defined by (10.4); it is possible to use another *smoothing operators*. In [BSu4], it was discussed in detail, when the smoothing operator can be eliminated; it is not always possible, if we want to preserve estimate (0.8). Also in [BSu4] the cases where $K_3 = 0$ were distinguished. At the same time, it was shown that in the *vector* problems of mathematical physics (elasticity theory, electrodynamics), in general, the term K_3 is nontrivial and should not be ignored.

0.6. The main goal of the present paper

is the proof of the estimate

$$\|(\widehat{\mathcal{A}}_\varepsilon + I)^{-1} - (\widehat{\mathcal{A}}^0 + I)^{-1} - \varepsilon K_1(\varepsilon)\|_{L_2(\mathbb{R}^d;\mathbb{C}^n)\to H^1(\mathbb{R}^d;\mathbb{C}^n)} \le C\varepsilon. \tag{0.10}$$

It is easily seen that the estimate (0.10) can be reduced to the inequality

$$\|\widehat{\mathcal{A}}_\varepsilon^{1/2}\left((\widehat{\mathcal{A}}_\varepsilon + I)^{-1} - (\widehat{\mathcal{A}}^0 + I)^{-1} - \varepsilon K_1(\varepsilon)\right)\|_{L_2(\mathbb{R}^d;\mathbb{C}^n)\to L_2(\mathbb{R}^d;\mathbb{C}^n)} \le C\varepsilon. \tag{0.11}$$

This inequality can be obtained on the same way as the estimate (0.8). For this, one should rely on the abstact results of §2.

Thus, we see that in (0.10) the role of corrector is played by the first term in (0.9). One may also use the full corrector $\widetilde{K}(\varepsilon)$ in (0.10), (0.11); this will change only the constants in the right-hand side. Using this version, one may interpolate between (0.10) and (0.8), which leads to the estimate

$$\|(\widehat{\mathcal{A}}_\varepsilon + I)^{-1} - (\widehat{\mathcal{A}}^0 + I)^{-1} - \varepsilon\widetilde{K}(\varepsilon)\|_{L_2(\mathbb{R}^d;\mathbb{C}^n)\to H^s(\mathbb{R}^d;\mathbb{C}^n)} \le C_s\varepsilon^{2-s}, \quad 0 \le s \le 1.$$

In what follows, we distinguish the cases, where the smoothing operator Π_ε in the corrector in (0.10) may be eliminated. The required restrictions are harder than the similar restrictions related to estimate (0.8).

Estimates of the type (0.10) give possibility to approximate the *fluxes*, i. e., the operators $g(\varepsilon^{-1}\mathbf{x})b(\mathbf{D})(\widehat{\mathcal{A}}_\varepsilon + I)^{-1}$. The corresponding error estimate of order ε in the $(L_2(\mathbb{R}^d;\mathbb{C}^n) \to L_2(\mathbb{R}^d;\mathbb{C}^m))$-norm is given (in somewhat different form) below in Theorem 12.1. In a number of important cases (acoustics, elasticity theory, electrodynamics), the fluxes are of direct interest.

Note that estimates of the form (0.10) for the acoustics operator and the operator of elasticity theory were obtained in recent papers [Zh, Pas, ZhPas]. In these papers, an essentially different (non-spectral) method was used. This method is related to including an additional parameter $\boldsymbol{\omega}$ in the problem, which corresponds to the shift of the vectors of the lattice Γ by an arbitrary vector $\boldsymbol{\omega} \in \Omega$. This way leads to the appearance of the *Steklov averaging operator* as a smoothing operator (different from our smoothing operator Π_ε). However, as well as in our method, afterwards this smoothing operator is eliminated for the acoustics operator and, in the case $d = 2$, also for the elasticity operator.

0.7. The generalized resolvent

What was said in Subsections 0.3–0.6 is related to the resolvent $(\widehat{\mathcal{A}}_\varepsilon + I)^{-1}$ of the operator (0.5). Besides, we study similar questions also for the *generalized resolvent* of the operator $\widehat{\mathcal{A}}_\varepsilon$. Namely, let $Q(\mathbf{x})$ be a positive Γ-periodic $(n \times n)$-matrix-valued function such that $Q, Q^{-1} \in L_\infty$, and let $Q^\varepsilon(\mathbf{x}) = Q(\varepsilon^{-1}\mathbf{x})$. The operator $(\widehat{\mathcal{A}}_\varepsilon + Q^\varepsilon)^{-1}$ is called a generalized resolvent of the operator $\widehat{\mathcal{A}}_\varepsilon$. The analogs of all the statements already proved for the ordinary resolvent $(\widehat{\mathcal{A}}_\varepsilon + I)^{-1}$ are proved afterwards for the generalized resolvent. This corresponds to the „two-level“ order of exposition, which has already been mentioned. Herewith, we rely on the abstract results of §3 and also on the corresponding material for the ordinary resolvent. In is important to note that the study of the resolvent $(\mathcal{A}_\varepsilon + I)^{-1}$ for an arbitrary operator of the form (0.4) is reduced to the study of the generalized resolvent $(\widehat{\mathcal{A}}_\varepsilon + Q^\varepsilon)^{-1}$ with the same matrix g and an appropriate matrix-valued function Q. However, the generalized resolvent is of its own interest.

0.8. The structure of the paper

The paper consists of four chapters and a small concluding §23. The necessary operator theoretic material is contained in Chapter 1 (§1–3). In Chapter 2 (§4–9) the threshold approximations (i. e., as $\varepsilon \to 0$) for $\widehat{\mathcal{A}}^{1/2}(\widehat{\mathcal{A}} + \varepsilon^2 I)^{-1}$ (in §8) and for $\widehat{\mathcal{A}}^{1/2}(\widehat{\mathcal{A}} + \varepsilon^2 Q)^{-1}$ (in §9) are studied. In Chapter 3 (§10–16), the homogenization results are deduced from the threshold approximations. Namely, approximations of the resolvents $(\widehat{\mathcal{A}}_\varepsilon + I)^{-1}$ and $(\widehat{\mathcal{A}}_\varepsilon + Q^\varepsilon)^{-1}$ in the $(L_2 \to H^1)$-norm and approximations of the corresponding fluxes in the $(L_2 \to L_2)$-norm are obtained. Besides, approximation for the operator $f^\varepsilon(\mathcal{A}_\varepsilon + I)^{-1}$ in the $(L_2 \to H^1)$-norm is obtained, where $\mathcal{A}_\varepsilon$ is of general type (0.4), and $f^\varepsilon(\mathbf{x}) = f(\varepsilon^{-1}\mathbf{x})$. In all cases, the corrector is taken into account in approximations, and the error term is of order ε. In Chapter 4 (§17–22), general results of Chapter 3 are interpreted for a number of operators of mathematical physics. The choice of examples was motivated by their own significance, as well as by the will to illustrate various exceptional and special cases, which were distinguished in the general theory. In more details, the content of the paper can be seen from the table of contents and from the prefaces in the beginnings of the chapters. A small separate §23 is devoted to additional comments and references.

Below, the constants in estimates either are controlled explicitly, or admit such a control (probably, bulky), in principal.

0.9. Notation

Let $\mathcal{G}$, $\mathcal{G}_*$ be separable Hilbert spaces. The symbols $(\cdot,\cdot)_\mathcal{G}$ and $\|\cdot\|_\mathcal{G}$ stand for the inner product and the norm in $\mathcal{G}$, correspondingly. The symbol $\|\cdot\|_{\mathcal{G}\to\mathcal{G}_*}$ stands for the norm of a linear continuous operator acting from $\mathcal{G}$ to $\mathcal{G}_*$. Sometimes we omit indices, if this does not lead to confusion. By $I = I_\mathcal{G}$ we denote the identity operator in $\mathcal{G}$. If $A : \mathcal{G} \to \mathcal{G}_*$ is a linear operator, then $\operatorname{Dom} A$ stands for its domain. If $\mathfrak{N}$ is a subspace in $\mathcal{G}$, then $\mathfrak{N}^\perp := \mathcal{G} \ominus \mathfrak{N}$. If P is the orthogonal projection of $\mathcal{G}$ onto $\mathfrak{N}$, then $P^\perp$ is the orthogonal projection onto $\mathfrak{N}^\perp$. The spectrum of a closed operator T in $\mathcal{G}$ is denoted by $\operatorname{spec} T$. The symbol $\langle\cdot,\cdot\rangle = \langle\cdot,\cdot\rangle_n$ is the standard inner product in $\mathbb{C}^n$, $|\cdot|$ is the norm of a vector in $\mathbb{C}^n$; $\mathbf{1} = \mathbf{1}_n$ is the unit $(n \times n)$-matrix. For $z \in \mathbb{C}$, by z^* we denote the complex conjugate number; we use such non-standard notation, since notation $\overline{g}$ is used for the mean value of the matrix-valued function g. For an $(m \times n)$-matrix a, the symbol $|a|$ stands for the norm of the corresponding linear operator from $\mathbb{C}^n$ to $\mathbb{C}^m$; the symbol a^t denotes the transposed matrix, and a^* is the Hermitian adjoint $(n \times m)$-matrix.

The L_p-classes of $\mathbb{C}^n$-valued functions in domain $\mathcal{O} \subset \mathbb{R}^d$ are denoted by $L_p(\mathcal{O};\mathbb{C}^n)$, $1 \le p \le \infty$. The Sobolev classes of $\mathbb{C}^n$-valued functions (in domain $\mathcal{O} \subseteq \mathbb{R}^d$) of order s and summability index p are denoted by $W_p^s(\mathcal{O};\mathbb{C}^n)$. If $p = 2$, we use the notation $H^s(\mathcal{O};\mathbb{C}^n)$, $s \in \mathbb{R}$. For $n = 1$, we use the simplified notation $L_p(\mathcal{O})$, $W_p^s(\mathcal{O})$, $H^s(\mathcal{O})$, etc. But sometimes (if this does not lead to confusion), we use such simplified notation also for the spaces of vector-valued or matrix-valued functions.

Next, $\mathbf{x} = (x^1, \dots, x^d) \in \mathbb{R}^d$, $iD_j = \partial_j = \partial/\partial x^j$, $j = 1, \dots, d$, $\nabla = \mathrm{grad} = (\partial_1, \dots, \partial_d)$, $\mathbf{D} = -i\nabla = (D_1, \dots, D_d)$.

Below C, c, $\mathcal{C}$, $\mathfrak{C}$ (probably, with indices and marks) denote various constants in estimates. By β with indices and marks we denote various *absolute constants.*

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Chapter 1. The abstract method

This Chapter (§1–3) contains the facts of the abstract theory of selfadjoint operators. These facts are basic for further constructions, and they are additional to the facts given in [BSu2, Ch. 1] and [BSu3]. We still rely on the spectral perturbation theory for factorized selfadjoint operator families which quadratically depend on the parameter. Part of the material is borrowed from [BSu2,3] and is given without proofs. New facts are completely proved.

The main results of Chapter 1 are contained in §2 and §3. Formally, the statements of §3 contain the statements of §2. However, for the proof of the results of §3, as well as for convenience of further applications, the two-level exposition is useful.

1 Preliminaries

1.1. A factorized family $A(t)$

Let $\mathfrak{H}$ and $\mathfrak{H}_*$ be complex separable Hilbert spaces. Let $X_0 : \mathfrak{H} \to \mathfrak{H}_*$ be a densely defined closed operator, and let $X_1 : \mathfrak{H} \to \mathfrak{H}_*$ be a bounded operator. We put

$$X(t) := X_0 + tX_1, \quad \mathrm{Dom}\, X(t) = \mathrm{Dom}\, X_0, \ \ t \in \mathbb{R}.$$

The family of selfadjoint operators

$$A(t) := X(t)^* X(t), \quad t \in \mathbb{R}, \tag{1.1}$$

in $\mathfrak{H}$ is our main object. The operator (1.1) is generated by the quadratic form

$$a(t)[u,u] = \|X(t)u\|^2_{\mathfrak{H}_*}, \quad u \in \mathrm{Dom}\, X_0,$$

which is closed in $\mathfrak{H}$. We denote $A(0) = X_0^* X_0 =: A_0$,

$$\mathfrak{N} := \operatorname{Ker} A_0 = \operatorname{Ker} X_0, \quad \mathfrak{N}_* := \operatorname{Ker} X_0^*.$$

By P and P_* we denote the orthogonal projections of $\mathfrak{H}$ onto $\mathfrak{N}$ and of $\mathfrak{H}_*$ onto $\mathfrak{N}_*$, respectively. We use notation of the type $P^\perp = I - P$, $\mathfrak{N}^\perp = P^\perp \mathfrak{H}$, etc.

It is assumed that the point $\lambda_0 = 0$ is an isolated point of the spectrum of the operator A_0 (an eigenvalue), and that

$$0 < n := \dim \mathfrak{N} < \infty, \quad n \le n_* := \dim \mathfrak{N}_* \le \infty.$$

By d^0 we denote the distance from the point $\lambda_0 = 0$ to the rest of the spectrum of A_0. Let $F(t,s)$ be the spectral projection of the operator $A(t)$ for an interval $[0,s]$. *We fix a number* $\delta > 0$ *such that* $8\delta < d^0$. It turns out that (see [BSu2, (1.1.3)])

$$F(t,\delta) = F(t, 3\delta), \quad \operatorname{rank} F(t,\delta) = n, \quad \text{for } |t| \le t^0 := \delta^{1/2} \|X_1\|^{-1}. \tag{1.2}$$

Below we usually write $F(t)$ instead of $F(t,\delta)$.

1.2. The operators Z and R

Let $\mathcal{D} := \operatorname{Dom} X_0 \cap \mathfrak{N}^\perp$, and let $u \in \mathfrak{H}_*$. We consider the following equation for an element $\psi \in \mathcal{D}$ (cf. [BSu2, (1.1.7)]):

$$(X_0 \psi, X_0 \zeta)_{\mathfrak{H}_*} = (u, X_0 \zeta)_{\mathfrak{H}_*}, \quad \forall \zeta \in \mathcal{D}. \tag{1.3}$$

There exists a unique solution ψ of the equation (1.3), moreover, $\|X_0 \psi\|_{\mathfrak{H}_*} \le \|u\|_{\mathfrak{H}_*}$. Now, let

$$\omega \in \mathfrak{N}, \quad u = -X_1 \omega; \tag{1.4}$$

the corresponding solution of the equation (1.3) is denoted by $\psi(\omega)$. We define the bounded operator $Z : \mathfrak{H} \to \mathfrak{H}$ by the following relations:

$$Z\omega = \psi(\omega), \ \ \omega \in \mathfrak{N}; \quad Zx = 0, \ \ x \in \mathfrak{N}^\perp. \tag{1.5}$$

Clearly, $\operatorname{rank} Z \le n$. As it was shown in [BSu3, (1.8)],

$$\|Z\|_{\mathfrak{H} \to \mathfrak{H}} \le (8\delta)^{-1/2} \|X_1\|_{\mathfrak{H} \to \mathfrak{H}_*} = (2\sqrt{2} t^0)^{-1}. \tag{1.6}$$

Note that

$$ZP = Z, \quad PZ = 0. \tag{1.7}$$

Suppose now that (1.4) is satisfied, and $\psi(\omega)$ is the solution of equation (1.3). We put

$$\omega_* = X_0 \psi(\omega) + X_1 \omega$$

and introduce the operator R (see [BSu2, Subsection 1.1.2]):

$$R : \mathfrak{N} \to \mathfrak{N}_*, \quad R\omega = \omega_* \in \mathfrak{N}_*.$$

Another description of R is given by the formula

$$R = P_* X_1|_{\mathfrak{N}}. \tag{1.8}$$

1.3. The spectral germ

The selfadjoint operator

$$S = R^* R : \mathfrak{N} \to \mathfrak{N} \tag{1.9}$$

is called the *spectral germ* of the operator family (1.1) at $t = 0$ (see [BSu2, Subsection 1.1.3]). From (1.8) and (1.9) it follows that

$$S = P X_1^* P_* X_1|_{\mathfrak{N}}.$$

The germ S is called *non-degenerate* if $\operatorname{Ker} S = \{0\}$, or, equivalently, if $\operatorname{rank} R = n$.

According to the general analytic perturbation theory (see [K]), for $|t| \le t^0$ there exist real-analytic functions $\lambda_l(t)$ and real-analytic $\mathfrak{H}$-valued functions $\varphi_l(t)$ such that

$$A(t)\varphi_l(t) = \lambda_l(t)\varphi_l(t), \ \ l = 1, \dots, n, \quad |t| \le t^0,$$

and the $\varphi_l(t)$, $l = 1, \dots, n$, form an orthonormal basis in $F(t)\mathfrak{H}$. For sufficiently small t_* $(\le t^0)$ and $|t| \le t_*$, we have the following convergent power series expansions:

$$\lambda_l(t) = \gamma_l t^2 + \mu_l t^3 + \cdots, \quad \gamma_l \ge 0, \ \ \mu_l \in \mathbb{R}, \ \ l = 1, \dots, n, \tag{1.10}$$

$$\varphi_l(t) = \omega_l + t\varphi_l^{(1)} + t^2 \varphi_l^{(2)} + \cdots, \quad l = 1, \dots, n. \tag{1.11}$$

The elements $\omega_l = \varphi_l(0)$, $l = 1, \dots, n$, form an orthonormal basis in $\mathfrak{N}$. As it was shown in [BSu2, Subsection 1.1.6], the numbers γ_l and the elements ω_l, $l = 1, \dots, n$, are *eigenvalues and eigenvectors of the operator* S:

$$S\omega_l = \gamma_l \omega_l, \quad l = 1, \dots, n.$$

1.4. Threshold approximations

In [BSu2, Theorem 1.4.1], it was shown that

$$\begin{gathered} F(t) - P = \Phi(t), \\ \|\Phi(t)\|_{\mathfrak{H}\to\mathfrak{H}} \le C_1|t|, \quad |t| \le t^0, \ \ C_1 = \beta_1 \delta^{-1/2}\|X_1\|. \end{gathered} \tag{1.12}$$

As it has already been mentioned, by β (with indices) we denote various *absolute constants.* For instance, one can put $\beta_1 = 25(1+\pi^{-1})(1+\sqrt{2})2^{-3/2}$.

Next, in [BSu2, Theorem 1.4.3], it was shown that

$$\begin{gathered} A(t)F(t) - t^2 SP = \Psi(t), \\ \|\Psi(t)\|_{\mathfrak{H}\to\mathfrak{H}} \le C_2|t|^3, \quad |t| \le t^0, \ \ C_2 = \beta_2 \delta^{-1/2}\|X_1\|^3. \end{gathered} \tag{1.13}$$

1.5. The contour Γ_δ. The difference of resolvents

We put $R_z(t) := (A(t) - zI)^{-1}$, $R_z(0) := (A(0) - zI)^{-1}$. We need to integrate the difference of resolvents over the complex contour Γ_δ that envelopes the interval $[0, \delta]$ equidistantly at the distance δ. By (1.2), for $|t| \le t^0$ the contour Γ_δ is separated from the spectrum of $A(t)$ by the distance δ. Therefore,

$$\|R_z(t)\|_{\mathfrak{H}\to\mathfrak{H}} \le \delta^{-1}, \ \ z \in \Gamma_\delta, \quad |t| \le t^0. \tag{1.14}$$

We recall the notation introduced in [BSu2, §1.2, 1.3]. Putting

$$\Omega_z(0) = I + (z + 2\delta)R_z(0), \tag{1.15}$$

we have (see [BSu2, (1.3.3)])

$$\|\Omega_z(0)\|_{\mathfrak{H}\to\mathfrak{H}} \leq 5, \;\; z \in \Gamma_\delta, \quad |t| \leq t^0.$$

As in [BSu2, Subsection 1.3.2], by $\mathfrak{d}$ we denote the Hilbert space $\operatorname{Dom} X_0$ with the metric form

$$\|u\|^2_{\mathfrak{d}} := \|X_0 u\|^2_{\mathfrak{H}_*} + 2\delta\|u\|^2_{\mathfrak{H}}. \tag{1.16}$$

Let $T^{(1)}_\delta$ and $T^{(2)}_\delta$ be the selfadjoint continuous operators in $\mathfrak{d}$ generated by the forms $2\mathrm{Re}\,(X_0 u, X_1 u)_{\mathfrak{H}_*}$ and $\|X_1 u\|^2_{\mathfrak{H}_*}$, $u \in \mathfrak{d}$, respectively. We have (see [BSu2, (1.3.7), (1.3.8)])

$$\|T^{(1)}_\delta\|_{\mathfrak{d}\to\mathfrak{d}} \leq (2\delta)^{-1/2}\|X_1\|, \quad \|T^{(2)}_\delta\|_{\mathfrak{d}\to\mathfrak{d}} \leq (2\delta)^{-1}\|X_1\|^2. \tag{1.17}$$

Denote

$$T_\delta(t) = tT^{(1)}_\delta + t^2 T^{(2)}_\delta.$$

Then (see [BSu2, (1.3.9)])

$$\|T_\delta(t)\|_{\mathfrak{d}\to\mathfrak{d}} \leq \frac{\sqrt{2}+1}{2}|t|\delta^{-1/2}\|X_1\|, \quad |t| \leq t^0. \tag{1.18}$$

We have the following representation (see [BSu2, (1.3.11)]):

$$R_z(0) - R_z(t) = \Omega_z(0)T_\delta(t)R_z(t), \quad z \in \Gamma_\delta. \tag{1.19}$$

Iterating (1.19), we distinguish the term of order t in the right-hand side:

$$\begin{aligned} R_z(0) - R_z(t) &= t\,\Omega_z(0)T^{(1)}_\delta R_z(t) + t^2\Omega_z(0)T^{(2)}_\delta R_z(t) \\ &= t\,\Omega_z(0)T^{(1)}_\delta(R_z(0) - \Omega_z(0)T_\delta(t)R_z(t)) + t^2\Omega_z(0)T^{(2)}_\delta R_z(t). \end{aligned}$$

Then

$$R_z(0) - R_z(t) = t\,\Omega_z(0)T^{(1)}_\delta R_z(0) + \Psi_1(t,z), \tag{1.20}$$

where

$$\Psi_1(t,z) = -t\,\Omega_z(0)T^{(1)}_\delta\Omega_z(0)T_\delta(t)R_z(t) + t^2\Omega_z(0)T^{(2)}_\delta R_z(t). \tag{1.21}$$

1.6. Representation for the projection $F(t)$

From the formula

$$F(t) - P = \frac{1}{2\pi i}\int_{\Gamma_\delta}(R_z(0) - R_z(t))\,dz$$

and representation (1.20) it follows that (cf. [BSu3, (2.10), (2.11)])

$$F(t) - P = tF_1 + F_2(t), \tag{1.22}$$

where

$$F_1 = \frac{1}{2\pi i}\int_{\Gamma_\delta}\Omega_z(0)T^{(1)}_\delta R_z(0)\,dz,$$

$$F_2(t) = \frac{1}{2\pi i}\int_{\Gamma_\delta}\Psi_1(t,z)\,dz. \tag{1.23}$$

In [BSu3, (2.15)] it was shown that

$$F_1 = ZP + PZ^*. \tag{1.24}$$

Combining (1.24) with (1.7), we see that

$$F_1 P = ZP. \tag{1.25}$$

2 Approximation for the operator-valued function $A(t)^{1/2}(A(t) + \varepsilon^2 I)^{-1}$

2.1

We assume that (cf. [BSu2, (1.5.1)]) for some $c_* > 0$ we have

$$A(t) \geq c_* t^2 I, \quad |t| \leq t^0. \tag{2.1}$$

This is equivalent to the fact that the eigenvalues $\lambda_l(t)$ of the operator $A(t)$ satisfy the estimates

$$\lambda_l(t) \geq c_* t^2, \quad |t| \leq t^0, \ \ l = 1, \dots, n. \tag{2.2}$$

Then (1.10) implies that

$$\gamma_l \geq c_*, \quad l = 1, \dots, n. \tag{2.3}$$

Thus, the *germ S is non-degenerate.* We introduce the notation

$$\Xi = \Xi(t, \varepsilon) := (t^2 S + \varepsilon^2 I_{\mathfrak{N}})^{-1} P. \tag{2.4}$$

Obviously, (2.1) and (2.3) imply the estimates

$$\begin{gathered} \|(A(t) + \varepsilon^2 I)^{-1}\|_{\mathfrak{H} \to \mathfrak{H}} \leq (c_* t^2 + \varepsilon^2)^{-1}, \quad |t| \leq t^0, \\ \|\Xi(t, \varepsilon)\|_{\mathfrak{H} \to \mathfrak{H}} \leq (c_* t^2 + \varepsilon^2)^{-1}. \end{gathered} \tag{2.5}$$

2.2

In [BSu1,2], it was shown that for small $\varepsilon > 0$ operator (2.4) gives the principal term of approximation for the resolvent $(A(t) + \varepsilon^2 I)^{-1}$. Herewith,

$$\|(A(t) + \varepsilon^2 I)^{-1} - \Xi(t, \varepsilon)\|_{\mathfrak{H} \to \mathfrak{H}} \leq C_3 \varepsilon^{-1}, \quad 0 < \varepsilon \leq 1, \ \ |t| \leq t^0. \tag{2.6}$$

Note that each of the operators $(A(t) + \varepsilon^2 I)^{-1}$ and $\Xi(t, \varepsilon)$ is of order ε^{-2}, while the difference is of order ε^{-1}. The constant C_3 depends only on δ, $\|X_1\|$ and c_*, and is given by

$$C_3 = C_1 c_*^{-1/2} + \frac{1}{2} C_2 c_*^{-3/2} + (3\delta)^{-1},$$

where C_1 and C_2 are defined by (1.12), (1.13). The estimate (2.6) is order-sharp.

In [BSu3], more accurate approximation with the „corrector“ for the resolvent $(A(t) + \varepsilon^2 I)^{-1}$ has been found. Now we formulate the corresponding result. We introduce the following operator in $\mathfrak{H}$ (see [BSu3, (4.13)]):

$$N = Z^* X_1^* R P + (RP)^* X_1 Z. \tag{2.7}$$

Note that (see [BSu3, (4.16)])

$$\|N\|_{\mathfrak{H}\to\mathfrak{H}} \le (2\delta)^{-1/2}\|X_1\|^3. \tag{2.8}$$

By Theorem 5.1 from [BSu3], we have

$$\|(A(t)+\varepsilon^2 I)^{-1} - \Xi - t(Z\Xi + \Xi Z^*) + t^3 \Xi N \Xi\|_{\mathfrak{H}\to\mathfrak{H}} \le C_4, \quad 0<\varepsilon\le 1,\ |t|\le t^0. \tag{2.9}$$

The constant C_4 depends only on δ, $\|X_1\|$ and on c_*, and is given by

$$C_4 = \delta^{-1}(\beta_1^\circ c_*^{-2}\|X_1\|^4 + \beta_2^\circ c_*^{-3}\|X_1\|^6 + \beta_3^\circ c_*^{-1}\|X_1\|^2 + 1/3).$$

The terms

$$t(Z\Xi + \Xi Z^*) - t^3 \Xi N \Xi \tag{2.10}$$

play the role of the „corrector“. We see that, in order to approximate the resolvent $(A(t)+\varepsilon^2 I)^{-1}$ in the operator norm in $\mathfrak{H}$ with error of order $O(1)$, one has to add corrector (2.10) which contains three terms to the principal term $\Xi(t,\varepsilon)$.

2.3

Now our goal is to approximate for small ε the operator-valued function

$$\mathfrak{A}_\varepsilon(t) := A(t)^{1/2}(A(t)+\varepsilon^2 I)^{-1}, \quad \varepsilon>0, \tag{2.11}$$

in the operator norm in $\mathfrak{H}$ with error $O(1)$. For this, it suffices to add only one term $tZ\Xi$ to Ξ (instead of the three-term corrector (2.10)). The following theorem is true.

Theorem 2.1. *Suppose that the operator family $A(t)$ satisfies conditions of Subsection* 1.1 *and also condition* (2.1). *Let Z be the operator defined by* (1.5), *and let S be the spectral germ of the family $A(t)$ at $t=0$. Let $\Xi(t,\varepsilon)$ be the operator* (2.4). *Then for $|t|\le t^0$ and $0<\varepsilon\le 1$ we have*

$$\|A(t)^{1/2}\left((A(t)+\varepsilon^2 I)^{-1} - (I+t\,Z)\Xi(t,\varepsilon)\right)\|_{\mathfrak{H}\to\mathfrak{H}} \le C_5. \tag{2.12}$$

The constant C_5 is defined below in (2.24)*; it depends only on δ, $\|X_1\|$ and c_*.*

2.4

We start the proof of Theorem 2.1. By (2.2),

$$\|\mathfrak{A}_\varepsilon(t)F(t)\|_{\mathfrak{H}\to\mathfrak{H}} = \sup_{1\le l\le n}\sqrt{\lambda_l(t)}(\lambda_l(t)+\varepsilon^2)^{-1} \le c_*^{-1/2}|t|^{-1}, \quad |t|\le t^0. \tag{2.13}$$

Combining this with (1.12), we see that

$$\|\mathfrak{A}_\varepsilon(t)F(t)(F(t)-P)\|_{\mathfrak{H}\to\mathfrak{H}} \le C_1 c_*^{-1/2}, \quad |t|\le t^0. \tag{2.14}$$

Note that, by (1.2),

$$\|\mathfrak{A}_\varepsilon(t)F(t)^\perp\|_{\mathfrak{H}\to\mathfrak{H}} \le (3\delta)^{-1/2}, \quad |t|\le t^0. \tag{2.15}$$

Now we write down the resolvent identity (see [BSu3, (5.1)])

$$F(t)(A(t)+\varepsilon^2 I)^{-1}P$$
$$= F(t)\Xi(t,\varepsilon) - F(t)(A(t)+\varepsilon^2 I)^{-1}(A(t)F(t) - t^2 SP)\Xi(t,\varepsilon).$$

Multiplying this identity by $A(t)^{1/2}$, we obtain

$$\mathfrak{A}_\varepsilon(t)F(t)P = F(t)A(t)^{1/2}\Xi(t,\varepsilon) - \mathfrak{A}_\varepsilon(t)F(t)(A(t)F(t) - t^2SP)\Xi(t,\varepsilon).$$

We estimate the second term on the right, using (2.13), (1.13), and (2.5). Then

$$\|\mathfrak{A}_\varepsilon(t)F(t)P - F(t)A(t)^{1/2}\Xi(t,\varepsilon)\|_{\mathfrak{H}\to\mathfrak{H}} \le C_2 c_*^{-3/2}, \quad |t| \le t^0. \tag{2.16}$$

Next, by (1.22) and (1.25),

$$\begin{aligned} F(t)A(t)^{1/2}\Xi(t,\varepsilon) &= A(t)^{1/2}(P + tF_1 + F_2(t))\Xi(t,\varepsilon) \\ &= A(t)^{1/2}(\Xi(t,\varepsilon) + tZ\Xi(t,\varepsilon) + F_2(t)\Xi(t,\varepsilon)). \end{aligned} \tag{2.17}$$

Obviously,

$$\mathfrak{A}_\varepsilon(t) = \mathfrak{A}_\varepsilon(t)F(t)^\perp + \mathfrak{A}_\varepsilon(t)F(t)(F(t)-P) + \mathfrak{A}_\varepsilon(t)F(t)P.$$

Combining this with (2.17) and using (2.11) and (2.14)–(2.16), for $|t| \le t^0$, we obtain:

$$\begin{aligned} &\|A(t)^{1/2}\left((A(t)+\varepsilon^2 I)^{-1} - (I+tZ)\Xi(t,\varepsilon)\right)\|_{\mathfrak{H}\to\mathfrak{H}} \\ &\le (3\delta)^{-1/2} + C_1 c_*^{-1/2} + C_2 c_*^{-3/2} + \|A(t)^{1/2}F_2(t)\Xi(t,\varepsilon)\|_{\mathfrak{H}\to\mathfrak{H}}. \end{aligned} \tag{2.18}$$

It remains to estimate the last term in the right-hand side of (2.18).

2.5. Estimate for the term $A(t)^{1/2}F_2(t)\Xi(t,\varepsilon)$

We want to use representation (1.21), (1.23) for $F_2(t)$. First, we prove four lemmas.

Lemma 2.2. *We have*

$$\|A(t)^{1/2}\|_{\mathfrak{d}\to\mathfrak{H}} \le \sqrt{2}, \quad |t| \le t^0. \tag{2.19}$$

Proof. For $u \in \mathfrak{d}$, by (1.1), (1.2), and (1.16), we have:

$$\begin{aligned} \|A(t)^{1/2}u\|_{\mathfrak{H}} &= \|(X_0 + tX_1)u\|_{\mathfrak{H}_*} \le \|X_0 u\|_{\mathfrak{H}_*} + t^0\|X_1\|\|u\|_{\mathfrak{H}} \\ &\le \|X_0 u\|_{\mathfrak{H}_*} + \delta^{1/2}\|u\|_{\mathfrak{H}} \le \sqrt{2}\|u\|_{\mathfrak{d}}, \quad |t| \le t^0. \quad \square \end{aligned}$$

Lemma 2.3. *For $z \in \Gamma_\delta$ and $|t| \le t^0$ the resolvent $R_z(t) = (A(t) - zI)^{-1}$ satisfies the estimate*

$$\|R_z(t)\|_{\mathfrak{H}\to\mathfrak{d}} \le \sqrt{10}\,\delta^{-1/2}. \tag{2.20}$$

Proof. For $u \in \mathfrak{H}$ and $|t| \le t^0$, by (1.1), (1.2), and (1.16), we have:

$$\begin{aligned} \|R_z(t)u\|_{\mathfrak{d}}^2 &= \|X_0 R_z(t)u\|_{\mathfrak{H}_*}^2 + 2\delta\|R_z(t)u\|_{\mathfrak{H}}^2 \\ &\le (\|(X_0 + tX_1)R_z(t)u\|_{\mathfrak{H}_*} + |t|\|X_1\|\|R_z(t)u\|_{\mathfrak{H}})^2 + 2\delta\|R_z(t)u\|_{\mathfrak{H}}^2 \\ &\le 2(A(t)R_z(t)u, R_z(t)u)_{\mathfrak{H}} + (2(t^0\|X_1\|)^2 + 2\delta)\|R_z(t)u\|_{\mathfrak{H}}^2 \\ &= 2(u + zR_z(t)u, R_z(t)u)_{\mathfrak{H}} + 4\delta\|R_z(t)u\|_{\mathfrak{H}}^2. \end{aligned}$$

Combining this with (1.14) and using the inequality $|z| \le 2\delta$ for $z \in \Gamma_\delta$, we obtain:

$$\|R_z(t)u\|_{\mathfrak{d}}^2 \le 2\|u\|_{\mathfrak{H}}\|R_z(t)u\|_{\mathfrak{H}} + 8\delta\|R_z(t)u\|_{\mathfrak{H}}^2 \le 10\delta^{-1}\|u\|_{\mathfrak{H}}^2.$$

This proves estimate (2.20). $\square$

Lemma 2.4. *For the operator* (1.15) *we have*

$$\|\Omega_z(0)\|_{\mathfrak{d}\to\mathfrak{d}} \le 1 + 4\sqrt{5}, \quad z \in \Gamma_\delta. \tag{2.21}$$

Proof. By (1.15) and (2.20) (with $t = 0$), for $u \in \mathfrak{d}$ and $z \in \Gamma_\delta$ we have:

$$\|\Omega_z(0)u\|_{\mathfrak{d}} \le \|u\|_{\mathfrak{d}} + |z + 2\delta| \|R_z(0)u\|_{\mathfrak{d}} \le \|u\|_{\mathfrak{d}} + 4\delta\sqrt{10}\delta^{-1/2}\|u\|_{\mathfrak{H}}.$$

Since $\|u\|_{\mathfrak{H}} \le (2\delta)^{-1/2}\|u\|_{\mathfrak{d}}$, (2.21) follows. □

Lemma 2.5. *Let* $\Psi_1(t, z)$ *be the operator* (1.21)*. Then for* $|t| \le t^0$ *and* $z \in \Gamma_\delta$ *we have:*

$$\|A(t)^{1/2}\Psi_1(t, z)\|_{\mathfrak{H}\to\mathfrak{H}} \le t^2\beta_3\delta^{-3/2}\|X_1\|^2. \tag{2.22}$$

Proof. We have:

$$A(t)^{1/2}\Psi_1(t, z) = A(t)^{1/2}\left(-t\,\Omega_z(0)T_\delta^{(1)}\Omega_z(0)T_\delta(t) + t^2\Omega_z(0)T_\delta^{(2)}\right)R_z(t).$$

Then

$$\|A(t)^{1/2}\Psi_1(t, z)\|_{\mathfrak{H}\to\mathfrak{H}} \le \|A(t)^{1/2}\|_{\mathfrak{d}\to\mathfrak{H}}\|R_z(t)\|_{\mathfrak{H}\to\mathfrak{d}}\|\Omega_z(0)\|_{\mathfrak{d}\to\mathfrak{d}}$$
$$\times\left(|t|\|\Omega_z(0)\|_{\mathfrak{d}\to\mathfrak{d}}\|T_\delta^{(1)}\|_{\mathfrak{d}\to\mathfrak{d}}\|T_\delta(t)\|_{\mathfrak{d}\to\mathfrak{d}} + t^2\|T_\delta^{(2)}\|_{\mathfrak{d}\to\mathfrak{d}}\right).$$

Combining this with (2.19)–(2.21), (1.17), and (1.18), we obtain:

$$\|A(t)^{1/2}\Psi_1(t, z)\|_{\mathfrak{H}\to\mathfrak{H}} \le 2(\sqrt{5} + 20)\delta^{-1/2}t^2$$
$$\times\left((1 + 4\sqrt{5})(\sqrt{2} + 1)2^{-3/2}\delta^{-1}\|X_1\|^2 + (2\delta)^{-1}\|X_1\|^2\right) = t^2\beta_3\delta^{-3/2}\|X_1\|^2,$$

where $\beta_3 = (\sqrt{5} + 20)((1 + 4\sqrt{5})(1 + 2^{-1/2}) + 1)$. □

Now, representation (1.23) and estimate (2.22) imply that

$$\|A(t)^{1/2}F_2(t)\|_{\mathfrak{H}\to\mathfrak{H}} \le (2\pi)^{-1}(2\delta + 2\pi\delta)t^2\beta_3\delta^{-3/2}\|X_1\|^2$$
$$= t^2\beta_3(1 + \pi^{-1})\delta^{-1/2}\|X_1\|^2, \quad |t| \le t^0. \tag{2.23}$$

Combining this with (2.5), we obtain:

$$\|A(t)^{1/2}F_2(t)\Xi(t, \varepsilon)\|_{\mathfrak{H}\to\mathfrak{H}} \le \beta_3(1 + \pi^{-1})c_*^{-1}\delta^{-1/2}\|X_1\|^2, \quad |t| \le t^0.$$

Together with (2.18), this implies estimate (2.12) with

$$C_5 = (3\delta)^{-1/2} + C_1c_*^{-1/2} + C_2c_*^{-3/2} + \beta_3(1 + \pi^{-1})c_*^{-1}\delta^{-1/2}\|X_1\|^2. \tag{2.24}$$

This completes the proof of Theorem 2.1. □

2.6. Interpolation

In (2.12) the one-term corrector $tZ\Xi$ can be replaced by the „full“ corrector (2.10). Only the constant in estimate will change.

Theorem 2.6. *Suppose that conditions of Theorem* 2.1 *are satisfied. Let* N *be the operator* (2.7)*. Then for* $0 < \varepsilon \le 1$ *and* $|t| \le t^0$ *we have*

$$\|A(t)^{1/2}\left((A(t) + \varepsilon^2 I)^{-1} - \Xi - t(Z\Xi + \Xi Z^*) + t^3\Xi N\Xi\right)\|_{\mathfrak{H}\to\mathfrak{H}} \le C_6. \tag{2.25}$$

The constant C_6 *is defined below in* (2.29)*; it depends only on* δ, $\|X_1\|$ *and* c_*.

Proof. By (2.12), it suffices to estimate the terms $tA(t)^{1/2}\Xi Z^*$ and $t^3A(t)^{1/2}\Xi N\Xi$. Note that

$$\|A(t)^{1/2}P\|_{\mathfrak{H}\to\mathfrak{H}} \le |t|\|X_1\|, \tag{2.26}$$

which follows from (1.1):

$$\|A(t)^{1/2}Pu\|_{\mathfrak{H}} = \|(X_0 + tX_1)Pu\|_{\mathfrak{H}_*} = \|tX_1Pu\|_{\mathfrak{H}_*} \le |t|\|X_1\|\|u\|_{\mathfrak{H}}.$$

Since $A(t)^{1/2}\Xi = A(t)^{1/2}P\Xi$, then (2.26), (2.5), and (1.6) imply that

$$\|tA(t)^{1/2}\Xi Z^*\|_{\mathfrak{H}\to\mathfrak{H}} \le t^2(8\delta)^{-1/2}\|X_1\|^2(c_*t^2+\varepsilon^2)^{-1} \le (8\delta)^{-1/2}\|X_1\|^2c_*^{-1}. \tag{2.27}$$

By (2.26), (2.5), and (2.8), the term $t^3A(t)^{1/2}\Xi N\Xi = t^3A(t)^{1/2}P\Xi N\Xi$ satisfies the estimate

$$\|t^3A(t)^{1/2}\Xi N\Xi\|_{\mathfrak{H}\to\mathfrak{H}} \le t^4(2\delta)^{-1/2}\|X_1\|^4(c_*t^2+\varepsilon^2)^{-2} \le (2\delta)^{-1/2}\|X_1\|^4c_*^{-2}. \tag{2.28}$$

Now, (2.12), (2.27), and (2.28) yield estimate (2.25) with the constant

$$C_6 = C_5 + (8\delta)^{-1/2}\|X_1\|^2c_*^{-1} + (2\delta)^{-1/2}\|X_1\|^4c_*^{-2}. \quad \square \tag{2.29}$$

A simple interpolation between (2.9) and (2.25) implies the following result.

Theorem 2.7. *Under conditions of Theorem* 2.6*, for* $0 < \varepsilon \le 1$ *and* $|t| \le t^0$ *we have*

$$\|A(t)^{s/2}\left((A(t)+\varepsilon^2I)^{-1} - \Xi - t(Z\Xi + \Xi Z^*) + t^3\Xi N\Xi\right)\|_{\mathfrak{H}\to\mathfrak{H}} \le C_4^{1-s}C_6^s, \quad 0 \le s \le 1.$$

2.7. The case of zero corrector

We distinguish the case where $Z = 0$. Then the corrector in (2.12) is equal to zero. The three-term corrector (2.10) is also equal to zero, since $N = 0$ in this case (see (2.7)). Correspondingly, (2.12) turns into the estimate

$$\|A(t)^{1/2}\left((A(t)+\varepsilon^2I)^{-1} - \Xi(t,\varepsilon)\right)\|_{\mathfrak{H}\to\mathfrak{H}} \le C_5, \quad 0 < \varepsilon \le 1, \ |t| \le t^0, \tag{2.30}$$

and (2.9) turns into

$$\|(A(t)+\varepsilon^2I)^{-1} - \Xi(t,\varepsilon)\|_{\mathfrak{H}\to\mathfrak{H}} \le C_4, \quad 0 < \varepsilon \le 1, \ |t| \le t^0. \tag{2.31}$$

Interpolating between (2.30) and (2.31), we obtain the following statement.

Theorem 2.8. *Suppose that conditions of Theorem* 2.1 *are satisfied. Suppose also that* $Z = 0$*. Then for* $0 < \varepsilon \le 1$ *and* $|t| \le t^0$ *we have*

$$\|A(t)^{s/2}\left((A(t)+\varepsilon^2I)^{-1} - \Xi(t,\varepsilon)\right)\|_{\mathfrak{H}\to\mathfrak{H}} \le C_4^{1-s}C_5^s, \quad 0 \le s \le 1.$$

3 Approximation for the operator-valued function $\widehat{A}(t)^{1/2}(\widehat{A}(t)+\varepsilon^2 Q)^{-1}$

3.1. Preliminaries

(See [BSu2, Subsections 1.1.5 and 1.5.3]). Let $\widehat{\mathfrak{H}}$ be yet another Hilbert space, and let $M : \mathfrak{H} \to \widehat{\mathfrak{H}}$ be an isomorphism. Let $\widehat{X}(t) = \widehat{X}_0 + t\widehat{X}_1 : \widehat{\mathfrak{H}} \to \mathfrak{H}_*$ be a family of the same type as $X(t)$. Suppose that

$$M \operatorname{Dom} X_0 = \operatorname{Dom} \widehat{X}_0, \quad X_0 = \widehat{X}_0 M, \ \ X_1 = \widehat{X}_1 M. \tag{3.1}$$

Then $X(t) = \widehat{X}(t)M$. Consider the family of operators

$$\widehat{A}(t) = \widehat{X}(t)^* \widehat{X}(t) : \widehat{\mathfrak{H}} \to \widehat{\mathfrak{H}}. \tag{3.2}$$

Obviously,

$$A(t) = M^* \widehat{A}(t) M. \tag{3.3}$$

In what follows, all the objects corresponding to the family (3.2) are marked by the upper index „$\frown$“. Note that

$$\begin{gathered} \widehat{\mathfrak{N}} = M\mathfrak{N}, \quad \widehat{n} = n, \\ \widehat{\mathfrak{N}}_* = \mathfrak{N}_*, \quad \widehat{n}_* = n_*, \quad \widehat{P}_* = P_*, \end{gathered} \tag{3.4}$$

$$R = \widehat{R}M|_{\mathfrak{N}}, \quad S = PM^*\widehat{S}M|_{\mathfrak{N}}, \quad \widehat{S} = \widehat{P}(M^*)^{-1}SM^{-1}|_{\widehat{\mathfrak{N}}}.$$

Denote

$$Q := (MM^*)^{-1} = (M^*)^{-1}M^{-1} : \widehat{\mathfrak{H}} \to \widehat{\mathfrak{H}}. \tag{3.5}$$

The operator Q is positive and continuous together with Q^{-1}.

Sometimes, it is convenient to assume that initially the operator $\widehat{A}(t)$ of the form (3.2) and the continuous positive definite operator Q in $\widehat{\mathfrak{H}}$ are given. Then, since Q admits a (non-unique) representation of the form (3.5), we can construct the operator $A(t)$ by the formula (3.3).

From condition (2.1) it follows that

$$\widehat{A}(t) \geq \widehat{c}_* t^2 I, \quad |t| \leq t^0, \tag{3.6}$$

where $\widehat{c}_* = c_* \|M\|^{-2}$.

The selfadjoint operator

$$(\widehat{A}(t) + \varepsilon^2 Q)^{-1}, \quad \varepsilon > 0, \tag{3.7}$$

in $\widehat{\mathfrak{H}}$ is called the *generalized resolvent* (*Q-resolvent*) of the family $\widehat{A}(t)$. From (3.3) and (3.5) it follows that

$$(\widehat{A}(t) + \varepsilon^2 Q)^{-1} = M(A(t) + \varepsilon^2 I)^{-1} M^*. \tag{3.8}$$

By (3.6), the germ $\widehat{S}$ is non-degenerate (as well as S).

3.2

Let $Q_{\widehat{\mathfrak{N}}}$ be the block of the operator Q in the subspace $\widehat{\mathfrak{N}}$:

$$Q_{\widehat{\mathfrak{N}}} = \widehat{P}Q|_{\widehat{\mathfrak{N}}} : \widehat{\mathfrak{N}} \to \widehat{\mathfrak{N}}.$$

Then the operator

$$(t^2\widehat{S} + \varepsilon^2 Q_{\widehat{\mathfrak{N}}})^{-1} : \widehat{\mathfrak{N}} \to \widehat{\mathfrak{N}}$$

exists. We put

$$\widehat{\Xi}_Q(t,\varepsilon) = (t^2\widehat{S} + \varepsilon^2 Q_{\widehat{\mathfrak{N}}})^{-1}\widehat{P}. \tag{3.9}$$

Then the operators (2.4) and (3.9) satisfy the relation (cf. [BSu2, (1.5.18)])

$$M\Xi(t,\varepsilon)M^* = \widehat{\Xi}_Q(t,\varepsilon). \tag{3.10}$$

In [BSu2, Subsection 1.5.3], it was shown that the operator (3.9) gives the principal term of approximation for the generalized resolvent (3.7). Herewith,

$$\|(\widehat{A}(t) + \varepsilon^2 Q)^{-1} - \widehat{\Xi}_Q(t,\varepsilon)\|_{\widehat{\mathfrak{H}}\to\widehat{\mathfrak{H}}} \le C_3\|M\|^2\varepsilon^{-1}, \quad 0 < \varepsilon \le 1, \ \ |t| \le t^0.$$

In [BSu3, §6], more accurate approximation for the generalized resolvent (3.7) was found. Now we describe the corresponding result. Let $\widehat{Z}_Q$ be the operator in $\widehat{\mathfrak{H}}$ which takes an element $\widehat{u} \in \widehat{\mathfrak{H}}$ to the solution $\widehat{\psi}_Q$ of the equation

$$\widehat{X}_0^*(\widehat{X}_0\widehat{\psi}_Q + \widehat{X}_1\widehat{\omega}) = 0, \quad Q\widehat{\psi}_Q \perp \widehat{\mathfrak{N}},$$

where $\widehat{\omega} = \widehat{P}\widehat{u} \in \widehat{\mathfrak{N}}$. Then (see [BSu3, Lemma 6.1])

$$\widehat{Z}_Q = MZM^{-1}\widehat{P}, \tag{3.11}$$

where Z is the operator (1.5). We put (see [BSu3, (6.18)])

$$\widehat{N}_Q = \widehat{P}(M^*)^{-1}NM^{-1}\widehat{P} = \widehat{Z}_Q^*\widehat{X}_1^*\widehat{R}\widehat{P} + (\widehat{R}\widehat{P})^*\widehat{X}_1\widehat{Z}_Q.$$

(Recall that the operator N is defined by (2.7).)

By Theorem 6.3 from [BSu3], for $0 < \varepsilon \le 1$ and $|t| \le t^0$ we have

$$\|(\widehat{A}(t) + \varepsilon^2 Q)^{-1} - \widehat{\Xi}_Q - t(\widehat{Z}_Q\widehat{\Xi}_Q + \widehat{\Xi}_Q\widehat{Z}_Q^*) + t^3\widehat{\Xi}_Q\widehat{N}_Q\widehat{\Xi}_Q\|_{\widehat{\mathfrak{H}}\to\widehat{\mathfrak{H}}} \le C_4\|M\|^2, \tag{3.12}$$

where $\widehat{\Xi}_Q$ is the operator (3.9), and C_4 is the constant from (2.9). The terms

$$t(\widehat{Z}_Q\widehat{\Xi}_Q + \widehat{\Xi}_Q\widehat{Z}_Q^*) - t^3\widehat{\Xi}_Q\widehat{N}_Q\widehat{\Xi}_Q \tag{3.13}$$

play the role of the „corrector“. Thus, in order to approximate the generalized resolvent $(\widehat{A}(t) + \varepsilon^2 Q)^{-1}$ in the operator norm in $\widehat{\mathfrak{H}}$ with error $O(1)$, one should add the three-term corrector (3.13) to the principal term $\widehat{\Xi}_Q$.

3.3

Now we want to approximate the operator-valued function

$$\widehat{A}(t)^{1/2}(\widehat{A}(t)+\varepsilon^2 Q)^{-1}, \quad \varepsilon>0,$$

in the operator norm in $\widehat{\mathfrak{H}}$ with error $O(1)$. For this, it suffices to add only one term $t\widehat{Z}_Q\widehat{\Xi}_Q$ to $\widehat{\Xi}_Q$ (instead of the three-term corrector (3.13)).

Theorem 3.1. *Under the assumptions of Subsections* 3.1 *and* 3.2*, we have*

$$\|\widehat{A}(t)^{1/2}\big((\widehat{A}(t)+\varepsilon^2 Q)^{-1}-\widehat{\Xi}_Q-t\widehat{Z}_Q\widehat{\Xi}_Q\big)\|_{\widehat{\mathfrak{H}}\to\widehat{\mathfrak{H}}}\le C_5\|M\|, \qquad 0<\varepsilon\le 1,\ \ |t|\le t^0. \tag{3.14}$$

Proof follows from estimate (2.12) by recalculation. By (3.8), (3.10), (3.11), (3.1), and (3.4), we have:

$$\begin{aligned}
&\|\widehat{A}(t)^{1/2}\big((\widehat{A}(t)+\varepsilon^2 Q)^{-1}-\widehat{\Xi}_Q-t\widehat{Z}_Q\widehat{\Xi}_Q\big)\|_{\widehat{\mathfrak{H}}\to\widehat{\mathfrak{H}}}\\
&=\|(\widehat{X}_0+t\widehat{X}_1)\big(M(A(t)+\varepsilon^2 I)^{-1}M^*-M\Xi M^*-tMZM^{-1}\widehat{P}M\Xi M^*\big)\|_{\widehat{\mathfrak{H}}\to\mathfrak{H}_*}\\
&=\|(X_0+tX_1)\left((A(t)+\varepsilon^2 I)^{-1}-\Xi-tZ\Xi\right)M^*\|_{\widehat{\mathfrak{H}}\to\mathfrak{H}_*}\\
&\le\|M\|\|A(t)^{1/2}\left((A(t)+\varepsilon^2 I)^{-1}-\Xi-tZ\Xi\right)\|_{\mathfrak{H}\to\mathfrak{H}}.
\end{aligned}$$

Combining this with (2.12), we obtain (3.14). □

3.4. Interpolation

In (3.14), the one-term corrector $t\widehat{Z}_Q\widehat{\Xi}_Q$ can be replaced by the full corrector (3.13). The next statement follows from (2.25) by recalculation.

Theorem 3.2. *For* $0<\varepsilon\le 1$ *and* $|t|\le t^0$ *we have*

$$\|\widehat{A}(t)^{1/2}\big((\widehat{A}(t)+\varepsilon^2 Q)^{-1}-\widehat{\Xi}_Q-t(\widehat{Z}_Q\widehat{\Xi}_Q+\widehat{\Xi}_Q\widehat{Z}_Q^*)+t^3\widehat{\Xi}_Q\widehat{N}_Q\widehat{\Xi}_Q\big)\|_{\widehat{\mathfrak{H}}\to\widehat{\mathfrak{H}}}\le C_6\|M\|. \tag{3.15}$$

Interpolating between (3.12) and (3.15), we arrive at the following result.

Theorem 3.3. *For* $0<\varepsilon\le 1$, $|t|\le t^0$ *and* $0\le s\le 1$ *we have:*

$$\|\widehat{A}(t)^{s/2}\big((\widehat{A}(t)+\varepsilon^2 Q)^{-1}-\widehat{\Xi}_Q-t(\widehat{Z}_Q\widehat{\Xi}_Q+\widehat{\Xi}_Q\widehat{Z}_Q^*)+t^3\widehat{\Xi}_Q\widehat{N}_Q\widehat{\Xi}_Q\big)\|_{\widehat{\mathfrak{H}}\to\widehat{\mathfrak{H}}} \le C_4^{1-s}C_6^s\|M\|^{2-s}.$$

3.5. The case of zero corrector

In the case where $\widehat{Z}_Q=0$ (which is equivalent to the condition $Z=0$), the corrector in (3.14) is equal to zero. Then the three-term corrector (3.13) is also equal to zero. In this case (3.14) turns into the estimate

$$\|\widehat{A}(t)^{1/2}\big((\widehat{A}(t)+\varepsilon^2 Q)^{-1}-\widehat{\Xi}_Q\big)\|_{\widehat{\mathfrak{H}}\to\widehat{\mathfrak{H}}}\le C_5\|M\|, \quad 0<\varepsilon\le 1,\ \ |t|\le t^0, \tag{3.16}$$

and (3.12) turns into the estimate

$$\|(\widehat{A}(t)+\varepsilon^2 Q)^{-1}-\widehat{\Xi}_Q\|_{\widehat{\mathfrak{H}}\to\widehat{\mathfrak{H}}}\le C_4\|M\|^2, \quad 0<\varepsilon\le 1,\ \ |t|\le t^0. \tag{3.17}$$

Interpolating between (3.16) and (3.17), we arrive at the following statement.

Theorem 3.4. *Suppose that conditions of Theorem* 3.1 *are satisfied. Suppose also that* $\widehat{Z}_Q=0$ (*or, equivalently,* $Z=0$). *Then for* $0<\varepsilon\le 1$ *and* $|t|\le t^0$ *we have*

$$\|\widehat{A}(t)^{s/2}\big((\widehat{A}(t)+\varepsilon^2 Q)^{-1}-\widehat{\Xi}_Q\big)\|_{\widehat{\mathfrak{H}}\to\widehat{\mathfrak{H}}}\le C_4^{1-s}C_5^s\|M\|^{2-s}, \quad 0\le s\le 1.$$

Chapter 2. Periodic differential operators in $L_2(\mathbb{R}^d;\mathbb{C}^n)$. Threshold approximations for the resolvents

In this Chapter, we consider periodic elliptic second order DO's acting in the space $L_2(\mathbb{R}^d;\mathbb{C}^n)$. We study the behavior of the resolvent near the spectral threshold. In §4, 5, we give the detailed description of the classes of periodic operators under consideration and their images under the Gelfand transform; the definition and the properties of the effective matrix g^0 are discussed. In §6, 7, the required threshold approximations for the operator family $\widehat{\mathcal{A}}(\mathbf{k})$ are obtained; this operator family acts in $L_2(\Omega;\mathbb{C}^n)$ and depends on the quasimomentum $\mathbf{k}$. In §8, 9, from these approximations we deduce approximations near the threshold for the resolvent and the generalized resolvent of the periodic operator $\widehat{\mathcal{A}}$ acting in $L_2(\mathbb{R}^d;\mathbb{C}^n)$. The approximations are accompanied by the error estimates in the operator norm.

4 Classes of periodic operators. Direct integral expansion

Here we recall the description of a class of matrix second order differential operators (DO's) admitting a factorization of the form $\mathcal{A} = \mathcal{X}^*\mathcal{X}$, where $\mathcal{X}$ is a homogeneous first order DO. This class was distinguished and studied in [BSu1,2].

4.1. Factorized second order operators

Let $b(\mathbf{D}) : L_2(\mathbb{R}^d;\mathbb{C}^n) \to L_2(\mathbb{R}^d;\mathbb{C}^m)$ be a homogeneous first order DO with constant coefficients. *We always assume that* $m \geq n$. The symbol $b(\boldsymbol{\xi})$, $\boldsymbol{\xi} \in \mathbb{R}^d$, of the operator $b(\mathbf{D})$ is an $(m \times n)$-matrix-valued linear homogeneous function of $\boldsymbol{\xi}$. Suppose that

$$\operatorname{rank} b(\boldsymbol{\xi}) = n, \quad 0 \neq \boldsymbol{\xi} \in \mathbb{R}^d. \tag{4.1}$$

From (4.1) it follows that

$$\alpha_0 \mathbf{1}_n \leq b(\boldsymbol{\theta})^* b(\boldsymbol{\theta}) \leq \alpha_1 \mathbf{1}_n, \quad \boldsymbol{\theta} \in \mathbb{S}^{d-1}, \ 0 < \alpha_0 \leq \alpha_1 < \infty, \tag{4.2}$$

with some constants α_0, α_1.

Suppose that an $(n \times n)$-matrix-valued function $f(\mathbf{x})$ and an $(m \times m)$-matrix-valued function $h(\mathbf{x})$ are bounded, together with their inverses:

$$f, \ f^{-1} \in L_\infty(\mathbb{R}^d); \quad h, \ h^{-1} \in L_\infty(\mathbb{R}^d). \tag{4.3}$$

We consider the DO

$$\mathcal{X} := hb(\mathbf{D})f : L_2(\mathbb{R}^d;\mathbb{C}^n) \to L_2(\mathbb{R}^d;\mathbb{C}^m),$$
$$\operatorname{Dom} \mathcal{X} := \{\mathbf{u} \in L_2(\mathbb{R}^d;\mathbb{C}^n) : \ f\mathbf{u} \in H^1(\mathbb{R}^d;\mathbb{C}^n)\}.$$

The operator $\mathcal{X}$ is closed. The selfadjoint operator $\mathcal{A}(g,f) = \mathcal{A} := \mathcal{X}^*\mathcal{X}$ in $L_2(\mathbb{R}^d;\mathbb{C}^n)$ is generated by the closed quadratic form

$$a[\mathbf{u},\mathbf{u}] := \|\mathcal{X}\mathbf{u}\|^2_{L_2(\mathbb{R}^d;\mathbb{C}^m)}, \quad \mathbf{u} \in \operatorname{Dom} \mathcal{X}.$$

Formally,

$$\mathcal{A}(g,f) = \mathcal{A} = f(\mathbf{x})^* b(\mathbf{D})^* g(\mathbf{x}) b(\mathbf{D}) f(\mathbf{x}), \quad g(\mathbf{x}) := h(\mathbf{x})^* h(\mathbf{x}). \tag{4.4}$$

By using the Fourier transformation and conditions (4.2) and (4.3), it is easy to show that

$$c_0 \int\limits_{\mathbb{R}^d} |\mathbf{D}(f\mathbf{u})|^2\, d\mathbf{x} \le a[\mathbf{u},\mathbf{u}] \le c_1 \int\limits_{\mathbb{R}^d} |\mathbf{D}(f\mathbf{u})|^2\, d\mathbf{x}, \quad \mathbf{u} \in \operatorname{Dom} \mathcal{X},$$

where

$$c_0 = \alpha_0 \|h^{-1}\|^{-2}_{L_\infty}, \quad c_1 = \alpha_1 \|h\|^2_{L_\infty}. \tag{4.5}$$

4.2. Lattices Γ and $\widetilde{\Gamma}$

In what follows, the *matrix-valued functions* f and h *are assumed to be periodic with respect to some lattice* $\Gamma \subset \mathbb{R}^d$. By $\Omega \subset \mathbb{R}^d$ we denote the elementary cell of the lattice Γ. Next, let $\widetilde{\Gamma}$ be the dual lattice, and let $\widetilde{\Omega}$ be the Brillouin zone of $\widetilde{\Gamma}$. (See [BSu2, Ch. 2, Subsection 1.2].) Note that $|\Omega||\widetilde{\Omega}| = (2\pi)^d$. Let r_0 denote the *radius of the ball inscribed in* $\operatorname{clos}\widetilde{\Omega}$, *and put* $r_1 = \max_{\mathbf{k}\in\partial\widetilde{\Omega}} |\mathbf{k}|$. Note that

$$2r_0 = \min |\mathbf{b}|, \quad 0 \ne \mathbf{b} \in \widetilde{\Gamma}. \tag{4.6}$$

We denote $\mathcal{B}(r) = \{\mathbf{k} \in \mathbb{R}^d : \ |\mathbf{k}| \le r\}$.

By $\widetilde{H}^s(\Omega)$ *we denote the subspace of all functions in* $H^s(\Omega)$ *such that the* Γ*-periodic extension of them to* $\mathbb{R}^d$ *belongs to* $H^s_{\mathrm{loc}}(\mathbb{R}^d)$.

4.3. The Gelfand transformation

Initially, the Gelfand transformation $\mathcal{V}$ is defined on the functions $\mathbf{v} \in \mathcal{S}(\mathbb{R}^d;\mathbb{C}^n)$ of the Schwartz class by the formula

$$\widetilde{\mathbf{v}}(\mathbf{k},\mathbf{x}) = (\mathcal{V}\mathbf{v})(\mathbf{k},\mathbf{x}) = |\widetilde{\Omega}|^{-1/2} \sum_{\mathbf{a}\in\Gamma} \exp(-i\langle \mathbf{k}, \mathbf{x}+\mathbf{a}\rangle)\mathbf{v}(\mathbf{x}+\mathbf{a}),$$
$$\mathbf{x} \in \mathbb{R}^d, \quad \mathbf{k} \in \mathbb{R}^d.$$

Since

$$\int\limits_{\widetilde{\Omega}} \int\limits_{\Omega} |\widetilde{\mathbf{v}}(\mathbf{k},\mathbf{x})|^2\, d\mathbf{x} d\mathbf{k} = \int\limits_{\mathbb{R}^d} |\mathbf{v}(\mathbf{x})|^2\, d\mathbf{x}, \quad \widetilde{\mathbf{v}} = \mathcal{V}\mathbf{v},$$

the transformation $\mathcal{V}$ extends by continuity to a *unitary* mapping

$$\mathcal{V} : L_2(\mathbb{R}^d;\mathbb{C}^n) \to \int\limits_{\widetilde{\Omega}} \oplus L_2(\Omega;\mathbb{C}^n)\, d\mathbf{k} =: \mathcal{K}. \tag{4.7}$$

The relation $\mathbf{v} \in H^1(\mathbb{R}^d;\mathbb{C}^n)$ is equivalent to the fact that $\widetilde{\mathbf{v}}(\mathbf{k},\cdot) \in \widetilde{H}^1(\Omega;\mathbb{C}^n)$ for almost every $\mathbf{k} \in \widetilde{\Omega}$ and

$$\int\limits_{\widetilde{\Omega}} \int\limits_{\Omega} \left(|(\mathbf{D}+\mathbf{k})\widetilde{\mathbf{v}}(\mathbf{k},\mathbf{x})|^2 + |\widetilde{\mathbf{v}}(\mathbf{k},\mathbf{x})|^2 \right) d\mathbf{x}\, d\mathbf{k} < \infty.$$

4.4. The forms $a(\mathbf{k})$ and the operators $\mathcal{A}(\mathbf{k})$

Putting

$$\mathfrak{H} = L_2(\Omega; \mathbb{C}^n), \quad \mathfrak{H}_* = L_2(\Omega; \mathbb{C}^m),$$

we consider the operator

$$\mathcal{X}(\mathbf{k}) : \mathfrak{H} \to \mathfrak{H}_*, \quad \mathbf{k} \in \mathbb{R}^d,$$

defined by the formula

$$\mathcal{X}(\mathbf{k}) = hb(\mathbf{D} + \mathbf{k})f$$

on the domain

$$\operatorname{Dom} \mathcal{X}(\mathbf{k}) = \{\mathbf{u} \in \mathfrak{H} : \ f\mathbf{u} \in \widetilde{H}^1(\Omega; \mathbb{C}^n)\} =: \mathfrak{d}.$$

The operator $\mathcal{X}(\mathbf{k})$ is closed. The selfadjoint operator

$$\mathcal{A}(\mathbf{k}) := \mathcal{X}(\mathbf{k})^* \mathcal{X}(\mathbf{k}) : \mathfrak{H} \to \mathfrak{H}$$

is generated by the closed quadratic form

$$a(\mathbf{k})[\mathbf{u}, \mathbf{u}] := \|\mathcal{X}(\mathbf{k})\mathbf{u}\|^2_{\mathfrak{H}_*}, \quad \mathbf{u} \in \mathfrak{d}.$$

It is easy to check that (cf. [BSu2, (2.2.6)])

$$c_0 \int_{\Omega} |(\mathbf{D} + \mathbf{k})\mathbf{v}|^2 \, d\mathbf{x} \le a(\mathbf{k})[\mathbf{u}, \mathbf{u}] \le c_1 \int_{\Omega} |(\mathbf{D} + \mathbf{k})\mathbf{v}|^2 \, d\mathbf{x}, \quad \mathbf{v} = f\mathbf{u} \in \widetilde{H}^1(\Omega; \mathbb{C}^n), \quad (4.8)$$

where c_0 and c_1 are defined by (4.5). From (4.8) and the compactness of the embedding of $\widetilde{H}^1(\Omega; \mathbb{C}^n)$ in $L_2(\Omega; \mathbb{C}^n)$ it follows that the spectrum of $\mathcal{A}(\mathbf{k})$ is discrete. Observe also that the resolvent of the operator $\mathcal{A}(\mathbf{k})$ is compact and depends on $\mathbf{k} \in \mathbb{R}^d$ continuously (in the operator norm). Let

$$\mathfrak{N} := \operatorname{Ker} \mathcal{A}(0) = \operatorname{Ker} \mathcal{X}(0). \quad (4.9)$$

Relations (4.8) with $\mathbf{k} = 0$ show that

$$\mathfrak{N} = \{\mathbf{u} \in L_2(\Omega; \mathbb{C}^n) : \ f\mathbf{u} = \mathbf{c} \in \mathbb{C}^n\}, \quad \dim \mathfrak{N} = n. \quad (4.10)$$

4.5. The direct integral for the operator $\mathcal{A}$

The operators $\mathcal{A}(\mathbf{k})$ allow us to partially diagonalize the operator $\mathcal{A}$ in the direct integral $\mathcal{K}$. Let $\widetilde{\mathbf{u}} = \mathcal{V}\mathbf{u}$, $\mathbf{u} \in \operatorname{Dom} a$. Then

$$\widetilde{\mathbf{u}}(\mathbf{k}, \cdot) \in \mathfrak{d} \quad \text{for a. e. } \mathbf{k} \in \widetilde{\Omega}, \quad (4.11)$$

$$a[\mathbf{u}, \mathbf{u}] = \int_{\widetilde{\Omega}} a(\mathbf{k})[\widetilde{\mathbf{u}}(\mathbf{k}, \cdot), \widetilde{\mathbf{u}}(\mathbf{k}, \cdot)] \, d\mathbf{k}. \quad (4.12)$$

Conversely, if $\widetilde{\mathbf{u}} \in \mathcal{K}$ satisfies (4.11) and the integral in (4.12) is finite, then $\mathbf{u} \in \operatorname{Dom} a$ and (4.12) is valid. The above arguments show that, in the direct integral $\mathcal{K}$, the operator $\mathcal{A}$ turns into multiplication by the operator-valued function $\mathcal{A}(\mathbf{k})$, $\mathbf{k} \in \widetilde{\Omega}$. All this can be expressed briefly by the formula

$$\mathcal{V}\mathcal{A}\mathcal{V}^{-1} = \int_{\widetilde{\Omega}} \oplus \mathcal{A}(\mathbf{k}) \, d\mathbf{k}. \quad (4.13)$$

4.6. Incorporation of the operators $\mathcal{A}(\mathbf{k})$ into the general scheme

For $\mathbf{k} \in \mathbb{R}^d$ we put

$$\mathbf{k} = t\boldsymbol{\theta}, \quad t = |\mathbf{k}|, \ |\boldsymbol{\theta}| = 1,$$

and consider t as a perturbation parameter. At the same time, we have to make our constructions and estimates uniform with respect to the parameter $\boldsymbol{\theta}$.

We apply the method of §1, putting $\mathfrak{H} = L_2(\Omega; \mathbb{C}^n)$, $\mathfrak{H}_* = L_2(\Omega; \mathbb{C}^m)$. The role of $X(t)$ is played by the operator

$$X(t, \boldsymbol{\theta}) = \mathcal{X}(t\boldsymbol{\theta}) = X_0 + tX_1(\boldsymbol{\theta}),$$

where $X_0 = \mathcal{X}(0) = hb(\mathbf{D})f$, $\operatorname{Dom} X_0 = \mathfrak{d}$; $X_1(\boldsymbol{\theta}) = hb(\boldsymbol{\theta})f$. The role of $A(t)$ is played by $A(t, \boldsymbol{\theta}) = \mathcal{A}(t\boldsymbol{\theta})$. By (4.9) and (4.10),

$$\mathfrak{N} = \operatorname{Ker} X_0 = \operatorname{Ker} \mathcal{X}(0), \quad \dim \mathfrak{N} = n.$$

Condition $m \geq n$ guarantees that $n \leq n_*$; and the following *alternative* realizes: either $n_* = \infty$ (if $m > n$), or $n_* = n$ (if $m = n$); see [BSu2, §2.3]. In [BSu2, §2.2, 2.3], it was shown that the distance d^0 from the point $\lambda_0 = 0$ to the rest of the spectrum of $\mathcal{A}(0)$ satisfies the estimate

$$d^0 \geq 4c_* r_0^2, \quad c_* = \alpha_0 \|f^{-1}\|^{-2}_{L_\infty} \|h^{-1}\|^{-2}_{L_\infty}. \tag{4.14}$$

In §1, it was required to choose $\delta < d^0/8$. Recalling (4.14), we fix δ so that

$$\delta = c_* r_0^2/4 = (r_0/2)^2 \alpha_0 \|f^{-1}\|^{-2}_{L_\infty} \|h^{-1}\|^{-2}_{L_\infty}. \tag{4.15}$$

Next, the estimate $\|X_1(\boldsymbol{\theta})\| \leq \alpha_1^{1/2} \|f\|_{L_\infty} \|h\|_{L_\infty}$ allows us to choose t^0 (see (1.2)) equal not to $\delta^{1/2}\|X_1(\boldsymbol{\theta})\|^{-1}$, but to a smaller number independent of $\boldsymbol{\theta}$. Namely, we put

$$\begin{aligned} t^0 &= \delta^{1/2} \alpha_1^{-1/2} \|f\|^{-1}_{L_\infty} \|h\|^{-1}_{L_\infty} \\ &= (r_0/2) \alpha_0^{1/2} \alpha_1^{-1/2} \|f\|^{-1}_{L_\infty} \|f^{-1}\|^{-1}_{L_\infty} \|h\|^{-1}_{L_\infty} \|h^{-1}\|^{-1}_{L_\infty}. \end{aligned} \tag{4.16}$$

Observe that, by (4.16), $t^0 \leq r_0/2$, whence $\mathcal{B}(t^0) \subset \mathcal{B}(r_0/2) \subset \widetilde{\Omega}$.

The next estimate easily follows from the variational estimates for the eigenvalues (see [BSu2, (2.2.13)]):

$$\mathcal{A}(\mathbf{k}) = A(t, \boldsymbol{\theta}) \geq c_* r^2 I, \quad \mathbf{k} \in \widetilde{\Omega} \setminus \mathcal{B}(r), \ 0 < r \leq r_0. \tag{4.17}$$

4.7. Non-degeneracy of the germ of the family $A(t, \boldsymbol{\theta})$

The analytic in t (see (1.10), (1.11)) branches of the eigenvalues $\lambda_l(t, \boldsymbol{\theta})$ and the branches of the eigenvectors $\varphi_l(t, \boldsymbol{\theta})$, $l = 1, \dots, n$, $|t| \leq t^0$, depend on $\boldsymbol{\theta}$. In [BSu2, §2.2, 2.3], it was shown that

$$\lambda_l(t, \boldsymbol{\theta}) \geq c_* t^2, \quad l = 1, \dots, n, \ t \in [0, t^0], \tag{4.18}$$

where c_* and t^0 are defined by (4.14) and (4.16). It is essential that in (4.18) c_* and t^0 do not depend on $\boldsymbol{\theta}$. From (4.18) it follows that condition (2.1) for $A(t, \boldsymbol{\theta})$ is satisfied, whence the *germ* $S(\boldsymbol{\theta})$ *of the family* $A(t, \boldsymbol{\theta})$ *is non-degenerate uniformly in* $\boldsymbol{\theta}$:

$$S(\boldsymbol{\theta}) \geq c_* I_{\mathfrak{N}}, \quad \boldsymbol{\theta} \in \mathbb{S}^{d-1}. \tag{4.19}$$

5 The effective matrix

The effective matrix g^0 (*with constant coefficients*) for the operator (4.4) is defined by the rule known in the homogenization theory. In fact, the matrix g^0 does not depend on f; therefore, it suffices to consider the operator (4.4) with $f = \mathbf{1}_n$.

5.1. The operator $\widehat{\mathcal{A}}(g) := \mathcal{A}(g, \mathbf{1}_n)$ and the matrix g^0

In the case where $f = \mathbf{1}_n$, we agree to mark all the objects by the upper index „$\frown$“. For the operator

$$\widehat{\mathcal{A}} = \widehat{\mathcal{A}}(g) = b(\mathbf{D})^* g(\mathbf{x}) b(\mathbf{D}),$$

the family $\widehat{\mathcal{A}}(\mathbf{k}) = \widehat{\mathcal{A}}(\mathbf{k}; g)$ is denoted by $\widehat{A}(t, \boldsymbol{\theta}) = \widehat{A}(t, \boldsymbol{\theta}; g)$. If $f = \mathbf{1}_n$, the kernel (4.10) takes the form

$$\widehat{\mathfrak{N}} = \{\mathbf{u} \in \mathfrak{H} : \ \mathbf{u} = \mathbf{c} \in \mathbb{C}^n\}. \tag{5.1}$$

Let $\widehat{P}$ be the orthogonal projection of $\mathfrak{H}$ onto the subspace (5.1). Then

$$\widehat{P}\mathbf{u} = |\Omega|^{-1} \int_\Omega \mathbf{u}(\mathbf{x})\, d\mathbf{x}, \quad \mathbf{u} \in \mathfrak{H}. \tag{5.2}$$

In other words, $\widehat{P}$ is the operator of averaging over the cell.

As it was shown in [BSu2, §3.1], the *spectral germ $\widehat{S}(\boldsymbol{\theta})$ of the family $\widehat{A}(t, \boldsymbol{\theta}; g)$ acting in $\widehat{\mathfrak{N}}$ is represented as*

$$\widehat{S}(\boldsymbol{\theta}) = b(\boldsymbol{\theta})^* g^0 b(\boldsymbol{\theta}), \quad \boldsymbol{\theta} \in \mathbb{S}^{d-1},$$

where g^0 is the constant positive $(m \times m)$-matrix called the *effective matrix.* We shall also use the notation

$$\widehat{S}(\mathbf{k}) := t^2 \widehat{S}(\boldsymbol{\theta}) = b(\mathbf{k})^* g^0 b(\mathbf{k}), \quad \mathbf{k} \in \mathbb{R}^d. \tag{5.3}$$

In order to define g^0, we introduce the operator $\Lambda : \mathbb{C}^m \to \mathfrak{H}$, which takes a vector $\mathbf{C} \in \mathbb{C}^m$ to the (weak) periodic solution $\mathbf{v}_{\mathbf{C}} \in \widetilde{H}^1(\Omega; \mathbb{C}^n)$ of the problem

$$b(\mathbf{D})^* g(\mathbf{x})(b(\mathbf{D})\mathbf{v}_{\mathbf{C}}(\mathbf{x}) + \mathbf{C}) = 0, \quad \int_\Omega \mathbf{v}_{\mathbf{C}}\, d\mathbf{x} = 0. \tag{5.4}$$

Let $\mathbf{e}_1, \ldots, \mathbf{e}_m$ be the standard orthonormal basis in $\mathbb{C}^m$, and let $\mathbf{v}_j = \mathbf{v}_{\mathbf{e}_j}$. In the standard basis $\widetilde{\mathbf{e}}_1, \ldots, \widetilde{\mathbf{e}}_n$ in $\mathbb{C}^n$ the vector-valued functions $\mathbf{v}_j(\mathbf{x})$ can be written as columns of length n. Let $\Lambda(\mathbf{x})$ be the periodic $(n \times m)$-matrix-valued function with the columns $\mathbf{v}_1(\mathbf{x}), \ldots, \mathbf{v}_m(\mathbf{x})$. Then the operator Λ acts as multiplication by the matrix $\Lambda(\mathbf{x})$. Note that the mean value of $\Lambda(\mathbf{x})$ over the cell Ω is equal to zero. We introduce the following periodic $(m \times m)$-matrix-valued function:

$$\widetilde{g}(\mathbf{x}) = g(\mathbf{x})(b(\mathbf{D})\Lambda(\mathbf{x}) + \mathbf{1}_m). \tag{5.5}$$

The *effective matrix g^0 is defined by the relation*

$$g^0 = |\Omega|^{-1} \int_\Omega \widetilde{g}(\mathbf{x})\, d\mathbf{x}.$$

We introduce the following operator with constant coefficients:

$$\widehat{\mathcal{A}}^0 = \widehat{\mathcal{A}}(g^0) = b(\mathbf{D})^* g^0 b(\mathbf{D}), \tag{5.6}$$

and the corresponding family

$$\widehat{\mathcal{A}}^0(\mathbf{k}) = b(\mathbf{D}+\mathbf{k})^* g^0 b(\mathbf{D}+\mathbf{k}).$$

It is essential that the germ $\widehat{S}^0(\boldsymbol{\theta})$ for the family $\widehat{A}^0(t,\boldsymbol{\theta}) = \widehat{\mathcal{A}}^0(t\boldsymbol{\theta})$ coincides with $\widehat{S}(\boldsymbol{\theta})$. The *operator* (5.6) *is called the effective operator for* $\widehat{\mathcal{A}}(g)$.

5.2. The properties of the effective matrix g^0

(See [BSu2, §3.1].)

Proposition 5.1. *The effective matrix satisfies the estimates*

$$\underline{g} \le g^0 \le \overline{g}, \tag{5.7}$$

where

$$\overline{g} := |\Omega|^{-1} \int_\Omega g(\mathbf{x})\, d\mathbf{x}, \quad \underline{g}^{-1} := |\Omega|^{-1} \int_\Omega g(\mathbf{x})^{-1}\, d\mathbf{x}.$$

If $m = n$, *the effective matrix* g^0 *coincides with* $\underline{g}$: $g^0 = \underline{g}$.

For specific DO's, estimates (5.7) are well known in the homogenization theory as the Voight-Reuss bracketing.

We distinguish the cases where one of the inequalities in (5.7) becomes an identity. The following statements were obtained in [BSu2, Propositions 3.1.6 and 3.1.7].

Proposition 5.2. *The identity* $g^0 = \overline{g}$ *is equivalent to the relations*

$$b(\mathbf{D})^* \mathbf{g}_k(\mathbf{x}) = 0, \quad k = 1, \dots, m, \tag{5.8}$$

where $\mathbf{g}_k(\mathbf{x})$, $k = 1, \dots, m$, *are the columns of the matrix* $g(\mathbf{x})$.

Note that, under condition (5.8), we have $\Lambda(\mathbf{x}) = 0$.

Proposition 5.3. *The identity* $g^0 = \underline{g}$ *is equivalent to the relations*

$$\mathbf{l}_k(\mathbf{x}) = \mathbf{l}_k^0 + b(\mathbf{D})\mathbf{w}_k,\ \ \mathbf{l}_k^0 \in \mathbb{C}^m, \quad \mathbf{w}_k \in \widetilde{H}^1(\Omega; \mathbb{C}^n),\ \ k = 1, \dots, m, \tag{5.9}$$

where $\mathbf{l}_k(\mathbf{x})$, $k = 1, \dots, m$, *are the columns of the matrix* $g(\mathbf{x})^{-1}$.

6 Approximation for the operator $\widehat{\mathcal{A}}(\mathbf{k})^{1/2}(\widehat{\mathcal{A}}(\mathbf{k}) + \varepsilon^2 I)^{-1}$

6.1. The main goal

of the present section is to prove the following theorem.

Theorem 6.1. *We have*

$$\|\widehat{\mathcal{A}}(\mathbf{k})^{1/2}\big((\widehat{\mathcal{A}}(\mathbf{k})+\varepsilon^2 I)^{-1}-(I+\Lambda b(\mathbf{D}+\mathbf{k}))(\widehat{\mathcal{A}}^0(\mathbf{k})+\varepsilon^2 I)^{-1}\widehat{P}\big)\|_{\mathfrak{H}\to\mathfrak{H}}\le \mathcal{C}_3,\quad \mathbf{k}\in\widetilde{\Omega},\quad 0<\varepsilon\le 1. \tag{6.1}$$

Here $\Lambda:\mathfrak{H}_*\to\mathfrak{H}$ *is the operator of multiplication by the matrix-valued function* $\Lambda(\mathbf{x})$ *defined in Subsection* 5.1, *and the orthogonal projection* $\widehat{P}$ *onto the subspace* $\widehat{\mathfrak{N}}$ *is defined by* (5.2). *The constant* $\mathcal{C}_3$ *depends only on* m, α_0, α_1, $\|g\|_{L_\infty}$, $\|g^{-1}\|_{L_\infty}$, r_0, *and* r_1.

The explicit expression for $\mathcal{C}_3$ will be found in the proof; see formula (6.24) below.

The estimate (6.1) is more informative for $|\mathbf{k}|\le \widehat{t}^{\,0}$, where the number $\widehat{t}^{\,0}$ is defined according to (4.16) with $f=\mathbf{1}_n$ (see (6.9) below). The estimate for $\mathbf{k}\in\widetilde{\Omega}\setminus\mathcal{B}(\widehat{t}^{\,0})$ is rougher, since in this case each term in (6.1) is estimated separately; however, the corresponding calculations also require attention.

Note that $b(\mathbf{k})\widehat{P}=b(\mathbf{D}+\mathbf{k})\widehat{P}$, whence

$$t b(\boldsymbol{\theta})\widehat{P}=b(\mathbf{k})\widehat{P}=b(\mathbf{D}+\mathbf{k})\widehat{P},\quad \mathbf{k}\in\widetilde{\Omega}. \tag{6.2}$$

Next, by (5.3),

$$t^2\widehat{S}(\boldsymbol{\theta})\widehat{P}=\widehat{S}(\mathbf{k})\widehat{P}=\widehat{P}b(\mathbf{k})^*g^0b(\mathbf{k})\widehat{P}=b(\mathbf{D}+\mathbf{k})^*g^0b(\mathbf{D}+\mathbf{k})\widehat{P}=\widehat{\mathcal{A}}^0(\mathbf{k})\widehat{P},\quad \mathbf{k}\in\widetilde{\Omega}, \tag{6.3}$$

therefore,

$$(\widehat{S}(\mathbf{k})+\varepsilon^2 I_{\widehat{\mathfrak{N}}})^{-1}\widehat{P}=(\widehat{\mathcal{A}}^0(\mathbf{k})+\varepsilon^2 I)^{-1}\widehat{P},\quad \mathbf{k}\in\widetilde{\Omega}. \tag{6.4}$$

6.2. Estimate for $|\mathbf{k}|\le \widehat{t}^{\,0}$

We apply Theorem 2.1 to the operator $\widehat{A}(t,\boldsymbol{\theta})=\widehat{\mathcal{A}}(\mathbf{k})$. Now the role of the operator (2.4) is played by the operator

$$\widehat{\Xi}=\widehat{\Xi}(t,\boldsymbol{\theta},\varepsilon)=\widehat{\Xi}(\mathbf{k},\varepsilon):=(t^2\widehat{S}(\boldsymbol{\theta})+\varepsilon^2 I_{\widehat{\mathfrak{N}}})^{-1}\widehat{P}=(\widehat{S}(\mathbf{k})+\varepsilon^2 I_{\widehat{\mathfrak{N}}})^{-1}\widehat{P}, \tag{6.5}$$

and the role of Z is played by the operator (see [BSu4, (4.2)])

$$\widehat{Z}(\boldsymbol{\theta})=\Lambda b(\boldsymbol{\theta})\widehat{P}. \tag{6.6}$$

We put $\widehat{Z}(\mathbf{k}):=t\widehat{Z}(\boldsymbol{\theta})$. Then $\widehat{Z}(\mathbf{k})=\Lambda b(\mathbf{k})\widehat{P}$.

Estimate (2.12) is applicable. We only have to specify the constants. The constants $\widehat{c}_*$, $\widehat{\delta}$, $\widehat{t}^{\,0}$ are defined according to (4.14)–(4.16) with $f=\mathbf{1}$. Namely,

$$\widehat{c}_*=\alpha_0\|h^{-1}\|^{-2}_{L_\infty}, \tag{6.7}$$

$$\widehat{\delta}=(r_0/2)^2\alpha_0\|h^{-1}\|^{-2}_{L_\infty}, \tag{6.8}$$

$$\widehat{t}^{\,0}=\widehat{\delta}^{1/2}\alpha_1^{-1/2}\|h\|^{-1}_{L_\infty}. \tag{6.9}$$

Taking the estimate $\|\widehat{X}_1(\boldsymbol{\theta})\|\le\alpha_1^{1/2}\|h\|_{L_\infty}$ into account, instead of the precise values $\widehat{C}_1(\boldsymbol{\theta})=\beta_1\widehat{\delta}^{-1/2}\|\widehat{X}_1(\boldsymbol{\theta})\|$ and $\widehat{C}_2(\boldsymbol{\theta})=\beta_2\widehat{\delta}^{-1/2}\|\widehat{X}_1(\boldsymbol{\theta})\|^3$ (see Subsection 1.4), we take the values

$$\widehat{C}_1=\beta_1\widehat{\delta}^{-1/2}\alpha_1^{1/2}\|h\|_{L_\infty},\quad \widehat{C}_2=\beta_2\widehat{\delta}^{-1/2}\alpha_1^{3/2}\|h\|^3_{L_\infty}.$$

Instead of the precise constant $\widehat{C}_5(\boldsymbol{\theta})$ (see (2.24)), we take the higher value

$$\widehat{C}_5 = (3\widehat{\delta})^{-1/2} + \widehat{C}_1 \widehat{c}_*^{-1/2} + \widehat{C}_2 \widehat{c}_*^{-3/2} + \beta_3 (1 + \pi^{-1}) \widehat{\delta}^{-1/2} \alpha_1 \|h\|^2_{L_\infty} \widehat{c}_*^{-1}.$$

Thus,

$$\widehat{C}_5 = \widehat{C}_5(\alpha_0, \alpha_1, \|h\|_{L_\infty}, \|h^{-1}\|_{L_\infty}, r_0).$$

Now, estimate (2.12) implies that

$$\|\widehat{\mathcal{A}}(\mathbf{k})^{1/2} \Big((\widehat{\mathcal{A}}(\mathbf{k}) + \varepsilon^2 I)^{-1} - (I + \Lambda b(\mathbf{k}))(\widehat{S}(\mathbf{k}) + \varepsilon^2 I_{\widehat{\mathfrak{N}}})^{-1} \widehat{P} \Big) \|_{\mathfrak{H} \to \mathfrak{H}} \le \widehat{C}_5, \quad |\mathbf{k}| \le \widehat{t}^{\,0}, \quad 0 < \varepsilon \le 1.$$

By (6.2) and (6.4), the last inequality can be written as

$$\|\widehat{\mathcal{A}}(\mathbf{k})^{1/2} \big((\widehat{\mathcal{A}}(\mathbf{k}) + \varepsilon^2 I)^{-1} - (I + \Lambda b(\mathbf{D} + \mathbf{k}))(\widehat{\mathcal{A}}^0(\mathbf{k}) + \varepsilon^2 I)^{-1} \widehat{P} \big) \|_{\mathfrak{H} \to \mathfrak{H}} \le \widehat{C}_5, \quad |\mathbf{k}| \le \widehat{t}^{\,0}, \quad 0 < \varepsilon \le 1. \tag{6.10}$$

Thus, estimate (6.1) is proved for $|\mathbf{k}| \le \widehat{t}^{\,0}$.

6.3. Estimate for $\mathbf{k} \in \widetilde{\Omega} \setminus \mathcal{B}(\widehat{t}^{\,0})$

Note that constants (6.7)–(6.9) are related not only to the family $\widehat{\mathcal{A}}(\mathbf{k})$, but also to the family $\widehat{\mathcal{A}}^0(\mathbf{k})$ (this follows from inequalities (5.7)). Then (4.17) with $r = \widehat{t}^{\,0}$ implies that

$$\widehat{\mathcal{A}}(\mathbf{k}) \ge \widehat{c}_* (\widehat{t}^{\,0})^2 I, \quad \widehat{\mathcal{A}}^0(\mathbf{k}) \ge \widehat{c}_* (\widehat{t}^{\,0})^2 I, \quad \mathbf{k} \in \widetilde{\Omega} \setminus \mathcal{B}(\widehat{t}^{\,0}). \tag{6.11}$$

The operator under the norm sign in (6.1) can be written as

$$\mathcal{E}(\mathbf{k}, \varepsilon) := \widehat{\mathcal{A}}(\mathbf{k})^{1/2} (\widehat{\mathcal{A}}(\mathbf{k}) + \varepsilon^2 I)^{-1} - (\widehat{\mathcal{A}}(\mathbf{k})^{1/2} \widehat{P})(\widehat{\mathcal{A}}^0(\mathbf{k}) + \varepsilon^2 I)^{-1} - (\widehat{\mathcal{A}}(\mathbf{k})^{1/2} \Lambda \widehat{P}_m)(b(\mathbf{D} + \mathbf{k}) \widehat{P})(\widehat{\mathcal{A}}^0(\mathbf{k}) + \varepsilon^2 I)^{-1}. \tag{6.12}$$

Here $\widehat{P}_m$ is the orthogonal projection of $\mathfrak{H}_* = L_2(\Omega; \mathbb{C}^m)$ onto the subspace of constants.

From (6.11) it follows that

$$\|\widehat{\mathcal{A}}(\mathbf{k})^{1/2} (\widehat{\mathcal{A}}(\mathbf{k}) + \varepsilon^2 I)^{-1}\|_{\mathfrak{H} \to \mathfrak{H}} \le \widehat{c}_*^{-1/2} (\widehat{t}^{\,0})^{-1}, \quad \mathbf{k} \in \widetilde{\Omega} \setminus \mathcal{B}(\widehat{t}^{\,0}), \tag{6.13}$$

$$\|(\widehat{\mathcal{A}}^0(\mathbf{k}) + \varepsilon^2 I)^{-1}\|_{\mathfrak{H} \to \mathfrak{H}} \le \widehat{c}_*^{-1} (\widehat{t}^{\,0})^{-2}, \quad \mathbf{k} \in \widetilde{\Omega} \setminus \mathcal{B}(\widehat{t}^{\,0}). \tag{6.14}$$

Now we estimate the norm of the operator $\widehat{\mathcal{A}}(\mathbf{k})^{1/2} \widehat{P}$. We have:

$$\|\widehat{\mathcal{A}}(\mathbf{k})^{1/2} \widehat{P} u\|^2_{\mathfrak{H}} = \|(\widehat{X}_0 + t \widehat{X}_1(\boldsymbol{\theta})) \widehat{P} u\|^2_{\mathfrak{H}_*} = |\mathbf{k}|^2 \|\widehat{X}_1(\boldsymbol{\theta}) \widehat{P} u\|^2_{\mathfrak{H}_*} \le \alpha_1 \|h\|^2_{L_\infty} |\mathbf{k}|^2 \|u\|^2_{\mathfrak{H}},$$

whence

$$\|\widehat{\mathcal{A}}(\mathbf{k})^{1/2} \widehat{P}\|_{\mathfrak{H} \to \mathfrak{H}} \le \alpha_1^{1/2} \|h\|_{L_\infty} r_1, \quad \mathbf{k} \in \widetilde{\Omega}. \tag{6.15}$$

Next, by (6.2) and (4.2),

$$\|b(\mathbf{D} + \mathbf{k}) \widehat{P}\|_{\mathfrak{H} \to \mathfrak{H}} \le |\mathbf{k}| |b(\boldsymbol{\theta})| \le \alpha_1^{1/2} r_1, \quad \mathbf{k} \in \widetilde{\Omega}. \tag{6.16}$$

It remains to estimate the norm of the operator $\widehat{\mathcal{A}}(\mathbf{k})^{1/2}\Lambda\widehat{P}_m$. Since $\Lambda(\mathbf{x})$ is the matrix with the columns $\mathbf{v}_j(\mathbf{x})$, $j = 1,\dots,m$, it suffices to estimate the norm $\|\widehat{\mathcal{A}}(\mathbf{k})^{1/2}\mathbf{v}_j\|_{\mathfrak{H}}$. Recall that the function $\mathbf{v}_j$ satisfies the relation

$$(g(b(\mathbf{D})\mathbf{v}_j + \mathbf{e}_j), b(\mathbf{D})\mathbf{w})_{\mathfrak{H}_*} = 0, \quad \forall \mathbf{w} \in \widetilde{H}^1(\Omega;\mathbb{C}^n), \tag{6.17}$$

and also the normalization condition $\int_\Omega \mathbf{v}_j\, d\mathbf{x} = 0$. From (6.17) it follows that

$$\|g^{1/2}b(\mathbf{D})\mathbf{v}_j\|_{\mathfrak{H}_*} \le \|g\|^{1/2}_{L_\infty}|\Omega|^{1/2}. \tag{6.18}$$

Next, decomposing $\mathbf{v}_j$ in the Fourier series under the normalization condition and taking (4.6) and (4.2) into account, we obtain that

$$\begin{aligned}\|\mathbf{v}_j\|^2_{\mathfrak{H}} &\le (2r_0)^{-2}\int_\Omega |\mathbf{D}\mathbf{v}_j|^2\, d\mathbf{x} \le (2r_0)^{-2}\alpha_0^{-1}\|b(\mathbf{D})\mathbf{v}_j\|^2_{\mathfrak{H}_*}\\ &\le (2r_0)^{-2}\alpha_0^{-1}\|g^{-1}\|_{L_\infty}\|g^{1/2}b(\mathbf{D})\mathbf{v}_j\|^2_{\mathfrak{H}_*}.\end{aligned}$$

Combining this with (6.18), we obtain the estimate

$$\|\mathbf{v}_j\|_{\mathfrak{H}} \le (2r_0)^{-1}\alpha_0^{-1/2}\|g^{-1}\|^{1/2}_{L_\infty}\|g\|^{1/2}_{L_\infty}|\Omega|^{1/2}. \tag{6.19}$$

By (4.2) and (6.19),

$$\begin{aligned}\|g^{1/2}b(\mathbf{k})\mathbf{v}_j\|_{\mathfrak{H}_*} &\le \|g\|^{1/2}_{L_\infty}\alpha_1^{1/2}r_1\|\mathbf{v}_j\|_{\mathfrak{H}}\\ &\le (2r_0)^{-1}r_1\alpha_1^{1/2}\alpha_0^{-1/2}\|g^{-1}\|^{1/2}_{L_\infty}\|g\|_{L_\infty}|\Omega|^{1/2}, \quad \mathbf{k} \in \widetilde{\Omega}.\end{aligned} \tag{6.20}$$

Now, since

$$\|\widehat{\mathcal{A}}(\mathbf{k})^{1/2}\mathbf{v}_j\|_{\mathfrak{H}} = \|g^{1/2}b(\mathbf{D}+\mathbf{k})\mathbf{v}_j\|_{\mathfrak{H}_*} \le \|g^{1/2}b(\mathbf{D})\mathbf{v}_j\|_{\mathfrak{H}_*} + \|g^{1/2}b(\mathbf{k})\mathbf{v}_j\|_{\mathfrak{H}_*},$$

relations (6.18) and (6.20) imply that

$$\begin{aligned}\|\widehat{\mathcal{A}}(\mathbf{k})^{1/2}\mathbf{v}_j\|_{\mathfrak{H}} \le \|g\|^{1/2}_{L_\infty}|\Omega|^{1/2}\big(1 + r_1(2r_0)^{-1}\alpha_1^{1/2}\alpha_0^{-1/2}\|g\|^{1/2}_{L_\infty}\|g^{-1}\|^{1/2}_{L_\infty}\big) =: \mathcal{C}_1,\\ \mathbf{k} \in \widetilde{\Omega}.\end{aligned} \tag{6.21}$$

From (6.21) it directly follows that

$$\begin{aligned}\|\widehat{\mathcal{A}}(\mathbf{k})^{1/2}\Lambda\widehat{P}_m\|_{\mathfrak{H}_*\to\mathfrak{H}} &\le m^{1/2}|\Omega|^{-1/2}\sup_{1\le j\le m}\|\widehat{\mathcal{A}}(\mathbf{k})^{1/2}\mathbf{v}_j\|_{\mathfrak{H}}\\ &\le m^{1/2}\mathcal{C}_1|\Omega|^{-1/2} =: \widetilde{\mathcal{C}}_1, \quad \mathbf{k} \in \widetilde{\Omega}.\end{aligned} \tag{6.22}$$

The inequalities (6.13)–(6.16) and (6.22) lead to the following estimate for the norm of the operator (6.12):

$$\begin{aligned}\|\mathcal{E}(\mathbf{k},\varepsilon)\|_{\mathfrak{H}\to\mathfrak{H}} \le \widehat{c}_*^{-1/2}(\widehat{t}^0)^{-1} + \alpha_1^{1/2}r_1\widehat{c}_*^{-1}(\widehat{t}^0)^{-2}(\|g\|^{1/2}_{L_\infty} + \widetilde{\mathcal{C}}_1) =: \mathcal{C}_2,\\ \mathbf{k} \in \widetilde{\Omega}\setminus\mathcal{B}(\widehat{t}^0).\end{aligned} \tag{6.23}$$

Combining estimates (6.10) and (6.23), we obtain (6.1) with the constant

$$\mathcal{C}_3 = \max\{\widehat{C}_5, \mathcal{C}_2\}. \tag{6.24}$$

This completes the proof of Theorem 6.1. $\square$

6.4. The case of zero corrector

If
$$\Lambda(\mathbf{x})b(\boldsymbol{\theta}) = 0, \quad \mathbf{x} \in \Omega, \ \boldsymbol{\theta} \in \mathbb{S}^{d-1}, \tag{6.25}$$
then $\widehat{Z}(\boldsymbol{\theta}) = 0$ for all $\boldsymbol{\theta}$ (see (6.6)). In paricular, (6.25) is satisfied if $g^0 = \overline{g}$ (then $\Lambda(\mathbf{x}) = 0$). Under condition (6.25), the term in (6.1) which corresponds to the corrector is equal to zero. Herewith, the constant in estimate (6.1) can be made more precise. In this case, estimate (6.10) yields that
$$\|\widehat{\mathcal{A}}(\mathbf{k})^{1/2}\big((\widehat{\mathcal{A}}(\mathbf{k}) + \varepsilon^2 I)^{-1} - (\widehat{\mathcal{A}}^0(\mathbf{k}) + \varepsilon^2 I)^{-1}\widehat{P}\big)\|_{\mathfrak{H}\to\mathfrak{H}} \le \widehat{C}_5,$$
$$|\mathbf{k}| \le \widehat{t}^{\,0}, \quad 0 < \varepsilon \le 1.$$

For $\mathbf{k} \in \widetilde{\Omega} \setminus \mathcal{B}(\widehat{t}^{\,0})$, we use estimates (6.13), (6.14), and (6.15). Then
$$\|\widehat{\mathcal{A}}(\mathbf{k})^{1/2}\big((\widehat{\mathcal{A}}(\mathbf{k}) + \varepsilon^2 I)^{-1} - (\widehat{\mathcal{A}}^0(\mathbf{k}) + \varepsilon^2 I)^{-1}\widehat{P}\big)\|_{\mathfrak{H}\to\mathfrak{H}}$$
$$\le \widehat{c}_*^{-1/2}(\widehat{t}^{\,0})^{-1} + \alpha_1^{1/2} r_1 \widehat{c}_*^{-1}(\widehat{t}^{\,0})^{-2}\|g\|_{L_\infty}^{1/2}, \quad \mathbf{k} \in \widetilde{\Omega} \setminus \mathcal{B}(\widehat{t}^{\,0}), \ 0 < \varepsilon \le 1.$$

As a result, we arrive at the following theorem.

Theorem 6.2. *Under condition* (6.25)*, we have:*
$$\|\widehat{\mathcal{A}}(\mathbf{k})^{1/2}\big((\widehat{\mathcal{A}}(\mathbf{k}) + \varepsilon^2 I)^{-1} - (\widehat{\mathcal{A}}^0(\mathbf{k}) + \varepsilon^2 I)^{-1}\widehat{P}\big)\|_{\mathfrak{H}\to\mathfrak{H}} \le \mathcal{C}_3^\circ,$$
$$\mathbf{k} \in \widetilde{\Omega}, \quad 0 < \varepsilon \le 1,$$
where
$$\mathcal{C}_3^\circ = \max\{\widehat{C}_5, \widehat{c}_*^{-1/2}(\widehat{t}^{\,0})^{-1} + \alpha_1^{1/2} r_1 \widehat{c}_*^{-1}(\widehat{t}^{\,0})^{-2}\|g\|_{L_\infty}^{1/2}\}.$$

6.5. Approximation with the three-term corrector

In what follows (in order to obtain the interpolational results), besides Theorem 6.1 which gives approximation for the operator $\widehat{\mathcal{A}}(\mathbf{k})^{1/2}(\widehat{\mathcal{A}}(\mathbf{k})+\varepsilon^2 I)^{-1}$ with the „one-term corrector“, we will need another approximation of this operator with the „three-term corrector“. We apply Theorem 2.6 to the operator $\widehat{A}(t, \boldsymbol{\theta}) = \widehat{\mathcal{A}}(\mathbf{k})$. Now Ξ is the operator $\widehat{\Xi}(\mathbf{k}, \varepsilon)$ defined by (6.5), and Z is the operator $\widehat{Z}(\boldsymbol{\theta})$ (see (6.6)). The operator N is realized (see [BSu4, (4.9)]) as
$$\widehat{N}(\boldsymbol{\theta}) = b(\boldsymbol{\theta})^* L(\boldsymbol{\theta}) b(\boldsymbol{\theta}) \widehat{P},$$
where
$$L(\boldsymbol{\theta}) = |\Omega|^{-1} \int_\Omega (\Lambda(\mathbf{x})^* b(\boldsymbol{\theta})^* \widetilde{g}(\mathbf{x}) + \widetilde{g}(\mathbf{x})^* b(\boldsymbol{\theta}) \Lambda(\mathbf{x}))\, d\mathbf{x}.$$
Recall that $\widetilde{g}(\mathbf{x})$ is the matrix defined by (5.5). Then
$$\widehat{N}(\mathbf{k}) := t^3 \widehat{N}(\boldsymbol{\theta}) = b(\mathbf{k})^* L(\mathbf{k}) b(\mathbf{k}) \widehat{P} = b(\mathbf{D} + \mathbf{k})^* L(\mathbf{D} + \mathbf{k}) b(\mathbf{D} + \mathbf{k}) \widehat{P}. \tag{6.26}$$

Now we specify the constant C_6 from (2.25). Instead of the precise value $\widehat{C}_6(\boldsymbol{\theta})$ (see (2.29)), we can take the rougher value
$$\widehat{C}_6 = \widehat{C}_5 + (8\widehat{\delta})^{-1/2} \widehat{c}_*^{-1} \alpha_1 \|g\|_{L_\infty} + (2\widehat{\delta})^{-1/2} \widehat{c}_*^{-2} \alpha_1^2 \|g\|_{L_\infty}^2.$$

Now estimate (2.25) means that

$$\|\widehat{\mathcal{A}}(\mathbf{k})^{1/2}\big((\widehat{\mathcal{A}}(\mathbf{k})+\varepsilon^2 I)^{-1}-(\widehat{\mathcal{A}}^0(\mathbf{k})+\varepsilon^2 I)^{-1}\widehat{P}-\mathcal{K}(\mathbf{k},\varepsilon)\big)\|_{\mathfrak{H}\to\mathfrak{H}}\le \widehat{C}_6,$$
$$|\mathbf{k}|\le \widehat{t}^0,\quad 0<\varepsilon\le 1. \tag{6.27}$$

Here

$$\begin{aligned}\mathcal{K}(\mathbf{k},\varepsilon) &:= \widehat{Z}(\mathbf{k})\widehat{\Xi}(\mathbf{k},\varepsilon)+\widehat{\Xi}(\mathbf{k},\varepsilon)\widehat{Z}(\mathbf{k})^*-\widehat{\Xi}(\mathbf{k},\varepsilon)\widehat{N}(\mathbf{k})\widehat{\Xi}(\mathbf{k},\varepsilon)\\ &= \Lambda b(\mathbf{k})\widehat{\Xi}(\mathbf{k},\varepsilon)+\widehat{\Xi}(\mathbf{k},\varepsilon)b(\mathbf{k})^*\Lambda^*-\widehat{\Xi}(\mathbf{k},\varepsilon)\widehat{N}(\mathbf{k})\widehat{\Xi}(\mathbf{k},\varepsilon).\end{aligned} \tag{6.28}$$

By (6.2), (6.4), (6.5), and (6.26), expression (6.28) can be rewritten as

$$\begin{aligned}\mathcal{K}(\mathbf{k},\varepsilon) &= \Lambda b(\mathbf{D}+\mathbf{k})(\widehat{\mathcal{A}}^0(\mathbf{k})+\varepsilon^2 I)^{-1}\widehat{P}+(\widehat{\mathcal{A}}^0(\mathbf{k})+\varepsilon^2 I)^{-1}\widehat{P}b(\mathbf{D}+\mathbf{k})^*\Lambda^*\\ &-(\widehat{\mathcal{A}}^0(\mathbf{k})+\varepsilon^2 I)^{-1}b(\mathbf{D}+\mathbf{k})^*L(\mathbf{D}+\mathbf{k})b(\mathbf{D}+\mathbf{k})(\widehat{\mathcal{A}}^0(\mathbf{k})+\varepsilon^2 I)^{-1}\widehat{P}.\end{aligned} \tag{6.29}$$

Now we obtain the estimate of the form (6.27) for $\mathbf{k}\in\widetilde{\Omega}\setminus\mathcal{B}(\widehat{t}^0)$, estimating each term in (6.27) separately. We write the operator under the norm sign in (6.27) as

$$\begin{aligned}&\widehat{\mathcal{A}}(\mathbf{k})^{1/2}\Big((\widehat{\mathcal{A}}(\mathbf{k})+\varepsilon^2 I)^{-1}-(\widehat{\mathcal{A}}^0(\mathbf{k})+\varepsilon^2 I)^{-1}\widehat{P}-\mathcal{K}(\mathbf{k},\varepsilon)\Big)\\ &=\widehat{\mathcal{A}}(\mathbf{k})^{1/2}\Big((\widehat{\mathcal{A}}(\mathbf{k})+\varepsilon^2 I)^{-1}-(\widehat{\mathcal{A}}^0(\mathbf{k})+\varepsilon^2 I)^{-1}\widehat{P}\\ &\qquad-\widehat{Z}(\mathbf{k})(\widehat{\mathcal{A}}^0(\mathbf{k})+\varepsilon^2 I)^{-1}\widehat{P}-\widehat{P}(\widehat{\mathcal{A}}^0(\mathbf{k})+\varepsilon^2 I)^{-1}\widehat{Z}(\mathbf{k})^*\\ &\qquad+(\widehat{\mathcal{A}}^0(\mathbf{k})+\varepsilon^2 I)^{-1}\widehat{N}(\mathbf{k})(\widehat{\mathcal{A}}^0(\mathbf{k})+\varepsilon^2 I)^{-1}\widehat{P}\Big).\end{aligned} \tag{6.30}$$

First of all, using (6.12), we rewrite inequality (6.23) as

$$\begin{aligned}&\|\widehat{\mathcal{A}}(\mathbf{k})^{1/2}\Big((\widehat{\mathcal{A}}(\mathbf{k})+\varepsilon^2 I)^{-1}-(\widehat{\mathcal{A}}^0(\mathbf{k})+\varepsilon^2 I)^{-1}\widehat{P}\\ &\qquad-\widehat{Z}(\mathbf{k})(\widehat{\mathcal{A}}^0(\mathbf{k})+\varepsilon^2 I)^{-1}\widehat{P}\Big)\|_{\mathfrak{H}\to\mathfrak{H}}\le \mathcal{C}_2,\quad \mathbf{k}\in\widetilde{\Omega}\setminus\mathcal{B}(\widehat{t}^0).\end{aligned} \tag{6.31}$$

Next, the norm of the operator $\widehat{Z}(\boldsymbol{\theta})$ satisfies the estimate (see (1.6))

$$\|\widehat{Z}(\boldsymbol{\theta})\|_{\mathfrak{H}\to\mathfrak{H}}\le(8\widehat{\delta})^{-1/2}\|\widehat{X}_1(\boldsymbol{\theta})\|\le(8\widehat{\delta})^{-1/2}\alpha_1^{1/2}\|g\|_{L_\infty}^{1/2},$$

whence

$$\|\widehat{Z}(\mathbf{k})\|_{\mathfrak{H}\to\mathfrak{H}}=|\mathbf{k}|\|\widehat{Z}(\boldsymbol{\theta})\|_{\mathfrak{H}\to\mathfrak{H}}\le r_1(8\widehat{\delta})^{-1/2}\alpha_1^{1/2}\|g\|_{L_\infty}^{1/2},\quad \mathbf{k}\in\widetilde{\Omega}. \tag{6.32}$$

As it was shown in [BSu4, (4.27)],

$$\|\widehat{N}(\boldsymbol{\theta})\|_{\mathfrak{H}\to\mathfrak{H}}\le(2\widehat{\delta})^{-1/2}\alpha_1^{3/2}\|g\|_{L_\infty}^{3/2},$$

whence

$$\|\widehat{N}(\mathbf{k})\|_{\mathfrak{H}\to\mathfrak{H}}=|\mathbf{k}|^3\|\widehat{N}(\boldsymbol{\theta})\|_{\mathfrak{H}\to\mathfrak{H}}\le r_1^3(2\widehat{\delta})^{-1/2}\alpha_1^{3/2}\|g\|_{L_\infty}^{3/2},\quad \mathbf{k}\in\widetilde{\Omega}. \tag{6.33}$$

The term $(\widehat{\mathcal{A}}(\mathbf{k})^{1/2}\widehat{P})(\widehat{\mathcal{A}}^0(\mathbf{k})+\varepsilon^2 I)^{-1}\widehat{Z}(\mathbf{k})^*$ can be estimated by using (6.14), (6.15), and (6.32):

$$\|\widehat{\mathcal{A}}(\mathbf{k})^{1/2}\widehat{P}(\widehat{\mathcal{A}}^0(\mathbf{k})+\varepsilon^2 I)^{-1}\widehat{Z}(\mathbf{k})^*\|_{\mathfrak{H}\to\mathfrak{H}}\le r_1^2(8\widehat{\delta})^{-1/2}\alpha_1\widehat{c}_*^{-1}(\widehat{t}^0)^{-2}\|g\|_{L_\infty},$$
$$\mathbf{k}\in\widetilde{\Omega}\setminus\mathcal{B}(\widehat{t}^0). \tag{6.34}$$

Finally, relations (6.14), (6.15), and (6.33) imply that

$$\|\widehat{\mathcal{A}}(\mathbf{k})^{1/2}\widehat{P}(\widehat{\mathcal{A}}^0(\mathbf{k})+\varepsilon^2 I)^{-1}\widehat{N}(\mathbf{k})(\widehat{\mathcal{A}}^0(\mathbf{k})+\varepsilon^2 I)^{-1}\widehat{P}\|_{\mathfrak{H}\to\mathfrak{H}} \\ \le r_1^4(2\widehat{\delta})^{-1/2}\alpha_1^2\widehat{c}_*^{-2}(\widehat{t}^0)^{-4}\|g\|^2_{L_\infty}, \quad \mathbf{k}\in\widetilde{\Omega}\setminus\mathcal{B}(\widehat{t}^0). \tag{6.35}$$

Now, (6.31), (6.34), and (6.35) yield the following estimate for the operator (6.30):

$$\|\widehat{\mathcal{A}}(\mathbf{k})^{1/2}\big((\widehat{\mathcal{A}}(\mathbf{k})+\varepsilon^2 I)^{-1}-(\widehat{\mathcal{A}}^0(\mathbf{k})+\varepsilon^2 I)^{-1}\widehat{P}-\mathcal{K}(\mathbf{k},\varepsilon)\big)\|_{\mathfrak{H}\to\mathfrak{H}}\le\mathcal{C}_4, \\ \mathbf{k}\in\widetilde{\Omega}\setminus\mathcal{B}(\widehat{t}^0), \tag{6.36}$$

where

$$\mathcal{C}_4=\mathcal{C}_2+r_1^2(8\widehat{\delta})^{-1/2}\alpha_1\widehat{c}_*^{-1}(\widehat{t}^0)^{-2}\|g\|_{L_\infty}+r_1^4(2\widehat{\delta})^{-1/2}\alpha_1^2\widehat{c}_*^{-2}(\widehat{t}^0)^{-4}\|g\|^2_{L_\infty}.$$

Relations (6.27) and (6.36) imply the following result.

Theorem 6.3. *Let $\mathcal{K}(\mathbf{k},\varepsilon)$ be the operator defined by* (6.29). *Then*

$$\|\widehat{\mathcal{A}}(\mathbf{k})^{1/2}\big((\widehat{\mathcal{A}}(\mathbf{k})+\varepsilon^2 I)^{-1}-(\widehat{\mathcal{A}}^0(\mathbf{k})+\varepsilon^2 I)^{-1}\widehat{P}-\mathcal{K}(\mathbf{k},\varepsilon)\big)\|_{\mathfrak{H}\to\mathfrak{H}} \\ \le\max\{\widehat{C}_6,\mathcal{C}_4\}=:\mathcal{C}_5, \quad \mathbf{k}\in\widetilde{\Omega},\ \ 0<\varepsilon\le 1.$$

7 Approximation for the operator $\widehat{\mathcal{A}}(\mathbf{k})^{1/2}(\widehat{\mathcal{A}}(\mathbf{k})+\varepsilon^2 Q)^{-1}$

7.1

Now we obtain approximation for the operator-valued function

$$\widehat{\mathcal{A}}(\mathbf{k})^{1/2}(\widehat{\mathcal{A}}(\mathbf{k})+\varepsilon^2 Q)^{-1}.$$

Here Q is the operator of multiplication by the Γ-periodic positive $(n\times n)$-matrix-valued function $Q(\mathbf{x})$ such that

$$Q,Q^{-1}\in L_\infty. \tag{7.1}$$

Let $\overline{Q}$ be the mean value of the matrix $Q(\mathbf{x})$ over the cell Ω. The following theorem is the main result of this section.

Theorem 7.1. *We have*

$$\|\widehat{\mathcal{A}}(\mathbf{k})^{1/2}\big((\widehat{\mathcal{A}}(\mathbf{k})+\varepsilon^2 Q)^{-1}-(I+\Lambda b(\mathbf{D}+\mathbf{k}))(\widehat{\mathcal{A}}^0(\mathbf{k})+\varepsilon^2\overline{Q})^{-1}\widehat{P}\big)\|_{\mathfrak{H}\to\mathfrak{H}}\le\check{\mathcal{C}}_3, \\ \mathbf{k}\in\widetilde{\Omega}, \quad 0<\varepsilon\le 1. \tag{7.2}$$

Here $\Lambda:\mathfrak{H}_*\to\mathfrak{H}$ is the operator of multiplication by the matrix-valued function $\Lambda(\mathbf{x})$ introduced in Subsection 5.1, *and $\widehat{P}$ is the orthogonal projection onto the subspace $\widehat{\mathfrak{N}}$ defined by* (5.2). *The constant $\check{\mathcal{C}}_3$ is defined below in* (7.17) *and depends only on m, α_0, α_1, $\|g\|_{L_\infty}$, $\|g^{-1}\|_{L_\infty}$, $\|Q\|_{L_\infty}$, $\|Q^{-1}\|_{L_\infty}$, r_0, and r_1.*

We use the following representation (cf. (3.5)) for the matrix $Q(\mathbf{x})$:

$$Q(\mathbf{x})=(f(\mathbf{x})f(\mathbf{x})^*)^{-1}.$$

Here $f(\mathbf{x})$ is a Γ-periodic $(n\times n)$-matrix-valued function such that $f,f^{-1}\in L_\infty$. Suppose that the number t^0 is defined by (4.16) (and corresponds to the operator $\mathcal{A}(\mathbf{k})=f^*\widehat{\mathcal{A}}(\mathbf{k})f$). Estimate (7.2) for $|\mathbf{k}|\le t^0$ is obtained by applying Theorem 3.1, and the estimate for $\mathbf{k}\in\widetilde{\Omega}\setminus\mathcal{B}(t^0)$ is rougher, since each term is estimated separately.

7.2. The estimate for $|\mathbf{k}| \le t^0$

We apply Theorem 3.1 to the family $\widehat{A}(t,\boldsymbol{\theta}) = \widehat{\mathcal{A}}(\mathbf{k})$ and the operator Q. The block $Q_{\widehat{\mathfrak{N}}}$ of the operator of multiplication by $Q(\mathbf{x})$ in the subspace $\widehat{\mathfrak{N}}$ (see (5.1)) is the operator of multiplication by the constant matrix

$$\overline{Q} = (\underline{ff^*})^{-1}.$$

The operator $\widehat{\Xi}_Q$ (see (3.9)) now takes the form

$$\widehat{\Xi}_Q(t,\boldsymbol{\theta},\varepsilon) = \widehat{\Xi}_Q(\mathbf{k},\varepsilon) = (t^2\widehat{S}(\boldsymbol{\theta}) + \varepsilon^2\overline{Q})^{-1}\widehat{P} = (\widehat{S}(\mathbf{k}) + \varepsilon^2\overline{Q})^{-1}\widehat{P}. \tag{7.3}$$

The role of the operator $\widehat{Z}_Q$ is played by the operator (see [BSu4, §5])

$$\widehat{Z}_Q(\boldsymbol{\theta}) = \Lambda_Q b(\boldsymbol{\theta})\widehat{P}. \tag{7.4}$$

Here Λ_Q is the operator of multiplication by the periodic $(n\times m)$-matrix $\Lambda_Q(\mathbf{x})$ with the columns $\mathbf{v}_j^{(Q)}(\mathbf{x})$, $j = 1,\dots,m$, which are the Γ-periodic solutions of the problem

$$b(\mathbf{D})^* g(\mathbf{x})(b(\mathbf{D})\mathbf{v}_j^{(Q)}(\mathbf{x}) + \mathbf{e}_j) = 0, \quad \int_\Omega Q(\mathbf{x})\mathbf{v}_j^{(Q)}(\mathbf{x})\,d\mathbf{x} = 0.$$

Then (see [BSu4, (5.4)])

$$\Lambda_Q(\mathbf{x}) = \Lambda(\mathbf{x}) + \Lambda_Q^0, \quad \Lambda_Q^0 = -(\overline{Q})^{-1}(\overline{Q\Lambda}). \tag{7.5}$$

By the presence of the projection $\widehat{P}$ and by (6.2) and (6.3), relations (7.3) and (7.4) imply that

$$(I + t\widehat{Z}_Q(\boldsymbol{\theta}))\widehat{\Xi}_Q(t,\boldsymbol{\theta},\varepsilon)\widehat{P} = (I + \Lambda_Q b(\mathbf{D}+\mathbf{k}))(\widehat{\mathcal{A}}^0(\mathbf{k}) + \varepsilon^2\overline{Q})^{-1}\widehat{P}. \tag{7.6}$$

The estimate (3.14) is applicable. We should only specify the constants. The constants c_*, δ, t^0 correspond to the operator $\mathcal{A}(\mathbf{k}) = f^*\widehat{\mathcal{A}}(\mathbf{k})f$ and are defined by (4.14)–(4.16). Using the estimate $\|X_1(\boldsymbol{\theta})\| \le \alpha_1^{1/2}\|h\|_{L_\infty}\|f\|_{L_\infty}$, instead of more precise values $C_1(\boldsymbol{\theta}) = \beta_1\delta^{-1/2}\|X_1(\boldsymbol{\theta})\|$ and $C_2(\boldsymbol{\theta}) = \beta_2\delta^{-1/2}\|X_1(\boldsymbol{\theta})\|^3$ (see Subsection 1.4), we take the following values:

$$C_1 = \beta_1\delta^{-1/2}\alpha_1^{1/2}\|h\|_{L_\infty}\|f\|_{L_\infty},$$
$$C_2 = \beta_2\delta^{-1/2}\alpha_1^{3/2}\|h\|_{L_\infty}^3\|f\|_{L_\infty}^3.$$

Instead of $C_5(\boldsymbol{\theta})$ (see (2.24)), we take the overstated value

$$C_5 = (3\delta)^{-1/2} + C_1 c_*^{-1/2} + C_2 c_*^{-3/2} + \beta_3(1+\pi^{-1})\delta^{-1/2}c_*^{-1}\alpha_1\|h\|_{L_\infty}^2\|f\|_{L_\infty}^2. \tag{7.7}$$

Thus,

$$C_5 = C_5(\alpha_0, \alpha_1, \|g\|_{L_\infty}, \|g^{-1}\|_{L_\infty}, \|Q\|_{L_\infty}, \|Q^{-1}\|_{L_\infty}, r_0).$$

Then the constant in the right-hand side of (3.14) can be replaced by the rougher value $C_5\|f\|_{L_\infty} = C_5\|Q^{-1}\|_{L_\infty}^{1/2}$. Applying Theorem 3.1 and taking (7.6) into account, we obtain:

$$\begin{aligned}&\|\widehat{\mathcal{A}}(\mathbf{k})^{1/2}\big((\widehat{\mathcal{A}}(\mathbf{k}) + \varepsilon^2 Q)^{-1} - (I + \Lambda_Q b(\mathbf{D}+\mathbf{k}))(\widehat{\mathcal{A}}^0(\mathbf{k}) + \varepsilon^2\overline{Q})^{-1}\widehat{P}\big)\|_{\mathfrak{H}\to\mathfrak{H}} \\ &\quad\le C_5\|Q^{-1}\|_{L_\infty}^{1/2}, \quad |\mathbf{k}| \le t^0, \ 0 < \varepsilon \le 1.\end{aligned} \tag{7.8}$$

Using (7.5), show that one can replace Λ_Q by Λ in (7.8). This will influence only on the constant in estimate. We have:

$$\begin{aligned}
&\|\widehat{\mathcal{A}}(\mathbf{k})^{1/2}\Lambda_Q^0 b(\mathbf{D}+\mathbf{k})(\widehat{\mathcal{A}}^0(\mathbf{k})+\varepsilon^2\overline{Q})^{-1}\widehat{P}\|_{\mathfrak{H}\to\mathfrak{H}} \\
&\quad= \|g^{1/2}b(\mathbf{D}+\mathbf{k})\Lambda_Q^0 b(\mathbf{D}+\mathbf{k})(\widehat{\mathcal{A}}^0(\mathbf{k})+\varepsilon^2\overline{Q})^{-1}\widehat{P}\|_{\mathfrak{H}\to\mathfrak{H}_*} \\
&\qquad\le \|g\|_{L_\infty}^{1/2}\sup_{\mathbf{k}}|b(\mathbf{k})\Lambda_Q^0 b(\mathbf{k})(\widehat{S}(\mathbf{k})+\varepsilon^2\overline{Q})^{-1}| \le \|g\|_{L_\infty}^{1/2}\alpha_1\widehat{c}_*^{-1}|\Lambda_Q^0|. \qquad (7.9)
\end{aligned}$$

We have taken into account the following relation, which is valid because of the presence of the projection $\widehat{P}$:

$$b(\mathbf{D}+\mathbf{k})\Lambda_Q^0 b(\mathbf{D}+\mathbf{k})(\widehat{\mathcal{A}}^0(\mathbf{k})+\varepsilon^2\overline{Q})^{-1}\widehat{P} = b(\mathbf{k})\Lambda_Q^0 b(\mathbf{k})(\widehat{S}(\mathbf{k})+\varepsilon^2\overline{Q})^{-1}\widehat{P},$$

and used the estimates $|b(\mathbf{k})| \le \alpha_1^{1/2}|\mathbf{k}|$ and $|(\widehat{S}(\mathbf{k})+\varepsilon^2\overline{Q})^{-1}| \le \widehat{c}_*^{-1}|\mathbf{k}|^{-2}$ (see (4.2), (4.19), and (5.3)).

Next, we have $|(\overline{Q})^{-1}| \le \|Q^{-1}\|_{L_\infty}$. Besides, as it was shown in [BSu4, (7.14)],

$$|\overline{Q\Lambda}| \le m^{1/2}(2r_0)^{-1}\alpha_0^{-1/2}\|g\|_{L_\infty}^{1/2}\|g^{-1}\|_{L_\infty}^{1/2}\|Q\|_{L_\infty}. \qquad (7.10)$$

Then, by (7.9) and (7.5), we obtain:

$$\begin{aligned}
&\|\widehat{\mathcal{A}}(\mathbf{k})^{1/2}\Lambda_Q^0 b(\mathbf{D}+\mathbf{k})(\widehat{\mathcal{A}}^0(\mathbf{k})+\varepsilon^2\overline{Q})^{-1}\widehat{P}\|_{\mathfrak{H}\to\mathfrak{H}} \\
&\quad\le m^{1/2}(2r_0)^{-1}\alpha_1\alpha_0^{-1/2}\widehat{c}_*^{-1}\|g\|_{L_\infty}\|g^{-1}\|_{L_\infty}^{1/2}\|Q\|_{L_\infty}\|Q^{-1}\|_{L_\infty}. \qquad (7.11)
\end{aligned}$$

Now, relations (7.8), (7.11), and (7.5) imply that

$$\begin{aligned}
&\|\widehat{\mathcal{A}}(\mathbf{k})^{1/2}\big((\widehat{\mathcal{A}}(\mathbf{k})+\varepsilon^2 Q)^{-1} - (I+\Lambda b(\mathbf{D}+\mathbf{k}))(\widehat{\mathcal{A}}^0(\mathbf{k})+\varepsilon^2\overline{Q})^{-1}\widehat{P}\big)\|_{\mathfrak{H}\to\mathfrak{H}} \\
&\quad\le C_5\|Q^{-1}\|_{L_\infty}^{1/2} + m^{1/2}(2r_0)^{-1}\alpha_1\alpha_0^{-1/2}\widehat{c}_*^{-1}\|g\|_{L_\infty}\|g^{-1}\|_{L_\infty}^{1/2}\|Q\|_{L_\infty}\|Q^{-1}\|_{L_\infty} \\
&\quad=: \check{\mathcal{C}}_0, \quad |\mathbf{k}| \le t^0,\ 0<\varepsilon\le 1. \qquad (7.12)
\end{aligned}$$

7.3. The estimate for $\mathbf{k} \in \widetilde{\Omega}\setminus\mathcal{B}(t^0)$

The operator under the norm sign in (7.2) can be represented as

$$\begin{aligned}
\mathcal{E}_Q(\mathbf{k},\varepsilon) &:= \widehat{\mathcal{A}}(\mathbf{k})^{1/2}(\widehat{\mathcal{A}}(\mathbf{k})+\varepsilon^2 Q)^{-1} - (\widehat{\mathcal{A}}(\mathbf{k})^{1/2}\widehat{P})(\widehat{\mathcal{A}}^0(\mathbf{k})+\varepsilon^2\overline{Q})^{-1} \\
&\quad - (\widehat{\mathcal{A}}(\mathbf{k})^{1/2}\Lambda\widehat{P}_m)(b(\mathbf{D}+\mathbf{k})\widehat{P})(\widehat{\mathcal{A}}^0(\mathbf{k})+\varepsilon^2\overline{Q})^{-1}. \qquad (7.13)
\end{aligned}$$

By (4.17) with $r = t^0$, we have (cf. (6.11))

$$\widehat{\mathcal{A}}(\mathbf{k}) \ge \widehat{c}_*(t^0)^2 I, \quad \widehat{\mathcal{A}}^0(\mathbf{k}) \ge \widehat{c}_*(t^0)^2 I, \quad \mathbf{k}\in\widetilde{\Omega}\setminus\mathcal{B}(t^0).$$

Hence,

$$\|\widehat{\mathcal{A}}(\mathbf{k})^{1/2}(\widehat{\mathcal{A}}(\mathbf{k})+\varepsilon^2 Q)^{-1}\|_{\mathfrak{H}\to\mathfrak{H}} \le \widehat{c}_*^{-1/2}(t^0)^{-1}, \quad \mathbf{k}\in\widetilde{\Omega}\setminus\mathcal{B}(t^0), \qquad (7.14)$$

$$\|(\widehat{\mathcal{A}}^0(\mathbf{k})+\varepsilon^2\overline{Q})^{-1}\|_{\mathfrak{H}\to\mathfrak{H}} \le \widehat{c}_*^{-1}(t^0)^{-2}, \quad \mathbf{k}\in\widetilde{\Omega}\setminus\mathcal{B}(t^0). \qquad (7.15)$$

The operator $\widehat{\mathcal{A}}(\mathbf{k})^{1/2}\widehat{P}$ is estimated in (6.15), the operator $b(\mathbf{D}+\mathbf{k})\widehat{P}$ is estimated in (6.16), while (6.22) gives the estimate for the operator $\widehat{\mathcal{A}}(\mathbf{k})^{1/2}\Lambda\widehat{P}_m$. As a result, using (7.13)–(7.15), we obtain:

$$\begin{aligned}
\|\mathcal{E}_Q(\mathbf{k},\varepsilon)\|_{\mathfrak{H}\to\mathfrak{H}} &\le \widehat{c}_*^{-1/2}(t^0)^{-1} + \alpha_1^{1/2}r_1\widehat{c}_*^{-1}(t^0)^{-2}(\|g\|_{L_\infty}^{1/2} + \widetilde{\mathcal{C}}_1) =: \check{\mathcal{C}}_2, \\
&\qquad \mathbf{k}\in\widetilde{\Omega}\setminus\mathcal{B}(t^0). \qquad (7.16)
\end{aligned}$$

Combining estimates (7.12) and (7.16), we arrive at (7.2) with

$$\check{\mathcal{C}}_3 = \max\{\check{\mathcal{C}}_0, \check{\mathcal{C}}_2\}. \qquad (7.17)$$

This completes the proof of Theorem 7.1. □

7.4. The case of zero corrector

If condition (6.25) is satisfied, then the term in (7.2) which corresponds to the corrector is equal to zero. Herewith, the constant in estimate (7.2) can be refined. Under condition (6.25), relation (7.5) implies that

$$\Lambda_Q(\mathbf{x})b(\boldsymbol{\theta}) = 0, \quad \mathbf{x} \in \Omega, \ \boldsymbol{\theta} \in \mathbb{S}^{d-1}.$$

Then (7.8) turns into the estimate

$$\|\widehat{\mathcal{A}}(\mathbf{k})^{1/2}\big((\widehat{\mathcal{A}}(\mathbf{k}) + \varepsilon^2 Q)^{-1} - (\widehat{\mathcal{A}}^0(\mathbf{k}) + \varepsilon^2 \overline{Q})^{-1}\widehat{P}\big)\|_{\mathfrak{H}\to\mathfrak{H}} \le C_5 \|Q^{-1}\|^{1/2}_{L_\infty}, \\ |\mathbf{k}| \le t^0, \quad 0 < \varepsilon \le 1.$$

For $\mathbf{k} \in \widetilde{\Omega} \setminus \mathcal{B}(t^0)$, we use estimates (7.14), (7.15), and (6.15):

$$\|\widehat{\mathcal{A}}(\mathbf{k})^{1/2}\big((\widehat{\mathcal{A}}(\mathbf{k}) + \varepsilon^2 Q)^{-1} - (\widehat{\mathcal{A}}^0(\mathbf{k}) + \varepsilon^2 \overline{Q})^{-1}\widehat{P}\big)\|_{\mathfrak{H}\to\mathfrak{H}} \\ \le \widehat{c}_*^{-1/2}(t^0)^{-1} + \alpha_1^{1/2} r_1 \widehat{c}_*^{-1}(t^0)^{-2}\|g\|^{1/2}_{L_\infty}, \quad \mathbf{k} \in \widetilde{\Omega} \setminus \mathcal{B}(t^0).$$

As a result, we arrive at the following theorem.

Theorem 7.2. *Under condition* (6.25), *we have*

$$\|\widehat{\mathcal{A}}(\mathbf{k})^{1/2}\big((\widehat{\mathcal{A}}(\mathbf{k}) + \varepsilon^2 Q)^{-1} - (\widehat{\mathcal{A}}^0(\mathbf{k}) + \varepsilon^2 \overline{Q})^{-1}\widehat{P}\big)\|_{\mathfrak{H}\to\mathfrak{H}} \le \check{\mathcal{C}}_3^\circ, \\ \mathbf{k} \in \widetilde{\Omega}, \quad 0 < \varepsilon \le 1,$$

where

$$\check{\mathcal{C}}_3^\circ = \max\{C_5\|Q^{-1}\|^{1/2}_{L_\infty}, \ \widehat{c}_*^{-1/2}(t^0)^{-1} + \alpha_1^{1/2} r_1 \widehat{c}_*^{-1}(t^0)^{-2}\|g\|^{1/2}_{L_\infty}\}.$$

7.5. Approximation with the three-term corrector

In what follows (with a view to interpolation), besides approximation (7.2), we will need another approximation for the operator $\widehat{\mathcal{A}}(\mathbf{k})^{1/2}(\widehat{\mathcal{A}}(\mathbf{k}) + \varepsilon^2 Q)^{-1}$, namely, the approximation with the „three-term corrector“. We apply Theorem 3.2 to the operator $\widehat{A}(t,\boldsymbol{\theta}) = \widehat{\mathcal{A}}(\mathbf{k})$. Now $\widehat{\Xi}_Q = \widehat{\Xi}_Q(\mathbf{k},\varepsilon)$ is the operator (7.3), and $\widehat{Z}_Q = \widehat{Z}_Q(\boldsymbol{\theta})$ is the operator (7.4). The operator $\widehat{N}_Q$ is realized (see [BSu4, (5.13)]) as

$$\widehat{N}_Q(\boldsymbol{\theta}) = b(\boldsymbol{\theta})^* L_Q(\boldsymbol{\theta}) b(\boldsymbol{\theta})\widehat{P},$$

where

$$L_Q(\boldsymbol{\theta}) = |\Omega|^{-1}\int_\Omega (\Lambda_Q(\mathbf{x})^* b(\boldsymbol{\theta})^* \widetilde{g}(\mathbf{x}) + \widetilde{g}(\mathbf{x})^* b(\boldsymbol{\theta})\Lambda_Q(\mathbf{x}))\, d\mathbf{x}.$$

We put $\widehat{Z}_Q(\mathbf{k}) = |\mathbf{k}|\widehat{Z}_Q(\boldsymbol{\theta})$, $\widehat{N}_Q(\mathbf{k}) = |\mathbf{k}|^3\widehat{N}_Q(\boldsymbol{\theta})$. Instead of the value

$$C_6(\boldsymbol{\theta}) = C_5(\boldsymbol{\theta}) + (8\delta)^{-1/2}\|X_1(\boldsymbol{\theta})\|^2 c_*^{-1} + (2\delta)^{-1/2}\|X_1(\boldsymbol{\theta})\|^4 c_*^{-2}$$

(see (2.29)), we take the value

$$C_6 = C_5 + (8\delta)^{-1/2} c_*^{-1}\alpha_1 \|g\|_{L_\infty}\|Q^{-1}\|_{L_\infty} + (2\delta)^{-1/2} c_*^{-2}\alpha_1^2 \|g\|^2_{L_\infty}\|Q^{-1}\|^2_{L_\infty},$$

where the constant C_5 is defined by (7.7).

Applying Theorem 3.2, we obtain the estimate

$$\|\widehat{\mathcal{A}}(\mathbf{k})^{1/2}\Big((\widehat{\mathcal{A}}(\mathbf{k})+\varepsilon^2 Q)^{-1}-(\widehat{\mathcal{A}}^0(\mathbf{k})+\varepsilon^2\overline{Q})^{-1}\widehat{P}-\mathcal{K}_Q(\mathbf{k},\varepsilon)\Big)\|_{\mathfrak{H}\to\mathfrak{H}} \\ \leq C_6\|Q^{-1}\|^{1/2}_{L_\infty}, \quad |\mathbf{k}|\leq t^0,\ 0<\varepsilon\leq 1. \tag{7.18}$$

Here

$$\mathcal{K}_Q(\mathbf{k},\varepsilon):=\widehat{Z}_Q(\mathbf{k})\widehat{\Xi}_Q(\mathbf{k},\varepsilon)+\widehat{\Xi}_Q(\mathbf{k},\varepsilon)\widehat{Z}_Q(\mathbf{k})^*-\widehat{\Xi}_Q(\mathbf{k},\varepsilon)\widehat{N}_Q(\mathbf{k})\widehat{\Xi}_Q(\mathbf{k},\varepsilon) \\ =\Lambda_Q b(\mathbf{D}+\mathbf{k})(\widehat{\mathcal{A}}^0(\mathbf{k})+\varepsilon^2\overline{Q})^{-1}\widehat{P}+(\widehat{\mathcal{A}}^0(\mathbf{k})+\varepsilon^2\overline{Q})^{-1}\widehat{P}b(\mathbf{D}+\mathbf{k})^*\Lambda_Q^* \\ -(\widehat{\mathcal{A}}^0(\mathbf{k})+\varepsilon^2\overline{Q})^{-1}b(\mathbf{D}+\mathbf{k})^*L_Q(\mathbf{D}+\mathbf{k})b(\mathbf{D}+\mathbf{k})(\widehat{\mathcal{A}}^0(\mathbf{k})+\varepsilon^2\overline{Q})^{-1}\widehat{P}.$$

This expression can be transformed to the following form (cf. [BSu4, (7.6)]):

$$\mathcal{K}_Q(\mathbf{k},\varepsilon)=\Lambda b(\mathbf{D}+\mathbf{k})(\widehat{\mathcal{A}}^0(\mathbf{k})+\varepsilon^2\overline{Q})^{-1}\widehat{P}+(\widehat{\mathcal{A}}^0(\mathbf{k})+\varepsilon^2\overline{Q})^{-1}\widehat{P}b(\mathbf{D}+\mathbf{k})^*\Lambda^* \\ -(\widehat{\mathcal{A}}^0(\mathbf{k})+\varepsilon^2\overline{Q})^{-1}b(\mathbf{D}+\mathbf{k})^*L(\mathbf{D}+\mathbf{k})b(\mathbf{D}+\mathbf{k})(\widehat{\mathcal{A}}^0(\mathbf{k})+\varepsilon^2\overline{Q})^{-1}\widehat{P} \\ -(\widehat{\mathcal{A}}^0(\mathbf{k})+\varepsilon^2\overline{Q})^{-1}\big(\varepsilon^2 b(\mathbf{D}+\mathbf{k})^*(\overline{Q\Lambda})^*+\varepsilon^2(\overline{Q\Lambda})b(\mathbf{D}+\mathbf{k})\big)(\widehat{\mathcal{A}}^0(\mathbf{k})+\varepsilon^2\overline{Q})^{-1}\widehat{P}. \tag{7.19}$$

Representation (7.19) contains Λ and $L(\mathbf{D}+\mathbf{k})$ instead of Λ_Q and $L_Q(\mathbf{D}+\mathbf{k})$.

Now we obtain the estimate of the form (7.18) for $\mathbf{k}\in\widetilde{\Omega}\setminus\mathcal{B}(t^0)$. Inequality (7.16) in combination with (7.13) means that

$$\|\widehat{\mathcal{A}}(\mathbf{k})^{1/2}\big((\widehat{\mathcal{A}}(\mathbf{k})+\varepsilon^2 Q)^{-1}-(I+\Lambda b(\mathbf{D}+\mathbf{k}))(\widehat{\mathcal{A}}^0(\mathbf{k})+\varepsilon^2\overline{Q})^{-1}\widehat{P}\big)\|_{\mathfrak{H}\to\mathfrak{H}}\leq\check{\mathcal{C}}_2, \\ \mathbf{k}\in\widetilde{\Omega}\setminus\mathcal{B}(t^0). \tag{7.20}$$

The estimate for the term

$$\widehat{\mathcal{A}}(\mathbf{k})^{1/2}(\widehat{\mathcal{A}}^0(\mathbf{k})+\varepsilon^2\overline{Q})^{-1}\widehat{P}b(\mathbf{D}+\mathbf{k})^*\Lambda^*=(\widehat{\mathcal{A}}(\mathbf{k})^{1/2}\widehat{P})(\widehat{\mathcal{A}}^0(\mathbf{k})+\varepsilon^2\overline{Q})^{-1}\widehat{Z}(\mathbf{k})^*$$

can be obtained on the basis of (6.15), (7.15) and (6.32):

$$\|\widehat{\mathcal{A}}(\mathbf{k})^{1/2}(\widehat{\mathcal{A}}^0(\mathbf{k})+\varepsilon^2\overline{Q})^{-1}\widehat{P}b(\mathbf{D}+\mathbf{k})^*\Lambda^*\|_{\mathfrak{H}\to\mathfrak{H}} \\ \leq r_1^2(8\widehat{\delta})^{-1/2}\alpha_1\|g\|_{L_\infty}\widehat{c}_*^{-1}(t^0)^{-2}, \quad \mathbf{k}\in\widetilde{\Omega}\setminus\mathcal{B}(t^0). \tag{7.21}$$

The term

$$\widehat{\mathcal{A}}(\mathbf{k})^{1/2}(\widehat{\mathcal{A}}^0(\mathbf{k})+\varepsilon^2\overline{Q})^{-1}b(\mathbf{D}+\mathbf{k})^*L(\mathbf{D}+\mathbf{k})b(\mathbf{D}+\mathbf{k})(\widehat{\mathcal{A}}^0(\mathbf{k})+\varepsilon^2\overline{Q})^{-1}\widehat{P} \\ =(\widehat{\mathcal{A}}(\mathbf{k})^{1/2}\widehat{P})(\widehat{\mathcal{A}}^0(\mathbf{k})+\varepsilon^2\overline{Q})^{-1}\widehat{N}(\mathbf{k})(\widehat{\mathcal{A}}^0(\mathbf{k})+\varepsilon^2\overline{Q})^{-1}\widehat{P}$$

is estimated with the help of (6.15), (7.15), and (6.33):

$$\|\widehat{\mathcal{A}}(\mathbf{k})^{1/2}(\widehat{\mathcal{A}}^0(\mathbf{k})+\varepsilon^2\overline{Q})^{-1}b(\mathbf{D}+\mathbf{k})^*L(\mathbf{D}+\mathbf{k})b(\mathbf{D}+\mathbf{k})(\widehat{\mathcal{A}}^0(\mathbf{k})+\varepsilon^2\overline{Q})^{-1}\widehat{P}\|_{\mathfrak{H}\to\mathfrak{H}} \\ \leq r_1^4(2\widehat{\delta})^{-1/2}\alpha_1^2\|g\|^2_{L_\infty}\widehat{c}_*^{-2}(t^0)^{-4}, \quad \mathbf{k}\in\widetilde{\Omega}\setminus\mathcal{B}(t^0). \tag{7.22}$$

Finally, the operator

$$\widehat{\mathcal{A}}(\mathbf{k})^{1/2}(\widehat{\mathcal{A}}^0(\mathbf{k})+\varepsilon^2\overline{Q})^{-1} \\ \times\big(\varepsilon^2 b(\mathbf{D}+\mathbf{k})^*(\overline{Q\Lambda})^*+\varepsilon^2(\overline{Q\Lambda})b(\mathbf{D}+\mathbf{k})\big)(\widehat{\mathcal{A}}^0(\mathbf{k})+\varepsilon^2\overline{Q})^{-1}\widehat{P}$$

is estimated by using of (6.15), (6.16), (7.10), and (7.15):

$$\begin{aligned}
&\|\widehat{\mathcal{A}}(\mathbf{k})^{1/2}(\widehat{\mathcal{A}}^0(\mathbf{k})+\varepsilon^2\overline{Q})^{-1} \\
&\quad\times\big(\varepsilon^2 b(\mathbf{D}+\mathbf{k})^*(\overline{Q\Lambda})^* + \varepsilon^2(\overline{Q\Lambda})b(\mathbf{D}+\mathbf{k})\big)(\widehat{\mathcal{A}}^0(\mathbf{k})+\varepsilon^2\overline{Q})^{-1}\widehat{P}\|_{\mathfrak{H}\to\mathfrak{H}} \\
&\quad\le r_1^2 r_0^{-1}\alpha_1\alpha_0^{-1/2}m^{1/2}\widehat{c}_*^{-2}(t^0)^{-4}\|g\|_{L_\infty}\|g^{-1}\|_{L_\infty}^{1/2}\|Q\|_{L_\infty}, \\
&\qquad\qquad \mathbf{k}\in\widetilde{\Omega}\setminus\mathcal{B}(t^0),\ \ 0<\varepsilon\le 1.
\end{aligned} \tag{7.23}$$

As a result, combining (7.20)–(7.23) with (7.19), we obtain the estimate

$$\begin{aligned}
&\|\widehat{\mathcal{A}}(\mathbf{k})^{1/2}\big((\widehat{\mathcal{A}}(\mathbf{k})+\varepsilon^2 Q)^{-1} - (\widehat{\mathcal{A}}^0(\mathbf{k})+\varepsilon^2\overline{Q})^{-1}\widehat{P} - \mathcal{K}_Q(\mathbf{k},\varepsilon)\big)\|_{\mathfrak{H}\to\mathfrak{H}} \le \check{\mathcal{C}}_4, \\
&\qquad\qquad\qquad \mathbf{k}\in\widetilde{\Omega}\setminus\mathcal{B}(t^0),\quad 0<\varepsilon\le 1,
\end{aligned} \tag{7.24}$$

where

$$\begin{aligned}
\check{\mathcal{C}}_4 &= \check{\mathcal{C}}_2 + r_1^2(8\widehat{\delta})^{-1/2}\alpha_1\|g\|_{L_\infty}\widehat{c}_*^{-1}(t^0)^{-2} + r_1^4(2\widehat{\delta})^{-1/2}\alpha_1^2\|g\|_{L_\infty}^2\widehat{c}_*^{-2}(t^0)^{-4} \\
&\quad + r_1^2 r_0^{-1}\alpha_1\alpha_0^{-1/2}m^{1/2}\widehat{c}_*^{-2}(t^0)^{-4}\|g\|_{L_\infty}\|g^{-1}\|_{L_\infty}^{1/2}\|Q\|_{L_\infty}.
\end{aligned}$$

Relations (7.18) and (7.24) imply the following theorem.

Theorem 7.3. *Let $\mathcal{K}_Q(\mathbf{k},\varepsilon)$ be the operator defined by* (7.19). *We have*

$$\begin{aligned}
&\|\widehat{\mathcal{A}}(\mathbf{k})^{1/2}\big((\widehat{\mathcal{A}}(\mathbf{k})+\varepsilon^2 Q)^{-1} - (\widehat{\mathcal{A}}^0(\mathbf{k})+\varepsilon^2\overline{Q})^{-1}\widehat{P} - \mathcal{K}_Q(\mathbf{k},\varepsilon)\big)\|_{\mathfrak{H}\to\mathfrak{H}} \\
&\quad\le \max\{C_6\|Q^{-1}\|_{L_\infty}^{1/2}, \check{\mathcal{C}}_4\} =: \check{\mathcal{C}}_5,\quad \mathbf{k}\in\widetilde{\Omega},\ \ 0<\varepsilon\le 1.
\end{aligned}$$

8 Approximation for the operator $\widehat{\mathcal{A}}^{1/2}(\widehat{\mathcal{A}}+\varepsilon^2 I)^{-1}$

8.1

We put

$$\mathfrak{G} = L_2(\mathbb{R}^d;\mathbb{C}^n),\quad \mathfrak{G}_* = L_2(\mathbb{R}^d;\mathbb{C}^m).$$

Now we return to the operator $\widehat{\mathcal{A}} = b(\mathbf{D})^* g(\mathbf{x}) b(\mathbf{D})$ acting in the space $\mathfrak{G}$. Let g^0 be the effective matrix, and let $\widehat{\mathcal{A}}^0 = b(\mathbf{D})^* g^0 b(\mathbf{D})$ be the effective operator. Let $\mathcal{V}$ be the Gelfand transform defined in Subsection 4.3. Using decomposition (4.13) for the operator $\widehat{\mathcal{A}}$, we represent the resolvent $(\widehat{\mathcal{A}}+\varepsilon^2 I)^{-1}$ as

$$(\widehat{\mathcal{A}}+\varepsilon^2 I)^{-1} = \mathcal{V}^{-1}\Big(\int_{\widetilde{\Omega}} \oplus(\widehat{\mathcal{A}}(\mathbf{k})+\varepsilon^2 I)^{-1}\,d\mathbf{k}\Big)\mathcal{V}.$$

The similar expansion is valid for $(\widehat{\mathcal{A}}^0+\varepsilon^2 I)^{-1}$. Under the Gelfand transformation, the operator $b(\mathbf{D})$ turns into $\int_{\widetilde{\Omega}}\oplus b(\mathbf{D}+\mathbf{k})\,d\mathbf{k}$, while the operator of multiplication by the periodic matrix-valued function $\Lambda(\mathbf{x})$ turns into the operator of multiplication by the same matrix in the fibers of the direct integral $\mathcal{K}$ (see (4.7)). Besides, we will need the operator

$$\Pi := \mathcal{V}^{-1}[\widehat{P}]\mathcal{V},$$

acting in $\mathfrak{G}$. Here $[\widehat{P}]$ is the projection in $\mathcal{K}$, which acts in the fibers as the operator $\widehat{P}$ (the operator of averaging over Ω). As it was shown in [BSu4, (6.8)], Π *is a pseudodifferential* *operator in* $\mathfrak{G}$ *with the symbol* $\chi_{\widetilde{\Omega}}(\boldsymbol{\xi})$, where $\chi_{\widetilde{\Omega}}$ is the characteristic function of the set $\widetilde{\Omega}$. Thus,

$$\Pi = \mathcal{F}^*[\chi_{\widetilde{\Omega}}]\mathcal{F}, \tag{8.1}$$

where $\mathcal{F}$ is the Fourier operator in $\mathfrak{G}$.

Applying the Gelfand transformation to the operators in (6.1), we arrive at the following result.

Theorem 8.1. *Let* $\widehat{\mathcal{A}} = b(\mathbf{D})^* g(\mathbf{x}) b(\mathbf{D})$*, and let* $\widehat{\mathcal{A}}^0 = b(\mathbf{D})^* g^0 b(\mathbf{D})$ *be the effective operator. Let* $\Lambda : \mathfrak{G}_* \to \mathfrak{G}$ *be the operator of multiplication by the matrix-valued function* $\Lambda(\mathbf{x})$ *introduced in Subsection* 5.1*. Let* Π *be the operator* (8.1)*. Then we have*

$$\|\widehat{\mathcal{A}}^{1/2}\big((\widehat{\mathcal{A}} + \varepsilon^2 I)^{-1} - (I + \Lambda b(\mathbf{D}))(\widehat{\mathcal{A}}^0 + \varepsilon^2 I)^{-1}\Pi\big)\|_{\mathfrak{G}\to\mathfrak{G}} \le \mathcal{C}_3, \quad 0 < \varepsilon \le 1. \tag{8.2}$$

The constant $\mathcal{C}_3$ *is defined by* (6.24) *and depends only on* m, α_0, α_1, $\|g\|_{L_\infty}$, $\|g^{-1}\|_{L_\infty}$, r_0, *and* r_1.

8.2. „Elimination“ of the operator Π

Now our goal is to find conditions, under which the pseudodifferential operator Π in (8.2) can be replaced by I. It turns out that, in the principal term of approximation, i. e., in the operator $\widehat{\mathcal{A}}^{1/2}(\widehat{\mathcal{A}}^0 + \varepsilon^2 I)^{-1}\Pi$, such replacement is always possible (only the constant in the remainder estimate will change), while in the corrector term it is not always possible to replace Π by I.

We consider the operator

$$\widehat{\mathcal{A}}^{1/2}(\widehat{\mathcal{A}}^0 + \varepsilon^2 I)^{-1}(I - \Pi). \tag{8.3}$$

The operator $(\widehat{\mathcal{A}}^0 + \varepsilon^2 I)^{-1}(I - \Pi)$ is the pseudodifferential operator of order (-2) with the symbol

$$(1 - \chi_{\widetilde{\Omega}}(\boldsymbol{\xi}))(b(\boldsymbol{\xi})^* g^0 b(\boldsymbol{\xi}) + \varepsilon^2 \mathbf{1}_n)^{-1}.$$

We estimate the norm of the operator (8.3). According to (4.19) and (5.3), we have

$$b(\boldsymbol{\xi})^* g^0 b(\boldsymbol{\xi}) \ge \widehat{c}_* |\boldsymbol{\xi}|^2 \mathbf{1}_n, \quad \boldsymbol{\xi} \in \mathbb{R}^d. \tag{8.4}$$

Together with (4.2), this implies that

$$|b(\boldsymbol{\xi})(b(\boldsymbol{\xi})^* g^0 b(\boldsymbol{\xi}) + \varepsilon^2 \mathbf{1}_n)^{-1}| \le \alpha_1^{1/2} \widehat{c}_*^{-1} |\boldsymbol{\xi}|^{-1}. \tag{8.5}$$

Then for $\mathbf{u} \in \mathfrak{G}$ we have:

$$\begin{aligned} &\|\widehat{\mathcal{A}}^{1/2}(\widehat{\mathcal{A}}^0 + \varepsilon^2 I)^{-1}(I - \Pi)\mathbf{u}\|_{\mathfrak{G}} \\ &= \|g^{1/2} b(\mathbf{D})(\widehat{\mathcal{A}}^0 + \varepsilon^2 I)^{-1}(I - \Pi)\mathbf{u}\|_{\mathfrak{G}_*} \\ &\le \|g\|_{L_\infty}^{1/2} \sup_{|\boldsymbol{\xi}| \ge r_0} |b(\boldsymbol{\xi})(b(\boldsymbol{\xi})^* g^0 b(\boldsymbol{\xi}) + \varepsilon^2 \mathbf{1}_n)^{-1}| \, \|\mathbf{u}\|_{\mathfrak{G}} \\ &\le \|g\|_{L_\infty}^{1/2} \alpha_1^{1/2} \widehat{c}_*^{-1} r_0^{-1} \|\mathbf{u}\|_{\mathfrak{G}}. \end{aligned}$$

Thus,

$$\|\widehat{\mathcal{A}}^{1/2}(\widehat{\mathcal{A}}^0 + \varepsilon^2 I)^{-1}(I - \Pi)\|_{\mathfrak{G}\to\mathfrak{G}} \le \|g\|_{L_\infty}^{1/2} \alpha_1^{1/2} \widehat{c}_*^{-1} r_0^{-1}. \tag{8.6}$$

Now Theorem 8.1 and estimate (8.6) lead to the following result.

Theorem 8.2. *Under the conditions of Theorem* 8.1, *we have*

$$\|\widehat{\mathcal{A}}^{1/2}\big((\widehat{\mathcal{A}}+\varepsilon^2 I)^{-1}-(\widehat{\mathcal{A}}^0+\varepsilon^2 I)^{-1}-\Lambda b(\mathbf{D})(\widehat{\mathcal{A}}^0+\varepsilon^2 I)^{-1}\Pi\big)\|_{\mathfrak{G}\to\mathfrak{G}}$$
$$\le \mathcal{C}_3+\|g\|^{1/2}_{L_\infty}\alpha_1^{1/2}\widehat{c}_*^{-1}r_0^{-1}=:\mathcal{C}_6,\quad 0<\varepsilon\le 1.$$

In order to eliminate Π in the corrector, we need additional assumptions on $\Lambda(\mathbf{x})$ (which in a number of cases are valid automatically). The operator

$$\Sigma(\varepsilon):=b(\mathbf{D})(\widehat{\mathcal{A}}^0+\varepsilon^2 I)^{-1}(I-\Pi)$$

continuously maps $\mathfrak{G}=L_2(\mathbb{R}^d;\mathbb{C}^n)$ to $\mathfrak{G}^1_*=H^1(\mathbb{R}^d;\mathbb{C}^m)$. By (8.5), we have

$$\|\Sigma(\varepsilon)\|_{\mathfrak{G}\to\mathfrak{G}^1_*}=\sup_{\boldsymbol{\xi}\in\mathbb{R}^d}(1+|\boldsymbol{\xi}|^2)^{1/2}|b(\boldsymbol{\xi})(b(\boldsymbol{\xi})^*g^0b(\boldsymbol{\xi})+\varepsilon^2\mathbf{1}_n)^{-1}|(1-\chi_{\widetilde{\Omega}}(\boldsymbol{\xi}))$$
$$\le\alpha_1^{1/2}\widehat{c}_*^{-1}\sup_{|\boldsymbol{\xi}|\ge r_0}(1+|\boldsymbol{\xi}|^2)^{1/2}|\boldsymbol{\xi}|^{-1}\le\alpha_1^{1/2}\widehat{c}_*^{-1}(1+r_0^{-2})^{1/2}. \tag{8.7}$$

Thus, for the replacement of Π by I in the term $\widehat{\mathcal{A}}^{1/2}\Lambda b(\mathbf{D})(\widehat{\mathcal{A}}^0+\varepsilon^2 I)^{-1}\Pi$, it suffices to assume that the operator $\widehat{\mathcal{A}}^{1/2}\Lambda$ continuosly maps $\mathfrak{G}^1_*$ to $\mathfrak{G}$. The last condition is equivalent to the condition that the operator $g^{1/2}b(\mathbf{D})\Lambda$ continuously maps $\mathfrak{G}^1_*$ to $\mathfrak{G}_*$. Since $\Lambda(\mathbf{x})$ is the matrix with the columns $\mathbf{v}_j(\mathbf{x})$, $j=1,\dots,m$, then

$$\|g^{1/2}b(\mathbf{D})\Lambda\|_{\mathfrak{G}^1_*\to\mathfrak{G}_*}\le\left(\sum_{j=1}^m\|g^{1/2}b(\mathbf{D})[\mathbf{v}_j]\|^2_{H^1(\mathbb{R}^d)\to\mathfrak{G}_*}\right)^{1/2}. \tag{8.8}$$

We prove the following lemma (cf. [Su2, Proposition 8.2]).

Lemma 8.3. *Suppose that the solutions $\mathbf{v}_j$ of the problem* (5.4) *with $\mathbf{C}=\mathbf{e}_j$ satisfy condition*

$$\mathbf{v}_j\in L_\infty,\quad j=1,\dots,m. \tag{8.9}$$

Then the operator $g^{1/2}b(\mathbf{D})[\mathbf{v}_j]$ continuously maps $H^1(\mathbb{R}^d)$ to $\mathfrak{G}_*$, and

$$\|g^{1/2}b(\mathbf{D})[\mathbf{v}_j]\|_{H^1(\mathbb{R}^d)\to\mathfrak{G}_*}\le C_j^\nabla, \tag{8.10}$$

$$C_j^\nabla:=\|g\|^{1/2}_{L_\infty}\left(\sqrt{2}+(\sqrt{8}+1)\alpha_1^{1/2}d^{1/2}\|\mathbf{v}_j\|_{L_\infty}+\sqrt{2}\alpha_1^{1/4}d^{1/2}\|\mathbf{v}_j\|^{1/2}_{L_\infty}\right). \tag{8.11}$$

Proof. We have

$$b(\mathbf{D})=\sum_{l=1}^d b_lD_l, \tag{8.12}$$

where b_l are constant $(m\times n)$-matrices. From (4.2) it follows that $|b_l|\le\alpha_1^{1/2}$, $l=1,\dots,d$. Let $u\in H^1(\mathbb{R}^d)$. Then

$$g^{1/2}b(\mathbf{D})(\mathbf{v}_ju)=g^{1/2}(b(\mathbf{D})\mathbf{v}_j)u+g^{1/2}\sum_{l=1}^d(b_lD_lu)\mathbf{v}_j. \tag{8.13}$$

By (8.9),

$$\|g^{1/2}\sum_{l=1}^d(b_lD_lu)\mathbf{v}_j\|_{\mathfrak{G}_*}\le\|g\|^{1/2}_{L_\infty}\|\mathbf{v}_j\|_{L_\infty}\left(\sum_{l=1}^d|b_l|^2\right)^{1/2}\|u\|_{H^1(\mathbb{R}^d)}$$
$$\le\|g\|^{1/2}_{L_\infty}\|\mathbf{v}_j\|_{L_\infty}\alpha_1^{1/2}d^{1/2}\|u\|_{H^1(\mathbb{R}^d)}. \tag{8.14}$$

Next, relation (5.4) with $\mathbf{C} = \mathbf{e}_j$ implies the following identity:

$$\int_{\mathbb{R}^d} \langle g(\mathbf{x})(b(\mathbf{D})\mathbf{v}_j + \mathbf{e}_j), b(\mathbf{D})\mathbf{w}\rangle \, d\mathbf{x} = 0 \tag{8.15}$$

for arbitrary $\mathbf{w} \in \mathfrak{G}^1 = H^1(\mathbb{R}^d; \mathbb{C}^n)$ such that $\mathbf{w}(\mathbf{x}) = 0$ for $|\mathbf{x}| > R$ (with some $R > 0$).

Let $u \in C_0^\infty(\mathbb{R}^d)$. We put $\mathbf{w}(\mathbf{x}) = |u(\mathbf{x})|^2 \mathbf{v}_j(\mathbf{x})$. Then

$$b(\mathbf{D})\mathbf{w} = |u|^2 b(\mathbf{D})\mathbf{v}_j + \sum_{l=1}^d b_l(D_l |u|^2)\mathbf{v}_j.$$

Substituting this in (8.15), we obtain that

$$\int_{\mathbb{R}^d} \langle g(\mathbf{x})(b(\mathbf{D})\mathbf{v}_j + \mathbf{e}_j), |u|^2 b(\mathbf{D})\mathbf{v}_j + \sum_{l=1}^d b_l(D_l|u|^2)\mathbf{v}_j\rangle \, d\mathbf{x} = 0.$$

Hence,

$$\begin{aligned}\int_{\mathbb{R}^d} |g^{1/2} b(\mathbf{D})\mathbf{v}_j|^2 |u|^2 \, d\mathbf{x} &= -\int_{\mathbb{R}^d} \langle g^{1/2}\mathbf{e}_j, g^{1/2} b(\mathbf{D})\mathbf{v}_j\rangle |u|^2 \, d\mathbf{x} \\ &- \int_{\mathbb{R}^d} \langle g^{1/2} b(\mathbf{D})\mathbf{v}_j, g^{1/2}\sum_{l=1}^d b_l(D_l|u|^2)\mathbf{v}_j\rangle \, d\mathbf{x} - \int_{\mathbb{R}^d} \langle g\mathbf{e}_j, \sum_{l=1}^d b_l(D_l|u|^2)\mathbf{v}_j\rangle \, d\mathbf{x}.\end{aligned}$$

The first term on the right can be estimated by

$$\begin{aligned}&\int_{\mathbb{R}^d} \left(|g^{1/2}\mathbf{e}_j|^2 + \frac{1}{4}|g^{1/2} b(\mathbf{D})\mathbf{v}_j|^2 \right) |u|^2 \, d\mathbf{x} \\ &\qquad \le \|g\|_{L_\infty} \int_{\mathbb{R}^d} |u|^2 \, d\mathbf{x} + \frac{1}{4}\int_{\mathbb{R}^d} |g^{1/2} b(\mathbf{D})\mathbf{v}_j|^2 |u|^2 \, d\mathbf{x},\end{aligned}$$

and the second term is estimated by

$$\begin{aligned}&\|g\|_{L_\infty}^{1/2} \int_{\mathbb{R}^d} |g^{1/2} b(\mathbf{D})\mathbf{v}_j| \left(\sum_{l=1}^d 2|b_l||D_l u||u| \right) |\mathbf{v}_j| \, d\mathbf{x} \\ &\le \frac{1}{4}\int_{\mathbb{R}^d} |g^{1/2} b(\mathbf{D})\mathbf{v}_j|^2 |u|^2 \, d\mathbf{x} + 4\|g\|_{L_\infty} \int_{\mathbb{R}^d} \left(\sum_{l=1}^d |b_l||D_l u| \right)^2 |\mathbf{v}_j|^2 \, d\mathbf{x} \\ &\le \frac{1}{4}\int_{\mathbb{R}^d} |g^{1/2} b(\mathbf{D})\mathbf{v}_j|^2 |u|^2 \, d\mathbf{x} + 4\|g\|_{L_\infty} \|\mathbf{v}_j\|_{L_\infty}^2 \alpha_1 d \int_{\mathbb{R}^d} |\nabla u|^2 \, d\mathbf{x}.\end{aligned}$$

The third term is estimated by

$$\|g\|_{L_\infty} \int_{\mathbb{R}^d} \left(\sum_{l=1}^d 2|b_l||D_l u||u| \right) |\mathbf{v}_j| \, d\mathbf{x} \le \|g\|_{L_\infty} \|\mathbf{v}_j\|_{L_\infty} \alpha_1^{1/2} \int_{\mathbb{R}^d} (|\nabla u|^2 + d|u|^2) \, d\mathbf{x}.$$

As a result, we obtain:

$$\frac{1}{2}\int\limits_{\mathbb{R}^d} |g^{1/2}b(\mathbf{D})\mathbf{v}_j|^2|u|^2\,d\mathbf{x} \le \|g\|_{L_\infty}\int\limits_{\mathbb{R}^d}|u|^2\,d\mathbf{x}$$

$$+\,4\|g\|_{L_\infty}\|\mathbf{v}_j\|^2_{L_\infty}\alpha_1 d\int\limits_{\mathbb{R}^d}|\nabla u|^2\,d\mathbf{x}+\|g\|_{L_\infty}\|\mathbf{v}_j\|_{L_\infty}\alpha_1^{1/2}\int\limits_{\mathbb{R}^d}(|\nabla u|^2+d|u|^2)\,d\mathbf{x}.$$

Hence,

$$\int\limits_{\mathbb{R}^d}|g^{1/2}b(\mathbf{D})\mathbf{v}_j|^2|u|^2\,d\mathbf{x}\le C_j^0\|u\|^2_{H^1(\mathbb{R}^d)},$$

$$C_j^0 = 2\|g\|_{L_\infty}(1+4\alpha_1 d\|\mathbf{v}_j\|^2_{L_\infty}+\alpha_1^{1/2}d\|\mathbf{v}_j\|_{L_\infty}).$$

Combining this with (8.13) and (8.14), we obtain that

$$\|g^{1/2}b(\mathbf{D})(\mathbf{v}_j u)\|_{\mathfrak{G}_*}\le \big(\|g\|^{1/2}_{L_\infty}\|\mathbf{v}_j\|_{L_\infty}\alpha_1^{1/2}d^{1/2}+(C_j^0)^{1/2}\big)\|u\|_{H^1(\mathbb{R}^d)},$$

which yields (8.10) and (8.11). □

We introduce the following condition.

Condition 8.4. *Suppose that the solutions* $\mathbf{v}_j$ *of the problem* (5.4) *with* $\mathbf{C}=\mathbf{e}_j$ *satisfy*

$$\mathbf{v}_j\in L_\infty,\quad j=1,\dots,m. \tag{8.16}$$

Consider the operator

$$\widehat{\mathcal{A}}^{1/2}\Lambda b(\mathbf{D})(\widehat{\mathcal{A}}^0+\varepsilon^2 I)^{-1}(I-\Pi)=\widehat{\mathcal{A}}^{1/2}\Lambda\Sigma(\varepsilon).$$

Using (8.7), (8.8), and Lemma 8.3, under Condition 8.4, we obtain the following estimate:

$$\begin{aligned}\|\widehat{\mathcal{A}}^{1/2}\Lambda\Sigma(\varepsilon)\|_{\mathfrak{G}\to\mathfrak{G}} &= \|g^{1/2}b(\mathbf{D})\Lambda\Sigma(\varepsilon)\|_{\mathfrak{G}\to\mathfrak{G}_*}\\ &\le \|\Sigma(\varepsilon)\|_{\mathfrak{G}\to\mathfrak{G}^1_*}\|g^{1/2}b(\mathbf{D})\Lambda\|_{\mathfrak{G}^1_*\to\mathfrak{G}_*}\\ &\le \alpha_1^{1/2}\widehat{c}_*^{-1}(1+r_0^{-2})^{1/2}\left(\sum_{j=1}^m (C_j^\nabla)^2\right)^{1/2}.\end{aligned} \tag{8.17}$$

Combining this with Theorem 8.2, we arrive at the following result.

Theorem 8.5. *Suppose that conditions of Theorem* 8.1 *are satisfied. Suppose also that Condition* 8.4 *is valid. Then*

$$\begin{aligned}&\|\widehat{\mathcal{A}}^{1/2}\big((\widehat{\mathcal{A}}+\varepsilon^2 I)^{-1}-(I+\Lambda b(\mathbf{D}))(\widehat{\mathcal{A}}^0+\varepsilon^2 I)^{-1}\big)\|_{\mathfrak{G}\to\mathfrak{G}}\\ &\le \mathcal{C}_6+\alpha_1^{1/2}\widehat{c}_*^{-1}(1+r_0^{-2})^{1/2}\left(\sum_{j=1}^m (C_j^\nabla)^2\right)^{1/2} =: \mathcal{C}_7,\quad 0<\varepsilon\le 1.\end{aligned}$$

Now we find conditions which guarantee the validity of Condition 8.4.

Condition 8.6. *Suppose that at least one of the following assumptions is true*:

1°) $d\le 2$;

2°) $d\ge 1$ *and* $\widehat{\mathcal{A}}=\mathbf{D}^* g(\mathbf{x})\mathbf{D}$, *where the matrix* $g(\mathbf{x})$ *has real entries*;

3°) $d\ge 1$ *and* $g^0=\underline{g}$ (*i. e., representations* (5.9) *are valid*).

Lemma 8.7. *Condition* 8.6 *guarantees that Condition* 8.4 *is satisfied.*

Proof. 1°. The solutions $\mathbf{v}_j$ satisfy condition $\mathbf{v}_j \in \widetilde{W}_p^1(\Omega)$ with some $p > 2$ (see [G]). If $d \leq 2$, then (8.16) follows, by the embedding $W_p^1(\Omega) \subset L_\infty$.
2°. For the operator $\widehat{\mathcal{A}} = \mathbf{D}^* g(\mathbf{x})\mathbf{D}$, where the matrix $g(\mathbf{x})$ has real entries, the boundedness of the solutions $\mathbf{v}_j$ follows from Theorem 13.1 of Chapter III in the book [LaU].
3°. In the case where $g^0 = \underline{g}$, the relation (8.16) was proved in Proposition 6.9 of [BSu4]. □

Remark 8.8. Under Condition 8.6(2°), the norms $\|\mathbf{v}_j\|_{L_\infty}$ are estimated by the constant depending only on $\|g\|_{L_\infty}$, $\|g^{-1}\|_{L_\infty}$, d, and Ω. Under Condition 8.6(3°), the norms $\|\mathbf{v}_j\|_{L_\infty}$ can be estimated by the constant depending on α_0, α_1, $\|g\|_{L_\infty}$, $\|g^{-1}\|_{L_\infty}$, m, n, d, and on parameters of the lattice Γ.

Remark 8.9. Condition 8.4 can be also guaranteed by some smoothness assumptions on the matrix-valued function $g(\mathbf{x})$.

8.3. The case of zero corrector

Under condition (6.25), by the Gelfand transformation, Theorem 6.2 yields the following estimate:

$$\|\widehat{\mathcal{A}}^{1/2}\big((\widehat{\mathcal{A}} + \varepsilon^2 I)^{-1} - (\widehat{\mathcal{A}}^0 + \varepsilon^2 I)^{-1}\Pi\big)\|_{\mathfrak{G}\to\mathfrak{G}} \leq \mathcal{C}_3^\circ, \quad 0 < \varepsilon \leq 1. \tag{8.18}$$

Using (8.6), we can „eliminate“ the operator Π in (8.18). As a result, we obtain the following theorem.

Theorem 8.10. *Under condition* (6.25), *we have*

$$\begin{aligned}&\|\widehat{\mathcal{A}}^{1/2}\big((\widehat{\mathcal{A}} + \varepsilon^2 I)^{-1} - (\widehat{\mathcal{A}}^0 + \varepsilon^2 I)^{-1}\big)\|_{\mathfrak{G}\to\mathfrak{G}} \\ &\quad \leq \mathcal{C}_3^\circ + \|g\|_{L_\infty}^{1/2}\alpha_1^{1/2}\widehat{c}_*^{-1} r_0^{-1} =: \mathcal{C}_6^\circ, \quad 0 < \varepsilon \leq 1.\end{aligned}$$

8.4. Approximation with the three-term corrector

By the Gelfand transformation, the following estimate is deduced from Theorem 6.3:

$$\|\widehat{\mathcal{A}}^{1/2}\big((\widehat{\mathcal{A}} + \varepsilon^2 I)^{-1} - (\widehat{\mathcal{A}}^0 + \varepsilon^2 I)^{-1}\Pi - \mathcal{K}(\varepsilon)\big)\|_{\mathfrak{G}\to\mathfrak{G}} \leq \mathcal{C}_5, \quad 0 < \varepsilon \leq 1, \tag{8.19}$$

where

$$\begin{aligned}\mathcal{K}(\varepsilon) &= \Lambda b(\mathbf{D})(\widehat{\mathcal{A}}^0 + \varepsilon^2 I)^{-1}\Pi + (\widehat{\mathcal{A}}^0 + \varepsilon^2 I)^{-1}\Pi b(\mathbf{D})^*\Lambda^* \\ &\quad - (\widehat{\mathcal{A}}^0 + \varepsilon^2 I)^{-1} b(\mathbf{D})^* L(\mathbf{D}) b(\mathbf{D})(\widehat{\mathcal{A}}^0 + \varepsilon^2 I)^{-1}\Pi.\end{aligned} \tag{8.20}$$

Using (8.6), we can replace Π by I in the term $\widehat{\mathcal{A}}^{1/2}(\widehat{\mathcal{A}}^0 + \varepsilon^2 I)^{-1}\Pi$ in (8.19). This will change only the constant in the estimate. We have:

$$\begin{aligned}&\|\widehat{\mathcal{A}}^{1/2}\big((\widehat{\mathcal{A}} + \varepsilon^2 I)^{-1} - (\widehat{\mathcal{A}}^0 + \varepsilon^2 I)^{-1} - \mathcal{K}(\varepsilon)\big)\|_{\mathfrak{G}\to\mathfrak{G}} \\ &\quad \leq \mathcal{C}_5 + \|g\|_{L_\infty}^{1/2}\alpha_1^{1/2}\widehat{c}_*^{-1} r_0^{-1}, \quad 0 < \varepsilon \leq 1.\end{aligned} \tag{8.21}$$

Similarly, Π can be replaced by I in the last term in $\mathcal{K}(\varepsilon)$. Indeed, we denote (cf. [BSu4, (6.16)])

$$\sigma(\boldsymbol{\xi}) = (1 - \chi_{\widetilde{\Omega}}(\boldsymbol{\xi}))(b(\boldsymbol{\xi})^* g^0 b(\boldsymbol{\xi}) + \varepsilon^2 \mathbf{1}_n)^{-1} b(\boldsymbol{\xi})^* L(\boldsymbol{\xi}) b(\boldsymbol{\xi})(b(\boldsymbol{\xi})^* g^0 b(\boldsymbol{\xi}) + \varepsilon^2 \mathbf{1}_n)^{-1}, \quad \boldsymbol{\xi} \in \mathbb{R}^d.$$

As it was shown in [BSu4, Subsection 6.3],

$$|\sigma(\boldsymbol{\xi})| \le \sqrt{2}|\boldsymbol{\xi}|^{-1} r_0^{-1} \alpha_1^{3/2} \alpha_0^{-5/2} \|g\|_{L_\infty}^{3/2} \|g^{-1}\|_{L_\infty}^{5/2}, \quad |\boldsymbol{\xi}| \ge r_0,$$

and $\sigma(\boldsymbol{\xi}) = 0$ for $|\boldsymbol{\xi}| < r_0$. We have:

$$\begin{aligned} &\|\widehat{\mathcal{A}}^{1/2}(\widehat{\mathcal{A}}^0 + \varepsilon^2 I)^{-1} b(\mathbf{D})^* L(\mathbf{D}) b(\mathbf{D})(\widehat{\mathcal{A}}^0 + \varepsilon^2 I)^{-1}(I - \Pi)\|_{\mathfrak{G}\to\mathfrak{G}} \\ &\quad = \|g^{1/2} b(\mathbf{D})(\widehat{\mathcal{A}}^0 + \varepsilon^2 I)^{-1} b(\mathbf{D})^* L(\mathbf{D}) b(\mathbf{D})(\widehat{\mathcal{A}}^0 + \varepsilon^2 I)^{-1}(I - \Pi)\|_{\mathfrak{G}\to\mathfrak{G}_*} \\ &\quad \le \|g\|_{L_\infty}^{1/2} \sup_{\boldsymbol{\xi}\in\mathbb{R}^d} |b(\boldsymbol{\xi})\sigma(\boldsymbol{\xi})| \le \sqrt{2} r_0^{-1} \alpha_1^2 \alpha_0^{-5/2} \|g\|_{L_\infty}^2 \|g^{-1}\|_{L_\infty}^{5/2}. \end{aligned} \tag{8.22}$$

The following result is a consequence of (8.20)–(8.22).

Theorem 8.11. *We have*

$$\|\widehat{\mathcal{A}}^{1/2}\big((\widehat{\mathcal{A}} + \varepsilon^2 I)^{-1} - (\widehat{\mathcal{A}}^0 + \varepsilon^2 I)^{-1} - \widetilde{\mathcal{K}}(\varepsilon)\big)\|_{\mathfrak{G}\to\mathfrak{G}} \le \mathcal{C}_8, \quad 0 < \varepsilon \le 1,$$

where

$$\begin{aligned} \widetilde{\mathcal{K}}(\varepsilon) &= \Lambda b(\mathbf{D})(\widehat{\mathcal{A}}^0 + \varepsilon^2 I)^{-1}\Pi + (\widehat{\mathcal{A}}^0 + \varepsilon^2 I)^{-1}\Pi b(\mathbf{D})^*\Lambda^* \\ &\quad - (\widehat{\mathcal{A}}^0 + \varepsilon^2 I)^{-1} b(\mathbf{D})^* L(\mathbf{D}) b(\mathbf{D})(\widehat{\mathcal{A}}^0 + \varepsilon^2 I)^{-1}, \\ \mathcal{C}_8 &= \mathcal{C}_5 + \|g\|_{L_\infty}^{1/2} \alpha_1^{1/2} \widehat{c}_*^{-1} r_0^{-1} + \sqrt{2} r_0^{-1} \alpha_1^2 \alpha_0^{-5/2} \|g\|_{L_\infty}^2 \|g^{-1}\|_{L_\infty}^{5/2}. \end{aligned}$$

It is possible to „eliminate“ the projection Π in the first two terms of the operator $\widetilde{\mathcal{K}}(\varepsilon)$ under Condition 8.4. As for the first term, estimate (8.17) required for this has been already obtained. In order to do this in the second term, we consider the operator

$$\widehat{\mathcal{A}}^{1/2}(\widehat{\mathcal{A}}^0 + \varepsilon^2 I)^{-1}(I - \Pi) b(\mathbf{D})^*\Lambda^*.$$

Under Condition 8.4, the operator of multiplication by $\Lambda^*(\mathbf{x})$ continuously maps $\mathfrak{G}$ to $\mathfrak{G}_*$, and

$$\|\Lambda^*\|_{\mathfrak{G}\to\mathfrak{G}_*} = \|\Lambda\|_{L_\infty}. \tag{8.23}$$

The operator $(\widehat{\mathcal{A}}^0 + \varepsilon^2 I)^{-1}(I - \Pi) b(\mathbf{D})^*$ is the pseudodifferential operator of order (-1) with the symbol

$$(1 - \chi_{\widetilde{\Omega}}(\boldsymbol{\xi}))(b(\boldsymbol{\xi})^* g^0 b(\boldsymbol{\xi}) + \varepsilon^2 \mathbf{1}_n)^{-1} b(\boldsymbol{\xi})^*,$$

therefore, it continuously maps $\mathfrak{G}_*$ to $\mathfrak{G}^1 = H^1(\mathbb{R}^d; \mathbb{C}^n)$, and (cf. (8.7))

$$\begin{aligned} &\|(\widehat{\mathcal{A}}^0 + \varepsilon^2 I)^{-1}(I - \Pi) b(\mathbf{D})^*\|_{\mathfrak{G}_*\to\mathfrak{G}^1} \\ &\quad \le \sup_{|\boldsymbol{\xi}|\ge r_0} |(b(\boldsymbol{\xi})^* g^0 b(\boldsymbol{\xi}) + \varepsilon^2 \mathbf{1}_n)^{-1} b(\boldsymbol{\xi})^*|(1 + |\boldsymbol{\xi}|^2)^{1/2} \le \alpha_1^{1/2} \widehat{c}_*^{-1}(1 + r_0^{-2})^{1/2}. \end{aligned} \tag{8.24}$$

Finally, the operator $\widehat{\mathcal{A}}^{1/2}$ is continuous from $\mathfrak{G}^1$ to $\mathfrak{G}$, and

$$\|\widehat{\mathcal{A}}^{1/2}\mathbf{u}\|_{\mathfrak{G}} = \|g^{1/2} b(\mathbf{D})\mathbf{u}\|_{\mathfrak{G}_*} \le \|g\|_{L_\infty}^{1/2} \alpha_1^{1/2} \|\mathbf{u}\|_{\mathfrak{G}^1},$$

whence

$$\|\widehat{\mathcal{A}}^{1/2}\|_{\mathfrak{G}^1\to\mathfrak{G}} \le \alpha_1^{1/2} \|g\|_{L_\infty}^{1/2}. \tag{8.25}$$

From (8.23)–(8.25) it follows that

$$\|\widehat{\mathcal{A}}^{1/2}(\widehat{\mathcal{A}}^0 + \varepsilon^2 I)^{-1}(I - \Pi) b(\mathbf{D})^*\Lambda^*\|_{\mathfrak{G}\to\mathfrak{G}} \le \alpha_1 \widehat{c}_*^{-1}(1 + r_0^{-2})^{1/2} \|g\|_{L_\infty}^{1/2} \|\Lambda\|_{L_\infty}. \tag{8.26}$$

Combining Theorem 8.11 with estimates (8.17) and (8.26), we arrive at the following result.

Theorem 8.12. *Under Condition* 8.4, *we have:*

$$\|\widehat{\mathcal{A}}^{1/2}\big((\widehat{\mathcal{A}}+\varepsilon^2 I)^{-1}-(\widehat{\mathcal{A}}^0+\varepsilon^2 I)^{-1}-\mathcal{K}^0(\varepsilon)\big)\|_{\mathfrak{G}\to\mathfrak{G}}\le \mathcal{C}_9,\quad 0<\varepsilon\le 1,$$

where

$$\mathcal{K}^0(\varepsilon)=\Lambda b(\mathbf{D})(\widehat{\mathcal{A}}^0+\varepsilon^2 I)^{-1}+(\widehat{\mathcal{A}}^0+\varepsilon^2 I)^{-1}b(\mathbf{D})^*\Lambda^* \\ -(\widehat{\mathcal{A}}^0+\varepsilon^2 I)^{-1}b(\mathbf{D})^*L(\mathbf{D})b(\mathbf{D})(\widehat{\mathcal{A}}^0+\varepsilon^2 I)^{-1},$$

$$\mathcal{C}_9=\mathcal{C}_8+\alpha_1^{1/2}\widehat{c}_*^{-1}(1+r_0^{-2})^{1/2}\bigg(\sum_{j=1}^m (C_j^\nabla)^2\bigg)^{1/2}+\alpha_1\widehat{c}_*^{-1}(1+r_0^{-2})^{1/2}\|g\|_{L_\infty}^{1/2}\|\Lambda\|_{L_\infty}.$$

9 Approximation for the operator $\widehat{\mathcal{A}}^{1/2}(\widehat{\mathcal{A}}+\varepsilon^2 Q)^{-1}$

9.1

Using Theorem 7.1 and applying the Gelfand transformation, we arrive at the following result.

Theorem 9.1. *Let* $\widehat{\mathcal{A}}=b(\mathbf{D})^*g(\mathbf{x})b(\mathbf{D})$, *and let* $\widehat{\mathcal{A}}^0=b(\mathbf{D})^*g^0b(\mathbf{D})$ *be the effective operator. Let* Q *be the operator of multiplication by the* Γ*-periodic positive* $(n\times n)$*-matrix-valued function* $Q(\mathbf{x})$ *satisfying condition* (7.1). *Let* $\overline{Q}$ *be the mean value of the matrix* $Q(\mathbf{x})$ *over the cell* Ω. *Let* $\Lambda:\mathfrak{G}_*\to\mathfrak{G}$ *be the operator of multiplication by the matrix-valued function* $\Lambda(\mathbf{x})$ *introduced in Subsection* 5.1. *Let* Π *be the operator* (8.1). *Then*

$$\|\widehat{\mathcal{A}}^{1/2}\big((\widehat{\mathcal{A}}+\varepsilon^2 Q)^{-1}-(I+\Lambda b(\mathbf{D}))(\widehat{\mathcal{A}}^0+\varepsilon^2\overline{Q})^{-1}\Pi\big)\|_{\mathfrak{G}\to\mathfrak{G}}\le\check{\mathcal{C}}_3,\quad 0<\varepsilon\le 1.$$

The constant $\check{\mathcal{C}}_3$ *is defined by* (7.17) *and depends only on* m, α_0, α_1, $\|g\|_{L_\infty}$, $\|g^{-1}\|_{L_\infty}$, $\|Q\|_{L_\infty}$, $\|Q^{-1}\|_{L_\infty}$, r_0, *and* r_1.

9.2. Elimination of the operator Π

The following estimate is obtained in the same way as estimate (8.6):

$$\|\widehat{\mathcal{A}}^{1/2}(\widehat{\mathcal{A}}^0+\varepsilon^2\overline{Q})^{-1}(I-\Pi)\|_{\mathfrak{G}\to\mathfrak{G}}\le\|g\|_{L_\infty}^{1/2}\alpha_1^{1/2}\widehat{c}_*^{-1}r_0^{-1}. \tag{9.1}$$

Combining Theorem 9.1 with (9.1), we arrive at the following statement.

Theorem 9.2. *Under conditions of Theorem* 9.1, *we have:*

$$\|\widehat{\mathcal{A}}^{1/2}\big((\widehat{\mathcal{A}}+\varepsilon^2 Q)^{-1}-(\widehat{\mathcal{A}}^0+\varepsilon^2\overline{Q})^{-1}-\Lambda b(\mathbf{D})(\widehat{\mathcal{A}}^0+\varepsilon^2\overline{Q})^{-1}\Pi\big)\|_{\mathfrak{G}\to\mathfrak{G}} \\ \le\check{\mathcal{C}}_3+\|g\|_{L_\infty}^{1/2}\alpha_1^{1/2}\widehat{c}_*^{-1}r_0^{-1}=\check{\mathcal{C}}_6,\quad 0<\varepsilon\le 1.$$

Next, similarly to (8.7), we obtain:

$$\|b(\mathbf{D})(\widehat{\mathcal{A}}^0+\varepsilon^2\overline{Q})^{-1}(I-\Pi)\|_{\mathfrak{G}\to\mathfrak{G}_*^1}\le\alpha_1^{1/2}\widehat{c}_*^{-1}(1+r_0^{-2})^{1/2}. \tag{9.2}$$

Under Condition 8.4, relations (9.2), (8.8), and Lemma 8.3 imply the following estimate (cf. (8.17)):

$$\|\widehat{\mathcal{A}}^{1/2}\Lambda b(\mathbf{D})(\widehat{\mathcal{A}}^0+\varepsilon^2\overline{Q})^{-1}(I-\Pi)\|_{\mathfrak{G}\to\mathfrak{G}} \\ \le\alpha_1^{1/2}\widehat{c}_*^{-1}(1+r_0^{-2})^{1/2}\bigg(\sum_{j=1}^m (C_j^\nabla)^2\bigg)^{1/2}. \tag{9.3}$$

Combining this with Theorem 9.2, we obtain the following result.

Theorem 9.3. *Suppose that conditions of Theorem* 9.1 *are satisfied. Besides, suppose that Condition* 8.4 *is valid. Then*

$$\|\widehat{\mathcal{A}}^{1/2}\big((\widehat{\mathcal{A}}+\varepsilon^2 Q)^{-1}-(I+\Lambda b(\mathbf{D}))(\widehat{\mathcal{A}}^0+\varepsilon^2\overline{Q})^{-1}\big)\|_{\mathfrak{G}\to\mathfrak{G}}\le \check{\mathcal{C}}_7,\quad 0<\varepsilon\le 1,$$

$$\check{\mathcal{C}}_7=\check{\mathcal{C}}_6+\alpha_1^{1/2}\widehat{c}_*^{-1}(1+r_0^{-2})^{1/2}\bigg(\sum_{j=1}^m (C_j^\nabla)^2\bigg)^{1/2}.$$

9.3. The case of zero corrector

Under condition (6.25), using Theorem 7.2 and applying the Gelfand transformation, we obtain the following estimate:

$$\|\widehat{\mathcal{A}}^{1/2}\big((\widehat{\mathcal{A}}+\varepsilon^2 Q)^{-1}-(\widehat{\mathcal{A}}^0+\varepsilon^2\overline{Q})^{-1}\Pi\big)\|_{\mathfrak{G}\to\mathfrak{G}}\le \check{\mathcal{C}}_3^\circ,\quad 0<\varepsilon\le 1.$$

Applying (9.1), we arrive at the following result.

Theorem 9.4. *Under condition* (6.25), *we have*

$$\|\widehat{\mathcal{A}}^{1/2}\big((\widehat{\mathcal{A}}+\varepsilon^2 Q)^{-1}-(\widehat{\mathcal{A}}^0+\varepsilon^2\overline{Q})^{-1}\big)\|_{\mathfrak{G}\to\mathfrak{G}}$$
$$\le \check{\mathcal{C}}_3^\circ+\|g\|_{L_\infty}^{1/2}\alpha_1^{1/2}\widehat{c}_*^{-1}r_0^{-1}=:\check{\mathcal{C}}_6^\circ,\quad 0<\varepsilon\le 1.$$

9.4. Approximation with the three-term corrector

Using Theorem 7.3, with the help of the Gelfand transformation, we obtain the following estimate:

$$\|\widehat{\mathcal{A}}^{1/2}\big((\widehat{\mathcal{A}}+\varepsilon^2 Q)^{-1}-(\widehat{\mathcal{A}}^0+\varepsilon^2\overline{Q})^{-1}\Pi-\mathcal{K}_Q(\varepsilon)\big)\|_{\mathfrak{G}\to\mathfrak{G}}\le \check{\mathcal{C}}_5,\quad 0<\varepsilon\le 1, \tag{9.4}$$

where

$$\mathcal{K}_Q(\varepsilon)=\Lambda b(\mathbf{D})(\widehat{\mathcal{A}}^0+\varepsilon^2\overline{Q})^{-1}\Pi+(\widehat{\mathcal{A}}^0+\varepsilon^2\overline{Q})^{-1}\Pi b(\mathbf{D})^*\Lambda^*$$
$$-(\widehat{\mathcal{A}}^0+\varepsilon^2\overline{Q})^{-1}\big(b(\mathbf{D})^*L(\mathbf{D})b(\mathbf{D})+\varepsilon^2 b(\mathbf{D})^*(\overline{Q\Lambda})^*+\varepsilon^2(\overline{Q\Lambda})b(\mathbf{D})\big)$$
$$\times(\widehat{\mathcal{A}}^0+\varepsilon^2\overline{Q})^{-1}\Pi. \tag{9.5}$$

Using (9.1), we can replace Π by I in the term $\widehat{\mathcal{A}}^{1/2}(\widehat{\mathcal{A}}^0+\varepsilon^2\overline{Q})^{-1}\Pi$ in (9.4):

$$\|\widehat{\mathcal{A}}^{1/2}\big((\widehat{\mathcal{A}}+\varepsilon^2 Q)^{-1}-(\widehat{\mathcal{A}}^0+\varepsilon^2\overline{Q})^{-1}-\mathcal{K}_Q(\varepsilon)\big)\|_{\mathfrak{G}\to\mathfrak{G}}$$
$$\le \check{\mathcal{C}}_5+\|g\|_{L_\infty}^{1/2}\alpha_1^{1/2}\widehat{c}_*^{-1}r_0^{-1},\quad 0<\varepsilon\le 1. \tag{9.6}$$

Now we show that it is possible to replace Π by I also in the last term of the corrector (9.5). By analogy with (8.22), we have

$$\|\widehat{\mathcal{A}}^{1/2}(\widehat{\mathcal{A}}^0+\varepsilon^2\overline{Q})^{-1}b(\mathbf{D})^*L(\mathbf{D})b(\mathbf{D})(\widehat{\mathcal{A}}^0+\varepsilon^2\overline{Q})^{-1}(I-\Pi)\|_{\mathfrak{G}\to\mathfrak{G}}$$
$$\le \sqrt{2}r_0^{-1}\alpha_1^2\alpha_0^{-5/2}\|g\|_{L_\infty}^2\|g^{-1}\|_{L_\infty}^{5/2}. \tag{9.7}$$

Relations (4.2) and (8.4) imply that

$$\|b(\mathbf{D})(\widehat{\mathcal{A}}^0+\varepsilon^2\overline{Q})^{-1}(I-\Pi)\|_{\mathfrak{G}\to\mathfrak{G}_*}$$
$$\le \sup_{|\boldsymbol{\xi}|\ge r_0}|b(\boldsymbol{\xi})(b(\boldsymbol{\xi})^*g^0 b(\boldsymbol{\xi})+\varepsilon^2\overline{Q})^{-1}|\le \alpha_1^{1/2}\widehat{c}_*^{-1}r_0^{-1}.$$

Combining this with (9.1) and (7.10), we obtain:

$$\|\widehat{\mathcal{A}}^{1/2}(\widehat{\mathcal{A}}^0+\varepsilon^2\overline{Q})^{-1}\varepsilon^2(\overline{Q\Lambda})b(\mathbf{D})(\widehat{\mathcal{A}}^0+\varepsilon^2\overline{Q})^{-1}(I-\Pi)\|_{\mathfrak{G}\to\mathfrak{G}} \\ \le 2^{-1}\alpha_1\alpha_0^{-1/2}m^{1/2}\widehat{c}_*^{-2}r_0^{-3}\|g\|_{L_\infty}\|g^{-1}\|_{L_\infty}^{1/2}\|Q\|_{L_\infty}, \quad 0<\varepsilon\le 1. \tag{9.8}$$

The term

$$\widehat{\mathcal{A}}^{1/2}(\widehat{\mathcal{A}}^0+\varepsilon^2\overline{Q})^{-1}\varepsilon^2 b(\mathbf{D})^*(\overline{Q\Lambda})^*(\widehat{\mathcal{A}}^0+\varepsilon^2\overline{Q})^{-1}(I-\Pi)$$

admits a similar estimate (with the same constant as in (9.8)). Combining this with (9.6)–(9.8), we obtain the following result.

Theorem 9.5. *We have*

$$\|\widehat{\mathcal{A}}^{1/2}\big((\widehat{\mathcal{A}}+\varepsilon^2 Q)^{-1}-(\widehat{\mathcal{A}}^0+\varepsilon^2\overline{Q})^{-1}-\widetilde{\mathcal{K}}_Q(\varepsilon)\big)\|_{\mathfrak{G}\to\mathfrak{G}}\le\check{\mathcal{C}}_8, \quad 0<\varepsilon\le 1,$$

where

$$\widetilde{\mathcal{K}}_Q(\varepsilon)=\Lambda b(\mathbf{D})(\widehat{\mathcal{A}}^0+\varepsilon^2\overline{Q})^{-1}\Pi+(\widehat{\mathcal{A}}^0+\varepsilon^2\overline{Q})^{-1}\Pi b(\mathbf{D})^*\Lambda^* \\ -(\widehat{\mathcal{A}}^0+\varepsilon^2\overline{Q})^{-1}(b(\mathbf{D})^*L(\mathbf{D})b(\mathbf{D})+\varepsilon^2 b(\mathbf{D})^*(\overline{Q\Lambda})^*+\varepsilon^2(\overline{Q\Lambda})b(\mathbf{D})) \\ \times(\widehat{\mathcal{A}}^0+\varepsilon^2\overline{Q})^{-1},$$

$$\check{\mathcal{C}}_8=\check{\mathcal{C}}_5+\|g\|_{L_\infty}^{1/2}\alpha_1^{1/2}\widehat{c}_*^{-1}r_0^{-1}+\sqrt{2}r_0^{-1}\alpha_1^2\alpha_0^{-5/2}\|g\|_{L_\infty}^2\|g^{-1}\|_{L_\infty}^{5/2} \\ +\alpha_1\alpha_0^{-1/2}m^{1/2}\widehat{c}_*^{-2}r_0^{-3}\|g\|_{L_\infty}\|g^{-1}\|_{L_\infty}^{1/2}\|Q\|_{L_\infty}.$$

It is possible to „eliminate“ Π in the first two terms in (9.5) under Condition 8.4. For this, we apply estimate (9.3) and the estimate

$$\|\widehat{\mathcal{A}}^{1/2}(\widehat{\mathcal{A}}^0+\varepsilon^2\overline{Q})^{-1}(I-\Pi)b(\mathbf{D})^*\Lambda^*\|_{\mathfrak{G}\to\mathfrak{G}}\le\alpha_1\widehat{c}_*^{-1}(1+r_0^{-2})^{1/2}\|g\|_{L_\infty}^{1/2}\|\Lambda\|_{L_\infty},$$

which can be proved in the same way as (8.26). As a result, we arrive at the following theorem.

Theorem 9.6. *Under Condition* 8.4*, we have*

$$\|\widehat{\mathcal{A}}^{1/2}\big((\widehat{\mathcal{A}}+\varepsilon^2 Q)^{-1}-(\widehat{\mathcal{A}}^0+\varepsilon^2\overline{Q})^{-1}-\mathcal{K}_Q^0(\varepsilon)\big)\|_{\mathfrak{G}\to\mathfrak{G}}\le\check{\mathcal{C}}_9, \quad 0<\varepsilon\le 1,$$

where

$$\mathcal{K}_Q^0(\varepsilon)=\Lambda b(\mathbf{D})(\widehat{\mathcal{A}}^0+\varepsilon^2\overline{Q})^{-1}+(\widehat{\mathcal{A}}^0+\varepsilon^2\overline{Q})^{-1}b(\mathbf{D})^*\Lambda^* \\ -(\widehat{\mathcal{A}}^0+\varepsilon^2\overline{Q})^{-1}(b(\mathbf{D})^*L(\mathbf{D})b(\mathbf{D})+\varepsilon^2 b(\mathbf{D})^*(\overline{Q\Lambda})^*+\varepsilon^2(\overline{Q\Lambda})b(\mathbf{D})) \\ \times(\widehat{\mathcal{A}}^0+\varepsilon^2\overline{Q})^{-1},$$

$$\check{\mathcal{C}}_9=\check{\mathcal{C}}_8+\alpha_1^{1/2}\widehat{c}_*^{-1}(1+r_0^{-2})^{1/2}\left(\sum_{j=1}^m(C_j^\nabla)^2\right)^{1/2}+\alpha_1\widehat{c}_*^{-1}(1+r_0^{-2})^{1/2}\|g\|_{L_\infty}^{1/2}\|\Lambda\|_{L_\infty}.$$

Chapter 3. Homogenization with corrector for periodic differential operators

We proceed to the homogenization problems in the small period limit for periodic DO's acting in $\mathfrak{G} = L_2(\mathbb{R}^d; \mathbb{C}^n)$. If $\phi(\mathbf{x})$ is a measurable Γ-periodic function in $\mathbb{R}^d$, we agree to denote $\phi^\varepsilon(\mathbf{x}) = \phi(\varepsilon^{-1}\mathbf{x})$, $\varepsilon > 0$. We consider the operators

$$\widehat{\mathcal{A}}_\varepsilon = \widehat{\mathcal{A}}(g^\varepsilon) = b(\mathbf{D})^* g^\varepsilon b(\mathbf{D}),$$
$$\mathcal{A}_\varepsilon = \mathcal{A}(g^\varepsilon, f^\varepsilon) = (f^\varepsilon)^* b(\mathbf{D})^* g^\varepsilon b(\mathbf{D}) f^\varepsilon$$

with rapidly oscillating (as $\varepsilon \to 0$) coefficients. Recall the notation $\mathfrak{G}^1 = H^1(\mathbb{R}^d; \mathbb{C}^n)$. In §10, we obtain approximation for the resolvent $(\widehat{\mathcal{A}}_\varepsilon + I)^{-1}$ in the operator norm from $\mathfrak{G}$ to $\mathfrak{G}^1$. In §11, by interpolation, we obtain approximation with the three-term corrector for $(\widehat{\mathcal{A}}_\varepsilon + I)^{-1}$ in the operator norm from $\mathfrak{G}$ to $\mathfrak{G}^s = H^s(\mathbb{R}^d; \mathbb{C}^n)$, $0 < s \le 1$. In §12, we study approximation for the fluxes in $\mathfrak{G}_* = L_2(\mathbb{R}^d; \mathbb{C}^m)$. Next, §13–15 are devoted to the study of the generalized resolvent $(\widehat{\mathcal{A}}_\varepsilon + Q^\varepsilon)^{-1}$: in §13, approximation in the operator norm from $\mathfrak{G}$ to $\mathfrak{G}^1$ is obtained, §14 contains the interpolational results, and §15 is devoted to approximation of the corresponding fluxes in $\mathfrak{G}_*$. Finally, the resolvent $(\mathcal{A}_\varepsilon + I)^{-1}$ is studied in §16.

10 Approximation for the resolvent $(\widehat{\mathcal{A}}_\varepsilon + I)^{-1}$ in the $(L_2 \to H^1)$-norm

10.1. Approximation for the operator $\widehat{\mathcal{A}}_\varepsilon^{1/2}(\widehat{\mathcal{A}}_\varepsilon + I)^{-1}$

Let T_ε be the unitary scaling transformation in $\mathfrak{G}$:

$$(T_\varepsilon \mathbf{u})(\mathbf{y}) = \varepsilon^{d/2} \mathbf{u}(\varepsilon \mathbf{y}), \quad \mathbf{y} \in \mathbb{R}^d. \tag{10.1}$$

Then we have:

$$\begin{gathered} \widehat{\mathcal{A}}_\varepsilon = \varepsilon^{-2} T_\varepsilon^* \widehat{\mathcal{A}} T_\varepsilon, \quad \widehat{\mathcal{A}}_\varepsilon^{1/2} = \varepsilon^{-1} T_\varepsilon^* \widehat{\mathcal{A}}^{1/2} T_\varepsilon, \\ \widehat{\mathcal{A}}_\varepsilon^{1/2}(\widehat{\mathcal{A}}_\varepsilon + I)^{-1} = \varepsilon T_\varepsilon^* \widehat{\mathcal{A}}^{1/2} (\widehat{\mathcal{A}} + \varepsilon^2 I)^{-1} T_\varepsilon, \\ \widehat{\mathcal{A}}_\varepsilon^{1/2}(\widehat{\mathcal{A}}^0 + I)^{-1} = \varepsilon T_\varepsilon^* \widehat{\mathcal{A}}^{1/2} (\widehat{\mathcal{A}}^0 + \varepsilon^2 I)^{-1} T_\varepsilon, \\ b(\mathbf{D}) = \varepsilon^{-1} T_\varepsilon^* b(\mathbf{D}) T_\varepsilon, \quad [\Lambda^\varepsilon] = T_\varepsilon^* [\Lambda] T_\varepsilon. \end{gathered} \tag{10.2}$$

Putting

$$\Pi_\varepsilon := T_\varepsilon^* \Pi T_\varepsilon, \tag{10.3}$$

and taking (8.1) into account, we see that Π_ε is the pseudodifferential operator in $\mathfrak{G}$ with the symbol $\chi_{\widetilde{\Omega}/\varepsilon}(\boldsymbol{\xi})$, i. e.,

$$\Pi_\varepsilon = \mathcal{F}^* [\chi_{\widetilde{\Omega}/\varepsilon}(\cdot)] \mathcal{F}. \tag{10.4}$$

From (10.2) and (10.3) it follows that

$$\begin{aligned} &\widehat{\mathcal{A}}_\varepsilon^{1/2} \big((\widehat{\mathcal{A}}_\varepsilon + I)^{-1} - (\widehat{\mathcal{A}}^0 + I)^{-1} - \varepsilon \Lambda^\varepsilon b(\mathbf{D}) (\widehat{\mathcal{A}}^0 + I)^{-1} \Pi_\varepsilon \big) \\ &\quad = \varepsilon T_\varepsilon^* \widehat{\mathcal{A}}^{1/2} \big((\widehat{\mathcal{A}} + \varepsilon^2 I)^{-1} - (\widehat{\mathcal{A}}^0 + \varepsilon^2 I)^{-1} - \Lambda b(\mathbf{D}) (\widehat{\mathcal{A}}^0 + \varepsilon^2 I)^{-1} \Pi \big) T_\varepsilon. \end{aligned} \tag{10.5}$$

Since T_ε is the *unitary* operator in $\mathfrak{G}$, Theorem 8.2 and identity (10.5) imply the following result.

Theorem 10.1. *Let* $\widehat{\mathcal{A}}_\varepsilon = b(\mathbf{D})^* g^\varepsilon(\mathbf{x}) b(\mathbf{D})$, *and let* $\widehat{\mathcal{A}}^0 = b(\mathbf{D})^* g^0 b(\mathbf{D})$ *be the effective operator. Let* $\Lambda^\varepsilon : \mathfrak{G}_* \to \mathfrak{G}$ *be the operator of multiplication by the matrix-valued function* $\Lambda^\varepsilon(\mathbf{x}) = \Lambda(\varepsilon^{-1}\mathbf{x})$, *where* $\Lambda(\mathbf{x})$ *is the matrix defined in Subsection* 5.1. *Let* Π_ε *be the pseudodifferential operator* (10.4). *Then*

$$\|\widehat{\mathcal{A}}_\varepsilon^{1/2}\big((\widehat{\mathcal{A}}_\varepsilon + I)^{-1} - (\widehat{\mathcal{A}}^0 + I)^{-1} - \varepsilon \Lambda^\varepsilon b(\mathbf{D})(\widehat{\mathcal{A}}^0 + I)^{-1}\Pi_\varepsilon\big)\|_{\mathfrak{G}\to\mathfrak{G}} \le \mathcal{C}_6 \varepsilon, \quad 0 < \varepsilon \le 1. \tag{10.6}$$

The operator

$$K_1(\varepsilon) := \Lambda^\varepsilon b(\mathbf{D})(\widehat{\mathcal{A}}^0 + I)^{-1}\Pi_\varepsilon \tag{10.7}$$

plays the role of the corrector.

Similarly, the following statement is deduced from Theorem 8.5.

Theorem 10.2. *Suppose that conditions of Theorem* 10.1 *are satisfied. Suppose also that Condition* 8.4 *is valid. Then*

$$\|\widehat{\mathcal{A}}_\varepsilon^{1/2}\big((\widehat{\mathcal{A}}_\varepsilon + I)^{-1} - (\widehat{\mathcal{A}}^0 + I)^{-1} - \varepsilon \Lambda^\varepsilon b(\mathbf{D})(\widehat{\mathcal{A}}^0 + I)^{-1}\big)\|_{\mathfrak{G}\to\mathfrak{G}} \le \mathcal{C}_7 \varepsilon, \quad 0 < \varepsilon \le 1. \tag{10.8}$$

The corrector

$$K_1^0(\varepsilon) := \Lambda^\varepsilon b(\mathbf{D})(\widehat{\mathcal{A}}^0 + I)^{-1}$$

in (10.8) does not contain the operator Π_ε.

Theorem 8.10 implies the following result, which distinguishes the case where the corrector is equal to zero.

Theorem 10.3. *Suppose that conditions of Theorem* 10.1 *are satisfied. Suppose also that condition* (6.25) *is valid. Then*

$$\|\widehat{\mathcal{A}}_\varepsilon^{1/2}\big((\widehat{\mathcal{A}}_\varepsilon + I)^{-1} - (\widehat{\mathcal{A}}^0 + I)^{-1}\big)\|_{\mathfrak{G}\to\mathfrak{G}} \le \mathcal{C}_6^\circ \varepsilon, \quad 0 < \varepsilon \le 1.$$

Theorem 8.11 yields approximation for the operator $\widehat{\mathcal{A}}_\varepsilon^{1/2}(\widehat{\mathcal{A}}_\varepsilon + I)^{-1}$ with the three-term corrector.

Theorem 10.4. *Suppose that conditions of Theorem* 10.1 *are satisfied. We put*

$$\widetilde{K}(\varepsilon) = \Lambda^\varepsilon b(\mathbf{D})(\widehat{\mathcal{A}}^0 + I)^{-1}\Pi_\varepsilon + (\widehat{\mathcal{A}}^0 + I)^{-1}\Pi_\varepsilon b(\mathbf{D})^*(\Lambda^\varepsilon)^* - (\widehat{\mathcal{A}}^0 + I)^{-1} b(\mathbf{D})^* L(\mathbf{D}) b(\mathbf{D})(\widehat{\mathcal{A}}^0 + I)^{-1}. \tag{10.9}$$

Then

$$\|\widehat{\mathcal{A}}_\varepsilon^{1/2}\big((\widehat{\mathcal{A}}_\varepsilon + I)^{-1} - (\widehat{\mathcal{A}}^0 + I)^{-1} - \varepsilon \widetilde{K}(\varepsilon)\big)\|_{\mathfrak{G}\to\mathfrak{G}} \le \mathcal{C}_8 \varepsilon, \quad 0 < \varepsilon \le 1. \tag{10.10}$$

Finally, Theorem 8.12 leads to the following result.

Theorem 10.5. *Suppose that conditions of Theorem* 10.1 *are satisfied. Suppose also that Condition* 8.4 *is valid. We put*

$$K^0(\varepsilon) = \Lambda^\varepsilon b(\mathbf{D})(\widehat{\mathcal{A}}^0 + I)^{-1} + (\widehat{\mathcal{A}}^0 + I)^{-1} b(\mathbf{D})^*(\Lambda^\varepsilon)^* - (\widehat{\mathcal{A}}^0 + I)^{-1} b(\mathbf{D})^* L(\mathbf{D}) b(\mathbf{D})(\widehat{\mathcal{A}}^0 + I)^{-1}. \tag{10.11}$$

Then

$$\|\widehat{\mathcal{A}}_\varepsilon^{1/2}\big((\widehat{\mathcal{A}}_\varepsilon + I)^{-1} - (\widehat{\mathcal{A}}^0 + I)^{-1} - \varepsilon K^0(\varepsilon)\big)\|_{\mathfrak{G}\to\mathfrak{G}} \le \mathcal{C}_9 \varepsilon, \quad 0 < \varepsilon \le 1.$$

10.2. Approximation for the resolvent $(\widehat{\mathcal{A}}_\varepsilon + I)^{-1}$ in the $(L_2 \to H^1)$-norm

Consider the equation

$$\widehat{\mathcal{A}}_\varepsilon \mathbf{u}_\varepsilon + \mathbf{u}_\varepsilon = \mathbf{F}, \quad \mathbf{F} \in \mathfrak{G}. \tag{10.12}$$

We apply Theorem 10.1, in order to approximate the solution $\mathbf{u}_\varepsilon$ in $\mathfrak{G}^1 = H^1(\mathbb{R}^d; \mathbb{C}^n)$. Let $\mathbf{u}_0$ be the solution of the „homogenized" equation

$$\widehat{\mathcal{A}}^0 \mathbf{u}_0 + \mathbf{u}_0 = \mathbf{F}. \tag{10.13}$$

Note that the operator (10.7) can be written as

$$K_1(\varepsilon) = \Lambda^\varepsilon \Pi_\varepsilon^{(m)} b(\mathbf{D})(\widehat{\mathcal{A}}^0 + I)^{-1},$$

where $\Pi_\varepsilon^{(m)}$ is the pseudodifferential operator with the symbol $\chi_{\widetilde{\Omega}/\varepsilon}(\boldsymbol{\xi})$ acting in $\mathfrak{G}_* = L_2(\mathbb{R}^d; \mathbb{C}^m)$. We put

$$\mathbf{u}_\varepsilon^{(1)} = K_1(\varepsilon)\mathbf{F} = \Lambda^\varepsilon \Pi_\varepsilon^{(m)} b(\mathbf{D})\mathbf{u}_0. \tag{10.14}$$

The estimate (10.6) means that

$$\|\widehat{\mathcal{A}}_\varepsilon^{1/2}(\mathbf{u}_\varepsilon - \mathbf{u}_0 - \varepsilon \mathbf{u}_\varepsilon^{(1)})\|_{\mathfrak{G}} \leq \mathcal{C}_6 \varepsilon \|\mathbf{F}\|_{\mathfrak{G}}, \quad 0 < \varepsilon \leq 1.$$

Since (see (4.2) and (6.7))

$$\begin{aligned}\|\widehat{\mathcal{A}}_\varepsilon^{1/2}(\mathbf{u}_\varepsilon - \mathbf{u}_0 - \varepsilon \mathbf{u}_\varepsilon^{(1)})\|^2_{\mathfrak{G}} &= \|g^{1/2} b(\mathbf{D})(\mathbf{u}_\varepsilon - \mathbf{u}_0 - \varepsilon \mathbf{u}_\varepsilon^{(1)})\|^2_{\mathfrak{G}_*} \\ &\geq \widehat{c}_* \int_{\mathbb{R}^d} |\mathbf{D}(\mathbf{u}_\varepsilon - \mathbf{u}_0 - \varepsilon \mathbf{u}_\varepsilon^{(1)})|^2 \, d\mathbf{x},\end{aligned}$$

then

$$\int_{\mathbb{R}^d} |\mathbf{D}(\mathbf{u}_\varepsilon - \mathbf{u}_0 - \varepsilon \mathbf{u}_\varepsilon^{(1)})|^2 \, d\mathbf{x} \leq \widehat{c}_*^{-1} \mathcal{C}_6^2 \varepsilon^2 \|\mathbf{F}\|^2_{\mathfrak{G}}, \quad 0 < \varepsilon \leq 1. \tag{10.15}$$

Besides, as it was shown in [BSu2, Theorem 2.1 of Ch. 4], we have

$$\|\mathbf{u}_\varepsilon - \mathbf{u}_0\|_{\mathfrak{G}} \leq \widehat{\mathcal{C}}_\times \varepsilon \|\mathbf{F}\|_{\mathfrak{G}}, \quad 0 < \varepsilon \leq 1. \tag{10.16}$$

The constant $\widehat{\mathcal{C}}_\times$ is defined by

$$\begin{aligned}\widehat{\mathcal{C}} &= \widehat{c}_*^{-1/2}\big(\beta_1^*(\widehat{t}^0)^{-1} + \beta_2^* \widehat{c}_*^{-1} \widehat{\delta} (\widehat{t}^0)^{-3}\big), \\ \widehat{\mathcal{C}}_\times &= \max\big\{\widehat{\mathcal{C}} + 2(3\widehat{\delta})^{-1}, 2\widehat{c}_*^{-1}(\widehat{t}^0)^{-2}\big\}.\end{aligned}$$

Now we estimate the norm of the function (10.14) in $\mathfrak{G}$. The operator $\Lambda^\varepsilon \Pi_\varepsilon^{(m)} : \mathfrak{G}_* \to \mathfrak{G}$ is unitarily equivalent to (see (10.2), (10.3)) the operator $\Lambda \Pi^{(m)}$, where $\Pi^{(m)}$ is the pseudodifferential operator in $\mathfrak{G}_*$ with the symbol $\chi_{\widetilde{\Omega}}(\boldsymbol{\xi})$. In its turn, by the Gelfand transformation, the operator $\Lambda \Pi^{(m)}$ is decomposed in the direct integral of the operators $\Lambda \widehat{P}_m$ acting from $\mathfrak{H}_*$ to $\mathfrak{H}$. Hence,

$$\begin{aligned}\|\Lambda^\varepsilon \Pi_\varepsilon^{(m)}\|_{\mathfrak{G}_* \to \mathfrak{G}} &= \|\Lambda \Pi^{(m)}\|_{\mathfrak{G}_* \to \mathfrak{G}} = \|\Lambda \widehat{P}_m\|_{\mathfrak{H}_* \to \mathfrak{H}} \\ &\leq |\Omega|^{-1/2} \Big(\int_\Omega |\Lambda(\mathbf{x})|^2 \, d\mathbf{x} \Big)^{1/2}.\end{aligned}$$

As it was shown in [BSu4, Subsection 7.3], we have

$$\Big(\int_\Omega |\Lambda(\mathbf{x})|^2\,d\mathbf{x}\Big)^{1/2} \le m^{1/2}(2r_0)^{-1}\alpha_0^{-1/2}|\Omega|^{1/2}\|g\|^{1/2}_{L_\infty}\|g^{-1}\|^{1/2}_{L_\infty},$$

whence

$$\|\Lambda^\varepsilon \Pi_\varepsilon^{(m)}\|_{\mathfrak{G}_*\to\mathfrak{G}} \le m^{1/2}(2r_0)^{-1}\alpha_0^{-1/2}\|g\|^{1/2}_{L_\infty}\|g^{-1}\|^{1/2}_{L_\infty}. \tag{10.17}$$

We estimate the norm of the function $b(\mathbf{D})\mathbf{u}_0$. From (10.13) it follows that

$$(g^0 b(\mathbf{D})\mathbf{u}_0, b(\mathbf{D})\mathbf{u}_0)_{\mathfrak{G}_*} + \|\mathbf{u}_0\|^2_{\mathfrak{G}} = (\mathbf{F},\mathbf{u}_0)_{\mathfrak{G}} \le \|\mathbf{u}_0\|^2_{\mathfrak{G}} + \frac{1}{4}\|\mathbf{F}\|^2_{\mathfrak{G}}.$$

Hence,

$$\|b(\mathbf{D})\mathbf{u}_0\|^2_{\mathfrak{G}_*} \le \frac{1}{4}|(g^0)^{-1}|\|\mathbf{F}\|^2_{\mathfrak{G}} \le \frac{1}{4}\|g^{-1}\|_{L_\infty}\|\mathbf{F}\|^2_{\mathfrak{G}}. \tag{10.18}$$

Relations (10.17) and (10.18) imply the following estimate for the norm of the function (10.14):

$$\|\mathbf{u}_\varepsilon^{(1)}\|_{\mathfrak{G}} \le m^{1/2}(4r_0)^{-1}\alpha_0^{-1/2}\|g\|^{1/2}_{L_\infty}\|g^{-1}\|_{L_\infty}\|\mathbf{F}\|_{\mathfrak{G}}. \tag{10.19}$$

Now (10.16) and (10.19) imply that

$$\|\mathbf{u}_\varepsilon - \mathbf{u}_0 - \varepsilon\mathbf{u}_\varepsilon^{(1)}\|_{\mathfrak{G}} \le \big(\widehat{\mathcal{C}}_\times + m^{1/2}(4r_0)^{-1}\alpha_0^{-1/2}\|g\|^{1/2}_{L_\infty}\|g^{-1}\|_{L_\infty}\big)\varepsilon\|\mathbf{F}\|_{\mathfrak{G}}, \quad 0<\varepsilon\le 1.$$

Combining this with (10.15), we obtain the estimate for the $\mathfrak{G}^1$-norm of the function $(\mathbf{u}_\varepsilon - \mathbf{u}_0 - \varepsilon\mathbf{u}_\varepsilon^{(1)})$:

$$\|\mathbf{u}_\varepsilon - \mathbf{u}_0 - \varepsilon\mathbf{u}_\varepsilon^{(1)}\|_{\mathfrak{G}^1} \le \mathcal{C}_{10}\varepsilon\|\mathbf{F}\|_{\mathfrak{G}}, \quad 0<\varepsilon\le 1,$$

where

$$\mathcal{C}_{10}^2 = \widehat{c}_*^{-1}\mathcal{C}_6^2 + \big(\widehat{\mathcal{C}}_\times + m^{1/2}(4r_0)^{-1}\alpha_0^{-1/2}\|g\|^{1/2}_{L_\infty}\|g^{-1}\|_{L_\infty}\big)^2. \tag{10.20}$$

Thus, we arrive at the following theorem.

Theorem 10.6. *Under conditions of Theorem* 10.1*, we have*

$$\|(\widehat{\mathcal{A}}_\varepsilon + I)^{-1} - (\widehat{\mathcal{A}}^0 + I)^{-1} - \varepsilon\Lambda^\varepsilon b(\mathbf{D})(\widehat{\mathcal{A}}^0 + I)^{-1}\Pi_\varepsilon\|_{\mathfrak{G}\to\mathfrak{G}^1} \le \mathcal{C}_{10}\varepsilon, \quad 0<\varepsilon\le 1. \tag{10.21}$$

The constant $\mathcal{C}_{10}$ is defined by (10.20) *and depends only on m, α_0, α_1, $\|g\|_{L_\infty}$, $\|g^{-1}\|_{L_\infty}$, and on parameters of the lattice Γ.*

Remark 10.7. One can show that the functions $\varepsilon\Lambda^\varepsilon b(\mathbf{D})\Pi_\varepsilon\mathbf{u}_0$ weakly tend to zero in $\mathfrak{G}^1$. Then the result of Theorem 10.6 agrees with that of Theorem 4.4.1(1°) from [BSu2] about weak ($\mathfrak{G}^1$)-convergence of the functions $\mathbf{u}_\varepsilon = (\widehat{\mathcal{A}}_\varepsilon + I)^{-1}\mathbf{F}$ to $\mathbf{u}_0 = (\widehat{\mathcal{A}}^0 + I)^{-1}\mathbf{F}$, where $\mathbf{F}\in\mathfrak{G}$.

10.3. Elimination of the operator Π_ε

Suppose now that Condition 8.4 is satisfied. Then estimate (10.8) is valid. We put

$$\widetilde{\mathbf{u}}_\varepsilon^{(1)} = K_1^0(\varepsilon)\mathbf{F} = \Lambda^\varepsilon b(\mathbf{D})\mathbf{u}_0. \tag{10.22}$$

Estimate (10.8) means that

$$\|\widehat{\mathcal{A}}_\varepsilon^{1/2}(\mathbf{u}_\varepsilon - \mathbf{u}_0 - \varepsilon\widetilde{\mathbf{u}}_\varepsilon^{(1)})\|_{\mathfrak{G}} \leq \mathcal{C}_7\varepsilon\|\mathbf{F}\|_{\mathfrak{G}}, \quad 0 < \varepsilon \leq 1. \tag{10.23}$$

Similarly to (10.15), this implies that

$$\int\limits_{\mathbb{R}^d} |\mathbf{D}(\mathbf{u}_\varepsilon - \mathbf{u}_0 - \varepsilon\widetilde{\mathbf{u}}_\varepsilon^{(1)})|^2\,d\mathbf{x} \leq \widehat{c}_*^{\,-1}\mathcal{C}_7^2\varepsilon^2\|\mathbf{F}\|_{\mathfrak{G}}^2, \quad 0 < \varepsilon \leq 1. \tag{10.24}$$

We estimate the $\mathfrak{G}$-norm of the function $\widetilde{\mathbf{u}}_\varepsilon^{(1)}$. By (10.18), (10.22), and Condition 8.4, we have:

$$\|\widetilde{\mathbf{u}}_\varepsilon^{(1)}\|_{\mathfrak{G}} \leq \|\Lambda\|_{L_\infty}\|b(\mathbf{D})\mathbf{u}_0\|_{\mathfrak{G}_*} \leq \frac{1}{2}\|\Lambda\|_{L_\infty}\|g^{-1}\|_{L_\infty}^{1/2}\|\mathbf{F}\|_{\mathfrak{G}}. \tag{10.25}$$

From (10.16) and (10.25) it follows that

$$\|\mathbf{u}_\varepsilon - \mathbf{u}_0 - \varepsilon\widetilde{\mathbf{u}}_\varepsilon^{(1)}\|_{\mathfrak{G}} \leq \Big(\widehat{\mathcal{C}}_\times + \frac{1}{2}\|\Lambda\|_{L_\infty}\|g^{-1}\|_{L_\infty}^{1/2}\Big)\varepsilon\|\mathbf{F}\|_{\mathfrak{G}}, \quad 0 < \varepsilon \leq 1. \tag{10.26}$$

Now (10.24) and (10.26) imply that

$$\|\mathbf{u}_\varepsilon - \mathbf{u}_0 - \varepsilon\widetilde{\mathbf{u}}_\varepsilon^{(1)}\|_{\mathfrak{G}^1} \leq \mathcal{C}_{11}\varepsilon\|\mathbf{F}\|_{\mathfrak{G}}, \quad 0 < \varepsilon \leq 1,$$

where

$$\mathcal{C}_{11}^2 = \widehat{c}_*^{\,-1}\mathcal{C}_7^2 + \Big(\widehat{\mathcal{C}}_\times + \frac{1}{2}\|\Lambda\|_{L_\infty}\|g^{-1}\|_{L_\infty}^{1/2}\Big)^2. \tag{10.27}$$

We have proved the following result.

Theorem 10.8. *Suppose that conditions of Theorem* 10.1 *are satisfied. Suppose also that Condition* 8.4 *is valid. Then*

$$\|(\widehat{\mathcal{A}}_\varepsilon + I)^{-1} - (\widehat{\mathcal{A}}^0 + I)^{-1} - \varepsilon\Lambda^\varepsilon b(\mathbf{D})(\widehat{\mathcal{A}}^0 + I)^{-1}\|_{\mathfrak{G}\to\mathfrak{G}^1} \leq \mathcal{C}_{11}\varepsilon, \quad 0 < \varepsilon \leq 1.$$

The constant $\mathcal{C}_{11}$ *is defined by* (10.27) *and depends only on* m, d, α_0, α_1, $\|g\|_{L_\infty}$, $\|g^{-1}\|_{L_\infty}$, *on parameters of the lattice* Γ, *and also on* $\|\Lambda\|_{L_\infty}$.

10.4. The case of zero corrector

Suppose that condition (6.25) is satisfied. Then, applying Theorem 10.3, we arrive at the estimate

$$\|\widehat{\mathcal{A}}_\varepsilon^{1/2}(\mathbf{u}_\varepsilon - \mathbf{u}_0)\|_{\mathfrak{G}} \leq \mathcal{C}_6^\circ\varepsilon\|\mathbf{F}\|_{\mathfrak{G}}, \quad 0 < \varepsilon \leq 1.$$

It follows that (cf. the proof of estimate (10.15))

$$\int\limits_{\mathbb{R}^d} |\mathbf{D}(\mathbf{u}_\varepsilon - \mathbf{u}_0)|^2\,d\mathbf{x} \leq \widehat{c}_*^{\,-1}(\mathcal{C}_6^\circ)^2\varepsilon^2\|\mathbf{F}\|_{\mathfrak{G}}^2, \quad 0 < \varepsilon \leq 1.$$

Combining this with (10.16), we arrive at the inequality

$$\|\mathbf{u}_\varepsilon - \mathbf{u}_0\|_{\mathfrak{G}^1} \leq \mathcal{C}_{12}\varepsilon\|\mathbf{F}\|_{\mathfrak{G}}, \quad 0 < \varepsilon \leq 1, \tag{10.28}$$

where

$$\mathcal{C}_{12} = \big(\widehat{\mathcal{C}}_\times^2 + \widehat{c}_*^{\,-1}(\mathcal{C}_6^\circ)^2\big)^{1/2}. \tag{10.29}$$

Thus, we have proved the following theorem.

Theorem 10.9. *Suppose that conditions of Theorem* 10.1 *are satisfied. Suppose also that condition* (6.25) *is valid. Then*

$$\|(\widehat{\mathcal{A}}_\varepsilon + I)^{-1} - (\widehat{\mathcal{A}}^0 + I)^{-1}\|_{\mathfrak{G}\to\mathfrak{G}^1} \le \mathcal{C}_{12}\varepsilon, \quad 0 < \varepsilon \le 1. \tag{10.30}$$

The constant $\mathcal{C}_{12}$ *is defined by* (10.29) *and depends only on* α_0, α_1, $\|g\|_{L_\infty}$, $\|g^{-1}\|_{L_\infty}$, *and on parameters of the lattice* Γ.

Remark 10.10. Conditions of Theorem 10.9 are a fortiori valid, if $g^0 = \overline{g}$. In [BSu2, Theorem 4.4.5] it was shown that, if $g^0 = \overline{g}$, then $\mathbf{u}_\varepsilon$ strongly converges in $\mathfrak{G}^1$ to $\mathbf{u}_0$, as $\varepsilon \to 0$. Theorem 10.9 strengthens this statement giving estimate (10.28).

11 Approximation with the three-term corrector for the resolvent $(\widehat{\mathcal{A}}_\varepsilon + I)^{-1}$. Interpolation

11.1

Now we apply Theorem 10.4. By (10.10),

$$\|\widehat{\mathcal{A}}_\varepsilon^{1/2}(\mathbf{u}_\varepsilon - \mathbf{u}_0 - \varepsilon\widetilde{K}(\varepsilon)\mathbf{F})\|_{\mathfrak{G}} \le \mathcal{C}_8\varepsilon\|\mathbf{F}\|_{\mathfrak{G}}, \quad 0 < \varepsilon \le 1.$$

As above (cf. the proof of (10.15)), this yields the estimate

$$\int_{\mathbb{R}^d} |\mathbf{D}(\mathbf{u}_\varepsilon - \mathbf{u}_0 - \varepsilon\widetilde{K}(\varepsilon)\mathbf{F})|^2\, d\mathbf{x} \le \widehat{c}_*^{-1}\mathcal{C}_8^2\varepsilon^2\|\mathbf{F}\|^2_{\mathfrak{G}}, \quad 0 < \varepsilon \le 1. \tag{11.1}$$

We estimate the $\mathfrak{G}$-norm of the function $\widetilde{K}(\varepsilon)\mathbf{F}$. Inequality (10.19) means that

$$\|\Lambda^\varepsilon b(\mathbf{D})(\widehat{\mathcal{A}}^0 + I)^{-1}\Pi_\varepsilon\|_{\mathfrak{G}\to\mathfrak{G}} \le m^{1/2}(4r_0)^{-1}\alpha_0^{-1/2}\|g\|^{1/2}_{L_\infty}\|g^{-1}\|_{L_\infty}. \tag{11.2}$$

The operator $(\widehat{\mathcal{A}}^0 + I)^{-1}\Pi_\varepsilon b(\mathbf{D})^*(\Lambda^\varepsilon)^*$ is adjoint to the operator from (11.2). Therefore, its norm satisfies the same estimate. Next, as it was shown in [BSu4, Subsection 6.3], we have

$$|b(\boldsymbol{\xi})^* L(\boldsymbol{\xi}) b(\boldsymbol{\xi})| \le |\boldsymbol{\xi}|^3 2^{1/2} r_0^{-1}\alpha_0^{-1/2}\alpha_1^{3/2}\|g\|^{3/2}_{L_\infty}\|g^{-1}\|^{1/2}_{L_\infty}, \quad \boldsymbol{\xi} \in \mathbb{R}^d.$$

By (8.4), this implies that

$$\begin{aligned}
&\|(\widehat{\mathcal{A}}^0 + I)^{-1} b(\mathbf{D})^* L(\mathbf{D}) b(\mathbf{D})(\widehat{\mathcal{A}}^0 + I)^{-1}\|_{\mathfrak{G}\to\mathfrak{G}} \\
&\quad = \sup_{\boldsymbol{\xi}\in\mathbb{R}^d} |(b(\boldsymbol{\xi})^* g^0 b(\boldsymbol{\xi}) + \mathbf{1}_n)^{-1} b(\boldsymbol{\xi})^* L(\boldsymbol{\xi}) b(\boldsymbol{\xi})(b(\boldsymbol{\xi})^* g^0 b(\boldsymbol{\xi}) + \mathbf{1}_n)^{-1}| \\
&\quad \le 2^{1/2} r_0^{-1}\alpha_0^{-1/2}\alpha_1^{3/2}\|g\|^{3/2}_{L_\infty}\|g^{-1}\|^{1/2}_{L_\infty} \sup_{\boldsymbol{\xi}\in\mathbb{R}^d} |\boldsymbol{\xi}|^3(\widehat{c}_*|\boldsymbol{\xi}|^2 + 1)^{-2} \\
&\quad \le 2^{1/2} r_0^{-1}\alpha_0^{-1/2}\alpha_1^{3/2}\|g\|^{3/2}_{L_\infty}\|g^{-1}\|^{1/2}_{L_\infty}\widehat{c}_*^{-3/2}.
\end{aligned} \tag{11.3}$$

From (10.9), (11.2), and (11.3) it follows that

$$\begin{aligned}
\|\widetilde{K}(\varepsilon)\|_{\mathfrak{G}\to\mathfrak{G}} \le\ & m^{1/2}(2r_0)^{-1}\alpha_0^{-1/2}\|g\|^{1/2}_{L_\infty}\|g^{-1}\|_{L_\infty} \\
&+ 2^{1/2} r_0^{-1}\alpha_0^{-1/2}\alpha_1^{3/2}\|g\|^{3/2}_{L_\infty}\|g^{-1}\|^{1/2}_{L_\infty}\widehat{c}_*^{-3/2}.
\end{aligned}$$

Combining this with (10.16) and (11.1), we obtain the estimate

$$\|\mathbf{u}_\varepsilon - \mathbf{u}_0 - \varepsilon \widetilde{K}(\varepsilon)\mathbf{F}\|_{\mathfrak{G}^1} \le \mathcal{C}_{13}\varepsilon\|\mathbf{F}\|_{\mathfrak{G}}, \quad 0 < \varepsilon \le 1,$$

where

$$\begin{aligned} \mathcal{C}_{13}^2 &= \widehat{c}_*^{-1}\mathcal{C}_8^2 \\ &\quad + \big(\widehat{\mathcal{C}}_\times + m^{1/2}(2r_0)^{-1}\alpha_0^{-1/2}\|g\|_{L_\infty}^{1/2}\|g^{-1}\|_{L_\infty} \\ &\qquad + 2^{1/2}r_0^{-1}\alpha_0^{-1/2}\alpha_1^{3/2}\|g\|_{L_\infty}^{3/2}\|g^{-1}\|_{L_\infty}^{1/2}\widehat{c}_*^{-3/2}\big)^2. \end{aligned} \tag{11.4}$$

We arrive at the following result.

Theorem 11.1. *Under conditions of Theorem* 10.4, *we have*

$$\|(\widehat{\mathcal{A}}_\varepsilon + I)^{-1} - (\widehat{\mathcal{A}}^0 + I)^{-1} - \varepsilon\widetilde{K}(\varepsilon)\|_{\mathfrak{G}\to\mathfrak{G}^1} \le \mathcal{C}_{13}\varepsilon, \quad 0 < \varepsilon \le 1. \tag{11.5}$$

The constant $\mathcal{C}_{13}$ is defined by (11.4) *and depends only on m, α_0, α_1, $\|g\|_{L_\infty}$, $\|g^{-1}\|_{L_\infty}$, and on parameters of the lattice Γ.*

11.2

Suppose now that Condition 8.4 is satisfied. Then, applying Theorem 10.5, we obtain:

$$\|\widehat{\mathcal{A}}_\varepsilon^{1/2}(\mathbf{u}_\varepsilon - \mathbf{u}_0 - \varepsilon K^0(\varepsilon)\mathbf{F})\|_{\mathfrak{G}} \le \mathcal{C}_9\varepsilon\|\mathbf{F}\|_{\mathfrak{G}}, \quad 0 < \varepsilon \le 1.$$

This implies the estimate

$$\int_{\mathbb{R}^d} |\mathbf{D}(\mathbf{u}_\varepsilon - \mathbf{u}_0 - \varepsilon K^0(\varepsilon)\mathbf{F})|^2\, d\mathbf{x} \le \widehat{c}_*^{-1}\mathcal{C}_9^2\varepsilon^2\|\mathbf{F}\|_{\mathfrak{G}}^2, \quad 0 < \varepsilon \le 1. \tag{11.6}$$

Now we estimate the norm of the function $K^0(\varepsilon)\mathbf{F}$ under Condition 8.4. Inequality (10.25) means that

$$\|\Lambda^\varepsilon b(\mathbf{D})(\widehat{\mathcal{A}}^0 + I)^{-1}\|_{\mathfrak{G}\to\mathfrak{G}} \le \frac{1}{2}\|\Lambda\|_{L_\infty}\|g^{-1}\|_{L_\infty}^{1/2}. \tag{11.7}$$

The adjoint operator $(\widehat{\mathcal{A}}^0 + I)^{-1}b(\mathbf{D})^*(\Lambda^\varepsilon)^*$ admits the same estimate. Then (10.11), (11.3), and (11.7) imply that

$$\|K^0(\varepsilon)\|_{\mathfrak{G}\to\mathfrak{G}} \le \|\Lambda\|_{L_\infty}\|g^{-1}\|_{L_\infty}^{1/2} + 2^{1/2}r_0^{-1}\alpha_0^{-1/2}\alpha_1^{3/2}\|g\|_{L_\infty}^{3/2}\|g^{-1}\|_{L_\infty}^{1/2}\widehat{c}_*^{-3/2}. \tag{11.8}$$

Now, by (10.16), (11.6), and (11.8), we obtain:

$$\|\mathbf{u}_\varepsilon - \mathbf{u}_0 - \varepsilon K^0(\varepsilon)\mathbf{F}\|_{\mathfrak{G}^1} \le \mathcal{C}_{14}\varepsilon\|\mathbf{F}\|_{\mathfrak{G}}, \quad 0 < \varepsilon \le 1,$$

where

$$\mathcal{C}_{14}^2 = \widehat{c}_*^{-1}\mathcal{C}_9^2 + \big(\widehat{\mathcal{C}}_\times + \|\Lambda\|_{L_\infty}\|g^{-1}\|_{L_\infty}^{1/2} + 2^{1/2}r_0^{-1}\alpha_0^{-1/2}\alpha_1^{3/2}\|g\|_{L_\infty}^{3/2}\|g^{-1}\|_{L_\infty}^{1/2}\widehat{c}_*^{-3/2}\big)^2. \tag{11.9}$$

As a result, we arrive at the following statement.

Theorem 11.2. *Under conditions of Theorem* 10.5, *we have:*

$$\|(\widehat{\mathcal{A}}_\varepsilon + I)^{-1} - (\widehat{\mathcal{A}}^0 + I)^{-1} - \varepsilon K^0(\varepsilon)\|_{\mathfrak{G}\to\mathfrak{G}^1} \le \mathcal{C}_{14}\varepsilon, \quad 0 < \varepsilon \le 1. \tag{11.10}$$

The constant $\mathcal{C}_{14}$ is defined by (11.9) *and depends on d, m, α_0, α_1, $\|g\|_{L_\infty}$, $\|g^{-1}\|_{L_\infty}$, on parameters of the lattice Γ, and also on $\|\Lambda\|_{L_\infty}$.*

11.3. Interpolation

In [BSu4, Theorem 8.1] it was proved that, under conditions of Theorem 10.4, we have

$$\|(\widehat{\mathcal{A}}_\varepsilon + I)^{-1} - (\widehat{\mathcal{A}}^0 + I)^{-1} - \varepsilon \widetilde{K}(\varepsilon)\|_{\mathfrak{G}\to\mathfrak{G}} \le \mathfrak{C}_1 \varepsilon^2, \quad 0 < \varepsilon \le 1. \qquad (11.11)$$

The constant $\mathfrak{C}_1$ depends on α_0, α_1, $\|g\|_{L_\infty}$, $\|g^{-1}\|_{L_\infty}$, and on parameters of the lattice Γ.

Interpolating between (11.11) and (11.5), we arrive at the following result.

Theorem 11.3. *Under conditions of Theorem* 10.4*, for* $0 \le s \le 1$ *we have*

$$\|(\widehat{\mathcal{A}}_\varepsilon + I)^{-1} - (\widehat{\mathcal{A}}^0 + I)^{-1} - \varepsilon \widetilde{K}(\varepsilon)\|_{\mathfrak{G}\to\mathfrak{G}^s} \le \mathfrak{C}_1^{1-s} \mathcal{C}_{13}^s \varepsilon^{2-s}, \quad 0 < \varepsilon \le 1.$$

As it follows from [BSu4, Theorem 8.2], under Condition 8.4 (precisely, under some weaker condition), we have

$$\|(\widehat{\mathcal{A}}_\varepsilon + I)^{-1} - (\widehat{\mathcal{A}}^0 + I)^{-1} - \varepsilon K^0(\varepsilon)\|_{\mathfrak{G}\to\mathfrak{G}} \le \mathfrak{C}_2 \varepsilon^2, \quad 0 < \varepsilon \le 1. \qquad (11.12)$$

The constant $\mathfrak{C}_2$ depends on α_0, α_1, $\|g\|_{L_\infty}$, $\|g^{-1}\|_{L_\infty}$, on parameters of the lattice Γ, and also on $\|\Lambda\|_{L_\infty}$.

Interpolating between (11.12) and (11.10), we arrive at the following theorem.

Theorem 11.4. *Under conditions of Theorem* 10.5*, for* $0 \le s \le 1$ *we have*

$$\|(\widehat{\mathcal{A}}_\varepsilon + I)^{-1} - (\widehat{\mathcal{A}}^0 + I)^{-1} - \varepsilon K^0(\varepsilon)\|_{\mathfrak{G}\to\mathfrak{G}^s} \le \mathfrak{C}_2^{1-s} \mathcal{C}_{14}^s \varepsilon^{2-s}, \quad 0 < \varepsilon \le 1.$$

Condition (6.25) distinguishes the case, where the corrector is equal to zero. Then the estimate (10.30), as well as the following estimate (cf. [BSu4, Theorem 8.3]), is satisfied:

$$\|(\widehat{\mathcal{A}}_\varepsilon + I)^{-1} - (\widehat{\mathcal{A}}^0 + I)^{-1}\|_{\mathfrak{G}\to\mathfrak{G}} \le \mathfrak{C}_3 \varepsilon^2, \quad 0 < \varepsilon \le 1. \qquad (11.13)$$

The constant $\mathfrak{C}_3$ depends on α_0, α_1, $\|g\|_{L_\infty}$, $\|g^{-1}\|_{L_\infty}$, and on parameters of the lattice Γ.

Interpolating between (11.13) and (10.30), we arrive at the following result.

Theorem 11.5. *Under condition* (6.25)*, for* $0 \le s \le 1$ *we have*

$$\|(\widehat{\mathcal{A}}_\varepsilon + I)^{-1} - (\widehat{\mathcal{A}}^0 + I)^{-1}\|_{\mathfrak{G}\to\mathfrak{G}^s} \le \mathfrak{C}_3^{1-s} \mathcal{C}_{12}^s \varepsilon^{2-s}, \quad 0 < \varepsilon \le 1.$$

12 Approximation of the fluxes for $(\widehat{\mathcal{A}}_\varepsilon + I)^{-1}$

12.1

In this section, we approximate the so called fluxes

$$\mathbf{p}_\varepsilon = g^\varepsilon b(\mathbf{D}) \mathbf{u}_\varepsilon \qquad (12.1)$$

in the norm of $\mathfrak{G}_* = L_2(\mathbb{R}^d; \mathbb{C}^m)$. Here $\mathbf{u}_\varepsilon$ is the solution of the equation (10.12). Now, it is convenient to rely on Theorem 8.1 which, by the scaling transformation, implies that

$$\|\widehat{\mathcal{A}}_\varepsilon^{1/2} \left((\widehat{\mathcal{A}}_\varepsilon + I)^{-1} - (I + \varepsilon \Lambda^\varepsilon b(\mathbf{D}))(\widehat{\mathcal{A}}^0 + I)^{-1} \Pi_\varepsilon \right) \|_{\mathfrak{G}\to\mathfrak{G}} \le \mathcal{C}_3 \varepsilon, \quad 0 < \varepsilon \le 1. \qquad (12.2)$$

We have: $(\widehat{\mathcal{A}}_\varepsilon + I)^{-1}\mathbf{F} = \mathbf{u}_\varepsilon$, $(\widehat{\mathcal{A}}^0 + I)^{-1}\Pi_\varepsilon \mathbf{F} = \Pi_\varepsilon \mathbf{u}_0$. Then (12.2) means that

$$\|\widehat{\mathcal{A}}_\varepsilon^{1/2} \left(\mathbf{u}_\varepsilon - (I + \varepsilon \Lambda^\varepsilon b(\mathbf{D}))\Pi_\varepsilon \mathbf{u}_0 \right) \|_{\mathfrak{G}} \le \mathcal{C}_3 \varepsilon \|\mathbf{F}\|_{\mathfrak{G}}, \quad 0 < \varepsilon \le 1.$$

This is equivalent to the estimate

$$\|(g^\varepsilon)^{1/2} b(\mathbf{D})\left(\mathbf{u}_\varepsilon - (I + \varepsilon\Lambda^\varepsilon b(\mathbf{D}))\Pi_\varepsilon \mathbf{u}_0\right)\|_{\mathfrak{G}_*} \le \mathcal{C}_3 \varepsilon \|\mathbf{F}\|_{\mathfrak{G}}, \quad 0 < \varepsilon \le 1.$$

Then

$$\|g^\varepsilon b(\mathbf{D})\mathbf{u}_\varepsilon - g^\varepsilon b(\mathbf{D})(\Pi_\varepsilon \mathbf{u}_0 + \varepsilon \Lambda^\varepsilon \Pi_\varepsilon^{(m)} b(\mathbf{D})\mathbf{u}_0)\|_{\mathfrak{G}_*} \le \mathcal{C}_3 \|g\|_{L_\infty}^{1/2} \varepsilon \|\mathbf{F}\|_{\mathfrak{G}}, \quad 0 < \varepsilon \le 1. \tag{12.3}$$

We have (see (8.12)):

$$\begin{aligned} &g^\varepsilon b(\mathbf{D})(\varepsilon \Lambda^\varepsilon \Pi_\varepsilon^{(m)} b(\mathbf{D})\mathbf{u}_0) \\ &\quad = \varepsilon g^\varepsilon (b(\mathbf{D})\Lambda^\varepsilon)\Pi_\varepsilon^{(m)} b(\mathbf{D})\mathbf{u}_0 + \varepsilon g^\varepsilon \sum_{l=1}^d b_l \Lambda^\varepsilon D_l(\Pi_\varepsilon^{(m)} b(\mathbf{D})\mathbf{u}_0) \\ &\quad = g^\varepsilon (b(\mathbf{D})\Lambda)^\varepsilon \Pi_\varepsilon^{(m)} b(\mathbf{D})\mathbf{u}_0 + \varepsilon g^\varepsilon \sum_{l=1}^d b_l \Lambda^\varepsilon \Pi_\varepsilon^{(m)} D_l(b(\mathbf{D})\mathbf{u}_0). \end{aligned} \tag{12.4}$$

Here we have used the obvious identity $\varepsilon(b(\mathbf{D})\Lambda^\varepsilon) = (b(\mathbf{D})\Lambda)^\varepsilon$. We estimate the second term in the right-hand side of (12.4). The norm of the operator $\Lambda^\varepsilon \Pi_\varepsilon^{(m)}$ is estimated in (10.17). We have:

$$D_l(b(\mathbf{D})\mathbf{u}_0) = D_l b(\mathbf{D})(\widehat{\mathcal{A}}^0 + I)^{-1}\mathbf{F}.$$

Combining this with (8.5), we see that

$$\|D_l(b(\mathbf{D})\mathbf{u}_0)\|_{\mathfrak{G}_*} \le \Big(\sup_{\boldsymbol{\xi} \in \mathbb{R}^d} |\xi_l b(\boldsymbol{\xi})(b(\boldsymbol{\xi})^* g^0 b(\boldsymbol{\xi}) + \mathbf{1}_n)^{-1}|\Big)\|\mathbf{F}\|_{\mathfrak{G}} \le \alpha_1^{1/2} \widehat{c}_*^{-1} \|\mathbf{F}\|_{\mathfrak{G}}. \tag{12.5}$$

From (10.17) and (12.5) it follows that

$$\begin{aligned} &\|g^\varepsilon \sum_{l=1}^d b_l \Lambda^\varepsilon \Pi_\varepsilon^{(m)} D_l(b(\mathbf{D})\mathbf{u}_0)\|_{\mathfrak{G}_*} \\ &\qquad \le d\, m^{1/2}(2r_0)^{-1} \alpha_0^{-1/2} \alpha_1 \widehat{c}_*^{-1} \|g\|_{L_\infty}^{3/2} \|g^{-1}\|_{L_\infty}^{1/2} \|\mathbf{F}\|_{\mathfrak{G}}. \end{aligned} \tag{12.6}$$

Now (12.1), (12.3), (12.4), and (12.6) imply that

$$\|\mathbf{p}_\varepsilon - g^\varepsilon \left(\mathbf{1}_m + (b(\mathbf{D})\Lambda)^\varepsilon\right) \Pi_\varepsilon^{(m)} b(\mathbf{D})\mathbf{u}_0\|_{\mathfrak{G}_*} \le \mathcal{C}_{15} \varepsilon \|\mathbf{F}\|_{\mathfrak{G}}, \tag{12.7}$$

where

$$\mathcal{C}_{15} = \mathcal{C}_3 \|g\|_{L_\infty}^{1/2} + d\, m^{1/2}(2r_0)^{-1} \alpha_0^{-1/2} \alpha_1 \widehat{c}_*^{-1} \|g\|_{L_\infty}^{3/2} \|g^{-1}\|_{L_\infty}^{1/2}. \tag{12.8}$$

Using the notation (5.5), we have $g^\varepsilon(\mathbf{1}_m + (b(\mathbf{D})\Lambda)^\varepsilon) = \widetilde{g}^\varepsilon$. Then, by (12.7), we obtain the following statement.

Theorem 12.1. *Let* $\mathbf{u}_\varepsilon = (\widehat{\mathcal{A}}_\varepsilon + I)^{-1}\mathbf{F}$, $\mathbf{u}_0 = (\widehat{\mathcal{A}}^0 + I)^{-1}\mathbf{F}$, *where* $\mathbf{F} \in \mathfrak{G}$. *We put* $\mathbf{p}_\varepsilon = g^\varepsilon b(\mathbf{D})\mathbf{u}_\varepsilon$ *and* $\widetilde{g} = g(\mathbf{1}_m + b(\mathbf{D})\Lambda)$. *Let* $\Pi_\varepsilon^{(m)}$ *be the pseudodifferential operator in* $\mathfrak{G}_*$ *with the symbol* $\chi_{\widetilde{\Omega}/\varepsilon}(\boldsymbol{\xi})$. *Then*

$$\|\mathbf{p}_\varepsilon - \widetilde{g}^\varepsilon \Pi_\varepsilon^{(m)} b(\mathbf{D})\mathbf{u}_0\|_{\mathfrak{G}_*} \le \mathcal{C}_{15} \varepsilon \|\mathbf{F}\|_{\mathfrak{G}}, \quad 0 < \varepsilon \le 1.$$

The constant $\mathcal{C}_{15}$ *is defined by* (12.8) *and depends only on* d, m, α_0, α_1, $\|g\|_{L_\infty}$, $\|g^{-1}\|_{L_\infty}$, *and on parameters of the lattice* Γ.

Note that the statement of Theorem 12.1 can be formulated in the operator terms. Namely, we have

$$\|g^\varepsilon b(\mathbf{D})(\widehat{\mathcal{A}}_\varepsilon + I)^{-1} - \widetilde{g}^\varepsilon \Pi_\varepsilon^{(m)} b(\mathbf{D})(\widehat{\mathcal{A}}^0 + I)^{-1}\|_{\mathfrak{G}\to\mathfrak{G}_*} \le \mathcal{C}_{15}\varepsilon, \quad 0 < \varepsilon \le 1.$$

Remark 12.2. It can be shown that the functions $\widetilde{g}^\varepsilon \Pi_\varepsilon^{(m)} b(\mathbf{D})\mathbf{u}_0$ weakly converge in $\mathfrak{G}_*$ to $\mathbf{p}_0 = g^0 b(\mathbf{D})\mathbf{u}_0$. Therefore, the result of Theorem 12.1 agrees with Theorem 4.4.1(2°) from [BSu2] about the weak $(\mathfrak{G}_*)$-convergence of the fluxes $\mathbf{p}_\varepsilon$ to $\mathbf{p}_0$.

12.2

Suppose now that Condition 8.4 is satisfied. Then estimate (10.23) is valid. This estimate is equivalent to the inequality

$$\|(g^\varepsilon)^{1/2} b(\mathbf{D})(\mathbf{u}_\varepsilon - \mathbf{u}_0 - \varepsilon \widetilde{\mathbf{u}}_\varepsilon^{(1)})\|_{\mathfrak{G}_*} \le \mathcal{C}_7 \varepsilon \|\mathbf{F}\|_{\mathfrak{G}}, \quad 0 < \varepsilon \le 1.$$

By (10.22), it follows that

$$\|g^\varepsilon b(\mathbf{D})(\mathbf{u}_\varepsilon - \mathbf{u}_0 - \varepsilon \Lambda^\varepsilon b(\mathbf{D})\mathbf{u}_0)\|_{\mathfrak{G}_*} \le \mathcal{C}_7 \|g\|_{L_\infty}^{1/2} \varepsilon \|\mathbf{F}\|_{\mathfrak{G}}, \quad 0 < \varepsilon \le 1. \tag{12.9}$$

Similarly to (12.4), we have:

$$g^\varepsilon b(\mathbf{D})(\varepsilon \Lambda^\varepsilon b(\mathbf{D})\mathbf{u}_0) = g^\varepsilon (b(\mathbf{D})\Lambda)^\varepsilon b(\mathbf{D})\mathbf{u}_0 + \varepsilon g^\varepsilon \sum_{l=1}^d b_l \Lambda^\varepsilon D_l(b(\mathbf{D})\mathbf{u}_0). \tag{12.10}$$

Under Condition 8.4, (12.5) implies that

$$\|\Lambda^\varepsilon D_l(b(\mathbf{D})\mathbf{u}_0)\|_{\mathfrak{G}_*} \le \|\Lambda\|_{L_\infty} \alpha_1^{1/2} \widehat{c}_*^{-1} \|\mathbf{F}\|_{\mathfrak{G}}.$$

Then

$$\|g^\varepsilon \sum_{l=1}^d b_l \Lambda^\varepsilon D_l(b(\mathbf{D})\mathbf{u}_0)\|_{\mathfrak{G}_*} \le d\, \alpha_1 \|g\|_{L_\infty} \|\Lambda\|_{L_\infty} \widehat{c}_*^{-1} \|\mathbf{F}\|_{\mathfrak{G}}. \tag{12.11}$$

From (12.1) and (12.9)–(12.11) it follows that

$$\|\mathbf{p}_\varepsilon - g^\varepsilon (\mathbf{1}_m + (b(\mathbf{D})\Lambda)^\varepsilon) b(\mathbf{D})\mathbf{u}_0\|_{\mathfrak{G}_*} \le \mathcal{C}_{16} \varepsilon \|\mathbf{F}\|_{\mathfrak{G}}, \quad 0 < \varepsilon \le 1,$$

where

$$\mathcal{C}_{16} = \mathcal{C}_7 \|g\|_{L_\infty}^{1/2} + d\, \alpha_1 \widehat{c}_*^{-1} \|g\|_{L_\infty} \|\Lambda\|_{L_\infty}. \tag{12.12}$$

We arrive at the following result.

Theorem 12.3. *Suppose that conditions of Theorem* 12.1 *are satisfied. Suppose also that Condition* 8.4 *is valid. Then*

$$\|\mathbf{p}_\varepsilon - \widetilde{g}^\varepsilon b(\mathbf{D})\mathbf{u}_0\|_{\mathfrak{G}_*} \le \mathcal{C}_{16} \varepsilon \|\mathbf{F}\|_{\mathfrak{G}}, \quad 0 < \varepsilon \le 1.$$

The constant $\mathcal{C}_{16}$ *is defined by* (12.12) *and depends only on* d, m, α_0, α_1, $\|g\|_{L_\infty}$, $\|g^{-1}\|_{L_\infty}$, *on parameters of the lattice* Γ, *and also on* $\|\Lambda\|_{L_\infty}$.

12.3. The case where $g^0 = \underline{g}$

If $g^0 = \underline{g}$ (i. e., conditions (5.9) are satisfied), then $\widetilde{g}(\mathbf{x}) = g^0 = \underline{g}$ (see [BSu4, Remark 3.5]). Then Condition 8.4 is also valid, and the norm $\|\Lambda\|_{L_\infty}$ is estimated by the constant depending only on α_0, α_1, $\|g\|_{L_\infty}$, $\|g^{-1}\|_{L_\infty}$, m, n, d, and on parameters of the lattice Γ (see Lemma 8.7 and Remark 8.8). Theorem 12.3 is applicable, and $\widetilde{g}^\varepsilon b(\mathbf{D})\mathbf{u}_0 = g^0 b(\mathbf{D})\mathbf{u}_0 =: \mathbf{p}_0$. We arrive at the following result.

Theorem 12.4. *Let $g^0 = \underline{g}$, i. e., conditions* (5.9) *are satisfied. Let $\mathbf{p}_\varepsilon = g^\varepsilon b(\mathbf{D})\mathbf{u}_\varepsilon$, $\mathbf{p}_0 = g^0 b(\mathbf{D})\mathbf{u}_0$. Then, as $\varepsilon \to 0$, the fluxes $\mathbf{p}_\varepsilon$ converge to $\mathbf{p}_0$ in the $\mathfrak{G}_*$-norm, and*

$$\|\mathbf{p}_\varepsilon - \mathbf{p}_0\|_{\mathfrak{G}_*} \le \mathcal{C}_{16}\varepsilon\|\mathbf{F}\|_{\mathfrak{G}}, \quad 0 < \varepsilon \le 1. \tag{12.13}$$

The constant $\mathcal{C}_{16}$ depends only on d, m, n, α_0, α_1, $\|g\|_{L_\infty}$, $\|g^{-1}\|_{L_\infty}$, and on parameters of the lattice Γ.

Remark 12.5. In [BSu2, Theorem 4.4.5(2°)], it was proved that, if $g^0 = \underline{g}$, then the fluxes $\mathbf{p}_\varepsilon$ tend to $\mathbf{p}_0$ strongly in $\mathfrak{G}_*$. Theorem 12.4 strengthens this statement giving estimate (12.13).

13 Approximation for the generalized resolvent $(\widehat{\mathcal{A}}_\varepsilon + Q^\varepsilon)^{-1}$ in the $(L_2 \to H^1)$-norm

13.1. Approximation of the operator $\widehat{\mathcal{A}}_\varepsilon^{1/2}(\widehat{\mathcal{A}}_\varepsilon + Q^\varepsilon)^{-1}$

The following result is deduced from Theorem 9.2, by the scaling transformation.

Theorem 13.1. *Let $\widehat{\mathcal{A}}_\varepsilon = b(\mathbf{D})^* g^\varepsilon b(\mathbf{D})$, and let $\widehat{\mathcal{A}}^0 = b(\mathbf{D})^* g^0 b(\mathbf{D})$ be the effective operator. Let $Q(\mathbf{x})$ be the Γ-periodic positive $(n\times n)$-matrix-valued function satisfying condition* (7.1)*. Let Q^ε be the operator of multiplication by the matrix $Q^\varepsilon(\mathbf{x}) = Q(\varepsilon^{-1}\mathbf{x})$, and let $\overline{Q}$ be the mean value of the matrix $Q(\mathbf{x})$ over the cell Ω. Let $\Lambda^\varepsilon : \mathfrak{G}_* \to \mathfrak{G}$ be the operator of multiplication by the matrix-valued function $\Lambda^\varepsilon(\mathbf{x}) = \Lambda(\varepsilon^{-1}\mathbf{x})$, where $\Lambda(\mathbf{x})$ is the matrix introduced in Subsection* 5.1*. Let Π_ε be the pseudodifferential operator* (10.4)*. Then*

$$\|\widehat{\mathcal{A}}_\varepsilon^{1/2}\big((\widehat{\mathcal{A}}_\varepsilon + Q^\varepsilon)^{-1} - (\widehat{\mathcal{A}}^0 + \overline{Q})^{-1} - \varepsilon\Lambda^\varepsilon b(\mathbf{D})(\widehat{\mathcal{A}}^0 + \overline{Q})^{-1}\Pi_\varepsilon\big)\|_{\mathfrak{G}\to\mathfrak{G}} \le \check{\mathcal{C}}_6\varepsilon, \quad 0 < \varepsilon \le 1. \tag{13.1}$$

Similarly, Theorem 9.3 implies the following statement.

Theorem 13.2. *Suppose that conditions of Theorem* 13.1 *are satisfied. Besides, suppose that Condition* 8.4 *is valid. Then*

$$\|\widehat{\mathcal{A}}_\varepsilon^{1/2}\big((\widehat{\mathcal{A}}_\varepsilon + Q^\varepsilon)^{-1} - (\widehat{\mathcal{A}}^0 + \overline{Q})^{-1} - \varepsilon\Lambda^\varepsilon b(\mathbf{D})(\widehat{\mathcal{A}}^0 + \overline{Q})^{-1}\big)\|_{\mathfrak{G}\to\mathfrak{G}} \le \check{\mathcal{C}}_7\varepsilon, \quad 0 < \varepsilon \le 1. \tag{13.2}$$

Theorem 9.4 implies the following statement, which distinguishes the case where the corrector is equal to zero.

Theorem 13.3. *Suppose that conditions of Theorem* 13.1 *are satisfied. Besides, suppose that condition* (6.25) *is valid. Then*

$$\|\widehat{\mathcal{A}}_\varepsilon^{1/2}\big((\widehat{\mathcal{A}}_\varepsilon + Q^\varepsilon)^{-1} - (\widehat{\mathcal{A}}^0 + \overline{Q})^{-1}\big)\|_{\mathfrak{G}\to\mathfrak{G}} \le \check{\mathcal{C}}_6^\circ\varepsilon, \quad 0 < \varepsilon \le 1.$$

Theorem 9.5 implies approximation for the operator $\widehat{\mathcal{A}}_\varepsilon^{1/2}(\widehat{\mathcal{A}}_\varepsilon+Q^\varepsilon)^{-1}$ with the three-term corrector.

Theorem 13.4. *Suppose that conditions of Theorem* 13.1 *are satisfied. Let*

$$\widetilde{K}_Q(\varepsilon) = \Lambda^\varepsilon b(\mathbf{D})(\widehat{\mathcal{A}}^0 + \overline{Q})^{-1}\Pi_\varepsilon + (\widehat{\mathcal{A}}^0 + \overline{Q})^{-1}\Pi_\varepsilon b(\mathbf{D})^*(\Lambda^\varepsilon)^* \\ - (\widehat{\mathcal{A}}^0 + \overline{Q})^{-1}(b(\mathbf{D})^* L(\mathbf{D}) b(\mathbf{D}) + b(\mathbf{D})^*(\overline{Q\Lambda})^* + (\overline{Q\Lambda}) b(\mathbf{D}))(\widehat{\mathcal{A}}^0 + \overline{Q})^{-1}. \tag{13.3}$$

Then

$$\|\widehat{\mathcal{A}}_\varepsilon^{1/2}\left((\widehat{\mathcal{A}}_\varepsilon + Q^\varepsilon)^{-1} - (\widehat{\mathcal{A}}^0 + \overline{Q})^{-1} - \varepsilon \widetilde{K}_Q(\varepsilon)\right)\|_{\mathfrak{G}\to\mathfrak{G}} \le \check{\mathcal{C}}_8\varepsilon, \quad 0 < \varepsilon \le 1.$$

Finally, Theorem 9.6 yields the following statement.

Theorem 13.5. *Suppose that conditions of Theorem* 13.1 *are satisfied. Besides, suppose that Condition* 8.4 *is valid. We put*

$$K_Q^0(\varepsilon) = \Lambda^\varepsilon b(\mathbf{D})(\widehat{\mathcal{A}}^0 + \overline{Q})^{-1} + (\widehat{\mathcal{A}}^0 + \overline{Q})^{-1} b(\mathbf{D})^*(\Lambda^\varepsilon)^* \\ - (\widehat{\mathcal{A}}^0 + \overline{Q})^{-1}\left(b(\mathbf{D})^* L(\mathbf{D}) b(\mathbf{D}) + b(\mathbf{D})^*(\overline{Q\Lambda})^* + (\overline{Q\Lambda}) b(\mathbf{D})\right)(\widehat{\mathcal{A}}^0 + \overline{Q})^{-1}. \tag{13.4}$$

Then

$$\|\widehat{\mathcal{A}}_\varepsilon^{1/2}\big((\widehat{\mathcal{A}}_\varepsilon + Q^\varepsilon)^{-1} - (\widehat{\mathcal{A}}^0 + \overline{Q})^{-1} - \varepsilon K_Q^0(\varepsilon)\big)\|_{\mathfrak{G}\to\mathfrak{G}} \le \check{\mathcal{C}}_9\varepsilon, \quad 0 < \varepsilon \le 1.$$

13.2. Approximation for the generalized resolvent $(\widehat{\mathcal{A}}_\varepsilon + Q^\varepsilon)^{-1}$ in the $(L_2 \to H^1)$-norm.

Consider equation

$$\widehat{\mathcal{A}}_\varepsilon \mathbf{v}_\varepsilon + Q^\varepsilon \mathbf{v}_\varepsilon = \mathbf{F}, \quad \mathbf{F} \in \mathfrak{G}. \tag{13.5}$$

Now, we apply Theorem 13.1, in order to approximate the solution $\mathbf{v}_\varepsilon$ in $\mathfrak{G}^1$. Let $\mathbf{v}_0$ be the solution of the „homogenized“ equation

$$\widehat{\mathcal{A}}^0 \mathbf{v}_0 + \overline{Q}\mathbf{v}_0 = \mathbf{F}. \tag{13.6}$$

We put

$$\mathbf{v}_\varepsilon^{(1)} = \Lambda^\varepsilon \Pi_\varepsilon^{(m)} b(\mathbf{D})\mathbf{v}_0. \tag{13.7}$$

Estimate (13.1) means that

$$\|\widehat{\mathcal{A}}_\varepsilon^{1/2}(\mathbf{v}_\varepsilon - \mathbf{v}_0 - \varepsilon \mathbf{v}_\varepsilon^{(1)})\|_{\mathfrak{G}} \le \check{\mathcal{C}}_6 \varepsilon \|\mathbf{F}\|_{\mathfrak{G}}, \quad 0 < \varepsilon \le 1.$$

By analogy with the proof of (10.15), it follows that

$$\int_{\mathbb{R}^d} |\mathbf{D}(\mathbf{v}_\varepsilon - \mathbf{v}_0 - \varepsilon \mathbf{v}_\varepsilon^{(1)})|^2\, d\mathbf{x} \le \widehat{c}_*^{-1} \check{\mathcal{C}}_6^2 \varepsilon^2 \|\mathbf{F}\|_{\mathfrak{G}}^2, \quad 0 < \varepsilon \le 1. \tag{13.8}$$

Besides, as it was shown in [BSu2, Theorem 2.4 of Ch. 4], we have

$$\|\mathbf{v}_\varepsilon - \mathbf{v}_0\|_{\mathfrak{G}} \le \varepsilon\, \mathcal{C}_\times \|Q^{-1}\|_{L_\infty} \|\mathbf{F}\|_{\mathfrak{G}}, \quad 0 < \varepsilon \le 1. \tag{13.9}$$

The constant $\mathcal{C}_\times$ is defined by the relations

$$\mathcal{C} = c_*^{-1/2}\left(\beta_1^*(t^0)^{-1} + \beta_2^* c_*^{-1}\delta(t^0)^{-3}\right),$$
$$\mathcal{C}_\times = \max\{\mathcal{C} + 2(3\delta)^{-1}, 2c_*^{-1}(t^0)^{-2}\}.$$

We estimate the norm of the function $\mathbf{v}_\varepsilon^{(1)}$ in $\mathfrak{G}$. By (13.6), we have

$$(g^0 b(\mathbf{D})\mathbf{v}_0, b(\mathbf{D})\mathbf{v}_0)_{\mathfrak{G}_*} + (\overline{Q}\mathbf{v}_0, \mathbf{v}_0)_{\mathfrak{G}} = (\mathbf{F}, \mathbf{v}_0)_{\mathfrak{G}} \leq (\overline{Q}\mathbf{v}_0, \mathbf{v}_0)_{\mathfrak{G}} + \frac{1}{4}((\overline{Q})^{-1}\mathbf{F}, \mathbf{F})_{\mathfrak{G}}.$$

Hence,

$$\|b(\mathbf{D})\mathbf{v}_0\|_{\mathfrak{G}_*}^2 \leq \frac{1}{4}|(g^0)^{-1}||(\overline{Q})^{-1}|\|\mathbf{F}\|_{\mathfrak{G}}^2 \leq \frac{1}{4}\|g^{-1}\|_{L_\infty}\|Q^{-1}\|_{L_\infty}\|\mathbf{F}\|_{\mathfrak{G}}^2. \tag{13.10}$$

Combining this with (10.17), we obtain the following estimate for the norm of the function (13.7):

$$\begin{aligned}\|\mathbf{v}_\varepsilon^{(1)}\|_{\mathfrak{G}} &\leq \|\Lambda^\varepsilon \Pi_\varepsilon^{(m)}\|_{\mathfrak{G}_* \to \mathfrak{G}}\|b(\mathbf{D})\mathbf{v}_0\|_{\mathfrak{G}_*} \\ &\leq m^{1/2}(4r_0)^{-1}\alpha_0^{-1/2}\|g\|_{L_\infty}^{1/2}\|g^{-1}\|_{L_\infty}\|Q^{-1}\|_{L_\infty}^{1/2}\|\mathbf{F}\|_{\mathfrak{G}}.\end{aligned} \tag{13.11}$$

Now, from (13.9) and (13.11) for $0 < \varepsilon \leq 1$ it follows that

$$\begin{aligned}&\|\mathbf{v}_\varepsilon - \mathbf{v}_0 - \varepsilon\mathbf{v}_\varepsilon^{(1)}\|_{\mathfrak{G}} \\ &\quad\leq \left(\mathcal{C}_\times\|Q^{-1}\|_{L_\infty} + m^{1/2}(4r_0)^{-1}\alpha_0^{-1/2}\|g\|_{L_\infty}^{1/2}\|g^{-1}\|_{L_\infty}\|Q^{-1}\|_{L_\infty}^{1/2}\right)\varepsilon\|\mathbf{F}\|_{\mathfrak{G}}.\end{aligned}$$

Combining this with (13.8), we arrive at the inequality

$$\|\mathbf{v}_\varepsilon - \mathbf{v}_0 - \varepsilon\mathbf{v}_\varepsilon^{(1)}\|_{\mathfrak{G}^1} \leq \check{\mathcal{C}}_{10}\varepsilon\|\mathbf{F}\|_{\mathfrak{G}}, \quad 0 < \varepsilon \leq 1,$$

where

$$\check{\mathcal{C}}_{10}^2 = \widehat{c}_*^{-1}\check{\mathcal{C}}_6^2 + \left(\mathcal{C}_\times\|Q^{-1}\|_{L_\infty} + m^{1/2}(4r_0)^{-1}\alpha_0^{-1/2}\|g\|_{L_\infty}^{1/2}\|g^{-1}\|_{L_\infty}\|Q^{-1}\|_{L_\infty}^{1/2}\right)^2. \tag{13.12}$$

Thus, we have proved the following theorem.

Theorem 13.6. *Under the conditions of Theorem* 13.1*, we have*

$$\|(\widehat{\mathcal{A}}_\varepsilon + Q^\varepsilon)^{-1} - (\widehat{\mathcal{A}}^0 + \overline{Q})^{-1} - \varepsilon\Lambda^\varepsilon b(\mathbf{D})(\widehat{\mathcal{A}}^0 + \overline{Q})^{-1}\Pi_\varepsilon\|_{\mathfrak{G} \to \mathfrak{G}^1} \leq \check{\mathcal{C}}_{10}\varepsilon, \quad 0 < \varepsilon \leq 1. \tag{13.13}$$

The constant $\check{\mathcal{C}}_{10}$ is defined by (13.12) *and depends only on m, α_0, α_1, $\|g\|_{L_\infty}$, $\|g^{-1}\|_{L_\infty}$, $\|Q\|_{L_\infty}$, $\|Q^{-1}\|_{L_\infty}$, and on parameters of the lattice Γ.*

Remark 13.7. It can be shown that, as $\varepsilon \to 0$, the weak $(\mathfrak{G}^1)$-limit of the functions $\varepsilon\Lambda^\varepsilon b(\mathbf{D})\Pi_\varepsilon\mathbf{v}_0$ is equal to zero. Therefore, the result of Theorem 13.6 agrees with Theorem 4.4.1(1°) of [BSu2] about the weak $(\mathfrak{G}^1)$-convergence of $\mathbf{v}_\varepsilon$ to $\mathbf{v}_0$.

13.3

Suppose now that Condition 8.4 is satisfied. Then the estimate (13.2) is valid. We put

$$\widetilde{\mathbf{v}}_\varepsilon^{(1)} = \Lambda^\varepsilon b(\mathbf{D})\mathbf{v}_0. \tag{13.14}$$

Estimate (13.2) means that

$$\|\widehat{\mathcal{A}}_\varepsilon^{1/2}(\mathbf{v}_\varepsilon - \mathbf{v}_0 - \varepsilon\widetilde{\mathbf{v}}_\varepsilon^{(1)})\|_{\mathfrak{G}} \leq \check{\mathcal{C}}_7\varepsilon\|\mathbf{F}\|_{\mathfrak{G}}, \quad 0 < \varepsilon \leq 1.$$

It follows that

$$\int_{\mathbb{R}^d} |\mathbf{D}(\mathbf{v}_\varepsilon - \mathbf{v}_0 - \varepsilon\widetilde{\mathbf{v}}_\varepsilon^{(1)})|^2\, d\mathbf{x} \leq \widehat{c}_*^{-1}\check{\mathcal{C}}_7^2\varepsilon^2\|\mathbf{F}\|_{\mathfrak{G}}^2, \quad 0 < \varepsilon \leq 1. \tag{13.15}$$

Under Condition 8.4, by (13.14) and (13.10), we have

$$\|\widetilde{\mathbf{v}}_\varepsilon^{(1)}\|_{\mathfrak{G}} \leq \|\Lambda\|_{L_\infty}\|b(\mathbf{D})\mathbf{v}_0\|_{\mathfrak{G}} \leq \frac{1}{2}\|\Lambda\|_{L_\infty}\|g^{-1}\|_{L_\infty}^{1/2}\|Q^{-1}\|_{L_\infty}^{1/2}\|\mathbf{F}\|_{\mathfrak{G}}. \tag{13.16}$$

Combining this with (13.9), for $0 < \varepsilon \leq 1$ we obtain that

$$\|\mathbf{v}_\varepsilon - \mathbf{v}_0 - \varepsilon\widetilde{\mathbf{v}}_\varepsilon^{(1)}\|_{\mathfrak{G}} \leq \varepsilon\Big(\mathcal{C}_\times\|Q^{-1}\|_{L_\infty} + \frac{1}{2}\|\Lambda\|_{L_\infty}\|g^{-1}\|_{L_\infty}^{1/2}\|Q^{-1}\|_{L_\infty}^{1/2}\Big)\|\mathbf{F}\|_{\mathfrak{G}}. \tag{13.17}$$

Now (13.15) and (13.17) imply that

$$\|\mathbf{v}_\varepsilon - \mathbf{v}_0 - \varepsilon\widetilde{\mathbf{v}}_\varepsilon^{(1)}\|_{\mathfrak{G}^1} \leq \check{\mathcal{C}}_{11}\varepsilon\|\mathbf{F}\|_{\mathfrak{G}}, \quad 0 < \varepsilon \leq 1,$$

where

$$\check{\mathcal{C}}_{11}^2 = \widehat{c}_*^{-1}\check{\mathcal{C}}_7^2 + \Big(\mathcal{C}_\times\|Q^{-1}\|_{L_\infty} + \frac{1}{2}\|\Lambda\|_{L_\infty}\|g^{-1}\|_{L_\infty}^{1/2}\|Q^{-1}\|_{L_\infty}^{1/2}\Big)^2. \tag{13.18}$$

As a result, we obtain the following theorem.

Theorem 13.8. *Suppose that conditions of Theorem* 13.1 *are satisfied. Besides, suppose that Condition* 8.4 *is valid. Then*

$$\|(\widehat{\mathcal{A}}_\varepsilon + Q^\varepsilon)^{-1} - (\widehat{\mathcal{A}}^0 + \overline{Q})^{-1} - \varepsilon\Lambda^\varepsilon b(\mathbf{D})(\widehat{\mathcal{A}}^0 + \overline{Q})^{-1}\|_{\mathfrak{G}\to\mathfrak{G}^1} \leq \check{\mathcal{C}}_{11}\varepsilon, \quad 0 < \varepsilon \leq 1.$$

The constant $\check{\mathcal{C}}_{11}$ *is defined by* (13.18) *and depends only on* m, d, α_0, α_1, $\|g\|_{L_\infty}$, $\|g^{-1}\|_{L_\infty}$, $\|Q\|_{L_\infty}$, $\|Q^{-1}\|_{L_\infty}$, *on parameters of the lattice* Γ, *and also on* $\|\Lambda\|_{L_\infty}$.

13.4. The case of zero corrector

Suppose now that Condition (6.25) is satisfied. Then, by Theorem 13.3, we have

$$\|\widehat{\mathcal{A}}_\varepsilon^{1/2}(\mathbf{v}_\varepsilon - \mathbf{v}_0)\|_{\mathfrak{G}} \leq \check{\mathcal{C}}_6^\circ\varepsilon\|\mathbf{F}\|_{\mathfrak{G}}, \quad 0 < \varepsilon \leq 1,$$

whence

$$\int_{\mathbb{R}^d} |\mathbf{D}(\mathbf{v}_\varepsilon - \mathbf{v}_0)|^2\, d\mathbf{x} \leq \widehat{c}_*^{-1}(\check{\mathcal{C}}_6^\circ)^2\varepsilon^2\|\mathbf{F}\|_{\mathfrak{G}}^2, \quad 0 < \varepsilon \leq 1.$$

Combining this with (13.9), we obtain

$$\|\mathbf{v}_\varepsilon - \mathbf{v}_0\|_{\mathfrak{G}^1} \leq \check{\mathcal{C}}_{12}\varepsilon\|\mathbf{F}\|_{\mathfrak{G}}, \quad 0 < \varepsilon \leq 1, \tag{13.19}$$

where

$$\check{\mathcal{C}}_{12}^2 = \widehat{c}_*^{-1}(\check{\mathcal{C}}_6^\circ)^2 + \mathcal{C}_\times^2\|Q^{-1}\|_{L_\infty}^2. \tag{13.20}$$

We have proved the following theorem.

Theorem 13.9. *Suppose that conditions of Theorem* 13.1 *are satisfied. Besides, suppose that condition* (6.25) *is valid. Then*

$$\|(\widehat{\mathcal{A}}_\varepsilon + Q^\varepsilon)^{-1} - (\widehat{\mathcal{A}}^0 + \overline{Q})^{-1}\|_{\mathfrak{G}\to\mathfrak{G}^1} \le \check{\mathcal{C}}_{12}\varepsilon, \quad 0 < \varepsilon \le 1. \tag{13.21}$$

The constant $\check{\mathcal{C}}_{12}$ *is defined by* (13.20) *and depends on* α_0, α_1, $\|g\|_{L_\infty}$, $\|g^{-1}\|_{L_\infty}$, $\|Q\|_{L_\infty}$, $\|Q^{-1}\|_{L_\infty}$ *and on parameters of the lattice* Γ.

Remark 13.10. In [BSu2, Theorem 4.4.5], it was shown that, if $g^0 = \overline{g}$, the solutions $\mathbf{v}_\varepsilon$ strongly converge to $\mathbf{v}_0$ in $\mathfrak{G}^1$. Theorem 13.9 gives a stronger result (i. e., estimate (13.19)).

§14. Approximation with the three-term corrector for the generalized resolvent $(\widehat{\mathcal{A}}_\varepsilon + Q^\varepsilon)^{-1}$. Interpolation

14.1

By Theorem 13.4, we have

$$\|\widehat{\mathcal{A}}_\varepsilon^{1/2}(\mathbf{v}_\varepsilon - \mathbf{v}_0 - \varepsilon\widetilde{K}_Q(\varepsilon)\mathbf{F})\|_{\mathfrak{G}} \le \check{\mathcal{C}}_8\varepsilon\|\mathbf{F}\|_{\mathfrak{G}}, \quad 0 < \varepsilon \le 1.$$

It follows that

$$\int_{\mathbb{R}^d} |\mathbf{D}(\mathbf{v}_\varepsilon - \mathbf{v}_0 - \varepsilon\widetilde{K}_Q(\varepsilon)\mathbf{F})|^2\, d\mathbf{x} \le \widehat{c}_*^{\,-1}\check{\mathcal{C}}_8^2\varepsilon^2\|\mathbf{F}\|_{\mathfrak{G}}^2, \quad 0 < \varepsilon \le 1. \tag{14.1}$$

We estimate the $\mathfrak{G}$-norm of the function $\widetilde{K}_Q(\varepsilon)\mathbf{F}$. Combining (13.11) with (13.7), we see that

$$\|\Lambda^\varepsilon b(\mathbf{D})(\widehat{\mathcal{A}}^0 + \overline{Q})^{-1}\Pi_\varepsilon\|_{\mathfrak{G}\to\mathfrak{G}} \le m^{1/2}(4r_0)^{-1}\alpha_0^{-1/2}\|g\|_{L_\infty}^{1/2}\|g^{-1}\|_{L_\infty}\|Q^{-1}\|_{L_\infty}^{1/2}. \tag{14.2}$$

The operator $(\widehat{\mathcal{A}}^0 + \overline{Q})^{-1}\Pi_\varepsilon b(\mathbf{D})^*(\Lambda^\varepsilon)^*$ is adjoint in $\mathfrak{G}$ to the operator from (14.2). Hence, it admits the same estimate. Next, similarly to (11.3), we have:

$$\begin{aligned}&\|(\widehat{\mathcal{A}}^0 + \overline{Q})^{-1}b(\mathbf{D})^*L(\mathbf{D})b(\mathbf{D})(\widehat{\mathcal{A}}^0 + \overline{Q})^{-1}\|_{\mathfrak{G}\to\mathfrak{G}}\\ &\quad \le 2^{1/2}r_0^{-1}\alpha_0^{-1/2}\alpha_1^{3/2}\|g\|_{L_\infty}^{3/2}\|g^{-1}\|_{L_\infty}^{1/2}\widehat{c}_*^{\,-3/2}\|Q^{-1}\|_{L_\infty}^{1/2}.\end{aligned} \tag{14.3}$$

Inequality (13.10) means that

$$\|b(\mathbf{D})(\widehat{\mathcal{A}}^0 + \overline{Q})^{-1}\|_{\mathfrak{G}\to\mathfrak{G}_*} \le \frac{1}{2}\|g^{-1}\|_{L_\infty}^{1/2}\|Q^{-1}\|_{L_\infty}^{1/2}. \tag{14.4}$$

Obviously,

$$\|(\widehat{\mathcal{A}}^0 + \overline{Q})^{-1}\|_{\mathfrak{G}\to\mathfrak{G}} \le \|Q^{-1}\|_{L_\infty}. \tag{14.5}$$

From (14.4), (14.5), and (7.10) it follows that

$$\begin{aligned}&\|(\widehat{\mathcal{A}}^0 + \overline{Q})^{-1}(\overline{Q\Lambda})b(\mathbf{D})(\widehat{\mathcal{A}}^0 + \overline{Q})^{-1}\|_{\mathfrak{G}\to\mathfrak{G}}\\ &\quad \le m^{1/2}(4r_0)^{-1}\alpha_0^{-1/2}\|g\|_{L_\infty}^{1/2}\|g^{-1}\|_{L_\infty}\|Q\|_{L_\infty}\|Q^{-1}\|_{L_\infty}^{3/2}.\end{aligned} \tag{14.6}$$

The operator $(\widehat{\mathcal{A}}^0 + \overline{Q})^{-1} b(\mathbf{D})^* (\overline{Q\Lambda})^* (\widehat{\mathcal{A}}^0 + \overline{Q})^{-1}$ admits the same estimate. As a result, relations (13.3), (14.2), (14.3), and (14.6) yield the estimate

$$\begin{aligned}
&\|\widetilde{K}_Q(\varepsilon)\|_{\mathfrak{G}\to\mathfrak{G}} \le m^{1/2}(2r_0)^{-1}\alpha_0^{-1/2}\|g\|_{L_\infty}^{1/2}\|g^{-1}\|_{L_\infty}\|Q^{-1}\|_{L_\infty}^{1/2} \\
&\quad + 2^{1/2} r_0^{-1}\alpha_0^{-1/2}\alpha_1^{3/2}\|g\|_{L_\infty}^{3/2}\|g^{-1}\|_{L_\infty}^{1/2}\widehat{c}_*^{-3/2}\|Q^{-1}\|_{L_\infty}^{1/2} \\
&\quad + m^{1/2}(2r_0)^{-1}\alpha_0^{-1/2}\|g\|_{L_\infty}^{1/2}\|g^{-1}\|_{L_\infty}\|Q\|_{L_\infty}\|Q^{-1}\|_{L_\infty}^{3/2} =: \mathcal{C}_Q.
\end{aligned}$$

Combining this with (13.9), we obtain that

$$\|\mathbf{v}_\varepsilon - \mathbf{v}_0 - \varepsilon \widetilde{K}_Q(\varepsilon)\mathbf{F}\|_{\mathfrak{G}} \le \varepsilon(\mathcal{C}_\times \|Q^{-1}\|_{L_\infty} + \mathcal{C}_Q)\|\mathbf{F}\|_{\mathfrak{G}}, \quad 0 < \varepsilon \le 1.$$

Taking (14.1) into account, we have

$$\|\mathbf{v}_\varepsilon - \mathbf{v}_0 - \varepsilon \widetilde{K}_Q(\varepsilon)\mathbf{F}\|_{\mathfrak{G}^1} \le \check{\mathcal{C}}_{13}\varepsilon\|\mathbf{F}\|_{\mathfrak{G}}, \quad 0 < \varepsilon \le 1,$$

where

$$\check{\mathcal{C}}_{13}^2 = \widehat{c}_*^{-1}\check{\mathcal{C}}_8^2 + (\mathcal{C}_\times\|Q^{-1}\|_{L_\infty} + \mathcal{C}_Q)^2. \tag{14.7}$$

We have proved the following theorem.

Theorem 14.1. *Under the conditions of Theorem* 13.4, *we have*

$$\|(\widehat{\mathcal{A}}_\varepsilon + Q^\varepsilon)^{-1} - (\widehat{\mathcal{A}}^0 + \overline{Q})^{-1} - \varepsilon \widetilde{K}_Q(\varepsilon)\|_{\mathfrak{G}\to\mathfrak{G}^1} \le \check{\mathcal{C}}_{13}\varepsilon, \quad 0 < \varepsilon \le 1. \tag{14.8}$$

The constant $\check{\mathcal{C}}_{13}$ *is defined by* (14.7) *and depends only on* m, α_0, α_1, $\|g\|_{L_\infty}$, $\|g^{-1}\|_{L_\infty}$, $\|Q\|_{L_\infty}$, $\|Q^{-1}\|_{L_\infty}$, *and on parameters of the lattice* Γ.

14.2

Suppose now that Condition 8.4 is satisfied. Then, by Theorem 13.5, we have

$$\|\widehat{\mathcal{A}}_\varepsilon^{1/2}(\mathbf{v}_\varepsilon - \mathbf{v}_0 - \varepsilon K_Q^0(\varepsilon)\mathbf{F})\|_{\mathfrak{G}} \le \check{\mathcal{C}}_9\varepsilon\|\mathbf{F}\|_{\mathfrak{G}}, \quad 0 < \varepsilon \le 1.$$

It follows that

$$\int_{\mathbb{R}^d} |\mathbf{D}(\mathbf{v}_\varepsilon - \mathbf{v}_0 - \varepsilon K_Q^0(\varepsilon)\mathbf{F})|^2\, d\mathbf{x} \le \widehat{c}_*^{-1}\check{\mathcal{C}}_9^2\varepsilon^2\|\mathbf{F}\|_{\mathfrak{G}}^2, \quad 0 < \varepsilon \le 1. \tag{14.9}$$

We estimate the $\mathfrak{G}$-norm of the function $K_Q^0(\varepsilon)\mathbf{F}$, under Condition 8.4. By (13.14), inequality (13.16) means that

$$\|\Lambda^\varepsilon b(\mathbf{D})(\widehat{\mathcal{A}}^0 + \overline{Q})^{-1}\|_{\mathfrak{G}\to\mathfrak{G}} \le \frac{1}{2}\|\Lambda\|_{L_\infty}\|g^{-1}\|_{L_\infty}^{1/2}\|Q^{-1}\|_{L_\infty}^{1/2}. \tag{14.10}$$

The adjoint operator $(\widehat{\mathcal{A}}^0 + \overline{Q})^{-1} b(\mathbf{D})^*(\Lambda^\varepsilon)^*$ satisfies the same estimate. Relations (13.4), (14.3), (14.6), and (14.10) imply the estimate

$$\begin{aligned}
&\|K_Q^0(\varepsilon)\|_{\mathfrak{G}\to\mathfrak{G}} \le \|\Lambda\|_{L_\infty}\|g^{-1}\|_{L_\infty}^{1/2}\|Q^{-1}\|_{L_\infty}^{1/2} \\
&\quad + 2^{1/2} r_0^{-1}\alpha_0^{-1/2}\alpha_1^{3/2}\|g\|_{L_\infty}^{3/2}\|g^{-1}\|_{L_\infty}^{1/2}\widehat{c}_*^{-3/2}\|Q^{-1}\|_{L_\infty}^{1/2} \\
&\quad + m^{1/2}(2r_0)^{-1}\alpha_0^{-1/2}\|g\|_{L_\infty}^{1/2}\|g^{-1}\|_{L_\infty}\|Q\|_{L_\infty}\|Q^{-1}\|_{L_\infty}^{3/2} =: \mathcal{C}_Q^0.
\end{aligned}$$

Together with (13.9), this implies that

$$\|\mathbf{v}_\varepsilon - \mathbf{v}_0 - \varepsilon K_Q^0(\varepsilon)\mathbf{F}\|_{\mathfrak{G}} \le (\mathcal{C}_\times \|Q^{-1}\|_{L_\infty} + \mathcal{C}_Q^0)\varepsilon \|\mathbf{F}\|_{\mathfrak{G}}, \quad 0 < \varepsilon \le 1.$$

Combining this with (14.9), we arrive at the estimate

$$\|\mathbf{v}_\varepsilon - \mathbf{v}_0 - \varepsilon K_Q^0(\varepsilon)\mathbf{F}\|_{\mathfrak{G}^1} \le \check{\mathcal{C}}_{14}\varepsilon \|\mathbf{F}\|_{\mathfrak{G}}, \quad 0 < \varepsilon \le 1,$$

where

$$\check{\mathcal{C}}_{14}^2 = \widehat{c}_*^{-1}\check{\mathcal{C}}_9^2 + (\mathcal{C}_\times \|Q^{-1}\|_{L_\infty} + \mathcal{C}_Q^0)^2. \tag{14.11}$$

Thus, we have proved the following theorem.

Theorem 14.2. *Under the conditions of Theorem* 13.5*, we have*

$$\|(\widehat{\mathcal{A}}_\varepsilon + Q^\varepsilon)^{-1} - (\widehat{\mathcal{A}}^0 + \overline{Q})^{-1} - \varepsilon K_Q^0(\varepsilon)\|_{\mathfrak{G}\to\mathfrak{G}^1} \le \check{\mathcal{C}}_{14}\varepsilon, \quad 0 < \varepsilon \le 1. \tag{14.12}$$

The constant $\check{\mathcal{C}}_{14}$ is defined by (14.11) *and depends on d, m, α_0, α_1, $\|g\|_{L_\infty}$, $\|g^{-1}\|_{L_\infty}$, $\|Q\|_{L_\infty}$, $\|Q^{-1}\|_{L_\infty}$, on parameters of the lattice Γ, and also on $\|\Lambda\|_{L_\infty}$.*

14.3. Interpolation

In [BSu4, Theorem 9.1], it was shown that, under conditions of Theorem 13.4, we have

$$\|(\widehat{\mathcal{A}}_\varepsilon + Q^\varepsilon)^{-1} - (\widehat{\mathcal{A}}^0 + \overline{Q})^{-1} - \varepsilon \widetilde{K}_Q(\varepsilon)\|_{\mathfrak{G}\to\mathfrak{G}} \le \check{\mathfrak{C}}_1\varepsilon^2, \quad 0 < \varepsilon \le 1. \tag{14.13}$$

The constant $\check{\mathfrak{C}}_1$ depends on m, α_0, α_1, $\|g\|_{L_\infty}$, $\|g^{-1}\|_{L_\infty}$, $\|Q\|_{L_\infty}$, $\|Q^{-1}\|_{L_\infty}$, and on parameters of the lattice Γ.

Interpolating between (14.13) and (14.8), we arrive at the following result.

Theorem 14.3. *Under conditions of Theorem* 13.4*, for $0 \le s \le 1$ we have*

$$\|(\widehat{\mathcal{A}}_\varepsilon + Q^\varepsilon)^{-1} - (\widehat{\mathcal{A}}^0 + \overline{Q})^{-1} - \varepsilon \widetilde{K}_Q(\varepsilon)\|_{\mathfrak{G}\to\mathfrak{G}^s} \le \check{\mathfrak{C}}_1^{1-s}\check{\mathcal{C}}_{13}^s\varepsilon^{2-s}, \quad 0 < \varepsilon \le 1. \tag{14.14}$$

Next, if Condition 8.4 is satisfied, then we have (see [BSu4, Theorem 9.3])

$$\|(\widehat{\mathcal{A}}_\varepsilon + Q^\varepsilon)^{-1} - (\widehat{\mathcal{A}}^0 + \overline{Q})^{-1} - \varepsilon K_Q^0(\varepsilon)\|_{\mathfrak{G}\to\mathfrak{G}} \le \check{\mathfrak{C}}_2\varepsilon^2, \quad 0 < \varepsilon \le 1. \tag{14.15}$$

The constant $\check{\mathfrak{C}}_2$ depends on α_0, α_1, $\|g\|_{L_\infty}$, $\|g^{-1}\|_{L_\infty}$, $\|Q\|_{L_\infty}$, $\|Q^{-1}\|_{L_\infty}$, on parameters of the lattice Γ, and also on $\|\Lambda\|_{L_\infty}$.

Interpolating between (14.15) and (14.12), we obtain the following statement.

Theorem 14.4. *Under conditions of Theorem* 13.5*, for $0 \le s \le 1$ we have*

$$\|(\widehat{\mathcal{A}}_\varepsilon + Q^\varepsilon)^{-1} - (\widehat{\mathcal{A}}^0 + \overline{Q})^{-1} - \varepsilon K_Q^0(\varepsilon)\|_{\mathfrak{G}\to\mathfrak{G}^s} \le \check{\mathfrak{C}}_2^{1-s}\check{\mathcal{C}}_{14}^s\varepsilon^{2-s}, \quad 0 < \varepsilon \le 1.$$

Condition (6.25) distinguishes the case, where the corrector is equal to zero. Then estimate (13.21) and also the following estimate (see [BSu4, Theorem 9.2]) are satisfied:

$$\|(\widehat{\mathcal{A}}_\varepsilon + Q^\varepsilon)^{-1} - (\widehat{\mathcal{A}}^0 + \overline{Q})^{-1}\|_{\mathfrak{G}\to\mathfrak{G}} \le \check{\mathfrak{C}}_3\varepsilon^2, \quad 0 < \varepsilon \le 1. \tag{14.16}$$

The constant $\check{\mathfrak{C}}_3$ depends on α_0, α_1, $\|g\|_{L_\infty}$, $\|g^{-1}\|_{L_\infty}$, $\|Q\|_{L_\infty}$, $\|Q^{-1}\|_{L_\infty}$, and on parameters of the lattice Γ.

Interpolating between (14.16) and (13.21), we obtain the following statement.

Theorem 14.5. *Under condition* (6.25)*, for $0 \le s \le 1$ we have*

$$\|(\widehat{\mathcal{A}}_\varepsilon + Q^\varepsilon)^{-1} - (\widehat{\mathcal{A}}^0 + \overline{Q})^{-1}\|_{\mathfrak{G}\to\mathfrak{G}^s} \le \check{\mathfrak{C}}_3^{1-s}\check{\mathcal{C}}_{12}^s\varepsilon^{2-s}, \quad 0 < \varepsilon \le 1.$$

§15. Approximation of the fluxes for the generalized resolvent $(\widehat{\mathcal{A}}_\varepsilon + Q^\varepsilon)^{-1}$

15.1

We consider the fluxes

$$\mathbf{q}_\varepsilon = g^\varepsilon b(\mathbf{D})\mathbf{v}_\varepsilon, \tag{15.1}$$

where $\mathbf{v}_\varepsilon$ is the solution of the equation (13.5). By the scaling transformation, Theorem 9.1 implies that

$$\|\widehat{\mathcal{A}}_\varepsilon^{1/2}\left((\widehat{\mathcal{A}}_\varepsilon + Q^\varepsilon)^{-1} - (I + \varepsilon\Lambda^\varepsilon b(\mathbf{D}))(\widehat{\mathcal{A}}^0 + \overline{Q})^{-1}\Pi_\varepsilon\right)\|_{\mathfrak{G}\to\mathfrak{G}} \le \check{\mathcal{C}}_3\varepsilon, \quad 0 < \varepsilon \le 1.$$

It means that

$$\|\widehat{\mathcal{A}}_\varepsilon^{1/2}(\mathbf{v}_\varepsilon - (I + \varepsilon\Lambda^\varepsilon b(\mathbf{D}))\Pi_\varepsilon\mathbf{v}_0)\|_{\mathfrak{G}} \le \check{\mathcal{C}}_3\varepsilon\|\mathbf{F}\|_{\mathfrak{G}}, \quad 0 < \varepsilon \le 1.$$

Similarly to (12.3), this implies that

$$\|g^\varepsilon b(\mathbf{D})\mathbf{v}_\varepsilon - g^\varepsilon b(\mathbf{D})(\Pi_\varepsilon\mathbf{v}_0 + \varepsilon\Lambda^\varepsilon\Pi_\varepsilon^{(m)}b(\mathbf{D})\mathbf{v}_0)\|_{\mathfrak{G}_*} \le \check{\mathcal{C}}_3\|g\|_{L_\infty}^{1/2}\varepsilon\|\mathbf{F}\|_{\mathfrak{G}}, \quad 0 < \varepsilon \le 1. \tag{15.2}$$

By analogy with (12.4), we have:

$$\begin{aligned} &g^\varepsilon b(\mathbf{D})(\varepsilon\Lambda^\varepsilon\Pi_\varepsilon^{(m)}b(\mathbf{D})\mathbf{v}_0) \\ &\quad = g^\varepsilon(b(\mathbf{D})\Lambda)^\varepsilon\Pi_\varepsilon^{(m)}b(\mathbf{D})\mathbf{v}_0 + \varepsilon g^\varepsilon\sum_{l=1}^d b_l\Lambda^\varepsilon\Pi_\varepsilon^{(m)}D_l(b(\mathbf{D})\mathbf{v}_0). \end{aligned} \tag{15.3}$$

We estimate the second term on the right. Similarly to (12.5), we obtain:

$$\|D_l(b(\mathbf{D})\mathbf{v}_0)\|_{\mathfrak{G}_*} \le \alpha_1^{1/2}\widehat{c}_*^{-1}\|\mathbf{F}\|_{\mathfrak{G}}. \tag{15.4}$$

From (10.17) and (15.4) it follows that

$$\begin{aligned} &\|g^\varepsilon\sum_{l=1}^d b_l\Lambda^\varepsilon\Pi_\varepsilon^{(m)}D_l(b(\mathbf{D})\mathbf{v}_0)\|_{\mathfrak{G}_*} \\ &\quad \le d\,m^{1/2}(2r_0)^{-1}\alpha_0^{-1/2}\alpha_1\widehat{c}_*^{-1}\|g\|_{L_\infty}^{3/2}\|g^{-1}\|_{L_\infty}^{1/2}\|\mathbf{F}\|_{\mathfrak{G}}. \end{aligned}$$

Combining this with (15.1)–(15.3), we have

$$\|\mathbf{q}_\varepsilon - g^\varepsilon(\mathbf{1}_m + (b(\mathbf{D})\Lambda)^\varepsilon)\Pi_\varepsilon^{(m)}b(\mathbf{D})\mathbf{v}_0\|_{\mathfrak{G}_*} \le \check{\mathcal{C}}_{15}\varepsilon\|\mathbf{F}\|_{\mathfrak{G}}, \quad 0 < \varepsilon \le 1,$$

where

$$\check{\mathcal{C}}_{15} = \check{\mathcal{C}}_3\|g\|_{L_\infty}^{1/2} + d\,m^{1/2}(2r_0)^{-1}\alpha_0^{-1/2}\alpha_1\widehat{c}_*^{-1}\|g\|_{L_\infty}^{3/2}\|g^{-1}\|_{L_\infty}^{1/2}. \tag{15.5}$$

We have proved the following theorem.

Theorem 15.1. *Let* $\mathbf{v}_\varepsilon = (\widehat{\mathcal{A}}_\varepsilon + Q^\varepsilon)^{-1}\mathbf{F}$, $\mathbf{v}_0 = (\widehat{\mathcal{A}}^0 + \overline{Q})^{-1}\mathbf{F}$, *where* $\mathbf{F} \in \mathfrak{G}$. *We put* $\mathbf{q}_\varepsilon = g^\varepsilon b(\mathbf{D})\mathbf{v}_\varepsilon$. *Let* $\widetilde{g}$ *be the matrix* (5.5), *and let* $\Pi_\varepsilon^{(m)}$ *be a pseudodifferential operator in* $\mathfrak{G}_*$ *with the symbol* $\chi_{\widetilde{\Omega}/\varepsilon}(\boldsymbol{\xi})$. *Then*

$$\|\mathbf{q}_\varepsilon - \widetilde{g}^\varepsilon\Pi_\varepsilon^{(m)}b(\mathbf{D})\mathbf{v}_0\|_{\mathfrak{G}_*} \le \check{\mathcal{C}}_{15}\varepsilon\|\mathbf{F}\|_{\mathfrak{G}}, \quad 0 < \varepsilon \le 1. \tag{15.6}$$

The constant $\check{\mathcal{C}}_{15}$ *is defined by* (15.5) *and depends only on* d, m, α_0, α_1, $\|g\|_{L_\infty}$, $\|g^{-1}\|_{L_\infty}$, $\|Q\|_{L_\infty}$, $\|Q^{-1}\|_{L_\infty}$, *and on parameters of the lattice* Γ.

Remark 15.2. It can be shown that the functions $\widetilde{g}^\varepsilon\Pi_\varepsilon^{(m)}b(\mathbf{D})\mathbf{v}_0$ weakly converge to $\mathbf{q}_0 = g^0 b(\mathbf{D})\mathbf{v}_0$ in $\mathfrak{G}_*$. Therefore, the result of Theorem 15.1 agrees with Theorem 4.4.1(2°) from [BSu2], where the weak ($\mathfrak{G}_*$)-convergence of the fluxes $\mathbf{q}_\varepsilon$ to $\mathbf{q}_0$ was established.

15.2

Suppose now that Condition 8.4 is satisfied. Then the estimate (13.2) is valid. Hence,

$$\|(g^\varepsilon)^{1/2} b(\mathbf{D})(\mathbf{v}_\varepsilon - \mathbf{v}_0 - \varepsilon \Lambda^\varepsilon b(\mathbf{D})\mathbf{v}_0)\|_{\mathfrak{G}_*} \le \check{\mathcal{C}}_7 \varepsilon \|\mathbf{F}\|_{\mathfrak{G}}, \quad 0 < \varepsilon \le 1.$$

Then

$$\|g^\varepsilon b(\mathbf{D})(\mathbf{v}_\varepsilon - \mathbf{v}_0 - \varepsilon \Lambda^\varepsilon b(\mathbf{D})\mathbf{v}_0)\|_{\mathfrak{G}_*} \le \check{\mathcal{C}}_7 \|g\|^{1/2}_{L_\infty} \varepsilon \|\mathbf{F}\|_{\mathfrak{G}}, \quad 0 < \varepsilon \le 1. \tag{15.7}$$

Similarly to (15.3), we have

$$g^\varepsilon b(\mathbf{D})(\varepsilon \Lambda^\varepsilon b(\mathbf{D})\mathbf{v}_0) = g^\varepsilon (b(\mathbf{D})\Lambda)^\varepsilon b(\mathbf{D})\mathbf{v}_0 + \varepsilon g^\varepsilon \sum_{l=1}^d b_l \Lambda^\varepsilon D_l(b(\mathbf{D})\mathbf{v}_0). \tag{15.8}$$

Under Condition 8.4, (15.4) implies that

$$\|\Lambda^\varepsilon D_l(b(\mathbf{D})\mathbf{v}_0)\|_{\mathfrak{G}_*} \le \|\Lambda\|_{L_\infty} \alpha_1^{1/2} \widehat{c}_*^{-1} \|\mathbf{F}\|_{\mathfrak{G}}. \tag{15.9}$$

Now from (15.1) and (15.7)–(15.9) it follows that

$$\|\mathbf{q}_\varepsilon - \widetilde{g}^\varepsilon b(\mathbf{D})\mathbf{v}_0\|_{\mathfrak{G}_*} \le \check{\mathcal{C}}_{16} \varepsilon \|\mathbf{F}\|_{\mathfrak{G}}, \quad 0 < \varepsilon \le 1, \tag{15.10}$$

where

$$\check{\mathcal{C}}_{16} = \check{\mathcal{C}}_7 \|g\|^{1/2}_{L_\infty} + d\, \alpha_1 \widehat{c}_*^{-1} \|g\|_{L_\infty} \|\Lambda\|_{L_\infty}. \tag{15.11}$$

We have proved the following theorem.

Theorem 15.3. *Suppose that conditions of Theorem*

5.1 *are satisfied. Besides, suppose that Condition* 8.4 *is valid. Then the estimate* (15.10) *holds, where the constant* $\check{\mathcal{C}}_{16}$ *is defined by* (15.11) *and depends on* d, m, α_0, α_1, $\|g\|_{L_\infty}$, $\|g^{-1}\|_{L_\infty}$, $\|Q\|_{L_\infty}$, $\|Q^{-1}\|_{L_\infty}$, *on parameters of the lattice* Γ, *and also on* $\|\Lambda\|_{L_\infty}$.

15.3. The case where $g^0 = \underline{g}$

In this case we have $\widetilde{g}(\mathbf{x}) = g^0 = \underline{g}$. By analogy with Theorem 12.4, Theorem 15.3 implies the following result.

Theorem 15.4. *Let* $g^0 = \underline{g}$, *i. e., conditions*

5.9) *are satisfied. Let* $\mathbf{q}_\varepsilon = g^\varepsilon b(\mathbf{D})\mathbf{v}_\varepsilon$, $\mathbf{q}_0 = g^0 b(\mathbf{D})\mathbf{v}_0$. *Then, as* $\varepsilon \to 0$, *the fluxes* $\mathbf{q}_\varepsilon$ *tend to* $\mathbf{q}_0$ *in the* $\mathfrak{G}_*$*-norm, and*

$$\|\mathbf{q}_\varepsilon - \mathbf{q}_0\|_{\mathfrak{G}_*} \le \check{\mathcal{C}}_{16} \varepsilon \|\mathbf{F}\|_{\mathfrak{G}}, \quad 0 < \varepsilon \le 1.$$

The constant $\check{\mathcal{C}}_{16}$ *depends only on* d, m, n, α_0, α_1, $\|g\|_{L_\infty}$, $\|g^{-1}\|_{L_\infty}$, $\|Q\|_{L_\infty}$, $\|Q^{-1}\|_{L_\infty}$, *and on parameters of the lattice* Γ.

Remark 15.5. Theorem 15.4 strengthens the result of Theorem 4.4.5(2°) of [BSu2], where the strong $\mathfrak{G}_*$-convergence of the fluxes was established under the condition $g^0 = \underline{g}$.

§16. The homogenization results for operators $\mathcal{A}_\varepsilon$

16.1

Now we consider the operator

$$\mathcal{A}_\varepsilon = (f^\varepsilon)^* b(\mathbf{D})^* g^\varepsilon b(\mathbf{D}) f^\varepsilon = (f^\varepsilon)^* \widehat{\mathcal{A}}_\varepsilon f^\varepsilon. \tag{16.1}$$

We have

$$(\mathcal{A}_\varepsilon + I)^{-1} = (f^\varepsilon)^{-1} (\widehat{\mathcal{A}}_\varepsilon + Q^\varepsilon)^{-1} ((f^\varepsilon)^*)^{-1}. \tag{16.2}$$

Here

$$Q(\mathbf{x}) = (f(\mathbf{x}) f(\mathbf{x})^*)^{-1}. \tag{16.3}$$

From Theorem 13.6, miltiplying the operators in (13.13) by $((f^\varepsilon)^*)^{-1}$ from the right, for $0 < \varepsilon \le 1$ we obtain

$$\begin{aligned} &\|(\widehat{\mathcal{A}}_\varepsilon + Q^\varepsilon)^{-1}((f^\varepsilon)^*)^{-1} - (I + \varepsilon \Lambda^\varepsilon b(\mathbf{D}) \Pi_\varepsilon)(\widehat{\mathcal{A}}^0 + \overline{Q})^{-1}((f^\varepsilon)^*)^{-1}\|_{\mathfrak{G} \to \mathfrak{G}^1} \\ &\quad \le \check{\mathcal{C}}_{10} \|f^{-1}\|_{L_\infty} \varepsilon. \end{aligned} \tag{16.4}$$

Relations (16.2) and (16.4) imply the following result.

Theorem 16.1. *Let $\mathcal{A}_\varepsilon$ be the operator* (16.1)*, and let $\widehat{\mathcal{A}}^0 = b(\mathbf{D})^* g^0 b(\mathbf{D})$. Let $Q(\mathbf{x})$ be the matrix defined by* (16.3)*, and let $\overline{Q}$ be the mean value of $Q(\mathbf{x})$ over the cell Ω. Let $\Lambda(\mathbf{x})$ be the matrix defined in Subsection* 5.1*. Let Π_ε be the pseudodifferential operator* (10.4)*. Then for $0 < \varepsilon \le 1$ we have*

$$\|f^\varepsilon(\mathcal{A}_\varepsilon + I)^{-1} - (I + \varepsilon \Lambda^\varepsilon b(\mathbf{D}) \Pi_\varepsilon)(\widehat{\mathcal{A}}^0 + \overline{Q})^{-1}((f^\varepsilon)^*)^{-1}\|_{\mathfrak{G} \to \mathfrak{G}^1} \le \check{\mathcal{C}}_{10} \|f^{-1}\|_{L_\infty} \varepsilon. \tag{16.5}$$

The constant $\check{\mathcal{C}}_{10}$ is defined by (13.12) *and depends on m, α_0, α_1, $\|g\|_{L_\infty}$, $\|g^{-1}\|_{L_\infty}$, $\|f\|_{L_\infty}$, $\|f^{-1}\|_{L_\infty}$, and on parameters of the lattice Γ.*

Now we formulate the result of Theorem 16.1 foe solutions of differential equations. Let $\mathbf{w}_\varepsilon$ be the solution of the equation

$$\mathcal{A}_\varepsilon \mathbf{w}_\varepsilon + \mathbf{w}_\varepsilon = \mathbf{F}, \quad \mathbf{F} \in \mathfrak{G}, \tag{16.6}$$

and let $\mathbf{w}_\varepsilon^0$ be the solution of the equation

$$\widehat{\mathcal{A}}^0 \mathbf{w}_\varepsilon^0 + \overline{Q} \mathbf{w}_\varepsilon^0 = ((f^\varepsilon)^*)^{-1} \mathbf{F}. \tag{16.7}$$

Then (16.5) means that

$$\|f^\varepsilon \mathbf{w}_\varepsilon - \mathbf{w}_\varepsilon^0 - \varepsilon \Lambda^\varepsilon \Pi_\varepsilon^{(m)} b(\mathbf{D}) \mathbf{w}_\varepsilon^0\|_{\mathfrak{G}^1} \le \check{\mathcal{C}}_{10} \|f^{-1}\|_{L_\infty} \varepsilon \|\mathbf{F}\|_{\mathfrak{G}}, \quad 0 < \varepsilon \le 1.$$

Remark 16.2. It can be shown that the functions $\varepsilon \Lambda^\varepsilon \Pi_\varepsilon^{(m)} b(\mathbf{D}) \mathbf{w}_\varepsilon^0$ converge to zero weakly in $\mathfrak{G}^1$. Besides, the functions $\mathbf{w}_\varepsilon^0$ tend to $\mathbf{w}_0 = (\widehat{\mathcal{A}}^0 + \overline{Q})^{-1} (\underline{f^*})^{-1} \mathbf{F}$ strongly in $\mathfrak{G}^1$:

$$(\mathfrak{G}^1)\text{-}\lim_{\varepsilon \to 0} (\mathbf{w}_\varepsilon^0 - \mathbf{w}_0) = 0. \tag{16.8}$$

Indeed,

$$\mathbf{w}_\varepsilon^0 - \mathbf{w}_0 = (\widehat{\mathcal{A}}^0 + \overline{Q})^{-1} \left(((f^\varepsilon)^*)^{-1} - (\underline{f^*})^{-1} \right) \mathbf{F},$$

whence
$$\|\mathbf{w}_\varepsilon^0 - \mathbf{w}_0\|_{\mathfrak{G}^1} \le 2\|(\widehat{\mathcal{A}}^0 + \overline{Q})^{-1}\|_{\mathfrak{G}\to\mathfrak{G}^1}\|f^{-1}\|_{L_\infty}\|\mathbf{F}\|_{\mathfrak{G}}.$$
Then it suffices to check (16.8) for $\mathbf{F} \in C_0^\infty(\mathbb{R}^d;\mathbb{C}^n)$. We fix a function $\zeta \in C_0^\infty(\mathbb{R}^d)$ such that $\mathbf{F}\zeta = \mathbf{F}$. Then
$$\mathbf{w}_\varepsilon^0 - \mathbf{w}_0 = (\widehat{\mathcal{A}}^0 + \overline{Q})^{-1}\zeta\left(((f^\varepsilon)^*)^{-1} - (\underline{f}^*)^{-1}\right)\mathbf{F}.$$
By the „mean value property", the functions $\left(((f^\varepsilon)^*)^{-1} - (\underline{f}^*)^{-1}\right)\mathbf{F}$ tend to zero weakly in $\mathfrak{G}$. Since the operator $(\widehat{\mathcal{A}}^0 + \overline{Q})^{-1}\zeta$ compactly maps $\mathfrak{G}$ to $\mathfrak{G}^1$, then (16.8) holds. From what was said it follows that the result of Theorem 16.1 agrees with the statement of [BSu2, Theorem 4.4.2] about the weak ($\mathfrak{G}^1$)-convergence of the functions $f^\varepsilon\mathbf{w}_\varepsilon$ to $\mathbf{w}_0$:
$$(w,\mathfrak{G}^1)\text{-}\lim_{\varepsilon\to 0} f^\varepsilon\mathbf{w}_\varepsilon = \mathbf{w}_0.$$

16.2

Under Condition 8.4, Theorem 13.8 implies the following result.

Theorem 16.3. *Suppose that conditions of Theorem* 16.1 *are satisfied. Besides, suppose that Condition* 8.4 *is valid. Then for* $0 < \varepsilon \le 1$ *we have*
$$\|f^\varepsilon(\mathcal{A}_\varepsilon + I)^{-1} - (I + \varepsilon\Lambda^\varepsilon b(\mathbf{D}))(\widehat{\mathcal{A}}^0 + \overline{Q})^{-1}((f^\varepsilon)^*)^{-1}\|_{\mathfrak{G}\to\mathfrak{G}^1} \le \check{\mathcal{C}}_{11}\|f^{-1}\|_{L_\infty}\varepsilon. \quad (16.9)$$
The constant $\check{\mathcal{C}}_{11}$ *is defined by* (13.18) *and depends on* m, d, α_0, α_1, $\|g\|_{L_\infty}$, $\|g^{-1}\|_{L_\infty}$, $\|f\|_{L_\infty}$, $\|f^{-1}\|_{L_\infty}$, *and on parameters of the lattice* Γ, *and also on* $\|\Lambda\|_{L_\infty}$.

In terms of solutions, the inequality (16.9) is equivalent to the estimate
$$\|f^\varepsilon\mathbf{w}_\varepsilon - \mathbf{w}_\varepsilon^0 - \varepsilon\Lambda^\varepsilon b(\mathbf{D})\mathbf{w}_\varepsilon^0\|_{\mathfrak{G}^1} \le \check{\mathcal{C}}_{11}\|f^{-1}\|_{L_\infty}\varepsilon\|\mathbf{F}\|_{\mathfrak{G}}, \quad 0 < \varepsilon \le 1. \quad (16.10)$$

16.3. The case of zero corrector

The following result is deduced from Theorem 13.9.

Theorem 16.4. *Suppose that conditions of Theorem* 16.1 *are satisfied. Suppose also that condition* (6.25) *is valid. Then for* $0 < \varepsilon \le 1$ *we have*
$$\|f^\varepsilon(\mathcal{A}_\varepsilon + I)^{-1} - (\widehat{\mathcal{A}}^0 + \overline{Q})^{-1}((f^\varepsilon)^*)^{-1}\|_{\mathfrak{G}\to\mathfrak{G}^1} \le \check{\mathcal{C}}_{12}\|f^{-1}\|_{L_\infty}\varepsilon. \quad (16.11)$$
The constant $\check{\mathcal{C}}_{12}$ *is defined by* (13.20) *and depends on* α_0, α_1, $\|g\|_{L_\infty}$, $\|g^{-1}\|_{L_\infty}$, $\|f\|_{L_\infty}$, $\|f^{-1}\|_{L_\infty}$, *and on parameters of the lattice* Γ.

The estimate (16.11) means that, under condition (6.25), we have
$$\|f^\varepsilon\mathbf{w}_\varepsilon - \mathbf{w}_\varepsilon^0\|_{\mathfrak{G}^1} \le \check{\mathcal{C}}_{12}\|f^{-1}\|_{L_\infty}\varepsilon\|\mathbf{F}\|_{\mathfrak{G}}, \quad 0 < \varepsilon \le 1. \quad (16.12)$$

Remark 16.5. In [BSu2, Ch. 4, Subsection 4.4], it was shown that, if $g^0 = \overline{g}$, then
$$(\mathfrak{G}^1)\text{-}\lim_{\varepsilon\to 0} f^\varepsilon\mathbf{w}_\varepsilon = \mathbf{w}_0, \quad (16.13)$$
where $\mathbf{w}_0 = (\widehat{\mathcal{A}}^0 + \overline{Q})^{-1}(\underline{f}^*)^{-1}\mathbf{F}$. By (16.8), this agrees with (16.12).

16.4. Interpolational results

From Theorem 14.3, multiplying operators in (14.14) by $((f^\varepsilon)^*)^{-1}$ from the right, we deduce the following result.

Theorem 16.6. *Suppose that conditions of Theorem* 16.1 *are satisfied. Let* $\widetilde{K}_Q(\varepsilon)$ *be the operator defined by* (13.3). *Then for* $0 \le s \le 1$ *we have*

$$\begin{aligned}&\|f^\varepsilon(\mathcal{A}_\varepsilon + I)^{-1} - \Big((\widehat{\mathcal{A}}^0 + \overline{Q})^{-1} + \varepsilon \widetilde{K}_Q(\varepsilon)\Big)((f^\varepsilon)^*)^{-1}\|_{\mathfrak{G}\to\mathfrak{G}^s}\\ &\quad\le \check{\mathfrak{C}}_1^{1-s}\check{\mathcal{C}}_{13}^s\|f^{-1}\|_{L_\infty}\varepsilon^{2-s}, \quad 0 < \varepsilon \le 1.\end{aligned}$$

Similarly, Theorem 14.4 implies the following statement.

Theorem 16.7. *Suppose that conditions of Theorem* 16.1 *are satisfied. Besides, let Condition* 8.4 *be valid. Let* $K_Q^0(\varepsilon)$ *be the operator defined by* (13.4). *Then for* $0 \le s \le 1$ *and* $0 < \varepsilon \le 1$ *we have*

$$\begin{aligned}&\|f^\varepsilon(\mathcal{A}_\varepsilon + I)^{-1} - \Big((\widehat{\mathcal{A}}^0 + \overline{Q})^{-1} + \varepsilon K_Q^0(\varepsilon)\Big)((f^\varepsilon)^*)^{-1}\|_{\mathfrak{G}\to\mathfrak{G}^s}\\ &\quad\le \check{\mathfrak{C}}_2^{1-s}\check{\mathcal{C}}_{14}^s\|f^{-1}\|_{L_\infty}\varepsilon^{2-s}.\end{aligned} \tag{16.14}$$

Finally, Theorem 14.5 implies the following result.

Theorem 16.8. *Suppose that conditions of Theorem* 16.1 *are satisfied. Besides, suppose that condition* (6.25) *is valid. Then for* $0 \le s \le 1$ *and* $0 < \varepsilon \le 1$ *we have*

$$\|f^\varepsilon(\mathcal{A}_\varepsilon + I)^{-1} - (\widehat{\mathcal{A}}^0 + \overline{Q})^{-1}((f^\varepsilon)^*)^{-1}\|_{\mathfrak{G}\to\mathfrak{G}^s} \le \check{\mathfrak{C}}_3^{1-s}\check{\mathcal{C}}_{12}^s\|f^{-1}\|_{L_\infty}\varepsilon^{2-s}. \tag{16.15}$$

The estimate (16.15) means that

$$\|f^\varepsilon \mathbf{w}_\varepsilon - \mathbf{w}_\varepsilon^0\|_{\mathfrak{G}^s} \le \check{\mathfrak{C}}_3^{1-s}\check{\mathcal{C}}_{12}^s\|f^{-1}\|_{L_\infty}\varepsilon^{2-s}\|\mathbf{F}\|_{\mathfrak{G}}, \quad 0 < \varepsilon \le 1.$$

16.5. Approximation of the fluxes

Now the role of the flux is played by the vector-valued function

$$\mathbf{r}_\varepsilon = g^\varepsilon b(\mathbf{D}) f^\varepsilon \mathbf{w}_\varepsilon = g^\varepsilon b(\mathbf{D}) f^\varepsilon(\mathcal{A}_\varepsilon + I)^{-1}\mathbf{F}. \tag{16.16}$$

By Theorem 15.1 (see (15.6)),

$$\|g^\varepsilon b(\mathbf{D})(\widehat{\mathcal{A}}_\varepsilon + Q^\varepsilon)^{-1} - \widetilde{g}^\varepsilon \Pi_\varepsilon^{(m)} b(\mathbf{D})(\widehat{\mathcal{A}}^0 + \overline{Q})^{-1}\|_{\mathfrak{G}\to\mathfrak{G}_*} \le \check{\mathcal{C}}_{15}\varepsilon, \quad 0 < \varepsilon \le 1. \tag{16.17}$$

Multiplying operators in (16.17) by $((f^\varepsilon)^*)^{-1}$ from the right and taking (16.2), (16.7), and (16.16) into account, we obtain:

$$\|\mathbf{r}_\varepsilon - \widetilde{g}^\varepsilon \Pi_\varepsilon^{(m)} b(\mathbf{D})\mathbf{w}_\varepsilon^0\|_{\mathfrak{G}_*} \le \check{\mathcal{C}}_{15}\|f^{-1}\|_{L_\infty}\varepsilon\|\mathbf{F}\|_{\mathfrak{G}}, \quad 0 < \varepsilon \le 1. \tag{16.18}$$

We have proved the following result.

Theorem 16.9. *Let* $\mathbf{r}_\varepsilon$ *be defined by* (16.16), *and let* $\mathbf{w}_\varepsilon^0$ *be defined by* (16.7). *Let* $\widetilde{g}$ *be the matrix defined by* (5.5), *and let* $\Pi_\varepsilon^{(m)}$ *be a pseudodifferential operator in* $\mathfrak{G}_*$ *with the symbol* $\chi_{\widetilde{\Omega}/\varepsilon}$. *Then the estimate* (16.18) *is valid.*

Remark 16.10. It can be shown that the functions $\widetilde{g}^\varepsilon \Pi_\varepsilon^{(m)} b(\mathbf{D})\mathbf{w}_\varepsilon^0$ converge to $g^0 b(\mathbf{D})\mathbf{w}_0 =: \mathbf{r}_0$ weakly in $\mathfrak{G}_*$. Therefore, the result of Theorem 16.9 agrees with the statement of Theorem 4.4.2 from [BSu2] about weak $(\mathfrak{G}_*)$-convergence of fluxes $\mathbf{r}_\varepsilon$ to $\mathbf{r}_0$.

Similarly, Theorem 15.3 (see estimate (15.10)) implies the following theorem.

Theorem 16.11. *Suppose that conditions of Theorem* 16.9 *are satisfied. Suppose also that Condition* 8.4 *is valid. Then*

$$\|\mathbf{r}_\varepsilon - \widetilde{g}^\varepsilon b(\mathbf{D})\mathbf{w}_\varepsilon^0\|_{\mathfrak{G}_*} \le \check{\mathcal{C}}_{16}\|f^{-1}\|_{L_\infty}\varepsilon\|\mathbf{F}\|_{\mathfrak{G}}, \quad 0 < \varepsilon \le 1. \tag{16.19}$$

If $g^0 = \underline{g}$, then we have $\widetilde{g} = g^0 = \underline{g}$. Therefore, (16.19) implies the following theorem.

Theorem 16.12. *Let $g^0 = \underline{g}$, i. e., conditions* (5.9) *are satisfied. Suppose that $\mathbf{r}_\varepsilon$ is defined by* (16.16)*, $\mathbf{w}_\varepsilon^0$ is defined by* (16.7)*, and $\mathbf{r}_\varepsilon^0 = g^0 b(\mathbf{D})\mathbf{w}_\varepsilon^0$. Then*

$$\|\mathbf{r}_\varepsilon - \mathbf{r}_\varepsilon^0\|_{\mathfrak{G}_*} \le \check{\mathcal{C}}_{16}\|f^{-1}\|_{L_\infty}\varepsilon\|\mathbf{F}\|_{\mathfrak{G}}, \quad 0 < \varepsilon \le 1. \tag{16.20}$$

Remark 16.13. In [BSu2, Theorem 4.4.8(2°)], it was shown that, under condition $g^0 = \underline{g}$, we have

$$(\mathfrak{G}_*)\text{-}\lim_{\varepsilon\to 0}\mathbf{r}_\varepsilon = \mathbf{r}_0 := g^0 b(\mathbf{D})\mathbf{w}_0.$$

This agrees with estimate (16.20), since, by (16.8), we have

$$(\mathfrak{G}_*)\text{-}\lim_{\varepsilon\to 0}(\mathbf{r}_\varepsilon^0 - \mathbf{r}_0) = (\mathfrak{G}_*)\text{-}\lim_{\varepsilon\to 0} g^0 b(\mathbf{D})(\mathbf{w}_\varepsilon^0 - \mathbf{w}_0) = 0.$$

16.6. Approximation of the generalized resolvent for $\mathcal{A}_\varepsilon$

(Cf. [BSu4, Subsection 9.4].) In conclusion of this section, we consider the question of approximation for the generalized resolvent of $\mathcal{A}_\varepsilon$. Let $\mathfrak{Q}(\mathbf{x})$ be a Γ-periodic positive $(n \times n)$-matrix-valued function such that $\mathfrak{Q}, \mathfrak{Q}^{-1} \in L_\infty$. We factorize $\mathfrak{Q}$ in the form

$$\mathfrak{Q}(\mathbf{x}) = (\varphi(\mathbf{x})\varphi(\mathbf{x})^*)^{-1}, \tag{16.21}$$

where the matrix-valued function φ is Γ-periodic. We use the notation

$$\psi(\mathbf{x}) := f(\mathbf{x})\varphi(\mathbf{x}), \quad Q_*(\mathbf{x}) = (\psi(\mathbf{x})\psi(\mathbf{x})^*)^{-1} = (f(\mathbf{x})^*)^{-1}\mathfrak{Q}(\mathbf{x})(f(\mathbf{x}))^{-1}. \tag{16.22}$$

By the identity

$$(\mathcal{A}_\varepsilon + \mathfrak{Q}^\varepsilon)^{-1} = (f^\varepsilon)^{-1}(\widehat{\mathcal{A}}_\varepsilon + Q_*^\varepsilon)^{-1}((f^\varepsilon)^*)^{-1}, \tag{16.23}$$

it is possible to obtain the results about approximation for $(\mathcal{A}_\varepsilon + \mathfrak{Q}^\varepsilon)^{-1}$ from the corresponding results for the generalized resolvent $(\widehat{\mathcal{A}}_\varepsilon + Q_*^\varepsilon)^{-1}$. The following statement is deduced from Theorem 13.6 (cf. the proof of Theorem 16.1).

Theorem 16.14. *Let $\mathcal{A}_\varepsilon$ be the operator* (16.1)*, and let $\widehat{\mathcal{A}}^0 = b(\mathbf{D})^* g^0 b(\mathbf{D})$. Let $\mathfrak{Q}(\mathbf{x})$ be a Γ-periodic positive $(n \times n)$-matrix-valued function, such that $\mathfrak{Q}, \mathfrak{Q}^{-1} \in L_\infty$. Suppose that relations* (16.21), (16.22) *are valid. Let $\Lambda(\mathbf{x})$ be the matrix defined in Subsection* 5.1*. Let Π_ε be the pseudodifferential operator* (10.4)*. Then for $0 < \varepsilon \le 1$ we have:*

$$\|f^\varepsilon(\mathcal{A}_\varepsilon + \mathfrak{Q}^\varepsilon)^{-1} - (I + \varepsilon\Lambda^\varepsilon b(\mathbf{D})\Pi_\varepsilon)(\widehat{\mathcal{A}}^0 + \overline{Q_*})^{-1}((f^\varepsilon)^*)^{-1}\|_{\mathfrak{G}\to\mathfrak{G}^1} \le \check{\mathcal{C}}_{10}^*\|f^{-1}\|_{L_\infty}\varepsilon. \tag{16.24}$$

The constant $\check{\mathcal{C}}_{10}^*$ is the analog of the constant $\check{\mathcal{C}}_{10}$ with Q replaced by Q_*. It depends on m, α_0, α_1, $\|g\|_{L_\infty}$, $\|g^{-1}\|_{L_\infty}$, $\|f\|_{L_\infty}$, $\|f^{-1}\|_{L_\infty}$, $\|\mathfrak{Q}\|_{L_\infty}$, $\|\mathfrak{Q}^{-1}\|_{L_\infty}$, and on parameters of the lattice Γ.

Let $\mathbf{z}_\varepsilon$ be the solution of the equation

$$\mathcal{A}_\varepsilon \mathbf{z}_\varepsilon + \mathfrak{Q}^\varepsilon \mathbf{z}_\varepsilon = \mathbf{F}, \quad \mathbf{F} \in \mathfrak{G}, \tag{16.25}$$

and let $\mathbf{z}_\varepsilon^0$ be the solution of the equation

$$\widehat{\mathcal{A}}^0 \mathbf{z}_\varepsilon^0 + \overline{Q_*} \mathbf{z}_\varepsilon^0 = ((f^\varepsilon)^*)^{-1} \mathbf{F}. \tag{16.26}$$

The estimate (16.24) means that

$$\| f^\varepsilon \mathbf{z}_\varepsilon - \mathbf{z}_\varepsilon^0 - \varepsilon \Lambda^\varepsilon \Pi_\varepsilon^{(m)} b(\mathbf{D}) \mathbf{z}_\varepsilon^0 \|_{\mathfrak{G}^1} \leq \check{\mathcal{C}}_{10}^* \| f^{-1} \|_{L_\infty} \varepsilon \| \mathbf{F} \|_{\mathfrak{G}}, \quad 0 < \varepsilon \leq 1. \tag{16.27}$$

By analogy with Remark 16.2, it can be shown that (16.27) agrees with the statement of Theorem 4.4.2 from [BSu2] about the weak $(\mathfrak{G}^1)$-convergence of the functions $f^\varepsilon \mathbf{z}_\varepsilon$:

$$(w, \mathfrak{G}^1)\text{-}\lim_{\varepsilon \to 0} f^\varepsilon \mathbf{z}_\varepsilon = \mathbf{z}_0, \tag{16.28}$$

where $\mathbf{z}_0$ is the solution of the equation

$$\widehat{\mathcal{A}}^0 \mathbf{z}_0 + \overline{Q_*} \mathbf{z}_0 = (\underline{f^*})^{-1} \mathbf{F}.$$

By the identity (16.23), Theorem 13.8 yields the following result.

Theorem 16.15. *Suppose that conditions of Theorem* 16.14 *are satisfied. Besides, suppose that Condition* 8.4 *is valid. Then for* $0 < \varepsilon \leq 1$ *we have:*

$$\| f^\varepsilon (\mathcal{A}_\varepsilon + \mathfrak{Q}^\varepsilon)^{-1} - (I + \varepsilon \Lambda^\varepsilon b(\mathbf{D})) (\widehat{\mathcal{A}}^0 + \overline{Q_*})^{-1} ((f^\varepsilon)^*)^{-1} \|_{\mathfrak{G} \to \mathfrak{G}^1} \leq \check{\mathcal{C}}_{11}^* \| f^{-1} \|_{L_\infty} \varepsilon.$$

The constant $\check{\mathcal{C}}_{11}^*$ *is the analog of the constant* $\check{\mathcal{C}}_{11}$ *with* Q *replaced by* Q_**. It depends on* m, α_0, α_1, $\|g\|_{L_\infty}$, $\|g^{-1}\|_{L_\infty}$, $\|f\|_{L_\infty}$, $\|f^{-1}\|_{L_\infty}$, $\|\mathfrak{Q}\|_{L_\infty}$, $\|\mathfrak{Q}^{-1}\|_{L_\infty}$, *on parameters of the lattice* Γ, *and also on* $\|\Lambda\|_{L_\infty}$.

The following theorem is deduced from Theorem 13.9.

Theorem 16.16. *Suppose that conditions of Theorem* 16.14 *are satisfied. Besides, suppose that condition* (6.25) *is valid. Then for* $0 < \varepsilon \leq 1$ *we have*

$$\| f^\varepsilon (\mathcal{A}_\varepsilon + \mathfrak{Q}^\varepsilon)^{-1} - (\widehat{\mathcal{A}}^0 + \overline{Q_*})^{-1} ((f^\varepsilon)^*)^{-1} \|_{\mathfrak{G} \to \mathfrak{G}^1} \leq \check{\mathcal{C}}_{12}^* \| f^{-1} \|_{L_\infty} \varepsilon. \tag{16.29}$$

The constant $\check{\mathcal{C}}_{12}^*$ *is the analog of the constant* $\check{\mathcal{C}}_{12}$ *with* Q *replaced by* Q_**. It depends on* α_0, α_1, $\|g\|_{L_\infty}$, $\|g^{-1}\|_{L_\infty}$, $\|f\|_{L_\infty}$, $\|f^{-1}\|_{L_\infty}$, $\|\mathfrak{Q}\|_{L_\infty}$, $\|\mathfrak{Q}^{-1}\|_{L_\infty}$, *and on parameters of the lattice* Γ.

The estimate (16.29) means that, under condition (6.25), we have

$$\| f^\varepsilon \mathbf{z}_\varepsilon - \mathbf{z}_\varepsilon^0 \|_{\mathfrak{G}^1} \leq \check{\mathcal{C}}_{12}^* \| f^{-1} \|_{L_\infty} \varepsilon \| \mathbf{F} \|_{\mathfrak{G}}, \quad 0 < \varepsilon \leq 1. \tag{16.30}$$

Remark 16.17. Similarly to (16.8), it can be shown that the functions $\mathbf{z}_\varepsilon^0$ strongly converge in $\mathfrak{G}^1$ to $\mathbf{z}_0$, as $\varepsilon \to 0$. Therefore, (16.30) agrees with the statement of Theorem 4.4.8(1°) from [BSu2] about the strong $(\mathfrak{G}^1)$-convergence of the functions $f^\varepsilon \mathbf{z}_\varepsilon$ to $\mathbf{z}_0$ (under the condition $g^0 = \overline{g}$):

$$(\mathfrak{G}^1)\text{-}\lim_{\varepsilon \to 0} f^\varepsilon \mathbf{z}_\varepsilon = \mathbf{z}_0, \quad \text{if } g^0 = \overline{g}.$$

16.7. Interpolational results for $(\mathcal{A}_\varepsilon + \mathfrak{Q}^\varepsilon)^{-1}$

The following result is deduced from (interpolational) Theorem 14.3 and identity (16.23).

Theorem 16.18. *Suppose that the conditions of Theorem* 16.14 *are satisfied. Let* $\widetilde{K}_{Q_*}(\varepsilon)$ *be the corrector* (13.3) *with* Q *replaced by* Q_*. *Then for* $0 \le s \le 1$ *and* $0 < \varepsilon \le 1$ *we have:*

$$\begin{aligned}&\| f^\varepsilon (\mathcal{A}_\varepsilon + \mathfrak{Q}^\varepsilon)^{-1} - \Big((\widehat{\mathcal{A}}^0 + \overline{Q_*})^{-1} + \varepsilon \widetilde{K}_{Q_*}(\varepsilon)\Big) ((f^\varepsilon)^*)^{-1} \|_{\mathfrak{G} \to \mathfrak{G}^s} \\ &\le (\check{\mathfrak{C}}_1^*)^{1-s} (\check{\mathcal{C}}_{13}^*)^s \| f^{-1}\|_{L_\infty} \varepsilon^{2-s}.\end{aligned}$$

Here $\check{\mathfrak{C}}_1^*$ *is the analog of the constant* $\check{\mathfrak{C}}_1$, *and* $\check{\mathcal{C}}_{13}^*$ *is the analog of* $\check{\mathcal{C}}_{13}$ *with* Q *replaced by* Q_*.

Similarly, the following statement is deduced from Theorem 14.4.

Theorem 16.19. *Suppose that conditions of Theorem* 16.14 *are satisfied. Suppose also that Condition* 8.4 *is valid. Let* $K^0_{Q_*}(\varepsilon)$ *be the corrector* (13.4) *with* Q *replaced by* Q_*. *Then for* $0 \le s \le 1$ *and* $0 < \varepsilon \le 1$ *we have:*

$$\begin{aligned}&\| f^\varepsilon (\mathcal{A}_\varepsilon + \mathfrak{Q}^\varepsilon)^{-1} - \Big((\widehat{\mathcal{A}}^0 + \overline{Q_*})^{-1} + \varepsilon K^0_{Q_*}(\varepsilon)\Big) ((f^\varepsilon)^*)^{-1} \|_{\mathfrak{G} \to \mathfrak{G}^s} \\ &\le (\check{\mathfrak{C}}_2^*)^{1-s} (\check{\mathcal{C}}_{14}^*)^s \| f^{-1}\|_{L_\infty} \varepsilon^{2-s}.\end{aligned}$$

Here $\check{\mathfrak{C}}_2^*$ *is the analog of the constant* $\check{\mathfrak{C}}_2$, *and* $\check{\mathcal{C}}_{14}^*$ *is the analog of* $\check{\mathcal{C}}_{14}$ *with* Q *replaced by* Q_*.

Theorem 14.5 leads to the following statement which distinguishes the case where the corrector is equal to zero.

Theorem 16.20. *Suppose that conditions of Theorem* 16.14 *are satisfied. Suppose also that condition* (6.25) *is valid. Then for* $0 \le s \le 1$ *and* $0 < \varepsilon \le 1$ *we have*

$$\| f^\varepsilon (\mathcal{A}_\varepsilon + \mathfrak{Q}^\varepsilon)^{-1} - (\widehat{\mathcal{A}}^0 + \overline{Q_*})^{-1} ((f^\varepsilon)^*)^{-1} \|_{\mathfrak{G} \to \mathfrak{G}^s} \le (\check{\mathfrak{C}}_3^*)^{1-s} (\check{\mathcal{C}}_{12}^*)^s \| f^{-1}\|_{L_\infty} \varepsilon^{2-s}.$$

Here $\check{\mathfrak{C}}_3^*$ *is the analog of the constant* $\check{\mathfrak{C}}_3$ *with* Q *replaced by* Q_*.

16.8. Approximation of the fluxes for $(\mathcal{A}_\varepsilon + \mathfrak{Q}^\varepsilon)^{-1}$

The role of the flux is now played by the vector-valued function

$$g^\varepsilon b(\mathbf{D}) f^\varepsilon \mathbf{z}_\varepsilon = g^\varepsilon b(\mathbf{D}) f^\varepsilon (\mathcal{A}_\varepsilon + \mathfrak{Q}^\varepsilon)^{-1} \mathbf{F} = g^\varepsilon b(\mathbf{D}) (\widehat{\mathcal{A}}_\varepsilon + Q_*^\varepsilon)^{-1} ((f^\varepsilon)^*)^{-1} \mathbf{F}.$$

In the last passage we used the identity (16.23).

Thr following statement is deduced from Theorem 15.1.

Theorem 16.21. *Suppose that conditions of Theorem* 16.14 *are satisfied. Let* $\mathbf{z}_\varepsilon$ *be the solution of the equation* (16.25), *and let* $\mathbf{z}_\varepsilon^0$ *be the solution of the equation* (16.26). *Let* $\widetilde{g}$ *be the matrix defined by* (5.5), *and let* $\Pi_\varepsilon^{(m)}$ *be the pseudodifferential operator in* $\mathfrak{G}_*$ *with the symbol* $\chi_{\widetilde{\Omega}/\varepsilon}$. *Then*

$$\| g^\varepsilon b(\mathbf{D}) f^\varepsilon \mathbf{z}_\varepsilon - \widetilde{g}^\varepsilon \Pi_\varepsilon^{(m)} b(\mathbf{D}) \mathbf{z}_\varepsilon^0 \|_{\mathfrak{G}_*} \le \check{\mathcal{C}}_{15}^* \| f^{-1}\|_{L_\infty} \varepsilon \| \mathbf{F} \|_{\mathfrak{G}}, \quad 0 < \varepsilon \le 1.$$

Here $\check{\mathcal{C}}_{15}^*$ *is the analog of the constant* $\check{\mathcal{C}}_{15}$ *with* Q *replaced by* Q_*.

In can be shown that the functions $\widetilde{g}^\varepsilon \Pi_\varepsilon^{(m)} b(\mathbf{D})\mathbf{z}_\varepsilon^0$ tend to $g^0 b(\mathbf{D})\mathbf{z}_0$ weakly in $\mathfrak{G}_*$. Therefore, the result of Theorem 16.21 agrees with the statement of Theorem 4.4.2 from [BSu2] about the weak $(\mathfrak{G}_*)$-convergence of the fluxes $g^\varepsilon b(\mathbf{D}) f^\varepsilon \mathbf{z}_\varepsilon$ to $g^0 b(\mathbf{D})\mathbf{z}_0$.

Similarly, Theorem 15.3 implies the following theorem.

Theorem 16.22. *Suppose that conditions of Theorem* 16.21 *are satisfied. Suppose also that Condition* 8.4 *is valid. Then*

$$\|g^\varepsilon b(\mathbf{D}) f^\varepsilon \mathbf{z}_\varepsilon - \widetilde{g}^\varepsilon b(\mathbf{D})\mathbf{z}_\varepsilon^0\|_{\mathfrak{G}_*} \le \check{\mathcal{C}}_{16}^* \|f^{-1}\|_{L_\infty} \varepsilon \|\mathbf{F}\|_{\mathfrak{G}}, \quad 0 < \varepsilon \le 1. \tag{16.31}$$

Here $\check{\mathcal{C}}_{16}^*$ *is the analog of the constant* $\check{\mathcal{C}}_{16}$ *with* Q *replaced by* Q_*.

If $g^0 = \underline{g}$, we have $\widetilde{g} = g^0 = \underline{g}$. Therefore, (16.31) implies the following statement.

Theorem 16.23. *Suppose that conditions of Theorem* 16.21 *are satisfied. Let* $g^0 = \underline{g}$, *i. e., conditions* (5.9) *are valid. Then*

$$\|g^\varepsilon b(\mathbf{D}) f^\varepsilon \mathbf{z}_\varepsilon - g^0 b(\mathbf{D})\mathbf{z}_\varepsilon^0\|_{\mathfrak{G}_*} \le \check{\mathcal{C}}_{16}^* \|f^{-1}\|_{L_\infty} \varepsilon \|\mathbf{F}\|_{\mathfrak{G}}, \quad 0 < \varepsilon \le 1.$$

Remark 16.24. As it has already been mentioned, the functions $\mathbf{z}_\varepsilon^0$ converge to $\mathbf{z}_0$ strongly in $\mathfrak{G}^1$ $\mathbf{z}_0$, whence the functions $g^0 b(\mathbf{D})\mathbf{z}_\varepsilon^0$ strongly converge in $\mathfrak{G}_*$ to $g^0 b(\mathbf{D})\mathbf{z}_0$. Therefore, the statement of Theorem 16.23 agrees with the result of [BSu2, Theorem 4.4.8(2°)] about the strong $(\mathfrak{G}_*)$-convergence of the fluxes $g^\varepsilon b(\mathbf{D}) f^\varepsilon \mathbf{z}_\varepsilon$ to $g^0 b(\mathbf{D})\mathbf{z}_0$ (under the condition $g^0 = \underline{g}$).

Chapter 4. Applications

We proceed to applications of the general results to specific periodic operators of mathematical physics. All the examples considered below have been studied before in [BSu4], where approximation for the resolvent in the L_2-operator norm with the three-term corrector was found. Now we obtain approximation for the resolvent in the operator norm from L_2 to H^1. We obtain also the interpolational results and approximation for the fluxes in L_2.

§17. The operator $\widehat{\mathcal{A}} = \mathbf{D}^* g \mathbf{D}$

17.1. The case where the matrix $g(\mathbf{x})$ has real entries

We consider the operator

$$\widehat{\mathcal{A}} = \mathbf{D}^* g(\mathbf{x})\mathbf{D} = -\operatorname{div} g(\mathbf{x})\nabla, \tag{17.1}$$

acting in $\mathfrak{G} = L_2(\mathbb{R}^d)$, $d \ge 1$ (cf. [BSu2, §5.1] and [BSu4, §10]). Here $g(\mathbf{x})$ is a Γ-periodic $(d \times d)$-matrix-valued function *with real entries* and such that

$$g(\mathbf{x}) > 0, \quad g, g^{-1} \in L_\infty. \tag{17.2}$$

The operator (17.1) describes a periodic acoustical medium; this operator is also useful in diffusion problems, etc. For us this example is also important as a basic object for the study of the periodic Schrödinger operator. Now we have $n = 1$, $m = d$, $b(\boldsymbol{\xi}) = \boldsymbol{\xi}$,

$\alpha_0 = \alpha_1 = 1$. The solutions $v_j \in \widetilde{H}^1(\Omega)$ of the equation (5.4) with $\mathbf{C} = \mathbf{e}_j$ are pure imaginary. Therefore, it is convenient to consider the solutions $\Phi_j \in \widetilde{H}^1(\Omega)$ of the problem

$$\operatorname{div} g(\mathbf{x})(\nabla\Phi_j(\mathbf{x}) + \mathbf{e}_j) = 0, \quad \int_\Omega \Phi_j(\mathbf{x})\, d\mathbf{x} = 0, \tag{17.3}$$

$j = 1, \dots, d$. Then $\Phi_j(\mathbf{x})$ are real-valued functions, and $v_j(\mathbf{x}) = i\Phi_j(\mathbf{x})$. Herewith, $\Lambda(\mathbf{x})$ is a row-matrix:

$$\Lambda(\mathbf{x}) = i(\Phi_1(\mathbf{x}), \dots, \Phi_d(\mathbf{x})),$$

$\widetilde{g}(\mathbf{x})$ is the real $(d \times d)$-matrix with the columns $g(\mathbf{x})(\nabla\Phi_j(\mathbf{x}) + \mathbf{e}_j)$, $j = 1, \dots, d$, and the effective matrix g^0 is defined by

$$g^0 = |\Omega|^{-1} \int_\Omega \widetilde{g}(\mathbf{x})\, d\mathbf{x}.$$

Next, $\Lambda b(\mathbf{D}) = \Lambda \mathbf{D} = \sum_{j=1}^d \Phi_j(\mathbf{x})\partial_j$. For the solutions Φ_j, we have $\Phi_j \in L_\infty$, and the norms $\|\Phi_j\|_{L_\infty}$ are estimated by the constant depending only on $\|g\|_{L_\infty}$, $\|g^{-1}\|_{L_\infty}$, on d and on parameters of the lattice Γ (see Remark 8.8).

We consider the operator $\widehat{\mathcal{A}}_\varepsilon = \mathbf{D}^* g^\varepsilon(\mathbf{x})\mathbf{D}$ with rapidly oscillating matrix $g^\varepsilon(\mathbf{x})$. Now Condition 8.6(2°), and then also Condition 8.4 is satisfied. Theorem 10.8 is applicable. By Remark 8.8, this theorem leads to the following statement.

Theorem 17.1. *Let $\widehat{\mathcal{A}}(g) = \mathbf{D}^* g(\mathbf{x})\mathbf{D}$, where $g(\mathbf{x})$ is a Γ-periodic matrix with real entries satisfying* (17.2)*, and let $\widehat{\mathcal{A}}^0 = \mathbf{D}^* g^0 \mathbf{D}$ be the effective operator. Let $\widehat{\mathcal{A}}_\varepsilon = \widehat{\mathcal{A}}(g^\varepsilon)$. Let $\Phi_j(\mathbf{x})$ be the Γ-periodic solution of the problem* (17.3)*, $j = 1, \dots, d$. Then*

$$\|(\widehat{\mathcal{A}}_\varepsilon + I)^{-1} - (\widehat{\mathcal{A}}^0 + I)^{-1} - \varepsilon \sum_{j=1}^d \Phi_j^\varepsilon \partial_j (\widehat{\mathcal{A}}^0 + I)^{-1}\|_{L_2(\mathbb{R}^d) \to H^1(\mathbb{R}^d)} \le \mathcal{C}_{11}\varepsilon, \quad 0 < \varepsilon \le 1, \tag{17.4}$$

where the constant $\mathcal{C}_{11}$ depends only on $\|g\|_{L_\infty}$, $\|g^{-1}\|_{L_\infty}$, on d, and on parameters of the lattice Γ.

We can also apply (interpolational) Theorem 11.4. Now the corrector $K^0(\varepsilon)$ (see (10.11)) takes the form

$$K^0(\varepsilon) = \sum_{j=1}^d \Phi_j^\varepsilon \partial_j (\widehat{\mathcal{A}}^0 + I)^{-1} - \sum_{j=1}^d (\widehat{\mathcal{A}}^0 + I)^{-1} \partial_j \Phi_j^\varepsilon, \tag{17.5}$$

since the third term of the corrector (10.11) for the operator (17.1) is equal to zero (see [BSu4, Proposition 8.4]). We arrive at the following result.

Theorem 17.2. *Suppose that conditions of Theorem* 17.1 *are satisfied. Let $K^0(\varepsilon)$ be the corrector defined by* (17.5)*. Then for $0 \le s \le 1$ we have*

$$\|(\widehat{\mathcal{A}}_\varepsilon + I)^{-1} - (\widehat{\mathcal{A}}^0 + I)^{-1} - \varepsilon K^0(\varepsilon)\|_{L_2(\mathbb{R}^d) \to H^s(\mathbb{R}^d)} \le C_s^0 \varepsilon^{2-s}, \quad 0 < \varepsilon \le 1,$$

where the constant C_s^0 depends on s, $\|g\|_{L_\infty}$, $\|g^{-1}\|_{L_\infty}$, on d, and on parameters of the lattice Γ.

Now we formulate the result of Theorem 17.1 in terms of solutions. Let u_ε be the solution of the equation

$$-\operatorname{div} g^\varepsilon(\mathbf{x})\nabla u_\varepsilon + u_\varepsilon = F, \quad F \in L_2(\mathbb{R}^d), \tag{17.6}$$

and let u_0 be the solution of the „homogenized" equation

$$-\operatorname{div} g^0 \nabla u_0 + u_0 = F. \tag{17.7}$$

Then (17.4) means that

$$\|u_\varepsilon - u_0 - \varepsilon \sum_{j=1}^d \Phi_j^\varepsilon \partial_j u_0\|_{H^1(\mathbb{R}^d)} \le \mathcal{C}_{11}\varepsilon \|F\|_{L_2(\mathbb{R}^d)}, \quad 0 < \varepsilon \le 1.$$

Note that u_ε weakly converges in $H^1(\mathbb{R}^d)$ to u_0, as $\varepsilon \to 0$ (see Remark 10.7).

Theorem 12.3 gives the following result about convergence of the fluxes.

Theorem 17.3. *Suppose that conditions of Theorem* 17.1 *are satisfied. Let u_ε be the solution of the equation* (17.6), *and let u_0 be the solution of the equation* (17.7). *Let $\widetilde{g}(\mathbf{x})$ be the matrix with the columns $g(\mathbf{x})(\nabla \Phi_j(\mathbf{x}) + \mathbf{e}_j)$, $j = 1, \dots, d$. Then*

$$\|g^\varepsilon \nabla u_\varepsilon - \widetilde{g}^\varepsilon \nabla u_0\|_{L_2(\mathbb{R}^d;\mathbb{C}^d)} \le \mathcal{C}_{16}\varepsilon \|F\|_{L_2(\mathbb{R}^d)}, \quad 0 < \varepsilon \le 1,$$

where the constant $\mathcal{C}_{16}$ depends on $\|g\|_{L_\infty}$, $\|g^{-1}\|_{L_\infty}$, on d, and on parameters of the lattice Γ.

Note that the fluxes $g^\varepsilon \nabla u_\varepsilon$ weakly converge in $L_2(\mathbb{R}^d;\mathbb{C}^d)$ to $g^0 \nabla u_0$ as $\varepsilon \to 0$ (see Remark 12.2).

17.2. Approximation of the generalized resolvent

Let $Q(\mathbf{x})$ be a real-valued Γ-periodic function such that

$$Q(\mathbf{x}) > 0, \quad Q, Q^{-1} \in L_\infty. \tag{17.8}$$

We consider a question about approximation of the generalized resolvent

$$(\mathbf{D}^* g^\varepsilon \mathbf{D} + Q^\varepsilon)^{-1}.$$

We apply Theorem 13.8, which leads to the following result.

Theorem 17.4. *Suppose that conditions of Theorem* 17.1 *are satisfied. Let $Q(\mathbf{x})$ be a Γ-periodic function satisfying condition* (17.8), *and let $\overline{Q}$ be the mean value of $Q(\mathbf{x})$ over the cell Ω. Let $Q^\varepsilon(\mathbf{x}) = Q(\varepsilon^{-1}\mathbf{x})$. Then*

$$\|(\widehat{\mathcal{A}}_\varepsilon + Q^\varepsilon)^{-1} - (\widehat{\mathcal{A}}^0 + \overline{Q})^{-1} - \varepsilon \sum_{j=1}^d \Phi_j^\varepsilon \partial_j (\widehat{\mathcal{A}}^0 + \overline{Q})^{-1}\|_{L_2(\mathbb{R}^d)\to H^1(\mathbb{R}^d)} \le \check{\mathcal{C}}_{11}\varepsilon,$$
$$0 < \varepsilon \le 1, \tag{17.9}$$

where the constant $\check{\mathcal{C}}_{11}$ depends only on $\|g\|_{L_\infty}$, $\|g^{-1}\|_{L_\infty}$, $\|Q\|_{L_\infty}$, $\|Q^{-1}\|_{L_\infty}$, on d, and on parameters of the lattice Γ.

We can also apply (interpolational) Theorem 14.4. Now the corrector $K_Q^0(\varepsilon)$ (see (13.4)) takes the form

$$K_Q^0(\varepsilon) = \sum_{j=1}^{d} \Phi_j^\varepsilon \partial_j (\widehat{\mathcal{A}}^0 + \overline{Q})^{-1} - \sum_{j=1}^{d} (\widehat{\mathcal{A}}^0 + \overline{Q})^{-1} \partial_j \Phi_j^\varepsilon, \qquad (17.10)$$

since the third term of the corrector (13.4) for the operator (17.1) is equal to zero (see [BSu4, Proposition 9.4]).

Theorem 17.5. *Suppose that conditions of Theorem* 17.4 *are satisfied. Let $K_Q^0(\varepsilon)$ be the corrector defined by* (17.10)*. Then for $0 \leq s \leq 1$ we have*

$$\|(\widehat{\mathcal{A}}_\varepsilon + Q^\varepsilon)^{-1} - (\widehat{\mathcal{A}}^0 + \overline{Q})^{-1} - \varepsilon K_Q^0(\varepsilon)\|_{L_2(\mathbb{R}^d)\to H^s(\mathbb{R}^d)} \leq C_{Q,s}^0 \varepsilon^{2-s}, \quad 0 < \varepsilon \leq 1, \quad (17.11)$$

where the constant $C_{Q,s}^0$ depends on s, $\|g\|_{L_\infty}$, $\|g^{-1}\|_{L_\infty}$, $\|Q\|_{L_\infty}$, $\|Q^{-1}\|_{L_\infty}$, on d, and on parameters of the lattice Γ.

Now we formulate the result of Theorem 17.4 in terms of solutions. Let v_ε be the solution of the equation

$$-\operatorname{div} g^\varepsilon(\mathbf{x})\nabla v_\varepsilon + Q^\varepsilon v_\varepsilon = F, \quad F \in L_2(\mathbb{R}^d), \qquad (17.12)$$

and let v_0 be the solution of the „homogenized" equation

$$-\operatorname{div} g^0 \nabla v_0 + \overline{Q} v_0 = F. \qquad (17.13)$$

Then (17.9) means that

$$\|v_\varepsilon - v_0 - \varepsilon \sum_{j=1}^{d} \Phi_j^\varepsilon \partial_j v_0\|_{H^1(\mathbb{R}^d)} \leq \check{\mathcal{C}}_{11} \varepsilon \|F\|_{L_2(\mathbb{R}^d)}, \quad 0 < \varepsilon \leq 1.$$

Herewith, v_ε weakly converges in $H^1(\mathbb{R}^d)$ to v_0 as $\varepsilon \to 0$ (see Remark 13.7).

Theorem 15.3 gives the following result about approximation of the fluxes.

Theorem 17.6. *Suppose that conditions of Theorem* 17.4 *are satisfied. Let v_ε be the solution of the equation* (17.12)*, and let v_0 be the solution of the equation* (17.13)*. Let $\widetilde{g}(\mathbf{x})$ be the matrix with the columns $g(\mathbf{x})(\nabla\Phi_j(\mathbf{x}) + \mathbf{e}_j)$, $j = 1, \dots, d$. Then*

$$\|g^\varepsilon \nabla v_\varepsilon - \widetilde{g}^\varepsilon \nabla v_0\|_{L_2(\mathbb{R}^d;\mathbb{C}^d)} \leq \check{\mathcal{C}}_{16} \varepsilon \|F\|_{L_2(\mathbb{R}^d)}, \quad 0 < \varepsilon \leq 1, \qquad (17.14)$$

where $\check{\mathcal{C}}_{16}$ depends on $\|g\|_{L_\infty}$, $\|g^{-1}\|_{L_\infty}$, $\|Q\|_{L_\infty}$, $\|Q^{-1}\|_{L_\infty}$, on d, and on parameters of the lattice Γ.

As $\varepsilon \to 0$, the weak $(L_2(\mathbb{R}^d;\mathbb{C}^d))$-limit of the fluxes $g^\varepsilon \nabla v_\varepsilon$ is equal to $g^0 \nabla v_0$ (cf. Remark 15.2).

The special cases ($g^0 = \overline{g}$ and $g^0 = \underline{g}$) are considered below in Subsection 17.7 in the general case of the matrix $g(\mathbf{x})$ with complex entries.

17.3. The case where the matrix $g(\mathbf{x})$ has complex entries

Now we consider operator (17.1) assuming that $g(\mathbf{x})$ is a periodic Hermitian matrix with complex entries and such that condition (17.2) is satisfied. Now the solutions $\Phi_j(\mathbf{x})$ of the problem (17.3) are complex-valued functions. Theorem 10.6 is applicable. Unlike the case of the real matrix $g(\mathbf{x})$, now we cannot rely on the boundedness of the solutions $\Phi_j(\mathbf{x})$. Therefore, in general case, we cannot replace Π_ε by I in (10.21). Obviously, such replacement is possible for $d \le 2$ (as well as for the general matrix operators), and also if $g^0 = \underline{g}$. Besides, the boundedness of the solutions $\Phi_j(\mathbf{x})$ is preserved under some additional conditions on $\operatorname{Im} g(\mathbf{x})$. We write the matrix $g(\mathbf{x})$ as

$$g(\mathbf{x}) = g_1(\mathbf{x}) + i g_2(\mathbf{x}),$$

where $g_1(\mathbf{x})$ is a symmetric matrix with real entries and $g_2(\mathbf{x})$ is an antisymmetric matrix with real entries. The solutions $\Phi_j(\mathbf{x})$ of the problem (17.3) are also represented as $\Phi_j(\mathbf{x}) = \Phi_j^{(1)}(\mathbf{x}) + i\Phi_j^{(2)}(\mathbf{x})$, where $\Phi_j^{(1)}(\mathbf{x})$, $\Phi_j^{(2)}(\mathbf{x})$ are real-valued functions. Then the problem (17.3) can be rewritten as the system of equations with real coefficients for $\Phi_j^{(1)}$ and $\Phi_j^{(2)}$:

$$\left.\begin{aligned} \operatorname{div} g_1(\mathbf{x})\nabla\Phi_j^{(1)}(\mathbf{x}) - \sum_{l=1}^{d} \mathfrak{A}_l(\mathbf{x})\partial_l \Phi_j^{(2)}(\mathbf{x}) &= -\operatorname{div} g_1(\mathbf{x})\mathbf{e}_j \\ \sum_{l=1}^{d} \mathfrak{A}_l(\mathbf{x})\partial_l \Phi_j^{(1)}(\mathbf{x}) + \operatorname{div} g_1(\mathbf{x})\nabla\Phi_j^{(2)}(\mathbf{x}) &= -\operatorname{div} g_2(\mathbf{x})\mathbf{e}_j \end{aligned}\right\} \tag{17.15}$$

under the conditions $\int_\Omega \Phi_j^{(1)}(\mathbf{x})\,d\mathbf{x} = \int_\Omega \Phi_j^{(2)}(\mathbf{x})\,d\mathbf{x} = 0$. Here

$$\mathfrak{A}_l(\mathbf{x}) = \sum_{k=1}^{d} \partial_k g_2^{(kl)}(\mathbf{x}) = \operatorname{div} \mathbf{g}_2^{(l)}(\mathbf{x}), \quad l = 1, \dots, d,$$

where $\mathbf{g}_2^{(l)}(\mathbf{x})$ are the columns of the matrix $g_2(\mathbf{x})$. *Suppose that, for some $q > d$, we have*

$$\operatorname{div} \mathbf{g}_2^{(l)} \in L_q(\Omega), \quad q > d, \ l = 1, \dots, d. \tag{17.16}$$

The system (17.15) is a system „with the diagonal principal part“. It satisfies conditions of Theorem 2.1 from [LaU, Ch. VII, §2]. By this theorem, $\Phi_j \in L_\infty$, and the norm $\|\Phi_j\|_{L_\infty}$ is estimated by the constant which depends on $\|g_1\|_{L_\infty}$, $\|g_1^{-1}\|_{L_\infty}$, on d, Ω, and on the norms $\|\mathfrak{A}_l\|_{L_q(\Omega)}$, $l = 1, \dots, d$. Thus, conditions (17.16) guarantee that Condition 8.4 is satisfied, and then Theorem 10.8 is applicable.

Applying Theorems 10.6 and 10.8 to the operator (17.1), we arrive at the following statement.

Theorem 17.7. 1) *Let $\widehat{\mathcal{A}}(g) = \mathbf{D}^* g(\mathbf{x})\mathbf{D}$, where $g(\mathbf{x})$ is a Γ-periodic matrix-valued function with complex entries satisfying* (17.2), *and let $\widehat{\mathcal{A}}^0 = \widehat{\mathcal{A}}(g^0)$ be the effective operator. Let $\widehat{\mathcal{A}}_\varepsilon = \widehat{\mathcal{A}}(g^\varepsilon)$. Let $\Phi_j \in \widetilde{H}^1(\Omega)$ be the solution of the problem* (17.3), *$j = 1, \dots, d$. Let Π_ε be the operator defined by* (10.4). *Then*

$$\|(\widehat{\mathcal{A}}_\varepsilon + I)^{-1} - (\widehat{\mathcal{A}}^0 + I)^{-1} - \varepsilon \sum_{j=1}^{d} \Phi_j^\varepsilon \partial_j (\widehat{\mathcal{A}}^0 + I)^{-1}\Pi_\varepsilon\|_{L_2(\mathbb{R}^d)\to H^1(\mathbb{R}^d)} \le \mathcal{C}_{10}\varepsilon,$$
$$0 < \varepsilon \le 1,$$

where the constant $\mathcal{C}_{10}$ *depends only on* $\|g\|_{L_\infty}$, $\|g^{-1}\|_{L_\infty}$, *on* d, *and on parameters of the lattice* Γ.

2) *Suppose that assumptions of statement* 1) *are satisfied. Suppose also that* $\Phi_j \in L_\infty$, $j = 1, \dots, d$. (*The latter condition is a fortiori valid if* $d \le 2$, *or if* $g^0 = \underline{g}$, *or if the columns* $\mathbf{g}_2^{(l)}(\mathbf{x})$, $l = 1, \dots, d$, *of the matrix* $g_2(\mathbf{x}) = \operatorname{Im} g(\mathbf{x})$ *satisfy condition* (17.16).) *Then*

$$\|(\widehat{\mathcal{A}}_\varepsilon + I)^{-1} - (\widehat{\mathcal{A}}^0 + I)^{-1} - \varepsilon \sum_{j=1}^{d} \Phi_j^\varepsilon \partial_j (\widehat{\mathcal{A}}^0 + I)^{-1}\|_{L_2(\mathbb{R}^d) \to H^1(\mathbb{R}^d)} \le \mathcal{C}_{11}\varepsilon, \quad 0 < \varepsilon \le 1,$$

where the constant $\mathcal{C}_{11}$ *depends only on* $\|g\|_{L_\infty}$, $\|g^{-1}\|_{L_\infty}$, *on* d, *on parameters of the lattice* Γ, *and also on the norms* $\|\Phi_j\|_{L_\infty}$, $j = 1, \dots, d$.

Note that, if $d \le 2$, or if $g^0 = \underline{g}$, the norms $\|\Phi_j\|_{L_\infty}$ themselves can be estimated by the constant depending only on the norms $\|g\|_{L_\infty}$, $\|g^{-1}\|_{L_\infty}$, on d, and on parameters of the lattice. While, under condition (17.16), the norms $\|\Phi_j\|_{L_\infty}$ depend also on the norms $\|\operatorname{div} \mathbf{g}_2^{(l)}\|_{L_q(\Omega)}$, $l = 1, \dots, d$.

17.4. Interpolational results

We can apply (interpolational) Theorem 11.3 to the operator (17.1) with complex matrix $g(\mathbf{x})$, while, under the condition $\Phi_j \in L_\infty$, $j = 1, \dots, d$, Theorem 11.4 is applicable. Now the third term of the correctors $\widetilde{K}(\varepsilon)$ and $K^0(\varepsilon)$, in general, is non-zero. The corrector (10.9) (see [BSu4, Subsection 10.3]) takes the form

$$\widetilde{K}(\varepsilon) = \sum_{j=1}^{d} \Phi_j^\varepsilon \partial_j (\widehat{\mathcal{A}}^0 + I)^{-1} \Pi_\varepsilon - \sum_{j=1}^{d} (\widehat{\mathcal{A}}^0 + I)^{-1} \Pi_\varepsilon \partial_j (\Phi_j^\varepsilon)^* - \sum_{j,l,s=1}^{d} (\widehat{\mathcal{A}}^0 + I)^{-1} (a_{jls} - a_{jls}^*) \partial_j \partial_l \partial_s (\widehat{\mathcal{A}}^0 + I)^{-1}, \tag{17.17}$$

where

$$a_{jls} = |\Omega|^{-1} \int_\Omega \Phi_j(\mathbf{x})^* \langle g(\mathbf{x})(\nabla \Phi_l(\mathbf{x}) + \mathbf{e}_l), \mathbf{e}_s \rangle \, d\mathbf{x}, \quad j, l, s = 1, \dots, d. \tag{17.18}$$

The corrector (10.11) is given by

$$K^0(\varepsilon) = \sum_{j=1}^{d} \Phi_j^\varepsilon \partial_j (\widehat{\mathcal{A}}^0 + I)^{-1} - \sum_{j=1}^{d} (\widehat{\mathcal{A}}^0 + I)^{-1} \partial_j (\Phi_j^\varepsilon)^* - \sum_{j,l,s=1}^{d} (\widehat{\mathcal{A}}^0 + I)^{-1} (a_{jls} - a_{jls}^*) \partial_j \partial_l \partial_s (\widehat{\mathcal{A}}^0 + I)^{-1}. \tag{17.19}$$

Applying Theorems 11.3 and 11.4, we arrive at the following result.

Theorem 17.8. 1) *Suppose that conditions of Theorem* 17.7(1) *are satisfied. Let* $\widetilde{K}(\varepsilon)$ *be the corrector defined by* (17.17), (17.18). *Then for* $0 \le s \le 1$ *we have*

$$\|(\widehat{\mathcal{A}}_\varepsilon + I)^{-1} - (\widehat{\mathcal{A}}^0 + I)^{-1} - \varepsilon \widetilde{K}(\varepsilon)\|_{L_2(\mathbb{R}^d) \to H^s(\mathbb{R}^d)} \le C_s \varepsilon^{2-s}, \quad 0 < \varepsilon \le 1,$$

where the constant C_s *depends on* s, *on* $\|g\|_{L_\infty}$, $\|g^{-1}\|_{L_\infty}$, *on* d, *and on parameters of the lattice* Γ.

2) *Suppose that conditions of Theorem* 17.7(2) *are satisfied. Let* $K^0(\varepsilon)$ *be the corrector defined by* (17.19). *Then for* $0 \le s \le 1$ *we have*

$$\|(\widehat{\mathcal{A}}_\varepsilon + I)^{-1} - (\widehat{\mathcal{A}}^0 + I)^{-1} - \varepsilon K^0(\varepsilon)\|_{L_2(\mathbb{R}^d) \to H^s(\mathbb{R}^d)} \le C_s^0 \varepsilon^{2-s}, \quad 0 < \varepsilon \le 1,$$

where the constant C_s^0 *depends on* s, *on* $\|g\|_{L_\infty}$, $\|g^{-1}\|_{L_\infty}$, *on* d, *on parameters of the lattice* Γ, *and also on the norms* $\|\Phi_j\|_{L_\infty}$, $j = 1, \dots, d$.

17.5. Approximation of the fluxes

We can apply Theorem 12.1 for the fluxes, and if $\Phi_j \in L_\infty$, $j = 1, \dots, d$, we can apply Theorem 12.3. This leads to the following result.

Theorem 17.9. *Let* u_ε *be the solution of the equation* (17.6), *and let* u_0 *be the solution of the equation* (17.7). *Let* $\widetilde{g}(\mathbf{x})$ *be the matrix with the columns* $g(\mathbf{x})(\nabla \Phi_j + \mathbf{e}_j)$, $j = 1, \dots, d$.

1) *Suppose that conditions of Theorem* 17.7(1) *are satisfied. Let* $\Pi_\varepsilon^{(d)}$ *be the pseudodifferential operator in* $L_2(\mathbb{R}^d; \mathbb{C}^d)$ *with the symbol* $\chi_{\widetilde{\Omega}/\varepsilon}(\boldsymbol{\xi})$. *Then*

$$\|g^\varepsilon \nabla u_\varepsilon - \widetilde{g}^\varepsilon \Pi_\varepsilon^{(d)} \nabla u_0\|_{L_2(\mathbb{R}^d; \mathbb{C}^d)} \le \mathcal{C}_{15} \varepsilon \|F\|_{L_2(\mathbb{R}^d)}, \quad 0 < \varepsilon \le 1,$$

where the constant $\mathcal{C}_{15}$ *depends on* d, *on* $\|g\|_{L_\infty}$, $\|g^{-1}\|_{L_\infty}$, *and on parameters of the lattice* Γ.

2) *Suppose that conditions of Theorem* 17.7(2) *are satisfied. Then*

$$\|g^\varepsilon \nabla u_\varepsilon - \widetilde{g}^\varepsilon \nabla u_0\|_{L_2(\mathbb{R}^d; \mathbb{C}^d)} \le \mathcal{C}_{16} \varepsilon \|F\|_{L_2(\mathbb{R}^d)}, \quad 0 < \varepsilon \le 1,$$

where the constant $\mathcal{C}_{16}$ *depends on* d, *on* $\|g\|_{L_\infty}$, $\|g^{-1}\|_{L_\infty}$, *on parameters of the lattice* Γ, *and also on the norms* $\|\Phi_j\|_{L_\infty}$, $j = 1, \dots, d$.

Note that, under conditions of Theorem 17.9(1), as $\varepsilon \to 0$, the solutions u_ε converge to u_0 weakly in $H^1(\mathbb{R}^d)$, while the fluxes $g^\varepsilon \nabla u_\varepsilon$ converge to $g^0 \nabla u_0$ weakly in $L_2(\mathbb{R}^d; \mathbb{C}^d)$ (see Remarks 10.7 and 12.2).

17.6. Approximation of the generalized resolvent

Let $Q(\mathbf{x})$ be a real-valued Γ-periodic function satisfying condition (17.8). We consider the question about approximation of the generalized resolvent $(\mathbf{D}^* g^\varepsilon \mathbf{D} + Q^\varepsilon)^{-1}$. Applying Theorems 13.6 and 13.8, we arrive at the following result.

Theorem 17.10. *Let* $Q(\mathbf{x})$ *be a* Γ*-periodic function satisfying condition* (17.8), *and let* $\overline{Q}$ *be the mean value of* $Q(\mathbf{x})$ *over the cell* Ω. *Let* $Q^\varepsilon(\mathbf{x}) = Q(\varepsilon^{-1}\mathbf{x})$.

1) *Under conditions of Theorem* 17.7(1), *we have*

$$\|(\widehat{\mathcal{A}}_\varepsilon + Q^\varepsilon)^{-1} - (\widehat{\mathcal{A}}^0 + \overline{Q})^{-1} - \varepsilon \sum_{j=1}^d \Phi_j^\varepsilon \partial_j (\widehat{\mathcal{A}}^0 + \overline{Q})^{-1} \Pi_\varepsilon\|_{L_2(\mathbb{R}^d) \to H^1(\mathbb{R}^d)} \le \check{\mathcal{C}}_{10} \varepsilon,$$

$$0 < \varepsilon \le 1,$$

where the constant $\check{\mathcal{C}}_{10}$ *depends only on* $\|g\|_{L_\infty}$, $\|g^{-1}\|_{L_\infty}$, $\|Q\|_{L_\infty}$, $\|Q^{-1}\|_{L_\infty}$, *on* d, *and on parameters of the lattice* Γ.

2) *Under conditions of Theorem* 17.7(2), *we have*

$$\|(\widehat{\mathcal{A}}_\varepsilon + Q^\varepsilon)^{-1} - (\widehat{\mathcal{A}}^0 + \overline{Q})^{-1} - \varepsilon \sum_{j=1}^d \Phi_j^\varepsilon \partial_j (\widehat{\mathcal{A}}^0 + \overline{Q})^{-1}\|_{L_2(\mathbb{R}^d)\to H^1(\mathbb{R}^d)} \le \check{\mathcal{C}}_{11}\varepsilon, \quad 0 < \varepsilon \le 1, \tag{17.20}$$

where the constant $\check{\mathcal{C}}_{11}$ *depends only on* $\|g\|_{L_\infty}$, $\|g^{-1}\|_{L_\infty}$, $\|Q\|_{L_\infty}$, $\|Q^{-1}\|_{L_\infty}$, *on* d, *on parameters of the lattice* Γ, *and also on the norms* $\|\Phi_j\|_{L_\infty}$, $j = 1, \dots, d$.

We can apply (interpolational) Theorem 14.3 to the generalized resolvent $(\widehat{\mathcal{A}}_\varepsilon + Q^\varepsilon)^{-1}$. If $\Phi_j \in L_\infty$, $j = 1, \dots, d$, then Theorem 14.4 is applicable. The corrector (13.3) takes the form

$$\begin{aligned} \widetilde{K}_Q(\varepsilon) &= \sum_{j=1}^d \Phi_j^\varepsilon \partial_j (\widehat{\mathcal{A}}^0 + \overline{Q})^{-1}\Pi_\varepsilon - \sum_{j=1}^d (\widehat{\mathcal{A}}^0 + \overline{Q})^{-1}\Pi_\varepsilon \partial_j (\Phi_j^\varepsilon)^* \\ &- (\widehat{\mathcal{A}}^0 + \overline{Q})^{-1}\Big(\sum_{j,l,s=1}^d (a_{jls} - a_{jls}^*)\partial_j\partial_l\partial_s + 2i \sum_{j=1}^d (\operatorname{Im} \overline{Q\Phi_j})\partial_j \Big)(\widehat{\mathcal{A}}^0 + \overline{Q})^{-1}, \end{aligned} \tag{17.21}$$

and the corrector (13.4) is given by

$$\begin{aligned} K_Q^0(\varepsilon) &= \sum_{j=1}^d \Phi_j^\varepsilon \partial_j (\widehat{\mathcal{A}}^0 + \overline{Q})^{-1} - \sum_{j=1}^d (\widehat{\mathcal{A}}^0 + \overline{Q})^{-1}\partial_j (\Phi_j^\varepsilon)^* \\ &- (\widehat{\mathcal{A}}^0 + \overline{Q})^{-1}\Big(\sum_{j,l,s=1}^d (a_{jls} - a_{jls}^*)\partial_j\partial_l\partial_s + 2i \sum_{j=1}^d (\operatorname{Im} \overline{Q\Phi_j})\partial_j \Big)(\widehat{\mathcal{A}}^0 + \overline{Q})^{-1}. \end{aligned} \tag{17.22}$$

(See [BSu4, Theorem 10.6].) Here the values a_{jls} are defined by (17.18).

Applying Theorems 14.3 and 14.4, we arrive at the following result.

Theorem 17.11. 1) *Suppose that conditions of Theorem* 17.10(1) *are satisfied. Let* $\widetilde{K}_Q(\varepsilon)$ *be the corrector defined by* (17.21). *Then for* $0 \le s \le 1$ *we have*

$$\|(\widehat{\mathcal{A}}_\varepsilon + Q^\varepsilon)^{-1} - (\widehat{\mathcal{A}}^0 + \overline{Q})^{-1} - \varepsilon \widetilde{K}_Q(\varepsilon)\|_{L_2(\mathbb{R}^d)\to H^s(\mathbb{R}^d)} \le C_{Q,s}\varepsilon^{2-s}, \quad 0 < \varepsilon \le 1,$$

where the constant $C_{Q,s}$ *depends on* s, *on* $\|g\|_{L_\infty}$, $\|g^{-1}\|_{L_\infty}$, $\|Q\|_{L_\infty}$, $\|Q^{-1}\|_{L_\infty}$, *on* d, *and on parameters of the lattice* Γ.

2) *Suppose that conditions of Theorem* 17.10(2) *are satisfied. Let* $K_Q^0(\varepsilon)$ *be the corrector defined by* (17.22). *Then for* $0 \le s \le 1$ *we have*

$$\|(\widehat{\mathcal{A}}_\varepsilon + Q^\varepsilon)^{-1} - (\widehat{\mathcal{A}}^0 + \overline{Q})^{-1} - \varepsilon K_Q^0(\varepsilon)\|_{L_2(\mathbb{R}^d)\to H^s(\mathbb{R}^d)} \le C_{Q,s}^0\varepsilon^{2-s}, \quad 0 < \varepsilon \le 1, \tag{17.23}$$

where the constant $C_{Q,s}^0$ *depends on* s, *on* $\|g\|_{L_\infty}$, $\|g^{-1}\|_{L_\infty}$, $\|Q\|_{L_\infty}$, $\|Q^{-1}\|_{L_\infty}$, *on* d, *on parameters of the lattice* Γ, *and also on the norms* $\|\Phi_j\|_{L_\infty}$, $j = 1, \dots, d$.

We can apply Theorem 15.1 for the fluxes, and under the condition $\Phi_j \in L_\infty$, $j = 1, \dots, d$, we can apply Theorem 15.3. We arrive at the following result.

Theorem 17.12. *Let v_ε be the solution of the equation* (17.12)*, and let v_0 be the solution of the equation* (17.13)*. Let $\widetilde{g}(\mathbf{x})$ be the matrix with the columns $g(\mathbf{x})(\nabla\Phi_j(\mathbf{x}) + \mathbf{e}_j)$, $j = 1, \dots, d$.*

1) *Suppose that conditions of Theorem* 17.10(1) *are satisfied. Let $\Pi_\varepsilon^{(d)}$ be the pseudodifferential operator in $L_2(\mathbb{R}^d; \mathbb{C}^d)$ with the symbol $\chi_{\widetilde{\Omega}/\varepsilon}(\boldsymbol{\xi})$. Then*

$$\|g^\varepsilon \nabla v_\varepsilon - \widetilde{g}^\varepsilon \Pi_\varepsilon^{(d)} \nabla v_0\|_{L_2(\mathbb{R}^d;\mathbb{C}^d)} \leq \check{\mathcal{C}}_{15} \varepsilon \|F\|_{L_2(\mathbb{R}^d)}, \quad 0 < \varepsilon \leq 1,$$

where the constant $\check{\mathcal{C}}_{15}$ depends on $\|g\|_{L_\infty}$, $\|g^{-1}\|_{L_\infty}$, $\|Q\|_{L_\infty}$, $\|Q^{-1}\|_{L_\infty}$, on d, and on parameters of the lattice Γ.

2) *Suppose that conditions of Theorem* 17.10(2) *are satisfied. Then*

$$\|g^\varepsilon \nabla v_\varepsilon - \widetilde{g}^\varepsilon \nabla v_0\|_{L_2(\mathbb{R}^d;\mathbb{C}^d)} \leq \check{\mathcal{C}}_{16} \varepsilon \|F\|_{L_2(\mathbb{R}^d)}, \quad 0 < \varepsilon \leq 1, \tag{17.24}$$

where the constant $\check{\mathcal{C}}_{16}$ depends on $\|g\|_{L_\infty}$, $\|g^{-1}\|_{L_\infty}$, $\|Q\|_{L_\infty}$, $\|Q^{-1}\|_{L_\infty}$, on d, on parameters of the lattice Γ, and also on the norms $\|\Phi_j\|_{L_\infty}$, $j = 1, \dots, d$.

Note that, under conditions of Theorem 17.12(1), as $\varepsilon \to 0$, the solutions v_ε converge to v_0 weakly in $H^1(\mathbb{R}^d)$, and the fluxes $g^\varepsilon \nabla v_\varepsilon$ converge to $g^0 \nabla v_0$ weakly in $L_2(\mathbb{R}^d; \mathbb{C}^d)$ (see Remarks 13.7 and 15.2).

17.7. Special cases

The case where the corrector is equal to zero is distinguished by Theorem 10.9, and for the generalized resolvent by Theorem 13.9. Condition $g^0 = \overline{g}$, which is equivalent to (5.8), now means that the columns $\mathbf{g}_k(\mathbf{x})$ of the matrix $g(\mathbf{x})$ are solenoidal vectors:

$$\operatorname{div} \mathbf{g}_k(\mathbf{x}) = 0, \quad k = 1, \dots, d. \tag{17.25}$$

We arrive at the following result.

Theorem 17.13. *Let $\widehat{\mathcal{A}}(g) = \mathbf{D}^* g(\mathbf{x}) \mathbf{D}$, where $g(\mathbf{x})$ is a Γ-periodic matrix-valued function with complex entries satisfying conditions* (17.2) *and* (17.25)*. Let $\widehat{\mathcal{A}}^0 = \widehat{\mathcal{A}}(\overline{g})$ be the effective operator. Let $\widehat{\mathcal{A}}_\varepsilon = \widehat{\mathcal{A}}(g^\varepsilon)$. Let $Q(\mathbf{x})$ be a Γ-periodic function satisfying condition* (17.8)*, and let $\overline{Q}$ be the mean value of $Q(\mathbf{x})$ over the cell Ω. Let $Q^\varepsilon(\mathbf{x}) = Q(\varepsilon^{-1}\mathbf{x})$. Then we have*

$$\|(\widehat{\mathcal{A}}_\varepsilon + I)^{-1} - (\widehat{\mathcal{A}}^0 + I)^{-1}\|_{L_2(\mathbb{R}^d) \to H^1(\mathbb{R}^d)} \leq \mathcal{C}_{12} \varepsilon, \quad 0 < \varepsilon \leq 1,$$

$$\|(\widehat{\mathcal{A}}_\varepsilon + Q^\varepsilon)^{-1} - (\widehat{\mathcal{A}}^0 + \overline{Q})^{-1}\|_{L_2(\mathbb{R}^d) \to H^1(\mathbb{R}^d)} \leq \check{\mathcal{C}}_{12} \varepsilon, \quad 0 < \varepsilon \leq 1, \tag{17.26}$$

where the constant $\mathcal{C}_{12}$ depends only on $\|g\|_{L_\infty}$, $\|g^{-1}\|_{L_\infty}$, on d, and on parameters of the lattice Γ, while the constant $\check{\mathcal{C}}_{12}$ depends on the same parameters and also on the norms $\|Q\|_{L_\infty}$, $\|Q^{-1}\|_{L_\infty}$.

Applying (interpolational) Theorems 11.5 and 14.5, we arrive at the following result.

Theorem 17.14. *Under conditions of Theorem* 17.13*, for $0 \leq s \leq 1$ we have*

$$\|(\widehat{\mathcal{A}}_\varepsilon + I)^{-1} - (\widehat{\mathcal{A}}^0 + I)^{-1}\|_{L_2(\mathbb{R}^d) \to H^s(\mathbb{R}^d)} \leq C'_s \varepsilon^{2-s}, \quad 0 < \varepsilon \leq 1,$$

$$\|(\widehat{\mathcal{A}}_\varepsilon + Q^\varepsilon)^{-1} - (\widehat{\mathcal{A}}^0 + \overline{Q})^{-1}\|_{L_2(\mathbb{R}^d) \to H^s(\mathbb{R}^d)} \leq C'_{Q,s} \varepsilon^{2-s}, \quad 0 < \varepsilon \leq 1, \tag{17.27}$$

where the constant C'_s depends on s, on $\|g\|_{L_\infty}$, $\|g^{-1}\|_{L_\infty}$, and on parameters of the lattice Γ, while the constant $C'_{Q,s}$ depends on the same parameters and also on $\|Q\|_{L_\infty}$, $\|Q^{-1}\|_{L_\infty}$.

Now we consider the case where $g^0 = \underline{g}$. Condition (5.9) means that the columns $\mathbf{l}_k(\mathbf{x})$ of the matrix $g(\mathbf{x})^{-1}$ are potential vectors:

$$\mathbf{l}_k(\mathbf{x}) = \mathbf{l}_k^0 + \nabla\phi_k, \quad \mathbf{l}_k^0 \in \mathbb{C}^d, \;\; \phi_k \in \widetilde{H}^1(\Omega), \;\; k = 1, \dots, d. \tag{17.28}$$

Theorems 12.4 and 15.4 lead to the following result.

Theorem 17.15. *Suppose that conditions of Theorem* 17.10(1) *are satisfied, and that* $g^0 = \underline{g}$, *i. e., conditions* (17.28) *for the columns* $\mathbf{l}_k(\mathbf{x})$ *of the matrix* $g(\mathbf{x})^{-1}$ *are valid. Let* u_ε *be the solution of the equation* (17.6), *and let* u_0 *be the solution of the equation* (17.7). *Let* v_ε *be the solution of the equation* (17.12), *and let* v_0 *be the solution of the equation* (17.13). *Then, as* $\varepsilon \to 0$, *the fluxes* $g^\varepsilon \nabla u_\varepsilon$ *tend to* $g^0 \nabla u_0$, *and* $g^\varepsilon \nabla v_\varepsilon$ *tend to* $g^0 \nabla v_0$ *in the* $L_2(\mathbb{R}^d; \mathbb{C}^d)$*-norm. We have*

$$\|g^\varepsilon \nabla u_\varepsilon - g^0 \nabla u_0\|_{L_2(\mathbb{R}^d;\mathbb{C}^d)} \le \mathcal{C}_{16} \varepsilon \|F\|_{L_2(\mathbb{R}^d)}, \quad 0 < \varepsilon \le 1,$$

$$\|g^\varepsilon \nabla v_\varepsilon - g^0 \nabla v_0\|_{L_2(\mathbb{R}^d;\mathbb{C}^d)} \le \check{\mathcal{C}}_{16} \varepsilon \|F\|_{L_2(\mathbb{R}^d)}, \quad 0 < \varepsilon \le 1. \tag{17.29}$$

The constant $\mathcal{C}_{16}$ *depends on* d, *on* $\|g\|_{L_\infty}$, $\|g^{-1}\|_{L_\infty}$, *and on parameters of the lattice* Γ, *while the constant* $\check{\mathcal{C}}_{16}$ *depends on the same parameters, and also on the norms* $\|Q\|_{L_\infty}$, $\|Q^{-1}\|_{L_\infty}$.

§18. The periodic Schrödinger operator

18.1. Preliminaries. Factorization

(See [BSu2, §6.1 and BSu4, §11].) In the space $L_2(\mathbb{R}^d)$, $d \ge 1$, we consider the periodic Schrödinger operator with the metric $\mathfrak{g}(\mathbf{x})$ and potential $p(\mathbf{x})$:

$$\mathcal{H} = \mathbf{D}^* \mathfrak{g}(\mathbf{x}) \mathbf{D} + p(\mathbf{x}), \quad \mathbf{x} \in \mathbb{R}^d. \tag{18.1}$$

Here $\mathfrak{g}(\mathbf{x})$ is a Γ-periodic $(d \times d)$-matrix *with real entries* satisfying the following conditions:

$$\mathfrak{g}(\mathbf{x}) > 0, \quad \mathfrak{g}, \mathfrak{g}^{-1} \in L_\infty, \tag{18.2}$$

and $p(\mathbf{x})$ is a *real-valued* Γ-periodic function such that

$$p \in L_s(\Omega), \quad 2s > d \quad \text{for } d \ge 2; \quad s = 1 \quad \text{for } d = 1. \tag{18.3}$$

Adding an appropriate constant to $p(\mathbf{x})$, we can always assume that the *point* $\lambda = 0$ *is the bottom of the spectrum of the operator* $\mathcal{H}$.

Let $\omega \in \widetilde{H}^1(\Omega)$ be a (weak) periodic solution of the equation

$$\mathbf{D}^* \mathfrak{g}(\mathbf{x}) \mathbf{D} \omega + p(\mathbf{x}) \omega = 0.$$

Solution ω is defined up to a constant factor, which may be fixed so that

$$\omega(\mathbf{x}) > 0, \quad \mathbf{x} \in \mathbb{R}^d,$$

$$\int_\Omega \omega^2(\mathbf{x})\, d\mathbf{x} = |\Omega|. \tag{18.4}$$

Under conditions (18.2) and (18.3), it turns out that $\omega, \omega^{-1} \in C^{\alpha}$ with some $\alpha > 0$. The *operator* (18.1) *admits a factorization of the form*

$$\mathcal{H} = \omega^{-1}\mathbf{D}^*\omega^2\mathfrak{g}\mathbf{D}\omega^{-1}. \tag{18.5}$$

Thus, the operator $\mathcal{H}$ is reduced to the form

$$\mathcal{H} = \mathcal{A}(g, f), \quad \text{with} \ \ b(\mathbf{D}) = \mathbf{D}, \quad g = \omega^2\mathfrak{g}, \quad f = \omega^{-1}.$$

Herewith, $n = 1$ and $m = d$.

Remark 18.1. We can view expression (18.5) as the definition of the operator $\mathcal{H}$, assuming that $\omega(\mathbf{x})$ is an arbitrary Γ-periodic function such that

$$\omega(\mathbf{x}) > 0, \quad \omega, \omega^{-1} \in L_\infty. \tag{18.6}$$

We take this definition as the initial one. The form (18.1) can be recovered by the formula $p = -\omega^{-1}(\mathbf{D}^*\mathfrak{g}\mathbf{D}\omega)$. The corresponding potential $p(\mathbf{x})$ may be strongly singular.

18.2. The homogenization problem for $\mathcal{H}_\varepsilon$

Now we consider the operator

$$\mathcal{H}_\varepsilon = (\omega^\varepsilon)^{-1}\mathbf{D}^*(\omega^\varepsilon)^2\mathfrak{g}^\varepsilon\mathbf{D}(\omega^\varepsilon)^{-1} = (\omega^\varepsilon)^{-1}\mathbf{D}^*g^\varepsilon\mathbf{D}(\omega^\varepsilon)^{-1} \tag{18.7}$$

with rapidly oscillating coefficients. In terms of (18.1), the operator (18.7) takes the form

$$\mathcal{H}_\varepsilon = \mathbf{D}^*\mathfrak{g}^\varepsilon(\mathbf{x})\mathbf{D} + \varepsilon^{-2}p^\varepsilon(\mathbf{x}).$$

We are interested in the behavior of the resolvent $(\mathcal{H}_\varepsilon + I)^{-1}$ for small ε.

Now Condition 8.6(2°), and then also Condition 8.4, is satisfied. Theorem 16.3 is applicable. Let $\widehat{\mathcal{A}}(g) = \mathbf{D}^*\omega^2\mathfrak{g}\mathbf{D} = \mathbf{D}^*g\mathbf{D}$, and let g^0 be the effective matrix for the operator $\widehat{\mathcal{A}}(g)$. Let $\widehat{\mathcal{A}}^0 = \widehat{\mathcal{A}}(g^0)$. Now $Q(\mathbf{x}) = \omega^2(\mathbf{x})$, and, by (18.4), $\overline{Q} = 1$. Taking what was said into account and applying Theorem 16.3, we obtain the following statement (cf. Theorem 17.4).

Theorem 18.2. *Let $\mathfrak{g}(\mathbf{x})$ be a Γ-periodic $(d \times d)$-matrix-valued function with real entries satisfying conditions* (18.2)*, and let $\omega(\mathbf{x})$ be a Γ-periodic function satisfying conditions* (18.6), (18.4). *Let $\mathcal{H}_\varepsilon$ be the operator defined by* (18.7). *We put $g(\mathbf{x}) := \omega^2(\mathbf{x})\mathfrak{g}(\mathbf{x})$. Let g^0 be the effective matrix for the operator $\mathbf{D}^*g\mathbf{D}$, and let $\widehat{\mathcal{A}}^0 = \mathbf{D}^*g^0\mathbf{D}$. Let $\Phi_j \in \widetilde{H}^1(\Omega)$ be the solution of the problem* (17.3), $j = 1, \dots, d$. *Then for $0 < \varepsilon \le 1$ we have*

$$\|(\omega^\varepsilon)^{-1}(\mathcal{H}_\varepsilon + I)^{-1} - (\widehat{\mathcal{A}}^0 + I)^{-1}\omega^\varepsilon - \varepsilon\sum_{j=1}^{d}\Phi_j^\varepsilon\partial_j(\widehat{\mathcal{A}}^0 + I)^{-1}\omega^\varepsilon\|_{L_2(\mathbb{R}^d)\to H^1(\mathbb{R}^d)} \le \widetilde{\mathcal{C}}_{11}\varepsilon, \tag{18.8}$$

where the constant $\widetilde{\mathcal{C}}_{11} = \check{\mathcal{C}}_{11}\|\omega\|_{L_\infty}$ depends on the norms $\|\mathfrak{g}\|_{L_\infty}$, $\|\mathfrak{g}^{-1}\|_{L_\infty}$, $\|\omega\|_{L_\infty}$, $\|\omega^{-1}\|_{L_\infty}$, on d, and on parameters of the lattice Γ. (*Here $\check{\mathcal{C}}_{11}$ is the constant from* (17.9) *with $Q = \omega^2$.*)

We can also apply (interpolational) Theorem 16.7. By (17.10) and by the identity $\overline{Q} = 1$, the operator $K_Q^0(\varepsilon)$ coincides now with $K^0(\varepsilon)$ and is given by (17.5). We arrive at the following result (cf. Theorem 17.5).

Theorem 18.3. *Suppose that conditions of Theorem* 18.2 *are satisfied. Let* $K^0(\varepsilon)$ *be the corrector defined by* (17.5)*. Then for* $0 \le s \le 1$ *and* $0 < \varepsilon \le 1$ *we have*

$$\|(\omega^\varepsilon)^{-1}(\mathcal{H}_\varepsilon + I)^{-1} - \Big((\widehat{\mathcal{A}}^0 + I)^{-1} + \varepsilon K^0(\varepsilon)\Big)\,\omega^\varepsilon\|_{L_2(\mathbb{R}^d)\to H^s(\mathbb{R}^d)} \le \widetilde{C}_s \varepsilon^{2-s}, \qquad (18.9)$$

where the constant $\widetilde{C}_s = C^0_{Q,s}\|\omega\|_{L_\infty}$ *depends only on* s*, on* $\|\mathfrak{g}\|_{L_\infty}$*,* $\|\mathfrak{g}^{-1}\|_{L_\infty}$*,* $\|\omega\|_{L_\infty}$*,* $\|\omega^{-1}\|_{L_\infty}$*, on* d*, and on parameters of the lattice* Γ*. (Here* $C^0_{Q,s}$ *is the constant from* (17.11) *with* $Q = \omega^2$*.)*

Remark 18.4 If the operator $\mathcal{H}$ is given in the form (18.1) under conditions (18.2) and (18.3), then the norms $\|\omega\|_{L_\infty}$, $\|\omega^{-1}\|_{L_\infty}$ are estimated by the constant depending only on $\|\mathfrak{g}\|_{L_\infty}$, $\|\mathfrak{g}^{-1}\|_{L_\infty}$, on d, Ω, and on $\|p\|_{L_s(\Omega)}$. Then the constant $\widetilde{\mathcal{C}}_{11}$ from (18.8) depends on $\|\mathfrak{g}\|_{L_\infty}$, $\|\mathfrak{g}^{-1}\|_{L_\infty}$, on d, $\|p\|_{L_s(\Omega)}$ and on parameters of the lattice Γ, while the constant $\widetilde{C}_s$ from (18.9) depends on the same parameters and also on s.

18.3. Approximation of the fluxes

We consider the solution w_ε of the equation

$$\mathcal{H}_\varepsilon w_\varepsilon + w_\varepsilon = F, \quad F \in L_2(\mathbb{R}^d). \qquad (18.10)$$

Let w^0_ε be the solution of the equation

$$\mathbf{D}^* g^0 \mathbf{D} w^0_\varepsilon + w^0_\varepsilon = \omega^\varepsilon F. \qquad (18.11)$$

(Cf. (16.6) and (16.7).) Then (18.8) means that

$$\|(\omega^\varepsilon)^{-1} w_\varepsilon - w^0_\varepsilon - \varepsilon \sum_{j=1}^d \Phi^\varepsilon_j \partial_j w^0_\varepsilon\|_{H^1(\mathbb{R}^d)} \le \widetilde{\mathcal{C}}_{11}\varepsilon \|F\|_{L_2(\mathbb{R}^d)}, \quad 0 < \varepsilon \le 1.$$

By Remark 16.2, under conditions of Theorem 18.2, there exists a weak (H^1)-limit of the functions $(\omega^\varepsilon)^{-1} w_\varepsilon$:

$$(w, H^1(\mathbb{R}^d))\text{-}\lim_{\varepsilon\to 0}(\omega^\varepsilon)^{-1} w_\varepsilon = w_0,$$

where w_0 is the solution of the equation

$$\mathbf{D}^* g^0 \mathbf{D} w_0 + w_0 = \overline{\omega} F. \qquad (18.12)$$

We can apply Theorem 16.11 for the fluxes. This leads to the following result.

Theorem 18.5. *Suppose that conditions of Theorem* 18.2 *are satisfied. Let* w_ε *be the solution of the equation* (18.10)*, and let* w^0_ε *be the solution of the equation* (18.11)*. Let* $\widetilde{g}(\mathbf{x})$ *be the matrix with the columns* $g(\mathbf{x})(\nabla\Phi_j + \mathbf{e}_j)$*,* $j = 1, \dots, d$*. Then*

$$\|g^\varepsilon \nabla (\omega^\varepsilon)^{-1} w_\varepsilon - \widetilde{g}^\varepsilon \nabla w^0_\varepsilon\|_{L_2(\mathbb{R}^d;\mathbb{C}^d)} \le \widetilde{\mathcal{C}}_{16}\varepsilon\|F\|_{L_2(\mathbb{R}^d)}, \quad 0 < \varepsilon \le 1,$$

where the constant $\widetilde{\mathcal{C}}_{16} = \check{\mathcal{C}}_{16}\|\omega\|_{L_\infty}$ *depends only on* $\|\mathfrak{g}\|_{L_\infty}$*,* $\|\mathfrak{g}^{-1}\|_{L_\infty}$*,* $\|\omega\|_{L_\infty}$*,* $\|\omega^{-1}\|_{L_\infty}$*, on* d*, and on parameters of the lattice* Γ*. (Here* $\check{\mathcal{C}}_{16}$ *is the constant from* (17.14) *with* $Q = \omega^2$*.)*

By Remark 16.10, under conditions of Theorem 18.5, the functions $g^\varepsilon \nabla(\omega^\varepsilon)^{-1} w_\varepsilon$ converge to $g^0 \nabla w_0$ weakly in $L_2(\mathbb{R}^d;\mathbb{C}^d)$, where w_0 is the solution of the equation (18.12).

18.4. Special cases

The case where the corrector is equal to zero is distinguished by Theorems 16.4 and 16.8. We arrive at the following result.

Theorem 18.6. *Suppose that conditions of Theorem* 18.2 *are satisfied, and that* $g^0 = \overline{g}$, *i. e., condition* (17.25) *for the columns* $\mathbf{g}_k(\mathbf{x})$ *of the matrix* $g(\mathbf{x}) = \mathfrak{g}(\mathbf{x})\omega(\mathbf{x})^2$ *is valid. Then we have*

$$\|(\omega^\varepsilon)^{-1}(\mathcal{H}_\varepsilon + I)^{-1} - (\widehat{\mathcal{A}}^0 + I)^{-1}\omega^\varepsilon\|_{L_2(\mathbb{R}^d)\to H^1(\mathbb{R}^d)} \le \widetilde{\mathcal{C}}_{12}\varepsilon, \quad 0 < \varepsilon \le 1. \tag{18.13}$$

Besides, for $0 \le s \le 1$ *we have*

$$\|(\omega^\varepsilon)^{-1}(\mathcal{H}_\varepsilon + I)^{-1} - (\widehat{\mathcal{A}}^0 + I)^{-1}\omega^\varepsilon\|_{L_2(\mathbb{R}^d)\to H^s(\mathbb{R}^d)} \le \widetilde{C}'_s\varepsilon^{2-s}, \quad 0 < \varepsilon \le 1. \tag{18.14}$$

The constant $\widetilde{\mathcal{C}}_{12} = \check{\mathcal{C}}_{12}\|\omega\|_{L_\infty}$ *depends on* $\|\mathfrak{g}\|_{L_\infty}$, $\|\mathfrak{g}^{-1}\|_{L_\infty}$, $\|\omega\|_{L_\infty}$, $\|\omega^{-1}\|_{L_\infty}$, *on* d, *and on parameters of the lattice* Γ. *The constant* $\widetilde{C}'_s = C'_{Q,s}\|\omega\|_{L_\infty}$ *depends on the same parameters and also on* s. (*Here* $\check{\mathcal{C}}_{12}$ *is the constant from* (17.26), *and* $C'_{Q,s}$ *is the constant from* (17.27) *with* $Q = \omega^2$.)

Now we consider the case where $g^0 = \underline{g}$. We can apply Theorem 16.12, which leads to the following statement.

Theorem 18.7. *Suppose that conditions of Theorem* 18.5 *are satisfied, and that* $g^0 = \underline{g}$, *i. e., relations* (17.28) *for the columns* $\mathbf{l}_k(\mathbf{x})$ *of the matrix* $g(\mathbf{x})^{-1}$ *are valid; here* $g = \mathfrak{g}\omega^2$. *Then we have*

$$\|g^\varepsilon\nabla(\omega^\varepsilon)^{-1}w_\varepsilon - g^0\nabla w_\varepsilon^0\|_{L_2(\mathbb{R}^d;\mathbb{C}^d)} \le \widetilde{\mathcal{C}}_{16}\varepsilon\|F\|_{L_2(\mathbb{R}^d)}, \quad 0 < \varepsilon \le 1,$$

where the constant $\widetilde{\mathcal{C}}_{16} = \check{\mathcal{C}}_{16}\|\omega\|_{L_\infty}$ *depends on* $\|\mathfrak{g}\|_{L_\infty}$, $\|\mathfrak{g}^{-1}\|_{L_\infty}$, $\|\omega\|_{L_\infty}$, $\|\omega^{-1}\|_{L_\infty}$, *on* d, *and on parameters of the lattice* Γ. (*Here* $\check{\mathcal{C}}_{16}$ *is the constant from* (17.29) *with* $Q = \omega^2$.)

By Remark 16.13, under conditions of Theorem 18.7, there exists a strong limit

$$(L_2(\mathbb{R}^d;\mathbb{C}^d))\text{-}\lim_{\varepsilon\to 0} g^\varepsilon\nabla(\omega^\varepsilon)^{-1}w_\varepsilon = g^0\nabla w_0,$$

where w_0 is the solution of the equation (18.12).

18.5. Approximation for the generalized resolvent

Now we consider the question about approximation of the generalized resolvent $(\mathcal{H}_\varepsilon + \mathfrak{Q}^\varepsilon)^{-1}$, where $\mathfrak{Q}(\mathbf{x})$ is a Γ-periodic function such that

$$\mathfrak{Q}(\mathbf{x}) > 0; \quad \mathfrak{Q}, \mathfrak{Q}^{-1} \in L_\infty. \tag{18.15}$$

We can apply Theorem 16.15, which gives the following result.

Theorem 18.8. *Suppose that conditions of Theorem* 18.2 *are satisfied. Let* $\mathfrak{Q}(\mathbf{x})$ *be the* Γ*-periodic function satisfying conditions* (18.15). *Let* $Q_*(\mathbf{x}) = \mathfrak{Q}(\mathbf{x})\omega^2(\mathbf{x})$. *Then for* $0 < \varepsilon \le 1$ *we have*

$$\begin{aligned}\|(\omega^\varepsilon)^{-1}&(\mathcal{H}_\varepsilon + \mathfrak{Q}^\varepsilon)^{-1} - (\widehat{\mathcal{A}}^0 + \overline{Q_*})^{-1}\omega^\varepsilon \\ &- \varepsilon\sum_{j=1}^d \Phi_j^\varepsilon\partial_j(\widehat{\mathcal{A}}^0 + \overline{Q_*})^{-1}\omega^\varepsilon\|_{L_2(\mathbb{R}^d)\to H^1(\mathbb{R}^d)} \le \widetilde{\mathcal{C}}_{11}^*\varepsilon,\end{aligned} \tag{18.16}$$

where the constant $\widetilde{\mathcal{C}}_{11}^* = \check{\mathcal{C}}_{11}^* \|\omega\|_{L_\infty}$ *depends on the norms* $\|\mathfrak{g}\|_{L_\infty}$, $\|\mathfrak{g}^{-1}\|_{L_\infty}$, $\|\omega\|_{L_\infty}$, $\|\omega^{-1}\|_{L_\infty}$, $\|\mathfrak{Q}\|_{L_\infty}$, $\|\mathfrak{Q}^{-1}\|_{L_\infty}$, *on* d, *and on parameters of the lattice* Γ. (*Here the constant* $\check{\mathcal{C}}_{11}^*$ *is the analog of the constant* $\check{\mathcal{C}}_{11}$ *from* (17.9) *with* Q *replaced by* Q_*.)

We can also apply (interpolational) Theorem 16.19. Now the corrector $K_{Q_*}^0(\varepsilon)$ takes the form (cf. (17.10)):

$$K_{Q_*}^0(\varepsilon) = \sum_{j=1}^d \Phi_j^\varepsilon \partial_j (\widehat{\mathcal{A}}^0 + \overline{Q_*})^{-1} - \sum_{j=1}^d (\widehat{\mathcal{A}}^0 + \overline{Q_*})^{-1} \partial_j \Phi_j^\varepsilon. \tag{18.17}$$

We obtain the following result (cf. Theorem 17.5).

Theorem 18.9. *Suppose that conditions of Theorem* 18.8 *are satisfied. Let* $K_{Q_*}^0(\varepsilon)$ *be the corrector defined by* (18.17). *Then for* $0 \le s \le 1$ *and* $0 < \varepsilon \le 1$ *we have*

$$\|(\omega^\varepsilon)^{-1}(\mathcal{H}_\varepsilon + \mathfrak{Q}^\varepsilon)^{-1} - \Big((\widehat{\mathcal{A}}^0 + \overline{Q_*})^{-1} + \varepsilon K_{Q_*}^0(\varepsilon)\Big)\omega^\varepsilon\|_{L_2(\mathbb{R}^d)\to H^s(\mathbb{R}^d)} \le \widetilde{C}_s^* \varepsilon^{2-s}, \tag{18.18}$$

where the constant $\widetilde{C}_s^* = C_{Q_*,s}^0 \|\omega\|_{L_\infty}$ *depends only on* s, *on* $\|\mathfrak{g}\|_{L_\infty}$, $\|\mathfrak{g}^{-1}\|_{L_\infty}$, $\|\omega\|_{L_\infty}$, $\|\omega^{-1}\|_{L_\infty}$, $\|\mathfrak{Q}\|_{L_\infty}$, $\|\mathfrak{Q}^{-1}\|_{L_\infty}$, *on* d, *and on parameters of the lattice* Γ. (*Here the constant* $C_{Q_*,s}^0$ *is the analog of the constant* $C_{Q,s}^0$ *from* (17.11) *with* Q *replaced by* Q_*.)

We consider the solution z_ε of the equation

$$\mathcal{H}_\varepsilon z_\varepsilon + \mathfrak{Q}^\varepsilon z_\varepsilon = F, \quad F \in L_2(\mathbb{R}^d). \tag{18.19}$$

Let z_ε^0 be the solution of the equation

$$\widehat{\mathcal{A}}^0 z_\varepsilon^0 + \overline{Q_*} z_\varepsilon^0 = \omega^\varepsilon F. \tag{18.20}$$

Estimate (18.16) shows that

$$\|(\omega^\varepsilon)^{-1} z_\varepsilon - z_\varepsilon^0 - \varepsilon \sum_{j=1}^d \Phi_j^\varepsilon \partial_j z_\varepsilon^0\|_{H^1(\mathbb{R}^d)} \le \widetilde{C}_{11}^* \varepsilon \|F\|_{L_2(\mathbb{R}^d)}, \quad 0 < \varepsilon \le 1.$$

By (16.28), there exists a weak (H^1)-limit of the functions $(\omega^\varepsilon)^{-1} z_\varepsilon$:

$$(w, H^1(\mathbb{R}^d))\text{-}\lim_{\varepsilon\to 0} (\omega^\varepsilon)^{-1} z_\varepsilon = z_0,$$

where z_0 is the solution of the equation

$$\widehat{\mathcal{A}}^0 z_0 + \overline{Q_*} z_0 = \overline{\omega} F. \tag{18.21}$$

We can apply Theorem 16.22 for the fluxes. This leads to the following result.

Theorem 18.10. *Suppose that conditions of Theorem* 18.8 *are satisfied. Let* z_ε *be the solution of the equation* (18.19), *and let* z_ε^0 *be the solution of the equation* (18.20). *Let* $\widetilde{g}(\mathbf{x})$ *be the matrix with the columns* $g(\mathbf{x})(\nabla \Phi_j + \mathbf{e}_j)$, $j = 1, \dots, d$. *Then*

$$\|g^\varepsilon \nabla (\omega^\varepsilon)^{-1} z_\varepsilon - \widetilde{g}^\varepsilon \nabla z_\varepsilon^0\|_{L_2(\mathbb{R}^d;\mathbb{C}^d)} \le \widetilde{\mathcal{C}}_{16}^* \varepsilon \|F\|_{L_2(\mathbb{R}^d)}, \quad 0 < \varepsilon \le 1,$$

where the constant $\widetilde{\mathcal{C}}_{16}^* = \check{\mathcal{C}}_{16}^* \|\omega\|_{L_\infty}$ *depends only on* $\|\mathfrak{g}\|_{L_\infty}$, $\|\mathfrak{g}^{-1}\|_{L_\infty}$, $\|\omega\|_{L_\infty}$, $\|\omega^{-1}\|_{L_\infty}$, $\|\mathfrak{Q}\|_{L_\infty}$, $\|\mathfrak{Q}^{-1}\|_{L_\infty}$, *on* d, *and on parameters of the lattice* Γ. (*Here the constant* $\check{\mathcal{C}}_{16}^*$ *is the analog of the constant* $\check{\mathcal{C}}_{16}$ *from* (17.14) *with* Q *replaced by* Q_*.)

The fluxes $g^\varepsilon \nabla(\omega^\varepsilon)^{-1} z_\varepsilon$ converge to $g^0 \nabla z_0$ weakly in $L_2(\mathbb{R}^d;\mathbb{C}^d)$.

The case where the corrector in (18.16) and in (18.18) is equal to zero is distinguished by Theorems 16.16 and 16.20. We obtain the folowing statement.

Theorem 18.11. *Suppose that conditions of Theorem* 18.8 *are satisfied, and that* $g^0 = \overline{g}$, *i. e., condition* (17.25) *for the columns* $\mathbf{g}_k(\mathbf{x})$ *of the matrix* $g(\mathbf{x}) = \mathfrak{g}(\mathbf{x})\omega(\mathbf{x})^2$ *is valid. Then for* $0 < \varepsilon \leq 1$ *we have*

$$\|(\omega^\varepsilon)^{-1}(\mathcal{H}_\varepsilon + \mathfrak{Q}^\varepsilon)^{-1} - (\widehat{\mathcal{A}}^0 + \overline{Q_*})^{-1}\omega^\varepsilon\|_{L_2(\mathbb{R}^d)\to H^1(\mathbb{R}^d)} \leq \widetilde{\mathcal{C}}_{12}^* \varepsilon. \tag{18.22}$$

Besides, for $0 \leq s \leq 1$ *and* $0 < \varepsilon \leq 1$ *we have*

$$\|(\omega^\varepsilon)^{-1}(\mathcal{H}_\varepsilon + \mathfrak{Q}^\varepsilon)^{-1} - (\widehat{\mathcal{A}}^0 + \overline{Q_*})^{-1}\omega^\varepsilon\|_{L_2(\mathbb{R}^d)\to H^s(\mathbb{R}^d)} \leq \widetilde{C}_{s,*}' \varepsilon^{2-s}. \tag{18.23}$$

The constant $\widetilde{\mathcal{C}}_{12}^* = \check{\mathcal{C}}_{12}^* \|\omega\|_{L_\infty}$ *from* (18.22) *depends on* $\|\mathfrak{g}\|_{L_\infty}$, $\|\mathfrak{g}^{-1}\|_{L_\infty}$, $\|\omega\|_{L_\infty}$, $\|\omega^{-1}\|_{L_\infty}$, $\|\mathfrak{Q}\|_{L_\infty}$, $\|\mathfrak{Q}^{-1}\|_{L_\infty}$, *on* d, *and on parameters of the lattice* Γ. *The constant* $\widetilde{C}_{s,*}' = C_{Q_*,s}' \|\omega\|_{L_\infty}$ *from* (18.23) *depends on the same parameters, and also on* s. *(Here* $\check{\mathcal{C}}_{12}^*$ *is the analog of the constant* $\check{\mathcal{C}}_{12}$ *from* (17.26), *and* $C_{Q_*,s}'$ *is the analog of the constant* $C_{Q,s}'$ *from* (17.27) *with* Q *replaced by* Q_*.)

In the case where $g^0 = \underline{g}$, we can apply Theorem 16.23, which leads to the folowing statement.

Theorem 18.12. *Suppose that conditions of Theorem* 18.10 *are satisfied, and that* $g^0 = \underline{g}$, *i. e., relations* (17.28) *for the columns* $\mathbf{l}_k(\mathbf{x})$ *of the matrix* $g(\mathbf{x})^{-1}$ *are valid, where* $g = \mathfrak{g}\omega^2$. *Then we have*

$$\|g^\varepsilon \nabla(\omega^\varepsilon)^{-1} z_\varepsilon - g^0 \nabla z_\varepsilon^0\|_{L_2(\mathbb{R}^d;\mathbb{C}^d)} \leq \widetilde{\mathcal{C}}_{16}^* \varepsilon \|F\|_{L_2(\mathbb{R}^d)}, \quad 0 < \varepsilon \leq 1,$$

where the constant $\widetilde{\mathcal{C}}_{16}^* = \check{\mathcal{C}}_{16}^* \|\omega\|_{L_\infty}$ *depends on* $\|\mathfrak{g}\|_{L_\infty}$, $\|\mathfrak{g}^{-1}\|_{L_\infty}$, $\|\omega\|_{L_\infty}$, $\|\omega^{-1}\|_{L_\infty}$, $\|\mathfrak{Q}\|_{L_\infty}$, $\|\mathfrak{Q}^{-1}\|_{L_\infty}$, *on* d, *and on parameters of the lattice* Γ.

Note that, under conditions of Theorem 18.12, as $\varepsilon \to 0$, the fluxes $g^\varepsilon \nabla(\omega^\varepsilon)^{-1} z_\varepsilon$ converge to $g^0 \nabla z_0$ strongly in $L_2(\mathbb{R}^d;\mathbb{C}^d)$, where z_0 is the solution of the equation (18.21).

§19. The magnetic Schrödinger operator

19.1

In $L_2(\mathbb{R}^d)$, $d \geq 2$, we consider the periodic magnetic Schrödinger operator $\mathcal{M}$ with the metric $\mathfrak{g}(\mathbf{x})$, magnetic potential $\mathbf{A}(\mathbf{x})$ and electric potential $p(\mathbf{x})$:

$$\mathcal{M} = (\mathbf{D} - \mathbf{A}(\mathbf{x}))^* \mathfrak{g}(\mathbf{x})(\mathbf{D} - \mathbf{A}(\mathbf{x})) + p(\mathbf{x}). \tag{19.1}$$

(Cf. [BSu4, Subsection 11.3].) Here $\mathfrak{g}(\mathbf{x})$ is a Γ-periodic $(d \times d)$-matrix-valued function with real entries satisfying conditions (18.2). Suppose also that, for $d \geq 3$, we have

$$\mathfrak{g} \in C^\alpha, \quad 0 < \alpha < 1, \ \ d \geq 3, \tag{19.2}$$

with some $\alpha \in (0,1)$. The vector-valued potential $\mathbf{A}(\mathbf{x})$ and the scalar-valued potential $p(\mathbf{x})$ are real and Γ-periodic. Assume that

$$\mathbf{A} \in L_{2s}(\Omega), \quad p \in L_s(\Omega), \ \ 2s > d. \tag{19.3}$$

Adding an appropriate constant to $p(\mathbf{x})$, we can assume that

$$\inf \operatorname{spec} \mathcal{M} = 0. \tag{19.4}$$

In the recent paper [Sh2] by R. G. Shterenberg, it was shown that, under the above assumptions and for sufficiently small (in the $L_{2s}(\Omega)$-norm) magnetic potential $\mathbf{A}$, the operator $\mathcal{M}$ admits a factorization appropriate for our goals. Now we describe this factorization. Let $\mathcal{M}(\mathbf{k})$ be the operators in $L_2(\Omega)$ occuring in the direct integral expansion for $\mathcal{M}$. Condition (19.4) means that, for some $\mathbf{k}_0 \in \widetilde{\Omega}$, the point $\lambda = 0$ is an eigenvalue of the operator $\mathcal{M}(\mathbf{k}_0)$. If the magnetic potential is sufficiently small, then this point $\mathbf{k}_0 \in \widetilde{\Omega}$ is unique and the eigenvalue $\lambda = 0$ is simple. Let $\phi(\mathbf{x})$ be the corresponding eigenfunction normalized by the condition

$$\int_{\Omega} |\phi(\mathbf{x})|^2 \, d\mathbf{x} = |\Omega| \tag{19.5}$$

(the phase factor of ϕ does not matter). Then we have: $\phi \in \widetilde{H}^1(\Omega)$, $\mathcal{M}(\mathbf{k}_0)\phi = 0$, and

$$\phi, \phi^{-1} \in L_\infty. \tag{19.6}$$

Moreover, as it was mentioned in [Sh2], we have

$$\phi \in \widetilde{W}^1_{2s}(\Omega), \quad 2s > d, \tag{19.7}$$

with the same number s, as in condition (19.3).

We denote

$$\widetilde{\mathcal{M}} = [e^{-i\langle \mathbf{k}_0, \cdot \rangle}] \mathcal{M} [e^{i\langle \mathbf{k}_0, \cdot \rangle}], \tag{19.8}$$

where $[e^{\pm i\langle \mathbf{k}_0, \cdot \rangle}]$ is the operator in $L_2(\mathbb{R}^d)$ of multiplication by the function $e^{\pm i\langle \mathbf{k}_0, \mathbf{x} \rangle}$. By Theorems 2.7 and 2.8 of [Sh2], if the norm $\|\mathbf{A}\|_{L_{2s}(\Omega)}$ is sufficiently small, then the periodic operator $\widetilde{\mathcal{M}}$ admits the following factorization:

$$\widetilde{\mathcal{M}} = (\phi(\mathbf{x})^*)^{-1} \mathbf{D}^* g(\mathbf{x}) \mathbf{D} (\phi(\mathbf{x}))^{-1}. \tag{19.9}$$

Relation (19.9) is similar to factorization (18.5) for the Schrödinger operator $\mathcal{H}$, but now the Hermitian matrix $g(\mathbf{x})$ has complex entries and the function $\phi(\mathbf{x})$ is complex-valued. Herewith, $g(\mathbf{x})$ is Γ-periodic and

$$g(\mathbf{x}) > 0, \quad g, g^{-1} \in L_\infty. \tag{19.10}$$

The matrix $g(\mathbf{x})$ has the form

$$g(\mathbf{x}) = \mathfrak{g}(\mathbf{x}) |\phi(\mathbf{x})|^2 + i g_2(\mathbf{x}),$$

where the Γ-periodic antisymmetric matrix $g_2(\mathbf{x})$ with real entries satisfies the equation

$$(\operatorname{div} g_2(\mathbf{x}))^t = -2|\phi(\mathbf{x})|^2 \mathfrak{g}(\mathbf{x})(\mathbf{A}(\mathbf{x}) - \mathbf{k}_0) + 2\operatorname{Im}(\phi(\mathbf{x})^* \mathfrak{g}(\mathbf{x}) \nabla \phi(\mathbf{x})). \tag{19.11}$$

Now we consider the homogenization problem for the operator $\mathcal{M}$. We put

$$\widetilde{\mathcal{M}}_\varepsilon = ((\phi^\varepsilon)^*)^{-1} \mathbf{D}^* g^\varepsilon \mathbf{D} (\phi^\varepsilon)^{-1}, \quad \mathcal{M}_\varepsilon = [e^{i\varepsilon^{-1}\langle \mathbf{k}_0, \cdot \rangle}] \widetilde{\mathcal{M}}_\varepsilon [e^{-i\varepsilon^{-1}\langle \mathbf{k}_0, \cdot \rangle}]. \tag{19.12}$$

In the initial terms,

$$\mathcal{M}_\varepsilon = (\mathbf{D} - \varepsilon^{-1}\mathbf{A}^\varepsilon)^* \mathfrak{g}^\varepsilon (\mathbf{D} - \varepsilon^{-1}\mathbf{A}^\varepsilon) + \varepsilon^{-2} p^\varepsilon. \tag{19.13}$$

19.2

The behavior of the resolvent $(\widetilde{\mathcal{M}}_\varepsilon + I)^{-1}$ is regulated by Theorem 16.3. Indeed, now $f(\mathbf{x}) = \phi(\mathbf{x})^{-1}$, $Q(\mathbf{x}) = |\phi(\mathbf{x})|^2$, and, by (19.5), $\overline{Q} = 1$. Let g^0 be the effective matrix for the operator $\widehat{\mathcal{A}} = \mathbf{D}^* g \mathbf{D}$, and let $\widehat{\mathcal{A}}^0 = \mathbf{D}^* g^0 \mathbf{D}$. The matrix $g_2(\mathbf{x}) = \operatorname{Im} g(\mathbf{x})$ satisfies condition (17.16) with $q = 2s > d$, since the function in the right-hand side of (19.11) belongs to $L_{2s}(\Omega)$, by conditions (18.2), (19.3), (19.6), and (19.7). This guarantees (see Subsection 17.3) that the solutions Φ_j of the problem (17.3) satisfy $\Phi_j \in L_\infty$, $j = 1, \dots, d$. Thus, Condition 8.4 is now satisfied, and, therefore, Theorem 16.3 is applicable. As a result, we obtain the estimate

$$\begin{aligned} &\|(\phi^\varepsilon)^{-1}(\widetilde{\mathcal{M}}_\varepsilon + I)^{-1} \\ &- \Big(I + \varepsilon \sum_{j=1}^d \Phi_j^\varepsilon \partial_j\Big)(\widehat{\mathcal{A}}^0 + I)^{-1}(\phi^\varepsilon)^* \|_{L_2(\mathbb{R}^d) \to H^1(\mathbb{R}^d)} \le \widetilde{\mathcal{C}}_{11}\varepsilon, \quad 0 < \varepsilon \le 1, \end{aligned} \tag{19.14}$$

where $\widetilde{\mathcal{C}}_{11} = \check{\mathcal{C}}_{11}\|\phi\|_{L_\infty}$. (Here $\check{\mathcal{C}}_{11}$ is the constant from (17.20) with $Q = |\phi|^2$.) The constant $\widetilde{\mathcal{C}}_{11}$ depends on d, $\|g\|_{L_\infty}$, $\|g^{-1}\|_{L_\infty}$, $\|\phi\|_{L_\infty}$, $\|\phi^{-1}\|_{L_\infty}$, and also on $\|\mathbf{A}\|_{L_{2s}(\Omega)}$, $\|\phi\|_{W^1_{2s}(\Omega)}$, and on parameters of the lattice Γ. We see that, for the operator $\mathcal{M}_\varepsilon$, it is harder to control the constants in estimates explicitly.

By (19.12), we have

$$(\widetilde{\mathcal{M}}_\varepsilon + I)^{-1} = [e^{-i\varepsilon^{-1}\langle \mathbf{k}_0, \cdot\rangle}](\mathcal{M}_\varepsilon + I)^{-1}[e^{i\varepsilon^{-1}\langle \mathbf{k}_0, \cdot\rangle}]. \tag{19.15}$$

Since the operator of multiplication by $e^{i\varepsilon^{-1}\langle \mathbf{k}_0, \mathbf{x}\rangle}$ is unitary in $L_2(\mathbb{R}^d)$, from (19.14) and (19.15) it follows that

$$\begin{aligned} &\|(\phi^\varepsilon)^{-1}[e^{-i\varepsilon^{-1}\langle \mathbf{k}_0, \cdot\rangle}](\mathcal{M}_\varepsilon + I)^{-1} \\ &- \Big(I + \varepsilon \sum_{j=1}^d \Phi_j^\varepsilon \partial_j\Big)(\widehat{\mathcal{A}}^0 + I)^{-1}(\phi^\varepsilon)^*[e^{-i\varepsilon^{-1}\langle \mathbf{k}_0, \cdot\rangle}]\|_{L_2(\mathbb{R}^d) \to H^1(\mathbb{R}^d)} \\ &\le \widetilde{\mathcal{C}}_{11}\varepsilon, \quad 0 < \varepsilon \le 1. \end{aligned} \tag{19.16}$$

We summarize the results.

Theorem 19.1. *Let $\mathcal{M}$ be the operator* (19.1) *with Γ-periodic real coefficients satisfying conditions* (18.2) *and* (19.2)–(19.4). *Let the norm $\|\mathbf{A}\|_{L_{2s}(\Omega)}$ be sufficiently small, so that factorization* (19.9) *for the operator* (19.8) *takes place, where the Γ-periodic matrix $g(\mathbf{x})$ and function $\phi(\mathbf{x})$ are subject to conditions* (19.10) *and* (19.5), (19.6). *Let $\mathcal{M}_\varepsilon$ be the operator defined by* (19.13). *Let g^0 be the effective matrix for the operator $\mathbf{D}^* g(\mathbf{x})\mathbf{D}$, and let $\widehat{\mathcal{A}}^0 = \mathbf{D}^* g^0 \mathbf{D}$. Let $\Phi_j \in \widetilde{H}^1(\Omega)$ be the solutions of the problem* (17.3), *$j = 1, \dots, d$. Then for $0 < \varepsilon \le 1$ estimate* (19.16) *is true.*

Now we formulate (19.16) in terms of solutions. Let w_ε be the solution of the equation

$$\mathcal{M}_\varepsilon w_\varepsilon + w_\varepsilon = F, \quad F \in L_2(\mathbb{R}^d), \tag{19.17}$$

and let w_ε^0 be the solution of the equation

$$\widehat{\mathcal{A}}^0 w_\varepsilon^0 + w_\varepsilon^0 = (\phi^\varepsilon(\mathbf{x}))^* e^{-i\varepsilon^{-1}\langle \mathbf{k}_0, \mathbf{x}\rangle} F(\mathbf{x}). \tag{19.18}$$

Then

$$\|(\phi^\varepsilon)^{-1} e^{-i\varepsilon^{-1}\langle \mathbf{k}_0, \cdot\rangle} w_\varepsilon - w_\varepsilon^0 - \varepsilon \sum_{j=1}^d \Phi_j^\varepsilon \partial_j w_\varepsilon^0\|_{H^1(\mathbb{R}^d)} \le \widetilde{\mathcal{C}}_{11}\varepsilon \|F\|_{L_2(\mathbb{R}^d)}.$$

19.3

We can apply (interpolational) Theorem 16.7 to the resolvent $(\widetilde{\mathcal{M}}_\varepsilon + I)^{-1}$. The operator $K_Q^0(\varepsilon)$ is defined according to (17.22). By $Q = |\phi|^2$ and $\overline{Q} = 1$, we have

$$\begin{aligned} K_Q^0(\varepsilon) &= \sum_{j=1}^d \Phi_j^\varepsilon \partial_j (\widehat{\mathcal{A}}^0 + I)^{-1} - \sum_{j=1}^d (\widehat{\mathcal{A}}^0 + I)^{-1} \partial_j (\Phi_j^\varepsilon)^* \\ &- (\widehat{\mathcal{A}}^0 + I)^{-1} \Big(\sum_{j,l,s=1}^d (a_{jls} - a_{jls}^*) \partial_j \partial_l \partial_s + 2i \sum_{j=1}^d (\operatorname{Im} \overline{|\phi|^2 \Phi_j}) \partial_j \Big) (\widehat{\mathcal{A}}^0 + I)^{-1}. \end{aligned} \tag{19.19}$$

The values a_{jls} are defined by (17.18).

Applying (16.14), for $0 \le s \le 1$ and $0 < \varepsilon \le 1$ we obtain:

$$\|(\phi^\varepsilon)^{-1}(\widetilde{\mathcal{M}}_\varepsilon + I)^{-1} - \Big((\widehat{\mathcal{A}}^0 + I)^{-1} + \varepsilon K_Q^0(\varepsilon) \Big) (\phi^\varepsilon)^* \|_{L_2(\mathbb{R}^d) \to H^s(\mathbb{R}^d)} \le \widetilde{C}_s \varepsilon^{2-s}.$$

Here $\widetilde{C}_s = C_{Q,s}^0 \|\phi\|_{L_\infty}$, and $C_{Q,s}^0$ is the constant from (17.23) with $Q = |\phi|^2$. Using (19.15), we arrive at the estimate

$$\begin{aligned} &\|(\phi^\varepsilon)^{-1}[e^{-i\varepsilon^{-1}\langle \mathbf{k}_0, \cdot \rangle}](\mathcal{M}_\varepsilon + I)^{-1} \\ &\quad - \Big((\widehat{\mathcal{A}}^0 + I)^{-1} + \varepsilon K_Q^0(\varepsilon) \Big) (\phi^\varepsilon)^* [e^{-i\varepsilon^{-1}\langle \mathbf{k}_0, \cdot \rangle}] \|_{L_2(\mathbb{R}^d) \to H^s(\mathbb{R}^d)} \le \widetilde{C}_s \varepsilon^{2-s}. \end{aligned} \tag{19.20}$$

We have obtained the following statement.

Theorem 19.2. *Suppose that conditions of Theorem* 19.1 *are satisfied. Let* $K_Q^0(\varepsilon)$ *be the corrector defined by* (19.19)*. Then for* $0 \le s \le 1$ *and* $0 < \varepsilon \le 1$ *estimate* (19.20) *is valid, where the constant* $\widetilde{C}_s$ *depends on* s, d, $\|g\|_{L_\infty}$, $\|g^{-1}\|_{L_\infty}$, $\|\phi\|_{L_\infty}$, $\|\phi^{-1}\|_{L_\infty}$, $\|\mathbf{A}\|_{L_{2s}(\Omega)}$, $\|\phi\|_{W_{2s}^1(\Omega)}$, *and on parameters of the lattice* Γ.

19.4

We can apply Theorem 16.11 for the fluxes. By this theorem, for $0 < \varepsilon \le 1$ we have

$$\|g^\varepsilon \nabla (\phi^\varepsilon)^{-1} (\widetilde{\mathcal{M}}_\varepsilon + I)^{-1} - \widetilde{g}^\varepsilon \nabla (\widehat{\mathcal{A}}^0 + I)^{-1} (\phi^\varepsilon)^* \|_{L_2(\mathbb{R}^d) \to L_2(\mathbb{R}^d; \mathbb{C}^d)} \le \widetilde{\mathcal{C}}_{16} \varepsilon.$$

Here $\widetilde{\mathcal{C}}_{16} = \check{\mathcal{C}}_{16} \|\phi\|_{L_\infty}$, and $\check{\mathcal{C}}_{16}$ is the constant from (17.24) with $Q = |\phi|^2$. By (19.15), this implies that

$$\begin{aligned} &\|g^\varepsilon \nabla (\phi^\varepsilon)^{-1} [e^{-i\varepsilon^{-1}\langle \mathbf{k}_0, \cdot \rangle}] (\mathcal{M}_\varepsilon + I)^{-1} \\ &\quad - \widetilde{g}^\varepsilon \nabla (\widehat{\mathcal{A}}^0 + I)^{-1} (\phi^\varepsilon)^* [e^{-i\varepsilon^{-1}\langle \mathbf{k}_0, \cdot \rangle}] \|_{L_2(\mathbb{R}^d) \to L_2(\mathbb{R}^d; \mathbb{C}^d)} \le \widetilde{\mathcal{C}}_{16} \varepsilon, \quad 0 < \varepsilon \le 1. \end{aligned}$$

In other words,

$$\|g^\varepsilon \nabla (\phi^\varepsilon)^{-1} e^{-i\varepsilon^{-1}\langle \mathbf{k}_0, \cdot \rangle} w_\varepsilon - \widetilde{g}^\varepsilon \nabla w_\varepsilon^0 \|_{L_2(\mathbb{R}^d; \mathbb{C}^d)} \le \widetilde{\mathcal{C}}_{16} \varepsilon \|F\|_{L_2(\mathbb{R}^d)}, \quad 0 < \varepsilon \le 1. \tag{19.21}$$

We arrive at the following result.

Theorem 19.3. *Suppose that conditions of Theorem* 19.1 *are satisfied. Let* w_ε *be the solution of the equation* (19.17)*, and let* w_ε^0 *be the solution of the equation* (19.18)*. Then estimate* (19.21) *is true, where the constant* $\widetilde{\mathcal{C}}_{16}$ *depends on* d, $\|g\|_{L_\infty}$, $\|g^{-1}\|_{L_\infty}$, $\|\phi\|_{L_\infty}$, $\|\phi^{-1}\|_{L_\infty}$, $\|\mathbf{A}\|_{L_{2s}(\Omega)}$, $\|\phi\|_{W_{2s}^1(\Omega)}$, *and on parameters of the lattice* Γ.

Now we distinguish special cases. Under the condition $g^0 = \overline{g}$, Theorems 16.4 and 16.8 are applicable. This leads to the following result.

Theorem 19.4. *Suppose that conditions of Theorem* 19.1 *are satisfied, and that* $g^0 = \overline{g}$, *i. e., relations* (17.25) *are valid. Then for* $0 < \varepsilon \leq 1$ *we have*

$$\|(\phi^\varepsilon)^{-1}[e^{-i\varepsilon^{-1}\langle \mathbf{k}_0, \cdot\rangle}](\mathcal{M}_\varepsilon + I)^{-1} - (\widehat{\mathcal{A}}^0 + I)^{-1}(\phi^\varepsilon)^*[e^{-i\varepsilon^{-1}\langle \mathbf{k}_0, \cdot\rangle}]\|_{L_2(\mathbb{R}^d)\to H^1(\mathbb{R}^d)} \leq \widetilde{\mathcal{C}}_{12}\varepsilon,$$

where the constant $\widetilde{\mathcal{C}}_{12} = \check{\mathcal{C}}_{12}\|\phi\|_{L_\infty}$ *depends on* d, $\|g\|_{L_\infty}$, $\|g^{-1}\|_{L_\infty}$, $\|\phi\|_{L_\infty}$, $\|\phi^{-1}\|_{L_\infty}$, *and on parameters of the lattice* Γ. (*Here* $\check{\mathcal{C}}_{12}$ *is the constant from* (17.26) *with* $Q = |\phi|^2$.) *For* $0 \leq s \leq 1$ *and* $0 < \varepsilon \leq 1$ *we have*

$$\|(\phi^\varepsilon)^{-1}[e^{-i\varepsilon^{-1}\langle \mathbf{k}_0, \cdot\rangle}](\mathcal{M}_\varepsilon + I)^{-1} - (\widehat{\mathcal{A}}^0 + I)^{-1}(\phi^\varepsilon)^*[e^{-i\varepsilon^{-1}\langle \mathbf{k}_0, \cdot\rangle}]\|_{L_2(\mathbb{R}^d)\to H^s(\mathbb{R}^d)} \leq \widetilde{C}'_s\varepsilon^{2-s},$$

where the constant $\widetilde{C}'_s = C'_{Q,s}\|\phi\|_{L_\infty}$ *depends on* s, d, $\|g\|_{L_\infty}$, $\|g^{-1}\|_{L_\infty}$, $\|\phi\|_{L_\infty}$, $\|\phi^{-1}\|_{L_\infty}$, *and on parameters of the lattice* Γ. (*Here* $C'_{Q,s}$ *is the constant from* (17.27) *with* $Q = |\phi|^2$.)

If $g^0 = \underline{g}$, then Theorem 16.12 is applicable. This leads to the following statement.

Theorem 19.5. *Suppose that conditions of Theorem* 19.3 *are satisfied, and that* $g^0 = \underline{g}$, *i. e., relations* (17.28) *for the columns of the matrix* $g(\mathbf{x})^{-1}$ *are valid. Then for* $0 < \varepsilon \leq 1$ *we have*

$$\|g^\varepsilon \nabla (\phi^\varepsilon)^{-1} e^{-i\varepsilon^{-1}\langle \mathbf{k}_0, \cdot\rangle} w_\varepsilon - g^0 \nabla w_\varepsilon^0\|_{L_2(\mathbb{R}^d;\mathbb{C}^d)} \leq \widetilde{\mathcal{C}}_{16}\varepsilon \|F\|_{L_2(\mathbb{R}^d)},$$

where the constant $\widetilde{\mathcal{C}}_{16} = \check{\mathcal{C}}_{16}\|\phi\|_{L_\infty}$ *depends on* d, $\|g\|_{L_\infty}$, $\|g^{-1}\|_{L_\infty}$, $\|\phi\|_{L_\infty}$, $\|\phi^{-1}\|_{L_\infty}$, *and on parameters of the lattice* Γ. (*Here* $\check{\mathcal{C}}_{16}$ *is the constant from* (17.29) *with* $Q = |\phi|^2$.)

For the magnetic Schrödinger operator, general results of Subsections 16.6–16.8 about approximation of the generalized resolvent $(\mathcal{M}_\varepsilon + \mathfrak{Q}^\varepsilon)^{-1}$ can be realized. We shall not dwell on the detailed formulations.

Remark 19.6. As it was shown in [Sh1], without the smallness condition on $\mathbf{A}$, the required factorization for the magnetic Schrödinger operator, in general, is destroyed.

§20. The twodimensional periodic Pauli operator

20.1. The operators $\widehat{\mathcal{B}}_\pm$

The examples considered in Subsections 20.1–20.4 (cf. [BSu2, Subsection 5.1.3], [BSu4, Subsections 12.1, 12.2]) are of preliminary nature. They will be useful for the study of the twodimensional Pauli operator in Subsections 20.5–20.7. However, these examples are interesting themselves.

Within Subsections 20.1 and 20.2, all formulas and statements should be read independently for upper and lower indices „±“.

Let $d = 2$, $m = n = 1$, and let $\omega_\pm(\mathbf{x})$ be a Γ-periodic function such that

$$\omega_\pm(\mathbf{x}) > 0; \quad \omega_\pm, \omega_\pm^{-1} \in L_\infty. \qquad (20.1\pm)$$

In $L_2(\mathbb{R}^2)$, we consider the operator

$$\widehat{\mathcal{B}}_\pm = \widehat{\mathcal{B}}_\pm(\omega_\pm^2) = \partial_\pm \omega_\pm^2 \partial_\mp, \tag{20.2$\pm$}$$

where $\partial_\pm = D_1 \pm iD_2$. (The operators $\widehat{\mathcal{B}}_+$ and $\widehat{\mathcal{B}}_-$ are of the same type, and it is convenient to consider them in parallel.) The operator $\widehat{\mathcal{B}}_\pm(\omega_\pm^2)$ is of the form $\widehat{\mathcal{A}}(g)$ with $b(\mathbf{D}) = D_1 \mp iD_2$ and $g = g_\pm = \omega_\pm^2$. Obviously, now we have $\alpha_0 = \alpha_1 = 1$. Since $m = n = 1$, the effective constant $g_\pm^0$ for the operator $\widehat{\mathcal{B}}_\pm$ is $\underline{g_\pm}$, i. e.,

$$g_\pm^0 = \underline{(\omega_\pm^2)} = \Big(|\Omega|^{-1} \int_\Omega \omega_\pm(\mathbf{x})^{-2}\, d\mathbf{x} \Big)^{-1}. \tag{20.3$\pm$}$$

Then the effective operator $\widehat{\mathcal{B}}_\pm^0$ for the operator $\widehat{\mathcal{B}}_\pm$ is given by

$$\widehat{\mathcal{B}}_\pm^0 = -g_\pm^0 \Delta. \tag{20.4$\pm$}$$

The solution $v_\pm \in \widetilde{H}^1(\Omega)$ of the problem

$$\partial_\pm \omega_\pm^2(\mathbf{x})(\partial_\mp v_\pm + 1) = 0, \quad \int_\Omega v_\pm(\mathbf{x})\, d\mathbf{x} = 0,$$

is simultaneously (see [BSu4, §12]) the solution of the problem

$$\partial_\mp v_\pm = g_\pm^0 \omega_\pm(\mathbf{x})^{-2} - 1, \quad \int_\Omega v_\pm(\mathbf{x})\, d\mathbf{x} = 0. \tag{20.5$\pm$}$$

For the operator $\widehat{\mathcal{B}}_\pm$, the role of the „matrix" $\Lambda(\mathbf{x})$ is played by the function $v_\pm(\mathbf{x})$.

We consider the operator

$$\widehat{\mathcal{B}}_{\pm,\varepsilon} = \widehat{\mathcal{B}}_\pm((\omega_\pm^\varepsilon)^2) = \partial_\pm (\omega_\pm^\varepsilon)^2 \partial_\mp. \tag{20.6$\pm$}$$

Now Condition 8.6 is satisfied, since $d = 2$ (however, we might also refer to the relation $g_\pm^0 = \underline{g_\pm}$). Hence, Condition 8.4 is also satisfied. By Remark 8.8, the norm $\|v_\pm\|_{L_\infty}$ is estimated by the constant depending only on $\|\omega_\pm\|_{L_\infty}$, $\|\omega_\pm^{-1}\|_{L_\infty}$, and on parameters of the lattice Γ. Theorem 10.8 is applicable. This leads to the following result.

Theorem 20.1($\pm$). *Suppose that $\omega_\pm(\mathbf{x})$ is a Γ-periodic function in $\mathbb{R}^2$ subject to condition* (20.1$\pm$), *and let $\widehat{\mathcal{B}}_{\pm,\varepsilon}$ be the operator* (20.6$\pm$). *Let $\widehat{\mathcal{B}}_\pm^0 = -g_\pm^0 \Delta$, where $g_\pm^0$ is defined by* (20.3$\pm$). *Let $v_\pm \in \widetilde{H}^1(\Omega)$ be the solution of the problem* (20.5$\pm$). *Then for $0 < \varepsilon \le 1$ we have*

$$\|(\widehat{\mathcal{B}}_{\pm,\varepsilon} + I)^{-1} - (\widehat{\mathcal{B}}_\pm^0 + I)^{-1} - \varepsilon v_\pm^\varepsilon \partial_\mp (\widehat{\mathcal{B}}_\pm^0 + I)^{-1}\|_{L_2(\mathbb{R}^2) \to H^1(\mathbb{R}^2)} \le \mathcal{C}_{11}^{(\pm)} \varepsilon, \tag{20.7$\pm$}$$

where the constant $\mathcal{C}_{11}^{(\pm)}$ depends only on $\|\omega_\pm\|_{L_\infty}$, $\|\omega_\pm^{-1}\|_{L_\infty}$, and on parameters of the lattice Γ.

We can also apply (interpolational) Theorem 11.4. The corrector (10.11) for the operator $\widehat{\mathcal{B}}_{\pm,\varepsilon}$ is of the form (cf. [BSu4, Proposition 12.1($\pm$)])

$$K_\pm^0(\varepsilon) = v_\pm^\varepsilon \partial_\mp (\widehat{\mathcal{B}}_\pm^0 + I)^{-1} + (\widehat{\mathcal{B}}_\pm^0 + I)^{-1} \partial_\pm (v_\pm^\varepsilon)^*. \tag{20.8$\pm$}$$

Note that the third term of the corrector (10.11) is now equal to zero. We arrive at the following result.

Theorem 20.2($\pm$). *Suppose that conditions of Theorem* 20.1($\pm$) *are satisfied. Let* $K^0_\pm(\varepsilon)$ *be the corrector defined by* (20.8$\pm$). *Then for* $0 \le s \le 1$ *and* $0 < \varepsilon \le 1$ *we have*

$$\|(\widehat{\mathcal{B}}_{\pm,\varepsilon} + I)^{-1} - (\widehat{\mathcal{B}}^0_\pm + I)^{-1} - \varepsilon K^0_\pm(\varepsilon)\|_{L_2(\mathbb{R}^2)\to H^s(\mathbb{R}^2)} \le C^{(\pm)}_s \varepsilon^{2-s}.$$

The constant $C^{(\pm)}_s$ *depends on* s, *on* $\|\omega_\pm\|_{L_\infty}$, $\|\omega_\pm^{-1}\|_{L_\infty}$, *and on parameters of the lattice* Γ.

Let $u_{\pm,\varepsilon}$ be the solution of the equation

$$\widehat{\mathcal{B}}_{\pm,\varepsilon} u_{\pm,\varepsilon} + u_{\pm,\varepsilon} = F_\pm, \quad F_\pm \in L_2(\mathbb{R}^2), \tag{20.9$\pm$}$$

and let $u^0_\pm$ be the solution of the „homogenized“ equation

$$\widehat{\mathcal{B}}^0_\pm u^0_\pm + u^0_\pm = F_\pm. \tag{20.10$\pm$}$$

Estimate (20.7$\pm$) means that

$$\|u_{\pm,\varepsilon} - u^0_\pm - \varepsilon v^\varepsilon_\pm \partial_\mp u^0_\pm\|_{H^1(\mathbb{R}^2)} \le \mathcal{C}^{(\pm)}_{11} \varepsilon \|F_\pm\|_{L_2(\mathbb{R}^2)}, \quad 0 < \varepsilon \le 1.$$

Note that, as $\varepsilon \to 0$, the functions $u_{\pm,\varepsilon}$ converge to $u^0_\pm$ weakly in $H^1(\mathbb{R}^2)$ (see Remark 10.7).

Since $g^0_\pm = \underline{g_\pm}$, then we can apply Theorem 12.4 for the fluxes. This leads to the following result.

Theorem 20.3($\pm$). *Suppose that conditions of Theorem* 20.1($\pm$) *are satisfied. Let* $u_{\pm,\varepsilon}$ *be the solution of the equation* (20.9$\pm$), *and let* $u^0_\pm$ *be the solution of the equation* (20.10$\pm$). *We put*

$$p_{\pm,\varepsilon} = (\omega^\varepsilon_\pm)^2 \partial_\mp u_{\pm,\varepsilon}, \quad p^0_\pm = g^0_\pm \partial_\mp u^0_\pm. \tag{20.11$\pm$}$$

Then

$$\|p_{\pm,\varepsilon} - p^0_\pm\|_{L_2(\mathbb{R}^2)} \le \mathcal{C}^{(\pm)}_{16} \varepsilon \|F_\pm\|_{L_2(\mathbb{R}^2)}, \quad 0 < \varepsilon \le 1,$$

where the constant $\mathcal{C}^{(\pm)}_{16}$ *depends on* $\|\omega_\pm\|_{L_\infty}$, $\|\omega_\pm^{-1}\|_{L_\infty}$, *and on parameters of the lattice* Γ.

20.2

Now we consider the generalized resolvent $(\widehat{\mathcal{B}}_{\pm,\varepsilon} + Q^\varepsilon_\pm)^{-1}$, where $Q_\pm(\mathbf{x})$ is a Γ-periodic function such that

$$Q_\pm(\mathbf{x}) > 0; \quad Q_\pm, Q_\pm^{-1} \in L_\infty. \tag{20.12$\pm$}$$

We apply Theorem 13.8, which leads to the folowing result.

Theorem 20.4($\pm$). *Suppose that conditions of Theorem* 20.1($\pm$) *are satisfied. Let* $Q_\pm(\mathbf{x})$ *be a* Γ*-periodic function satisfying conditions* (20.12$\pm$), *and let* $\overline{Q_\pm}$ *be the mean value of* $Q_\pm(\mathbf{x})$ *over the cell* Ω. *Then for* $0 < \varepsilon \le 1$ *we have*

$$\|(\widehat{\mathcal{B}}_{\pm,\varepsilon} + Q^\varepsilon_\pm)^{-1} - (\widehat{\mathcal{B}}^0_\pm + \overline{Q_\pm})^{-1} - \varepsilon v^\varepsilon_\pm \partial_\mp (\widehat{\mathcal{B}}^0_\pm + \overline{Q_\pm})^{-1}\|_{L_2(\mathbb{R}^2)\to H^1(\mathbb{R}^2)} \le \check{\mathcal{C}}^{(\pm)}_{11} \varepsilon, \tag{20.13$\pm$}$$

where the constant $\check{\mathcal{C}}^{(\pm)}_{11}$ *depends only on* $\|\omega_\pm\|_{L_\infty}$, $\|\omega_\pm^{-1}\|_{L_\infty}$, $\|Q_\pm\|_{L_\infty}$, $\|Q_\pm^{-1}\|_{L_\infty}$, *and on parameters of the lattice* Γ.

We can apply (interpolational) Theorem 14.4. The corrector (13.4) for the operator $\widehat{\mathcal{B}}_{\pm,\varepsilon}$ takes the form (cf. [BSu4, Proposition 12.2(±)])

$$\begin{aligned} K^0_{\pm,Q_\pm}(\varepsilon) &= v^\varepsilon_\pm \partial_\mp (\widehat{\mathcal{B}}^0_\pm + \overline{Q_\pm})^{-1} + (\widehat{\mathcal{B}}^0_\pm + \overline{Q_\pm})^{-1} \partial_\pm (v^\varepsilon_\pm)^* \\ &- (\widehat{\mathcal{B}}^0_\pm + \overline{Q_\pm})^{-1} \left(2D_1 (\operatorname{Re} \overline{Q_\pm v_\pm}) \pm 2D_2 (\operatorname{Im} \overline{Q_\pm v_\pm}) \right) (\widehat{\mathcal{B}}^0_\pm + \overline{Q_\pm})^{-1}. \end{aligned} \tag{20.14±}$$

As a result, we arrive at the following statement.

Theorem 20.5(±). *Suppose that conditions of Theorem 20.4(±) are satisfied. Let $K^0_{\pm,Q_\pm}(\varepsilon)$ be the corrector defined by (20.14±). Then for $0 \le s \le 1$ and $0 < \varepsilon \le 1$ we have*

$$\| (\widehat{\mathcal{B}}_{\pm,\varepsilon} + Q^\varepsilon_\pm)^{-1} - (\widehat{\mathcal{B}}^0_\pm + \overline{Q_\pm})^{-1} - \varepsilon K^0_{\pm,Q_\pm}(\varepsilon) \|_{L_2(\mathbb{R}^2) \to H^s(\mathbb{R}^2)} \le C^{(\pm)}_{s,Q_\pm} \varepsilon^{2-s}, \tag{20.15±}$$

where the constant $C^{(\pm)}_{s,Q_\pm}$ depends on s, on $\|\omega_\pm\|_{L_\infty}$, $\|\omega_\pm^{-1}\|_{L_\infty}$, $\|Q_\pm\|_{L_\infty}$, $\|Q_\pm^{-1}\|_{L_\infty}$, and on parameters of the lattice Γ.

Let $a_{\pm,\varepsilon}$ be the solution of the equation

$$\widehat{\mathcal{B}}_{\pm,\varepsilon} a_{\pm,\varepsilon} + Q^\varepsilon_\pm a_{\pm,\varepsilon} = F_\pm, \quad F_\pm \in L_2(\mathbb{R}^2), \tag{20.16±}$$

and let $a^0_\pm$ be the solution of the „homogenized“ equation

$$\widehat{\mathcal{B}}^0_\pm a^0_\pm + \overline{Q_\pm} a^0_\pm = F_\pm. \tag{20.17±}$$

Estimate (20.13±) means that

$$\| a_{\pm,\varepsilon} - a^0_\pm - \varepsilon v^\varepsilon_\pm \partial_\mp a^0_\pm \|_{H^1(\mathbb{R}^2)} \le \check{\mathcal{C}}^{(\pm)}_{11} \varepsilon \| F_\pm \|_{L_2(\mathbb{R}^2)}.$$

Note that, as $\varepsilon \to 0$, the weak limit of the functions $a_{\pm,\varepsilon}$ in $H^1(\mathbb{R}^2)$ is equal to $a^0_\pm$ (see Remark 13.7).

Since $g^0_\pm = \underline{g_\pm}$, we can apply Theorem 15.4 for the fluxes. This leads to the following result.

Theorem 20.6(±). *Suppose that conditions of Theorem 20.4(±) are satisfied. Let $a_{\pm,\varepsilon}$ be the solution of the equation (20.16±), and let $a^0_\pm$ be the solution of the equation (20.17±). We put*

$$q_{\pm,\varepsilon} = (\omega^\varepsilon_\pm)^2 \partial_\mp a_{\pm,\varepsilon}, \quad q^0_\pm = g^0_\pm \partial_\mp a^0_\pm.$$

Then

$$\| q_{\pm,\varepsilon} - q^0_\pm \|_{L_2(\mathbb{R}^2)} \le \check{\mathcal{C}}^{(\pm)}_{16} \varepsilon \| F_\pm \|_{L_2(\mathbb{R}^2)}, \quad 0 < \varepsilon \le 1, \tag{20.18±}$$

where the constant $\check{\mathcal{C}}^{(\pm)}_{16}$ depends on $\|\omega_\pm\|_{L_\infty}$, $\|\omega_\pm^{-1}\|_{L_\infty}$, $\|Q_\pm\|_{L_\infty}$, $\|Q_\pm^{-1}\|_{L_\infty}$, and on parameters of the lattice Γ.

20.3. The operator $\widehat{\mathcal{B}}_\times$

Now we consider the matrix operator consisting of the blocks $\widehat{\mathcal{B}}_-(\omega_-^2)$, $\widehat{\mathcal{B}}_+(\omega_+^2)$. We have $d = 2$ and $m = n = 2$. Let $\omega_\pm(\mathbf{x})$ be two Γ-periodic functions subject to conditions (20.1±). We put

$$h_\times = \operatorname{diag}\{\omega_+, \omega_-\}, \quad g_\times = h_\times^2 = \operatorname{diag}\{\omega_+^2, \omega_-^2\}. \tag{20.19}$$

In $L_2(\mathbb{R}^2;\mathbb{C}^2)$, we consider the operators

$$b_\times(\mathbf{D}) = \begin{pmatrix} 0 & \partial_- \\ \partial_+ & 0 \end{pmatrix}, \quad \widehat{\mathcal{B}}_\times = b_\times(\mathbf{D}) g_\times b_\times(\mathbf{D}). \tag{20.20}$$

Then

$$\widehat{\mathcal{B}}_\times = \operatorname{diag}\{\partial_- \omega_-^2 \partial_+, \partial_+ \omega_+^2 \partial_-\} = \operatorname{diag}\{\widehat{\mathcal{B}}_-(\omega_-^2), \widehat{\mathcal{B}}_+(\omega_+^2)\}. \tag{20.21}$$

The effective matrix for the operator (20.21) takes the form

$$g_\times^0 = \underline{g_\times} = \operatorname{diag}\{g_+^0, g_-^0\}, \quad g_\pm^0 = \underline{(\omega_\pm^2)}, \tag{20.22}$$

and the effective operator is given by

$$\widehat{\mathcal{B}}_\times^0 = b_\times(\mathbf{D}) g_\times^0 b_\times(\mathbf{D}) = \operatorname{diag}\{\widehat{\mathcal{B}}_-^0, \widehat{\mathcal{B}}_+^0\}, \quad \widehat{\mathcal{B}}_\pm^0 = -g_\pm^0 \Delta. \tag{20.23}$$

The role of the matrix $\Lambda(\mathbf{x})$ for the operator $\widehat{\mathcal{B}}_\times$ is played by the matrix

$$\Lambda_\times(\mathbf{x}) = \begin{pmatrix} 0 & v_-(\mathbf{x}) \\ v_+(\mathbf{x}) & 0 \end{pmatrix}, \tag{20.24}$$

where $v_\pm$ is the Γ-periodic solution of the problem (20.5±).

Now we consider the operator

$$\widehat{\mathcal{B}}_{\times,\varepsilon} = b_\times(\mathbf{D}) g_\times^\varepsilon b_\times(\mathbf{D}) = \operatorname{diag}\{\widehat{\mathcal{B}}_{-,\varepsilon}, \widehat{\mathcal{B}}_{+,\varepsilon}\}. \tag{20.25}$$

We apply Theorem 10.8, which leads to the following result.

Theorem 20.7. *Let $\omega_\pm$ be two Γ-periodic functions in $\mathbb{R}^2$ suject to conditions* (20.1±)*. Let $\widehat{\mathcal{B}}_{\times,\varepsilon}$ be the operator defined according to* (20.19), (20.20), (20.25), *and let $\widehat{\mathcal{B}}_\times^0$ be the effective operator defined by* (20.22), (20.23)*. Let $v_\pm \in \widetilde{H}^1(\Omega)$ be the solution of the problem* (20.5±)*, and let $\Lambda_\times(\mathbf{x})$ be the matrix* (20.24)*. Then for $0 < \varepsilon \le 1$ we have*

$$\|(\widehat{\mathcal{B}}_{\times,\varepsilon} + I)^{-1} - (\widehat{\mathcal{B}}_\times^0 + I)^{-1} - \varepsilon \Lambda_\times^\varepsilon b_\times(\mathbf{D})(\widehat{\mathcal{B}}_\times^0 + I)^{-1}\|_{L_2(\mathbb{R}^2;\mathbb{C}^2)\to H^1(\mathbb{R}^2;\mathbb{C}^2)} \le \mathcal{C}_{11}^\times \varepsilon. \tag{20.26}$$

The constant $\mathcal{C}_{11}^\times$ depends only on the norms $\|\omega_+\|_{L_\infty}$, $\|\omega_+^{-1}\|_{L_\infty}$, $\|\omega_-\|_{L_\infty}$, $\|\omega_-^{-1}\|_{L_\infty}$, and on parameters of the lattice Γ.

Note that all operators in (20.26) are diagonal:

$$\begin{aligned} &(\widehat{\mathcal{B}}_{\times,\varepsilon} + I)^{-1} - (I + \varepsilon \Lambda_\times^\varepsilon b_\times(\mathbf{D}))(\widehat{\mathcal{B}}_\times^0 + I)^{-1} \\ &\quad = \operatorname{diag}\{(\widehat{\mathcal{B}}_{-,\varepsilon} + I)^{-1} - (I + \varepsilon v_-^\varepsilon \partial_+)(\widehat{\mathcal{B}}_-^0 + I)^{-1}, \\ &\qquad\qquad (\widehat{\mathcal{B}}_{+,\varepsilon} + I)^{-1} - (I + \varepsilon v_+^\varepsilon \partial_-)(\widehat{\mathcal{B}}_+^0 + I)^{-1}\}. \end{aligned} \tag{20.27}$$

Therefore, the result of Theorem 20.7 could be also obtained from Theorems 20.1(+) and 20.1(−).

We can also apply (interpolational) Theorem 11.4 to the operator (20.25). The corrector (10.11) for the operator $\widehat{\mathcal{B}}_{\times,\varepsilon}$ takes the form (cf. [BSu4, Proposition 12.3])

$$K_\times^0(\varepsilon) = \Lambda_\times^\varepsilon b_\times(\mathbf{D})(\widehat{\mathcal{B}}_\times^0 + I)^{-1} + (\widehat{\mathcal{B}}_\times^0 + I)^{-1} b_\times(\mathbf{D})(\Lambda_\times^\varepsilon)^*. \tag{20.28}$$

Note that

$$K_\times^0(\varepsilon) = \operatorname{diag}\{K_-^0(\varepsilon), K_+^0(\varepsilon)\},$$

where the operators $K_\pm^0(\varepsilon)$ are defined by (20.8±). We arrive at the following statement.

Theorem 20.8. *Suppose that conditions of Theorem* 20.7 *are satisfied. Let* $K^0_\times(\varepsilon)$ *be the corrector defined by* (20.28). *Then for* $0 \le s \le 1$ *and* $0 < \varepsilon \le 1$ *we have*

$$\|(\widehat{\mathcal{B}}_{\times,\varepsilon} + I)^{-1} - (\widehat{\mathcal{B}}^0_\times + I)^{-1} - \varepsilon K^0_\times(\varepsilon)\|_{L_2(\mathbb{R}^2;\mathbb{C}^2)\to H^s(\mathbb{R}^2;\mathbb{C}^2)} \le C^\times_s \varepsilon^{2-s}. \tag{20.29}$$

The constant $C^\times_s$ *depends on* s, *on the norms* $\|\omega_+\|_{L_\infty}$, $\|\omega_+^{-1}\|_{L_\infty}$, $\|\omega_-\|_{L_\infty}$, $\|\omega_-^{-1}\|_{L_\infty}$, *and on parameters of the lattice* Γ.

Similarly to (20.27), the operator under the norm sign in (20.29) is diagonal, and the statement of Theorem 20.8 could be deduced from Theorems 20.2(+) and 20.2(−).

Let $\mathbf{u}_{\times,\varepsilon}$ be the solution of the equation

$$\widehat{\mathcal{B}}_{\times,\varepsilon}\mathbf{u}_{\times,\varepsilon} + \mathbf{u}_{\times,\varepsilon} = \mathbf{F}, \quad \mathbf{F} = \begin{pmatrix} F_- \\ F_+ \end{pmatrix} \in L_2(\mathbb{R}^2;\mathbb{C}^2), \tag{20.30}$$

and let $\mathbf{u}^0_\times$ be the solution of the equation

$$\widehat{\mathcal{B}}^0_\times \mathbf{u}^0_\times + \mathbf{u}^0_\times = \mathbf{F}. \tag{20.31}$$

Estimate (20.26) means that

$$\|\mathbf{u}_{\times,\varepsilon} - \mathbf{u}^0_\times - \varepsilon\Lambda^\varepsilon_\times b_\times(\mathbf{D})\mathbf{u}^0_\times\|_{H^1(\mathbb{R}^2;\mathbb{C}^2)} \le \mathcal{C}^\times_{11}\varepsilon\|\mathbf{F}\|_{L_2(\mathbb{R}^2;\mathbb{C}^2)}, \quad 0 < \varepsilon \le 1.$$

Note that, as $\varepsilon \to 0$, the weak limit of the functions $\mathbf{u}_{\times,\varepsilon}$ in $H^1(\mathbb{R}^2;\mathbb{C}^2)$ is equal to $\mathbf{u}^0_\times$.

Since $g^0_\times = \underline{g_\times}$, we can apply Theorem 12.4 for the fluxes. This leads to the folowing result.

Theorem 20.9. *Suppose that conditions of Theorem* 20.7 *are satisfied. Let* $\mathbf{u}_{\times,\varepsilon}$ *be the solution of the equation* (20.30), *and let* $\mathbf{u}^0_\times$ *be the solution of the equation* (20.31). *We put*

$$\mathbf{p}_{\times,\varepsilon} = g^\varepsilon_\times b_\times(\mathbf{D})\mathbf{u}_{\times,\varepsilon}, \quad \mathbf{p}^0_\times = g^0_\times b_\times(\mathbf{D})\mathbf{u}^0_\times.$$

Then

$$\|\mathbf{p}_{\times,\varepsilon} - \mathbf{p}^0_\times\|_{L_2(\mathbb{R}^2;\mathbb{C}^2)} \le \mathcal{C}^\times_{16}\varepsilon\|\mathbf{F}\|_{L_2(\mathbb{R}^2;\mathbb{C}^2)}, \quad 0 < \varepsilon \le 1. \tag{20.32}$$

The constant $\mathcal{C}^\times_{16}$ *depends on the norms* $\|\omega_+\|_{L_\infty}$, $\|\omega_+^{-1}\|_{L_\infty}$, $\|\omega_-\|_{L_\infty}$, $\|\omega_-^{-1}\|_{L_\infty}$, *and on parameters of the lattice* Γ.

Note that the fluxes $\mathbf{p}_{\times,\varepsilon}$ and $\mathbf{p}^0_\times$ are represented as

$$\mathbf{p}_{\times,\varepsilon} = \begin{pmatrix} p_{+,\varepsilon} \\ p_{-,\varepsilon} \end{pmatrix}, \quad \mathbf{p}^0_\times = \begin{pmatrix} p^0_+ \\ p^0_- \end{pmatrix},$$

where $p_{\pm,\varepsilon}$ and $p^0_\pm$ are defined by (20.11±). Therefore, the statement of Theorem 20.9 could be deduced from Theorems 20.3(+) and 20.3(−).

20.4

We proceed to the problem of approximation for the generalized resolvent $(\widehat{\mathcal{B}}_{\times,\varepsilon} + Q^\varepsilon_\times)^{-1}$, where $Q_\times(\mathbf{x})$ is a Γ-periodic Hermitian (2×2)-matrix (in general, with complex entries) such that

$$Q_\times(\mathbf{x}) > 0; \quad Q_\times, Q_\times^{-1} \in L_\infty. \tag{20.33}$$

We apply Theorem 13.8, which yields the following result.

Theorem 20.10. *Suppose that conditions of Theorem* 20.7 *are satisfied. Let* $Q_\times(\mathbf{x})$ *be a* Γ*-periodic* (2×2)*-matrix-valued function subject to conditions* (20.33)*. Let* $\overline{Q_\times}$ *be the mean value of* $Q_\times$ *over the cell* Ω*. Then for* $0<\varepsilon\le 1$ *we have*

$$\begin{aligned}&\|(\widehat{\mathcal{B}}_{\times,\varepsilon}+Q^\varepsilon_\times)^{-1}-(\widehat{\mathcal{B}}^0_\times+\overline{Q_\times})^{-1}\\&\quad-\varepsilon\Lambda^\varepsilon_\times b_\times(\mathbf{D})(\widehat{\mathcal{B}}^0_\times+\overline{Q_\times})^{-1}\|_{L_2(\mathbb{R}^2;\mathbb{C}^2)\to H^1(\mathbb{R}^2;\mathbb{C}^2)}\le\check{\mathcal{C}}^\times_{11}\varepsilon.\end{aligned}\tag{20.34}$$

The constant $\check{\mathcal{C}}^\times_{11}$ *depends only on* $\|\omega_+\|_{L_\infty}$, $\|\omega_+^{-1}\|_{L_\infty}$, $\|\omega_-\|_{L_\infty}$, $\|\omega_-^{-1}\|_{L_\infty}$, $\|Q_\times\|_{L_\infty}$, $\|Q_\times^{-1}\|_{L_\infty}$*, and on parameters of the lattice* Γ*.*

We also apply (interpolational) Theorem 14.4. The corrector (13.4) related to the operator $\widehat{\mathcal{B}}_\times$ and to the matrix $Q_\times$ has the form (cf. [BSu4, Proposition 12.4])

$$\begin{aligned}K^0_{\times,Q_\times}(\varepsilon)&=\Lambda^\varepsilon_\times b_\times(\mathbf{D})(\widehat{\mathcal{B}}^0_\times+\overline{Q_\times})^{-1}+(\widehat{\mathcal{B}}^0_\times+\overline{Q_\times})^{-1}b_\times(\mathbf{D})(\Lambda^\varepsilon_\times)^*\\&\quad-(\widehat{\mathcal{B}}^0_\times+\overline{Q_\times})^{-1}\left(b_\times(\mathbf{D})(\overline{Q_\times\Lambda_\times})^*+(\overline{Q_\times\Lambda_\times})b_\times(\mathbf{D})\right)(\widehat{\mathcal{B}}^0_\times+\overline{Q_\times})^{-1}.\end{aligned}\tag{20.35}$$

We arrive at the following statement.

Theorem 20.11. *Suppose that conditions of Theorem* 20.10 *are satisfied. Let* $K^0_{\times,Q_\times}(\varepsilon)$ *be the corrector defined by* (20.35)*. Then for* $0\le s\le 1$ *and* $0<\varepsilon\le 1$ *we have*

$$\|(\widehat{\mathcal{B}}_{\times,\varepsilon}+Q^\varepsilon_\times)^{-1}-(\widehat{\mathcal{B}}^0_\times+\overline{Q_\times})^{-1}-\varepsilon K^0_{\times,Q_\times}(\varepsilon)\|_{L_2(\mathbb{R}^2;\mathbb{C}^2)\to H^s(\mathbb{R}^2;\mathbb{C}^2)}\le C^\times_{s,Q_\times}\varepsilon^{2-s}.\tag{20.36}$$

The constant $C^\times_{s,Q_\times}$ *depends on* s*, on* $\|\omega_+\|_{L_\infty}$, $\|\omega_+^{-1}\|_{L_\infty}$, $\|\omega_-\|_{L_\infty}$, $\|\omega_-^{-1}\|_{L_\infty}$, $\|Q_\times\|_{L_\infty}$, $\|Q_\times^{-1}\|_{L_\infty}$*, and on parameters of the lattice* Γ*.*

Let $\mathbf{a}_{\times,\varepsilon}$ be the solution of the equation

$$\widehat{\mathcal{B}}_{\times,\varepsilon}\mathbf{a}_{\times,\varepsilon}+Q^\varepsilon_\times\mathbf{a}_{\times,\varepsilon}=\mathbf{F},\quad \mathbf{F}\in L_2(\mathbb{R}^2;\mathbb{C}^2),\tag{20.37}$$

and let $\mathbf{a}^0_\times$ be the solution of the homogenized equation

$$\widehat{\mathcal{B}}^0_\times\mathbf{a}^0_\times+\overline{Q_\times}\mathbf{a}^0_\times=\mathbf{F}.\tag{20.38}$$

Estimate (20.34) means that

$$\|\mathbf{a}_{\times,\varepsilon}-\mathbf{a}^0_\times-\varepsilon\Lambda^\varepsilon_\times b_\times(\mathbf{D})\mathbf{a}^0_\times\|_{H^1(\mathbb{R}^2;\mathbb{C}^2)}\le\check{\mathcal{C}}^\times_{11}\varepsilon\|\mathbf{F}\|_{L_2(\mathbb{R}^2;\mathbb{C}^2)},\quad 0<\varepsilon\le 1.$$

Note that, as $\varepsilon\to 0$, the weak limit of the functions $\mathbf{a}_{\times,\varepsilon}$ in $H^1(\mathbb{R}^2;\mathbb{C}^2)$ is equal to $\mathbf{a}^0_\times$.

Since $g^0_\times=\underline{g_\times}$, we can apply Theorem 15.4 for the fluxes. This implies the following statement.

Theorem 20.12. *Suppose that conditions of Theorem* 20.10 *are satisfied. Let* $\mathbf{a}_{\times,\varepsilon}$ *be the solution of the equation* (20.37)*, and let* $\mathbf{a}^0_\times$ *be the solution of the equation* (20.38)*. We put*

$$\mathbf{q}_{\times,\varepsilon}=g^\varepsilon_\times b_\times(\mathbf{D})\mathbf{a}_{\times,\varepsilon},\quad \mathbf{q}^0_\times=g^0_\times b_\times(\mathbf{D})\mathbf{a}^0_\times.$$

Then

$$\|\mathbf{q}_{\times,\varepsilon}-\mathbf{q}^0_\times\|_{L_2(\mathbb{R}^2;\mathbb{C}^2)}\le\check{\mathcal{C}}^\times_{16}\varepsilon\|\mathbf{F}\|_{L_2(\mathbb{R}^2;\mathbb{C}^2)},\quad 0<\varepsilon\le 1.\tag{20.39}$$

The constant $\check{\mathcal{C}}^\times_{16}$ *depends on* $\|\omega_+\|_{L_\infty}$, $\|\omega_+^{-1}\|_{L_\infty}$, $\|\omega_-\|_{L_\infty}$, $\|\omega_-^{-1}\|_{L_\infty}$, $\|Q_\times\|_{L_\infty}$, $\|Q_\times^{-1}\|_{L_\infty}$*, and on parameters of the lattice* Γ*.*

20.5. Definition and factorization for the Pauli operator

(See [BSu2, §6.2], [BSu4, Subsection 12.3].) Suppose that the *magnetic potential* $\mathbf{A} = \{A_1, A_2\}$ is a Γ-periodic real vector-valued function in $\mathbb{R}^2$ such that

$$\mathbf{A} \in L_p(\Omega; \mathbb{C}^2), \quad p > 2. \tag{20.40}$$

Recall the standard notation for the Pauli matrices:

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

In the space $\mathfrak{G} = L_2(\mathbb{R}^2; \mathbb{C}^2)$, we consider the operator

$$\mathcal{D} = (D_1 - A_1)\sigma_1 + (D_2 - A_2)\sigma_2, \quad \operatorname{Dom} \mathcal{D} = \mathfrak{G}^1 = H^1(\mathbb{R}^2; \mathbb{C}^2).$$

By definition, the Pauli operator $\mathcal{P}$ is the square of the operator $\mathcal{D}$:

$$\mathcal{P} = \mathcal{D}^2 = \begin{pmatrix} P_- & 0 \\ 0 & P_+ \end{pmatrix}. \tag{20.41}$$

The precise definition of the operator $\mathcal{P}$ is given in terms of the quadratic form $\|\mathcal{D}\mathbf{u}\|^2_{\mathfrak{G}}$, $\mathbf{u} \in \operatorname{Dom} \mathcal{D}$, which is closed in $\mathfrak{G}$. If the potential $\mathbf{A}$ is sufficiently smooth, then the blocks $P_\pm$ of the operator (20.41) are given by

$$P_\pm = (\mathbf{D} - \mathbf{A})^2 \pm B, \quad B = \partial_1 A_2 - \partial_2 A_1.$$

The expression B corresponds to the strength of the magnetic field.

We use the known (see, e. g., [BSu1,2]) factorization for the Pauli operator. A gauge transformation allows us to assume that the potential $\mathbf{A}$ is subject to the conditions

$$\operatorname{div} \mathbf{A} = 0, \quad \int_\Omega \mathbf{A}(\mathbf{x})\, d\mathbf{x} = 0, \tag{20.42}$$

and still satisfies (20.40). Under conditions (20.40) and (20.42), there exists a (unique) Γ-periodic real-valued function φ such that

$$\nabla \varphi = \{A_2, -A_1\}, \quad \int_\Omega \varphi(\mathbf{x})\, d\mathbf{x} = 0.$$

It turns out that $\varphi \in \widetilde{W}^1_p(\Omega) \subset C^\alpha$, $\alpha = 1 - 2p^{-1}$.

We introduce the notation

$$\omega_\pm(\mathbf{x}) = e^{\pm\varphi(\mathbf{x})}.$$

Then $\omega_\pm \in \widetilde{W}^1_p(\Omega)$, and we have

$$\omega_+(\mathbf{x})\omega_-(\mathbf{x}) = 1, \quad \mathbf{x} \in \mathbb{R}^2. \tag{20.43}$$

We consider the matrices $h_\times$ and $g_\times$ defined by (20.19). The operators $\mathcal{D}$ and $\mathcal{P}$ admit the factorization

$$\mathcal{D} = h_\times b_\times(\mathbf{D}) h_\times, \tag{20.44}$$

$$\mathcal{P} = h_\times b_\times(\mathbf{D}) g_\times b_\times(\mathbf{D}) h_\times. \tag{20.45}$$

The blocks $P_\pm$ of the operator $\mathcal{P}$ admit the representations

$$P_+ = \omega_-\partial_+\omega_+^2\partial_-\omega_-, \quad P_- = \omega_+\partial_-\omega_-^2\partial_+\omega_+. \tag{20.46}$$

It is convenient to view formulas (20.44)–(20.46) as the *definition of the operators* $\mathcal{D}$, $\mathcal{P}$, and $P_\pm$, assuming that the $\omega_\pm$ are arbitrary Γ-periodic functions satisfying conditions (20.1±) and condition (20.43). More precisely, the operator $\mathcal{D}$ is given by (20.44) on the domain

$$\operatorname{Dom}\mathcal{D} = \{\mathbf{u} \in \mathfrak{G} : \ h_\times \mathbf{u} \in H^1(\mathbb{R}^2;\mathbb{C}^2)\}.$$

The operator $\mathcal{P}$ is defined via the quadratic form $\|\mathcal{D}\mathbf{u}\|^2_{\mathfrak{G}}$, $\mathbf{u} \in \operatorname{Dom}\mathcal{D}$. The blocks $P_\pm$ are defined via the quadratic forms

$$\|\omega_\pm\partial_\mp\omega_\mp u\|^2_{L_2(\mathbb{R}^2)}, \quad \omega_\mp u \in H^1(\mathbb{R}^2).$$

Note that the operators P_+ and P_- are unitarily equivalent. Moreover, the operators $P_+(\mathbf{k})$ and $P_-(\mathbf{k})$, occuring in the direct integral expansion for $P_\pm$, are unitarily equivalent for each quasimomentum $\mathbf{k}$.

20.6. The operators $P_\pm$

The operators $P_\pm$ are of the form $\mathcal{A}(g, f)$ with $d = 2$, $m = n = 1$, $b(\mathbf{D}) = \partial_\mp$, $g = \omega_\pm^2$, and $f = \omega_\mp$. The role of the corresponding operator $\widehat{\mathcal{A}}(g)$ is played by the operator $\widehat{\mathcal{B}}_\pm$ (see (20.2±)).

We consider the operators

$$P_{\pm,\varepsilon} = \omega_\mp^\varepsilon\partial_\pm(\omega_\pm^\varepsilon)^2\partial_\mp\omega_\mp^\varepsilon. \tag{20.47±}$$

If the magnetic potential is sufficiently smooth, then

$$P_{\pm,\varepsilon} = (\mathbf{D} - \varepsilon^{-1}\mathbf{A}^\varepsilon)^2 \pm \varepsilon^{-2}B^\varepsilon.$$

Theorem 16.3 is applicable. Now $Q_\pm = \omega_\mp^2$ and (cf. [BSu4, (12.30±)]) we have

$$(\widehat{\mathcal{B}}_\pm^0 + \overline{Q_\pm})^{-1} = (-\underline{\omega_\pm^2}\Delta + \overline{\omega_\mp^2})^{-1} = (\overline{\omega_\mp^2})^{-1}(-\gamma\Delta + I)^{-1}, \tag{20.48±}$$

where

$$\gamma = (\overline{\omega_+^2})^{-1}(\overline{\omega_-^2})^{-1} = |\Omega|^2\Bigl(\int_\Omega \omega_+^2\,d\mathbf{x}\Bigr)^{-1}\Bigl(\int_\Omega \omega_-^2\,d\mathbf{x}\Bigr)^{-1}. \tag{20.49}$$

As a result, we arrive at the following statement.

Theorem 20.13. *Let $P_{\pm,\varepsilon}$ be the operators* (20.47±)*, where ω_+ and ω_- are two real-valued Γ-periodic functions in $\mathbb{R}^2$ satisfying conditions* (20.1±) *and* (20.43)*. Suppose that γ is the number defined by* (20.49)*, and that $v_\pm \in \widetilde{H}^1(\Omega)$ is the solution of the problem* (20.5±)*. Then for $0 < \varepsilon \le 1$ we have*

$$\|\omega_\mp^\varepsilon(P_{\pm,\varepsilon} + I)^{-1} - (I + \varepsilon v_\pm^\varepsilon\partial_\mp)(\overline{\omega_\pm^2})^{-1}(-\gamma\Delta + I)^{-1}\omega_\pm^\varepsilon\|_{L_2(\mathbb{R}^2)\to H^1(\mathbb{R}^2)} \le \widetilde{\mathcal{C}}_{11}^{(\pm)}\varepsilon. \tag{20.50±}$$

The constant $\widetilde{\mathcal{C}}_{11}^{(\pm)} = \check{\mathcal{C}}_{11}^{(\pm)}\|\omega_\pm\|_{L_\infty}$ depends on $\|\omega_+\|_{L_\infty}$, $\|\omega_-\|_{L_\infty}$, and on parameters of the lattice Γ. (*Here $\check{\mathcal{C}}_{11}^{(\pm)}$ is the constant from* (20.13±) *with $Q_\pm = \omega_\mp^2$.*)

We can also apply (interpolational) Theorem 16.7. Now the corrector (13.4) is given by relation (20.14±), and $Q_\pm = \omega_\pm^2$. By (20.48±), we have

$$\begin{aligned} K^0_{\pm,Q_\pm}(\varepsilon) &= (\overline{\omega_\pm^2})^{-1}\left(v^\varepsilon_\pm \partial_\mp(-\gamma\Delta+I)^{-1} + (-\gamma\Delta+I)^{-1}\partial_\pm(v^\varepsilon_\pm)^*\right) \\ &- (\overline{\omega_\pm^2})^{-2}(-\gamma\Delta+I)^{-1}\left(2D_1(\operatorname{Re}\overline{\omega_\pm^2 v_\pm}) \pm 2D_2(\operatorname{Im}\overline{\omega_\pm^2 v_\pm})\right)(-\gamma\Delta+I)^{-1}. \end{aligned} \tag{20.51±}$$

We arrive at the following result.

Theorem 20.14. *Suppose that conditions of Theorem* 20.13 *are satisfied. Let* $K^0_{\pm,Q_\pm}(\varepsilon)$ *be the corrector defined by* (20.51±). *Then for* $0 \le s \le 1$ *and* $0 < \varepsilon \le 1$ *we have*

$$\begin{aligned} &\|\omega^\varepsilon_\mp(P_{\pm,\varepsilon}+I)^{-1} \\ &\quad - \left((\overline{\omega_\pm^2})^{-1}(-\gamma\Delta+I)^{-1} + \varepsilon K^0_{\pm,Q_\pm}(\varepsilon)\right)\omega^\varepsilon_\pm\|_{L_2(\mathbb{R}^2)\to H^s(\mathbb{R}^2)} \le \widetilde{C}^{(\pm)}_s \varepsilon^{2-s}. \end{aligned}$$

The constant $\widetilde{C}^{(\pm)}_s = C^{(\pm)}_{s,Q_\pm}\|\omega_\pm\|_{L_\infty}$ *depends on* s, *on* $\|\omega_+\|_{L_\infty}$, $\|\omega_-\|_{L_\infty}$, *and on parameters of the lattice* Γ. (*Here* $C^{(\pm)}_{s,Q_\pm}$ *is the constant from* (20.15±) *with* $Q_\pm = \omega_\pm^2$.)

Let $w_{\pm,\varepsilon}$ be the solution of the equation

$$P_{\pm,\varepsilon} w_{\pm,\varepsilon} + w_{\pm,\varepsilon} = F_\pm, \quad F_\pm \in L_2(\mathbb{R}^2), \tag{20.52±}$$

and let $w^0_{\pm,\varepsilon}$ be the solution of the equation

$$-\gamma\Delta w^0_{\pm,\varepsilon} + w^0_{\pm,\varepsilon} = (\overline{\omega_\pm^2})^{-1}\omega^\varepsilon_\pm F_\pm. \tag{20.53±}$$

Estimate (20.50±) means that for $0 < \varepsilon \le 1$ we have

$$\|\omega^\varepsilon_\mp w_{\pm,\varepsilon} - w^0_{\pm,\varepsilon} - \varepsilon v^\varepsilon_\pm \partial_\mp w^0_{\pm,\varepsilon}\|_{H^1(\mathbb{R}^2)} \le \widetilde{\mathcal{C}}^{(\pm)}_{11}\varepsilon\|F_\pm\|_{L_2(\mathbb{R}^2)}.$$

Note that (see Remark 16.2), as $\varepsilon \to 0$, the weak $(H^1(\mathbb{R}^2))$-limit of the functions $\omega^\varepsilon_\mp w_{\pm,\varepsilon}$ is equal to $w^0_\pm$, where $w^0_\pm$ is the solution of the equation

$$-\gamma w^0_\pm + w^0_\pm = (\overline{\omega_\pm^2})^{-1}\overline{\omega_\pm}F_\pm. \tag{20.54±}$$

By relation (20.3±), we can apply Theorem 16.12 for the fluxes. The role of the fluxes for equation (20.52±) is played by the functions

$$r_{\pm,\varepsilon} = (\omega^\varepsilon_\pm)^2\partial_\mp(\omega^\varepsilon_\mp w_{\pm,\varepsilon}). \tag{20.55±}$$

Theorem 16.12 implies the following result.

Theorem 20.15. *Suppose that conditions of Theorem* 20.13 *are satisfied. Let* $w_{\pm,\varepsilon}$ *be the solution of the equation* (20.52±), *and let* $w^0_{\pm,\varepsilon}$ *be the solution of the equation* (20.53±). *Let* $r_{\pm,\varepsilon}$ *be defined by* (20.55±). *Then for* $0 < \varepsilon \le 1$ *we have*

$$\|r_{\pm,\varepsilon} - (\underline{\omega_\pm^2})\partial_\mp w^0_{\pm,\varepsilon}\|_{L_2(\mathbb{R}^2)} \le \widetilde{\mathcal{C}}^{(\pm)}_{16}\varepsilon\|F_\pm\|_{L_2(\mathbb{R}^2)}.$$

The constant $\widetilde{\mathcal{C}}^{(\pm)}_{16} = \check{\mathcal{C}}^{(\pm)}_{16}\|\omega_\pm\|_{L_\infty}$ *depends on* $\|\omega_+\|_{L_\infty}$, $\|\omega_-\|_{L_\infty}$, *and on parameters of the lattice* Γ. (*Here* $\check{\mathcal{C}}^{(\pm)}_{16}$ *is the constant from* (20.18±) *with* $Q_\pm = \omega_\pm^2$.)

Note that (see Remark 16.13), under conditions of Theorem 20.15, there exists the strong limit

$$(L_2(\mathbb{R}^2))\text{-}\lim_{\varepsilon\to 0} r_{\pm,\varepsilon} = r^0_\pm := (\underline{\omega_\pm^2})\partial_\mp w^0_\pm,$$

where $w^0_\pm$ is the solution of the equation (20.54±).

20.7. The operator $\mathcal{P}$

The operator $\mathcal{P}$ (see (20.45)) is of the form $\mathcal{A}(g,f)$ with $d=2$, $m=n=2$, $b(\mathbf{D})=b_\times(\mathbf{D})$, $g=g_\times$, and $f=h_\times$ (see (20.19), (20.20)). The role of the corresponding operator $\widehat{\mathcal{A}}(g)$ is played by the operator $\widehat{\mathcal{B}}_\times$ defined by (20.20). As above, we assume that the Γ-periodic functions ω_+, ω_- are subject to conditions (20.1±) and (20.43).

We consider the operator

$$\mathcal{P}_\varepsilon = h_\times^\varepsilon b_\times(\mathbf{D}) g_\times^\varepsilon b_\times(\mathbf{D}) h_\times^\varepsilon. \tag{20.56}$$

Then

$$\mathcal{P}_\varepsilon = \begin{pmatrix} P_{-,\varepsilon} & 0 \\ 0 & P_{+,\varepsilon} \end{pmatrix}, \tag{20.57}$$

where the operators $P_{\pm,\varepsilon}$ are defined by (20.47±).

Theorem 16.3 is applicable. Now the role of Q is played by the matrix

$$Q_\times = g_\times^{-1} = \operatorname{diag}\{\omega_-^2, \omega_+^2\}. \tag{20.58}$$

The operator $(\widehat{\mathcal{B}}_\times^0 + \overline{Q_\times})^{-1}$ takes the form (see (20.3±), (20.4±), (20.23), (20.48±)):

$$\begin{aligned} \widehat{\mathcal{R}}_\times^0 &:= (\widehat{\mathcal{B}}_\times^0 + \overline{Q_\times})^{-1} = \operatorname{diag}\{(\widehat{\mathcal{B}}_-^0 + \overline{\omega_-^2} I)^{-1}, (\widehat{\mathcal{B}}_+^0 + \overline{\omega_+^2} I)^{-1}\} \\ &= \operatorname{diag}\{(\overline{\omega_-^2})^{-1}, (\overline{\omega_+^2})^{-1}\}(-\gamma\Delta + I)^{-1}. \end{aligned} \tag{20.59}$$

As a result, we arrive at the following theorem.

Theorem 20.16. *Let ω_+ and ω_- be two Γ-periodic functions subject to conditions* (20.1±) *and* (20.43). *Let $b_\times(\mathbf{D})$ be the operator defined by* (20.20), *and let $h_\times, g_\times$ be the matrices defined by* (20.19). *Let $\mathcal{P}_\varepsilon$ be the operator* (20.56), *and let $\Lambda_\times(\mathbf{x})$ be the matrix* (20.24). *Finally, let $\widehat{\mathcal{R}}_\times^0$ be the operator* (20.59). *Then for $0<\varepsilon\le 1$ we have*

$$\| h_\times^\varepsilon (\mathcal{P}_\varepsilon + I)^{-1} - (I + \varepsilon \Lambda_\times^\varepsilon b_\times(\mathbf{D})) \widehat{\mathcal{R}}_\times^0 (h_\times^\varepsilon)^{-1} \|_{L_2(\mathbb{R}^2;\mathbb{C}^2)\to H^1(\mathbb{R}^2;\mathbb{C}^2)} \le \widetilde{\mathcal{C}}_{11}^\times \varepsilon. \tag{20.60}$$

The constant $\widetilde{\mathcal{C}}_{11}^\times = \check{\mathcal{C}}_{11}^\times \|h_\times^{-1}\|_{L_\infty}$ depends only on $\|\omega_+\|_{L_\infty}$, $\|\omega_-\|_{L_\infty}$, and on parameters of the lattice Γ. (*Here $\check{\mathcal{C}}_{11}^\times$ is the constant from* (20.34) *with $Q_\times = g_\times^{-1}$.*)

Note that the operator under the norm sign in (20.60) is diagonal:

$$\begin{aligned} & h_\times^\varepsilon (\mathcal{P}_\varepsilon + I)^{-1} - (I + \varepsilon \Lambda_\times^\varepsilon b_\times(\mathbf{D})) \widehat{\mathcal{R}}_\times^0 (h_\times^\varepsilon)^{-1} \\ &\quad = \operatorname{diag}\{\omega_+^\varepsilon (P_{-,\varepsilon} + I)^{-1} - (I + \varepsilon v_-^\varepsilon \partial_+)(\overline{\omega_-^2})^{-1}(-\gamma\Delta + I)^{-1}\omega_-^\varepsilon, \\ &\qquad\qquad \omega_-^\varepsilon (P_{+,\varepsilon} + I)^{-1} - (I + \varepsilon v_+^\varepsilon \partial_-)(\overline{\omega_+^2})^{-1}(-\gamma\Delta + I)^{-1}\omega_+^\varepsilon\}. \end{aligned}$$

Therefore, the result of Theorem 20.16 could be obtained also from Theorem 20.13.

We can also apply (interpolational) Theorem 16.7. Now the corrector (13.4) is given by relation (20.35), and $Q_\times$ is defined by (20.58). Then

$$K_{\times,Q_\times}^0(\varepsilon) = \operatorname{diag}\{K_{-,Q_-}^0(\varepsilon), K_{+,Q_+}^0(\varepsilon)\}, \tag{20.61}$$

where the operators $K_{\pm,Q_\pm}^0(\varepsilon)$ are given by (20.51±). We arrive at the following result.

Theorem 20.17. *Suppose that conditions of Theorem* 20.16 *are satisfied. Suppose that* $K^0_{\times,Q_\times}(\varepsilon)$ *is the corrector defined according to* (20.61), (20.51±). *Then for* $0 \le s \le 1$ *and* $0 < \varepsilon \le 1$ *we have*

$$\|h^\varepsilon_\times(\mathcal{P}_\varepsilon + I)^{-1} - (\widehat{\mathcal{R}}^0_\times + \varepsilon K^0_{\times,Q_\times}(\varepsilon))(h^\varepsilon_\times)^{-1}\|_{L_2(\mathbb{R}^2;\mathbb{C}^2)\to H^s(\mathbb{R}^2;\mathbb{C}^2)} \le \widetilde{C}^\times_s \varepsilon^{2-s}. \tag{20.62}$$

The constant $\widetilde{C}^\times_s = C^\times_{s,Q_\times}\|h_\times^{-1}\|_{L_\infty}$ *depends on* s, *on* $\|\omega_+\|_{L_\infty}$, $\|\omega_-\|_{L_\infty}$, *and on parameters of the lattice* Γ. (*Here* $C^\times_{s,Q_\times}$ *is the constant from* (20.36) *with* $Q_\times = g_\times^{-1}$.)

Taking relations (20.19), (20.57), (20.59), and (20.61) into account, we see that the operator under the norm sign in (20.62) is diagonal:

$$\begin{aligned} &h^\varepsilon_\times(\mathcal{P}_\varepsilon + I)^{-1} - (\widehat{\mathcal{R}}^0_\times + \varepsilon K^0_{\times,Q_\times}(\varepsilon))(h^\varepsilon_\times)^{-1} \\ &\quad = \operatorname{diag}\Big\{\omega^\varepsilon_+(P_{-,\varepsilon} + I)^{-1} - \Big((\overline{\omega_-^2})^{-1}(-\gamma\Delta + I)^{-1} + \varepsilon K^0_{-,Q_-}(\varepsilon)\Big)\,\omega^\varepsilon_-, \\ &\qquad\quad \omega^\varepsilon_-(P_{+,\varepsilon} + I)^{-1} - \Big((\overline{\omega_+^2})^{-1}(-\gamma\Delta + I)^{-1} + \varepsilon K^0_{+,Q_+}(\varepsilon)\Big)\,\omega^\varepsilon_+\Big\}. \end{aligned}$$

Therefore, the result of Theorem 20.17 could be deduced from Theorem 20.14.

Let $\mathbf{w}_{\times,\varepsilon}$ be the solution of the equation

$$\mathcal{P}_\varepsilon \mathbf{w}_{\times,\varepsilon} + \mathbf{w}_{\times,\varepsilon} = \mathbf{F}, \quad \mathbf{F} = \begin{pmatrix} F_- \\ F_+ \end{pmatrix} \in L_2(\mathbb{R}^2;\mathbb{C}^2), \tag{20.63}$$

and let $\mathbf{w}^0_{\times,\varepsilon}$ be the solution of the equation

$$-\gamma\Delta \mathbf{w}^0_{\times,\varepsilon} + \mathbf{w}^0_{\times,\varepsilon} = \operatorname{diag}\{(\overline{\omega_-^2})^{-1}, (\overline{\omega_+^2})^{-1}\}(h^\varepsilon_\times)^{-1}\mathbf{F}. \tag{20.64}$$

Note that

$$\mathbf{w}_{\times,\varepsilon} = \begin{pmatrix} w_{-,\varepsilon} \\ w_{+,\varepsilon} \end{pmatrix}, \quad \mathbf{w}^0_{\times,\varepsilon} = \begin{pmatrix} w^0_{-,\varepsilon} \\ w^0_{+,\varepsilon} \end{pmatrix}, \tag{20.65}$$

where $w_{\pm,\varepsilon}$ is the solution of the equation (20.52±), and $w^0_{\pm,\varepsilon}$ is the solution of the equation (20.53±). Estimate (20.60) means that

$$\|h^\varepsilon_\times \mathbf{w}_{\times,\varepsilon} - \mathbf{w}^0_{\times,\varepsilon} - \varepsilon\Lambda^\varepsilon_\times b_\times(\mathbf{D})\mathbf{w}^0_{\times,\varepsilon}\|_{H^1(\mathbb{R}^2;\mathbb{C}^2)} \le \widetilde{\mathcal{C}}^\times_{11}\varepsilon\|\mathbf{F}\|_{L_2(\mathbb{R}^2;\mathbb{C}^2)}.$$

Herewith, as $\varepsilon \to 0$, the weak $(H^1(\mathbb{R}^2;\mathbb{C}^2))$-limit of the functions $h^\varepsilon_\times \mathbf{w}_{\times,\varepsilon}$ is equal to $\mathbf{w}^0_\times$, where $\mathbf{w}^0_\times$ is the solution of the equation

$$-\gamma \mathbf{w}^0_\times + \mathbf{w}^0_\times = \operatorname{diag}\{(\overline{\omega_-^2})^{-1}\overline{\omega_-},\, (\overline{\omega_+^2})^{-1}\overline{\omega_+}\}\mathbf{F}. \tag{20.66}$$

We can apply Theorem 16.12 for the fluxes. The role of the flux for equation (20.63) is played by the vector-valued function

$$\mathbf{r}_{\times,\varepsilon} = g^\varepsilon_\times b_\times(\mathbf{D})(h^\varepsilon_\times \mathbf{w}_{\times,\varepsilon}). \tag{20.67}$$

Note that

$$\mathbf{r}_{\times,\varepsilon} = \begin{pmatrix} r_{+,\varepsilon} \\ r_{-,\varepsilon} \end{pmatrix}, \tag{20.68}$$

where the functions $r_{\pm,\varepsilon}$ are defined by (20.55±). We arrive at the following result.

Theorem 20.18. *Suppose that conditions of Theorem* 20.16 *are satisfied. Let* $\mathbf{w}_{\times,\varepsilon}$ *be the solution of the equation* (20.63), *and let* $\mathbf{w}^0_{\times,\varepsilon}$ *be the solution of the equation* (20.64). *Let* $\mathbf{r}_{\times,\varepsilon}$ *be defined by* (20.67). *Then for* $0 < \varepsilon \le 1$ *we have*

$$\|\mathbf{r}_{\times,\varepsilon} - g^0_\times b_\times(\mathbf{D})\mathbf{w}^0_{\times,\varepsilon}\|_{L_2(\mathbb{R}^2;\mathbb{C}^2)} \le \widetilde{\mathcal{C}}^\times_{16}\varepsilon\|\mathbf{F}\|_{L_2(\mathbb{R}^2;\mathbb{C}^2)}.$$

The constant $\widetilde{\mathcal{C}}^\times_{16} = \check{\mathcal{C}}^\times_{16}\|h_\times^{-1}\|_{L_\infty}$ *depends on* $\|\omega_+\|_{L_\infty}$, $\|\omega_-\|_{L_\infty}$, *and on parameters of the lattice* Γ. (*Here* $\check{\mathcal{C}}^\times_{16}$ *is the constant from* (20.39) *with* $Q_\times = g_\times^{-1}$.)

By (20.68), (20.65), (20.19), and (20.20), we have

$$\mathbf{r}_{\times,\varepsilon} - g^0_\times b_\times(\mathbf{D})\mathbf{w}^0_{\times,\varepsilon} = \begin{pmatrix} r_{+,\varepsilon} - (\underline{\omega_+^2})\partial_- w^0_{+,\varepsilon} \\ r_{-,\varepsilon} - (\underline{\omega_-^2})\partial_+ w^0_{-,\varepsilon} \end{pmatrix},$$

and, therefore, the result of Theorem 20.18 could be deduced from Theorem 20.15.

By Remark 16.13, under conditions of Theorem 20.18, there exists the strong limit

$$(L_2(\mathbb{R}^2;\mathbb{C}^2))\text{-}\lim_{\varepsilon\to 0}\mathbf{r}_{\times,\varepsilon} = \mathbf{r}^0_\times := g^0_\times b_\times(\mathbf{D})\mathbf{w}^0_\times,$$

where $\mathbf{w}^0_\times$ is the solution of the equation (20.66).

For the operators $P_{\pm,\varepsilon}$ and $\mathcal{P}_\varepsilon$, we could also apply the results of Subsections 16.6–16.8 about approximation of the generalized resolvent. We shall not give the detailed formulations here. We only note that, for the case of the general matrix potential $\mathfrak{Q}(\mathbf{x})$, we could not refer to diagonalization of the operator $(\mathcal{P}_\varepsilon + \mathfrak{Q}^\varepsilon)^{-1}$.

In [BSu2], besides the ordinary Pauli operator $\mathcal{P}$, the „Pauli operator with metric“ $\mathcal{P}_{\mathfrak{g}} = \mathcal{D}\mathfrak{g}\mathcal{D}$ was considered. The results of §16 can be applied also to this operator. We shall not dwell on details.

§21. The operator of elasticity theory

21.1. Description of the operator

In this section, we assume that $d \ge 2$. We represent the operator of elasticity theory as in [BSu2, §5.2], [BSu4, §13]. Let ζ be an orthogonal second rank tensor in $\mathbb{R}^d$; in the standard orthonormal basis in $\mathbb{R}^d$, it can be represented by a matrix $\zeta = \{\zeta_{jl}\}_{j,l=1}^d$. We shall consider *symmetric* tensors ζ, which will be identified with vectors $\zeta_* \in \mathbb{C}^m$, $2m = d(d+1)$, by the following rule. The vector ζ_* is formed by all components ζ_{jl}, $j \le l$, and the pairs (j,l) are put in order in some fixed way.

For the *displacement vector* $\mathbf{u} \in \mathfrak{G}^1 = H^1(\mathbb{R}^d;\mathbb{C}^d)$, we introduce the *deformation tensor*

$$e(\mathbf{u}) = \frac{1}{2}\Big\{\frac{\partial u_j}{\partial x_l} + \frac{\partial u_l}{\partial x_j}\Big\}.$$

Let $e_*(\mathbf{u})$ be the vector corresponding to the tensor $e(\mathbf{u})$ in accordance with the rule described above. The relation

$$b(\mathbf{D})\mathbf{u} = -ie_*(\mathbf{u})$$

determines an $(m\times d)$-matrix homogeneous DO $b(\mathbf{D})$ uniquely; the symbol of this DO is the matrix with real entries. For instance, with an appropriate ordering, we have

$$b(\boldsymbol{\xi}) = \begin{pmatrix} \xi^1 & 0 \\ \frac{1}{2}\xi^2 & \frac{1}{2}\xi^1 \\ 0 & \xi^2 \end{pmatrix}, \quad d = 2.$$

Let $\sigma(\mathbf{u})$ be the *stress tensor*, and let $\sigma_*(\mathbf{u})$ be the corresponding vector. For the accepted way of writing, the *Hooke law* about proportionality of stresses and deformations can be expressed by the relation

$$\sigma_*(\mathbf{u}) := g(\mathbf{x})e_*(\mathbf{u}),$$

where $g(\mathbf{x})$ is an $(m \times m)$-matrix (which gives a „concise“ description of the Hooke tensor). The matrix $g(\mathbf{x})$ characterizes the parameters of elastic (in general, *anisotropic*) medium. We assume that the matrix-valued function $g(\mathbf{x})$ is Γ-periodic and such that

$$g(\mathbf{x}) > 0; \quad g, g^{-1} \in L_\infty.$$

The energy of elastic deformations is given by the quadratic form

$$w[\mathbf{u},\mathbf{u}] = \frac{1}{2}\int_{\mathbb{R}^d} \langle \sigma_*(\mathbf{u}), e_*(\mathbf{u})\rangle_{\mathbb{C}^m}\, d\mathbf{x} = \frac{1}{2}\int_{\mathbb{R}^d} \langle g(\mathbf{x})b(\mathbf{D})\mathbf{u}, b(\mathbf{D})\mathbf{u}\rangle_{\mathbb{C}^m}\, d\mathbf{x}, \quad \mathbf{u} \in \mathfrak{G}^1. \tag{21.1}$$

The operator $\mathcal{W}$, acting in $\mathfrak{G} = L_2(\mathbb{R}^d; \mathbb{C}^d)$ and generated by this form, is the *operator of elasticity theory.* Thus,

$$2\mathcal{W} = b(\mathbf{D})^* g b(\mathbf{D}) = \widehat{\mathcal{A}}(g).$$

Now $n = d$ and $m = d(d+1)/2$.

In the case of isotropic medium, the matrix $g(\mathbf{x})$ depends only on two functional *Lamé parameters* $\lambda(\mathbf{x})$ and $\mu(\mathbf{x})$. The parameter μ is the *shear modulus.* Often another parameter $K(\mathbf{x})$ is introduced instead of $\lambda(\mathbf{x})$; $K(\mathbf{x})$ is called the *modulus of volume compression.* We shall need yet another modulus $\beta(\mathbf{x})$. Here are the relations:

$$K(\mathbf{x}) = \lambda(\mathbf{x}) + \frac{2\mu(\mathbf{x})}{d}, \quad \beta(\mathbf{x}) = \mu(\mathbf{x}) + \frac{\lambda(\mathbf{x})}{2}.$$

The modulus $\lambda(\mathbf{x})$ may be negative. In the isotropic case, the conditions that ensure the positive definiteness of the matrix $g(\mathbf{x})$ are

$$\mu(\mathbf{x}) \geq \mu_0 > 0, \quad K(\mathbf{x}) \geq K_0 > 0.$$

For instance, we write down the matrix g in the isotropic case for $d = 2$:

$$g(\mathbf{x}) = \begin{pmatrix} K+\mu & 0 & K-\mu \\ 0 & 4\mu & 0 \\ K-\mu & 0 & K+\mu \end{pmatrix}, \quad d = 2.$$

All general results related to operators of the form $\widehat{\mathcal{A}}(g)$ can be applied to the operator $\mathcal{W}$: Theorems 10.6, 11.3, and 12.1 are applicable, and for $d = 2$ Theorems 10.8, 11.4, and 12.3 are applicable. No simplification in general formulations occurs (even for the isotropic case with variable λ and μ).

21.2. The Hill body

In mechanics (see, e. g., [ZhKO]), the elastic isotropic medium with the constant shear modulus $\mu(\mathbf{x}) = \mu_0 = \text{const}$ is called the Hill body. In this case, a simpler factorization for the operator $\mathcal{W}$ is possible (see [BSu2, Subsection 5.2.3], [BSu4, Subsection 13.4]). Now the energy form (21.1) can be represented as

$$w[\mathbf{u},\mathbf{u}] = \mu_0 \int_{\mathbb{R}^d} |r(\mathbf{u})|^2 \, d\mathbf{x} + \int_{\mathbb{R}^d} \beta(\mathbf{x}) |\operatorname{div} \mathbf{u}|^2 \, d\mathbf{x}, \quad \mathbf{u} \in \mathfrak{G}^1, \tag{21.2}$$

where

$$r(\mathbf{u}) = \frac{1}{2} \Big\{ \frac{\partial u_j}{\partial x_l} - \frac{\partial u_l}{\partial x_j} \Big\}.$$

The form (21.2) admits (cf. [BSu2, §5.2]) more economic description than in general case:

$$w[\mathbf{u},\mathbf{u}] = \int_{\mathbb{R}^d} \langle g_\wedge b_\wedge(\mathbf{D})\mathbf{u}, b_\wedge(\mathbf{D})\mathbf{u} \rangle_{\mathbb{C}^{m_\wedge}} \, d\mathbf{x}.$$

Here $m_\wedge = 1 + d(d-1)/2$. The $(m_\wedge \times d)$-matrix $b_\wedge(\boldsymbol{\xi})$ can be described as follows. The first row in $b_\wedge(\boldsymbol{\xi})$ is $(\xi_1, \xi_2, \dots, \xi_d)$. The other rows correspond to (different) pairs of indices (j,l), $1 \le j < l \le d$. The element standing in the (j,l)-th row and the j-th column is ξ_l, and the element in the (j,l)-th row and the l-th column is $(-\xi_j)$; all other elements of the (j,l)-th row are equal to zero. The order of the rows is irrelevant. Finally,

$$g_\wedge(\mathbf{x}) = \operatorname{diag}\{\beta(\mathbf{x}), \mu_0/2, \mu_0/2, \dots, \mu_0/2\}. \tag{21.3}$$

Thus,

$$\mathcal{W} = b_\wedge(\mathbf{D})^* g_\wedge(\mathbf{x}) b_\wedge(\mathbf{D}).$$

As it was shown in [BSu2, Subsection 5.2.3], the effective matrix $g_\wedge^0$ coincides with $\underline{g_\wedge}$:

$$g_\wedge^0 = \underline{g_\wedge} = \operatorname{diag}\{\underline{\beta}, \mu_0/2, \mu_0/2, \dots, \mu_0/2\}. \tag{21.4}$$

The solutions $\mathbf{v}_j \in \widetilde{H}^1(\Omega;\mathbb{C}^d)$, $j = 1, \dots, m_\wedge$, of the problem

$$b_\wedge(\mathbf{D})^* g_\wedge(\mathbf{x}) (b_\wedge(\mathbf{D})\mathbf{v}_j + \mathbf{e}_j) = 0, \qquad \int_\Omega \mathbf{v}_j(\mathbf{x}) \, d\mathbf{x} = 0,$$

are constructed in [BSu4, §13]. The role of the matrix $\Lambda(\mathbf{x})$ is played by the $(d \times m_\wedge)$-matrix $\Lambda_\wedge(\mathbf{x})$. The first column of the matrix $\Lambda_\wedge(\mathbf{x})$ is $\mathbf{v}_1 = i\nabla\varphi(\mathbf{x})$, where φ is the periodic solution of the equation $\Delta\varphi = \underline{\beta}(\beta(\mathbf{x}))^{-1} - 1$. The other columns are equal to zero. Then $\Lambda_\wedge(\mathbf{x}) b_\wedge(\mathbf{D}) = (\nabla\varphi(\mathbf{x}))\operatorname{div}$. By (21.4), Condition 8.6(3°), and then also Condition 8.4, is satisfied . By Remark 8.8, the norm $\|\Lambda_\wedge\|_{L_\infty} = \|\mathbf{v}_1\|_{L_\infty}$ can be estimated by the constant depending only on d, on $\|\beta\|_{L_\infty}$, $\|\beta^{-1}\|_{L_\infty}$, and on parameters of the lattice Γ. We are under assumptions of Theorem 10.8, which leads to the following statement.

Theorem 21.1. *Let $\mu = \mu_0 = \mathrm{const}$, and let $\beta(\mathbf{x})$ be a positive Γ-periodic function in $\mathbb{R}^d$ such that $\beta, \beta^{-1} \in L_\infty$. Let $g_\wedge(\mathbf{x})$ be the matrix defined by* (21.3)*. We put*

$$\mathcal{W}_\varepsilon = b_\wedge(\mathbf{D})^* g_\wedge^\varepsilon b_\wedge(\mathbf{D}).$$

Let $\mathcal{W}^0 = b_\wedge(\mathbf{D})^* \underline{g_\wedge} b_\wedge(\mathbf{D})$. Let $\varphi \in \widetilde{H}^1(\Omega)$ be the solution of the equation $\Delta\varphi = \underline{\beta}(\beta(\mathbf{x}))^{-1} - 1$, and let $\mathbf{p} = \nabla\varphi$. Then for $0 < \varepsilon \le 1$ we have

$$\|(\mathcal{W}_\varepsilon + I)^{-1} - (\mathcal{W}^0 + I)^{-1} - \varepsilon \mathbf{p}^\varepsilon \operatorname{div}(\mathcal{W}^0 + I)^{-1}\|_{\mathfrak{G} \to \mathfrak{G}^1} \le \mathcal{C}_{11}^\wedge \varepsilon. \tag{21.5}$$

The constant $\mathcal{C}_{11}^\wedge$ depends on d, on the norms $\|\beta\|_{L_\infty}$, $\|\beta^{-1}\|_{L_\infty}$, and on parameters of the lattice Γ.

We can also apply (interpolational) Theorem 11.4. The corrector (10.11) now takes the form (cf. [BSu4, Theorem 13.1])

$$K^0_\wedge(\varepsilon) = \mathbf{p}^\varepsilon \mathrm{div}\,(\mathcal{W}^0 + I)^{-1} - (\mathcal{W}^0 + I)^{-1}\nabla(\mathbf{p}^\varepsilon)^t. \tag{21.6}$$

Note that now the third term of the corrector (10.11) is equal to zero (because of (21.4)). We arrive at the following result.

Theorem 21.2. *Suppose that conditions of Theorem* 21.1 *are satisfied. Let* $K^0_\wedge(\varepsilon)$ *be the corrector defined by* (21.6)*. Then for* $0 \le s \le 1$ *and* $0 < \varepsilon \le 1$ *we have*

$$\|(\mathcal{W}_\varepsilon + I)^{-1} - (\mathcal{W}^0 + I)^{-1} - \varepsilon K^0_\wedge(\varepsilon)\|_{\mathfrak{G}\to\mathfrak{G}^s} \le C^\wedge_s \varepsilon.$$

Here $\mathfrak{G}^s = H^s(\mathbb{R}^d;\mathbb{C}^d)$*. The constant* $C^\wedge_s$ *depends on* s*,* d*, on the norms* $\|\beta\|_{L_\infty}$*,* $\|\beta^{-1}\|_{L_\infty}$*, and on parameters of the lattice* Γ*.*

Let $\mathbf{u}_\varepsilon$ be the solution of the equation

$$\mathcal{W}_\varepsilon \mathbf{u}_\varepsilon + \mathbf{u}_\varepsilon = \mathbf{F}, \quad \mathbf{F} \in \mathfrak{G}, \tag{21.7}$$

and let $\mathbf{u}_0$ be the solution of the homogenized equation

$$\mathcal{W}^0 \mathbf{u}_0 + \mathbf{u}_0 = \mathbf{F}. \tag{21.8}$$

Estimate (21.5) means that

$$\|\mathbf{u}_\varepsilon - \mathbf{u}_0 - \varepsilon \mathbf{p}^\varepsilon \mathrm{div}\,\mathbf{u}_0\|_{\mathfrak{G}^1} \le \mathcal{C}^\wedge_{11}\varepsilon\|\mathbf{F}\|_{\mathfrak{G}}, \quad 0 < \varepsilon \le 1.$$

Note that (see Remark 10.7), as $\varepsilon \to 0$, the functions $\mathbf{u}_\varepsilon$ tend to $\mathbf{u}_0$ weakly in $\mathfrak{G}^1$.

Due to relation (21.4), we can apply Theorem 12.4 for the fluxes. It gives the following result.

Theorem 21.3. *Suppose that conditions of Theorem* 21.1 *are satisfied. Let* $\mathbf{u}_\varepsilon$ *be the solution of the equation* (21.7)*, and let* $\mathbf{u}_0$ *be the solution of the equation* (21.8)*. Then for* $0 < \varepsilon \le 1$ *we have*

$$\|g^\varepsilon_\wedge b_\wedge(\mathbf{D})\mathbf{u}_\varepsilon - \underline{g_\wedge} b_\wedge(\mathbf{D})\mathbf{u}_0\|_{\mathfrak{G}_*} \le \mathcal{C}^\wedge_{16}\varepsilon\|\mathbf{F}\|_{\mathfrak{G}}.$$

The constant $\mathcal{C}^\wedge_{16}$ *depends on* d*, on the norms* $\|\beta\|_{L_\infty}$*,* $\|\beta^{-1}\|_{L_\infty}$*, and on parameters of the lattice* Γ*.*

§22. The model operator of electrodynamics

22.1. Definition of the operator

In [BSu2, Chapter 7], in the study of the homogenization problem for the stationary periodic Maxwell system, the auxiliary second order operator $\mathcal{L}$ was considered. This operator was also studied in [BSu4, §14]. The operator $\mathcal{L}$ acts in the space $\mathfrak{G} = L_2(\mathbb{R}^3;\mathbb{C}^3)$ and is given by the expression

$$\mathcal{L} = \mathcal{L}(\eta,\nu) = \mathrm{rot}\,(\eta(\mathbf{x}))^{-1}\mathrm{rot}\, - \nabla\nu(\mathbf{x})\mathrm{div}\,. \tag{22.1}$$

Here the (3×3)-matrix-valued function $\eta(\mathbf{x})$ with real entries and the real-valued function $\nu(\mathbf{x})$ are Γ-periodic and such that

$$\eta(\mathbf{x}) > 0; \quad \eta, \eta^{-1} \in L_\infty, \tag{22.2}$$

$$\nu(\mathbf{x}) > 0; \quad \nu, \nu^{-1} \in L_\infty. \tag{22.3}$$

The precise definition of the operator $\mathcal{L}$ is given in terms of the closed quadratic form

$$\int_{\mathbb{R}^3} (\langle \eta(\mathbf{x})^{-1} \mathrm{rot}\, \mathbf{u}, \mathrm{rot}\, \mathbf{u}\rangle + \nu(\mathbf{x})|\mathrm{div}\, \mathbf{u}|^2)\, d\mathbf{x}, \quad \mathbf{u} \in \mathfrak{G}^1 = H^1(\mathbb{R}^3; \mathbb{C}^3).$$

The operator $\mathcal{L}$ has the form $\widehat{\mathcal{A}}(g) = b(\mathbf{D})^* g(\mathbf{x}) b(\mathbf{D})$ with $n = 3$, $m = 4$,

$$b(\mathbf{D}) = \begin{pmatrix} -i\,\mathrm{rot} \\ -i\,\mathrm{div} \end{pmatrix}, \quad g(\mathbf{x}) = \begin{pmatrix} (\eta(\mathbf{x}))^{-1} & 0 \\ 0 & \nu(\mathbf{x}) \end{pmatrix}.$$

The symbol $b(\boldsymbol{\xi})$ of the operator $b(\mathbf{D})$ is given by

$$b(\boldsymbol{\xi}) = \begin{pmatrix} 0 & -\xi_3 & \xi_2 \\ \xi_3 & 0 & -\xi_1 \\ -\xi_2 & \xi_1 & 0 \\ \xi_1 & \xi_2 & \xi_3 \end{pmatrix}.$$

The effective matrix g^0 has the form (see [BSu2, §7.2])

$$g^0 = \begin{pmatrix} (\eta^0)^{-1} & 0 \\ 0 & \underline{\nu} \end{pmatrix},$$

where η^0 is the effective matrix for the scalar elliptic operator $-\mathrm{div}\, \eta \nabla = \mathbf{D}^* \eta \mathbf{D}$, and $\underline{\nu}$ is defined by the formula

$$\underline{\nu} = \Big(|\Omega|^{-1} \int_\Omega \nu(\mathbf{x})^{-1}\, d\mathbf{x} \Big)^{-1}.$$

The effective operator $\mathcal{L}^0$ is given by

$$\mathcal{L}^0 = \mathcal{L}(\eta^0, \underline{\nu}) = \mathrm{rot}\, (\eta^0)^{-1} \mathrm{rot}\, - \nabla \underline{\nu} \mathrm{div}\,. \tag{22.4}$$

Let $\mathbf{v}_j \in \widetilde{H}^1(\Omega; \mathbb{C}^3)$ be the Γ-periodic solution of the problem

$$b(\mathbf{D})^* g(\mathbf{x})(b(\mathbf{D})\mathbf{v}_j + \mathbf{e}_j) = 0, \quad \int_\Omega \mathbf{v}_j(\mathbf{x})\, d\mathbf{x} = 0,$$

$j = 1, 2, 3, 4$. Here $\{\mathbf{e}_j\}$ is the standard orthonormal basis in $\mathbb{C}^4$. As it was shown in [BSu4, §14], the solutions $\mathbf{v}_j$, $j = 1, 2, 3$, are defined as follows. Let $\widetilde{\Phi}_j(\mathbf{x})$ be the Γ-periodic solution of the problem

$$\mathrm{div}\, \eta(\mathbf{x})(\nabla \widetilde{\Phi}_j(\mathbf{x}) + \mathbf{c}_j) = 0, \quad \int_\Omega \widetilde{\Phi}_j(\mathbf{x})\, d\mathbf{x} = 0, \tag{22.5}$$

$j = 1, 2, 3$, where $\mathbf{c}_j = (\eta^0)^{-1}\widetilde{\mathbf{e}}_j$, and $\{\widetilde{\mathbf{e}}_j\}$ is the standard orthonormal basis in $\mathbb{C}^3$. Let $\mathbf{q}_j$ be the Γ-periodic solution of the problem

$$\Delta \mathbf{q}_j = \eta(\nabla \widetilde{\Phi}_j + \mathbf{c}_j) - \widetilde{\mathbf{e}}_j, \quad \int_\Omega \mathbf{q}_j \, d\mathbf{x} = 0. \tag{22.6}$$

Then

$$\mathbf{v}_j = i \operatorname{rot} \mathbf{q}_j, \quad j = 1, 2, 3.$$

Next, we have

$$\mathbf{v}_4 = i \nabla \varphi,$$

where φ is the Γ-periodic solution of the problem

$$\Delta \varphi = \underline{\nu}(\nu(\mathbf{x}))^{-1} - 1, \quad \int_\Omega \varphi \, d\mathbf{x} = 0. \tag{22.7}$$

The matrix $\Lambda(\mathbf{x})$ is the (3×4)-matrix with the columns $i \operatorname{rot} \mathbf{q}_1$, $i \operatorname{rot} \mathbf{q}_2$, $i \operatorname{rot} \mathbf{q}_3$, $i \nabla \varphi$. By $\Psi(\mathbf{x})$ we denote the real (3×3)-matrix with the columns $\operatorname{rot} \mathbf{q}_1$, $\operatorname{rot} \mathbf{q}_2$, $\operatorname{rot} \mathbf{q}_3$. We put $\mathbf{w} = \nabla \varphi$. Then

$$\Lambda(\mathbf{x}) b(\mathbf{D}) = \Psi(\mathbf{x}) \operatorname{rot} + \mathbf{w}(\mathbf{x}) \operatorname{div}.$$

22.2

We apply Theorem 10.6, which gives the following result.

Theorem 22.1. *Suppose that the matrix-valued function $\eta(\mathbf{x})$ with real entries and the real-valued function $\nu(\mathbf{x})$ are Γ-periodic and satisfy conditions* (22.2) *and* (22.3). *Let $\mathcal{L}(\eta, \nu)$ be the operator* (22.1) *and let*

$$\mathcal{L}_\varepsilon = \mathcal{L}(\eta^\varepsilon, \nu^\varepsilon) = \operatorname{rot} (\eta^\varepsilon)^{-1} \operatorname{rot} - \nabla \nu^\varepsilon \operatorname{div}.$$

Let $\mathcal{L}^0$ be the operator (22.4), *where η^0 is the effective matrix for the operator $\mathbf{D}^*\eta\mathbf{D}$. Let $\mathbf{q}_j$, $j = 1, 2, 3$, be the Γ-periodic solution of the problem* (22.6), *and let $\varphi(\mathbf{x})$ be the Γ-periodic solution of the problem* (22.7). *Let $\Psi(\mathbf{x})$ be the (3×3)-matrix with the columns* $\operatorname{rot} \mathbf{q}_j$, $j = 1, 2, 3$, *and let $\mathbf{w} = \nabla \varphi$. Let Π_ε be the pseudodifferential operator* (10.4) *acting in $\mathfrak{G} = L_2(\mathbb{R}^3; \mathbb{C}^3)$. Then for $0 < \varepsilon \le 1$ we have*

$$\| (\mathcal{L}_\varepsilon + I)^{-1} - (\mathcal{L}^0 + I)^{-1} - \varepsilon (\Psi^\varepsilon \operatorname{rot} + \mathbf{w}^\varepsilon \operatorname{div}) (\mathcal{L}^0 + I)^{-1} \Pi_\varepsilon \|_{\mathfrak{G} \to \mathfrak{G}^1} \le \mathcal{C}_{10} \varepsilon. \tag{22.8}$$

Here $\mathfrak{G}^1 = H^1(\mathbb{R}^3; \mathbb{C}^3)$. The constant $\mathcal{C}_{10}$ depends only on $\|\eta\|_{L_\infty}$, $\|\eta^{-1}\|_{L_\infty}$, $\|\nu\|_{L_\infty}$, $\|\nu^{-1}\|_{L_\infty}$, and on parameters of the lattice Γ.

We can also apply (interpolational) Theorem 11.3. Now the corrector (10.9) takes the form (cf. [BSu4, Theorem 14.1])

$$\begin{aligned} \widetilde{K}(\varepsilon) = {} & (\Psi^\varepsilon \operatorname{rot} + \mathbf{w}^\varepsilon \operatorname{div}) (\mathcal{L}^0 + I)^{-1} \Pi_\varepsilon + (\mathcal{L}^0 + I)^{-1} \Pi_\varepsilon (\operatorname{rot} (\Psi^\varepsilon)^t - \nabla (\mathbf{w}^\varepsilon)^t) \\ & - (\mathcal{L}^0 + I)^{-1} \mathcal{E}(\mathbf{D}) (\mathcal{L}^0 + I)^{-1}, \end{aligned} \tag{22.9}$$

where

$$\mathcal{E}(\mathbf{D}) = \sum_{s=1}^{3} \left((\alpha_{12s} - \alpha_{21s}) \partial_3 \partial_s + (\alpha_{31s} - \alpha_{13s}) \partial_2 \partial_s + (\alpha_{23s} - \alpha_{32s}) \partial_1 \partial_s \right), \tag{22.10}$$

$$\alpha_{jks} = |\Omega|^{-1} \int\limits_{\Omega} \widetilde{\Phi}_j(\mathbf{x}) \langle \eta(\mathbf{x})(\nabla \widetilde{\Phi}_k(\mathbf{x}) + \mathbf{c}_k), \mathbf{e}_s \rangle \, d\mathbf{x}. \tag{22.11}$$

We arrive at the following statement.

Theorem 22.2. *Suppose that conditions of Theorem* 22.1 *are satisfied. Let* $\widetilde{K}(\varepsilon)$ *be the corrector defined according to* (22.9)–(22.11)*. Then for* $0 \le s \le 1$ *and* $0 < \varepsilon \le 1$ *we have*

$$\|(\mathcal{L}_\varepsilon + I)^{-1} - (\mathcal{L}^0 + I)^{-1} - \varepsilon \widetilde{K}(\varepsilon)\|_{\mathfrak{G} \to \mathfrak{G}^s} \le \mathcal{C}_s \varepsilon^{2-s}. \tag{22.12}$$

Here $\mathfrak{G}^s = H^s(\mathbb{R}^3; \mathbb{C}^3)$*. The constant* $\mathcal{C}_s$ *depends only on* s*, on* $\|\eta\|_{L_\infty}$*,* $\|\eta^{-1}\|_{L_\infty}$*,* $\|\nu\|_{L_\infty}$*,* $\|\nu^{-1}\|_{L_\infty}$*, and on parameters of the lattice* Γ*.*

22.3

Let $\mathbf{u}_\varepsilon$ be the solution of the equation

$$\mathcal{L}_\varepsilon \mathbf{u}_\varepsilon + \mathbf{u}_\varepsilon = \mathbf{F}, \quad \mathbf{F} \in \mathfrak{G}, \tag{22.13}$$

and let $\mathbf{u}_0$ be the solution of the homogenized equation

$$\mathcal{L}^0 \mathbf{u}_0 + \mathbf{u}_0 = \mathbf{F}. \tag{22.14}$$

Estimate (22.8) means that

$$\|\mathbf{u}_\varepsilon - \mathbf{u}_0 - \varepsilon(\Psi^\varepsilon \Pi_\varepsilon \mathrm{rot}\, \mathbf{u}_0 + \mathbf{w}^\varepsilon \Pi_\varepsilon^{(1)} \mathrm{div}\, \mathbf{u}_0)\|_{\mathfrak{G}^1} \le \mathcal{C}_{10} \varepsilon \|\mathbf{F}\|_{\mathfrak{G}}, \quad 0 < \varepsilon \le 1.$$

Herewith (see Remark 10.7), as $\varepsilon \to 0$, the functions $\mathbf{u}_\varepsilon$ converge to $\mathbf{u}_0$ weakly in $\mathfrak{G}^1$.

The role of the flux for the equation (22.13) is played by the vector-valued function

$$\mathbf{p}_\varepsilon = g^\varepsilon b(\mathbf{D}) \mathbf{u}_\varepsilon = -i \begin{pmatrix} (\eta^\varepsilon)^{-1} \mathrm{rot}\, \mathbf{u}_\varepsilon \\ \nu^\varepsilon \mathrm{div}\, \mathbf{u}_\varepsilon \end{pmatrix}.$$

We apply Theorem 12.1 about approximation of the fluxes. The matrix $\widetilde{g} = g(\mathbf{1} + b(\mathbf{D})\Lambda)$ has a block-diagonal structure (see [BSu4, Subsection 14.3]): the upper left (3×3)-block is represented by the matrix with the columns $\nabla \widetilde{\Phi}_j(\mathbf{x}) + \mathbf{c}_j$, $j = 1, 2, 3$. We denote this block by $a(\mathbf{x})$. The element in the right lower corner is equal to $\underline{\nu}$. The other elements are equal to zero. We denote

$$\widetilde{\mathbf{p}}_\varepsilon = \widetilde{g}^\varepsilon \Pi_\varepsilon^{(4)} b(\mathbf{D}) \mathbf{u}_0 = -i \begin{pmatrix} a^\varepsilon \Pi_\varepsilon \mathrm{rot}\, \mathbf{u}_0 \\ \underline{\nu}\, \Pi_\varepsilon^{(1)} \mathrm{div}\, \mathbf{u}_0 \end{pmatrix}.$$

Here $\Pi_\varepsilon^{(4)}$ is the pseudodifferential operator in $\mathfrak{G}_* = L_2(\mathbb{R}^3; \mathbb{C}^4)$ with the symbol $\chi_{\widetilde{\Omega}/\varepsilon}(\boldsymbol{\xi})$, and $\Pi_\varepsilon^{(1)}$ is the pseudodifferential operator in $L_2(\mathbb{R}^3)$ with the same symbol. By Theorem 12.1, we have

$$\|\mathbf{p}_\varepsilon - \widetilde{\mathbf{p}}_\varepsilon\|_{\mathfrak{G}_*} \le \mathcal{C}_{15} \varepsilon \|\mathbf{F}\|_{\mathfrak{G}}, \quad 0 < \varepsilon \le 1. \tag{22.15}$$

The constant $\mathcal{C}_{15}$ depends on $\|\eta\|_{L_\infty}$, $\|\eta^{-1}\|_{L_\infty}$, $\|\nu\|_{L_\infty}$, $\|\nu^{-1}\|_{L_\infty}$, and on parameters of the lattice Γ. Estimate (22.15) means that

$$\|(\eta^\varepsilon)^{-1} \mathrm{rot}\, \mathbf{u}_\varepsilon - a^\varepsilon \Pi_\varepsilon \mathrm{rot}\, \mathbf{u}_0\|_{\mathfrak{G}} \le \mathcal{C}_{15} \varepsilon \|\mathbf{F}\|_{\mathfrak{G}}, \quad 0 < \varepsilon \le 1, \tag{22.16}$$

$$\|\nu^\varepsilon \mathrm{div}\,\mathbf{u}_\varepsilon - \underline{\nu}\Pi_\varepsilon^{(1)}\mathrm{div}\,\mathbf{u}_0\|_{L_2(\mathbb{R}^3)} \le \mathcal{C}_{15}\varepsilon\|\mathbf{F}\|_{\mathfrak{G}}, \quad 0<\varepsilon\le 1. \tag{22.17}$$

We show that in (22.16) and (22.17) the operator Π_ε can be eliminated (replaced by I). Indeed, consider the operator

$$a^\varepsilon(I-\Pi_\varepsilon)\mathrm{rot}\,(\mathcal{L}^0+I)^{-1} = \varepsilon T_\varepsilon^* a(I-\Pi)\mathrm{rot}\,(\mathcal{L}^0+\varepsilon^2 I)^{-1}T_\varepsilon.$$

Here T_ε is the scaling transformation defined by (10.1). We have used relations (10.2) and (10.3). Since the operator T_ε is unitary in $\mathfrak{G}$, then

$$\|a^\varepsilon(I-\Pi_\varepsilon)\mathrm{rot}\,(\mathcal{L}^0+I)^{-1}\|_{\mathfrak{G}\to\mathfrak{G}} = \varepsilon\|a(I-\Pi)\mathrm{rot}\,(\mathcal{L}^0+\varepsilon^2 I)^{-1}\|_{\mathfrak{G}\to\mathfrak{G}}. \tag{22.18}$$

The operator $(I-\Pi)\mathrm{rot}\,(\mathcal{L}^0+\varepsilon^2 I)^{-1}$ is the pseudodifferential operator of order (-1) (with constant coefficients), therefore, it maps $\mathfrak{G}$ to $\mathfrak{G}^1$ continuously:

$$\|(I-\Pi)\mathrm{rot}\,(\mathcal{L}^0+\varepsilon^2 I)^{-1}\|_{\mathfrak{G}\to\mathfrak{G}^1} \le \sup_{|\boldsymbol{\xi}|\ge r_0} |r(\boldsymbol{\xi})(b(\boldsymbol{\xi})^* g^0 b(\boldsymbol{\xi})+\varepsilon^2\mathbf{1})^{-1}|(1+|\boldsymbol{\xi}|^2)^{1/2}. \tag{22.19}$$

Here

$$r(\boldsymbol{\xi}) = \begin{pmatrix} 0 & -\xi_3 & \xi_2 \\ \xi_3 & 0 & -\xi_1 \\ -\xi_2 & \xi_1 & 0 \end{pmatrix}$$

is the symbol of the operator $-i\mathrm{rot}$. It can be elementarily checked that

$$\begin{aligned} |(b(\boldsymbol{\xi})^* g^0 b(\boldsymbol{\xi})+\varepsilon^2\mathbf{1})^{-1}| \le{}& c(\eta^0,\underline{\nu})|\boldsymbol{\xi}|^{-2}, \quad \boldsymbol{\xi}\in\mathbb{R}^3\setminus\{0\}, \\ & c(\eta^0,\underline{\nu}) = \max\{|\eta^0|,\underline{\nu}^{-1}\}. \end{aligned} \tag{22.20}$$

Besides, $|r(\boldsymbol{\xi})|\le|\boldsymbol{\xi}|$. Then (22.19) implies that

$$\|(I-\Pi)\mathrm{rot}\,(\mathcal{L}^0+\varepsilon^2 I)^{-1}\|_{\mathfrak{G}\to\mathfrak{G}^1} \le c(\eta^0,\underline{\nu})(r_0^{-2}+1)^{1/2}. \tag{22.21}$$

Next, by Proposition 8.2 of [Su2], the columns of the matrix $a(\mathbf{x})$ (the vector-valued functions $\nabla\widetilde{\Phi}_j+\mathbf{c}_j$, $j=1,2,3$) are multipliers from H^1 to L_2. Thus,

$$\|[a]\|_{\mathfrak{G}^1\to\mathfrak{G}} \le C_*, \tag{22.22}$$

where the constant C_* depends only on $\|\eta\|_{L_\infty}$, $\|\eta^{-1}\|_{L_\infty}$, and on parameters of the lattice Γ.

As a result, relations (22.18), (22.21), and (22.22) imply that

$$\|a^\varepsilon(I-\Pi_\varepsilon)\mathrm{rot}\,(\mathcal{L}^0+I)^{-1}\|_{\mathfrak{G}\to\mathfrak{G}} \le C_* c(\eta^0,\underline{\nu})(r_0^{-2}+1)^{1/2}\varepsilon.$$

This estimate allows us to replace Π_ε by I in (22.16):

$$\begin{aligned} \|(\eta^\varepsilon)^{-1}\mathrm{rot}\,\mathbf{u}_\varepsilon - a^\varepsilon\mathrm{rot}\,\mathbf{u}_0\|_{\mathfrak{G}} \le \widetilde{\mathcal{C}}_{15}\varepsilon\|\mathbf{F}\|_{\mathfrak{G}}, \quad 0<\varepsilon\le 1, \\ \widetilde{\mathcal{C}}_{15} = \mathcal{C}_{15} + C_* c(\eta^0,\underline{\nu})(r_0^{-2}+1)^{1/2}. \end{aligned}$$

It is yet easier to replace $\Pi_\varepsilon^{(1)}$ by I in (22.17). We consider the operator

$$\underline{\nu}(I-\Pi_\varepsilon^{(1)})\mathrm{div}\,(\mathcal{L}^0+I)^{-1}.$$

By analogy with (22.18), taking (22.20) into account, we have:

$$\begin{aligned}\|\underline{\nu}(I-\Pi_\varepsilon^{(1)})\mathrm{div}\,(\mathcal{L}^0+I)^{-1}\|_{\mathfrak{G}\to L_2(\mathbb{R}^3)} &= \varepsilon\|\underline{\nu}(I-\Pi^{(1)})\mathrm{div}\,(\mathcal{L}^0+\varepsilon^2 I)^{-1}\|_{\mathfrak{G}\to L_2(\mathbb{R}^3)} \\ &\le \varepsilon\underline{\nu}\sup_{|\boldsymbol{\xi}|\ge r_0}|\boldsymbol{\xi}||(b(\boldsymbol{\xi})^*g^0b(\boldsymbol{\xi})+\varepsilon^2\mathbf{1})^{-1}| \le \varepsilon\underline{\nu}\,c(\eta^0,\underline{\nu})r_0^{-1}.\end{aligned} \tag{22.23}$$

From (22.17) and (22.23) it follows that

$$\|\nu^\varepsilon\mathrm{div}\,\mathbf{u}_\varepsilon - \underline{\nu}\mathrm{div}\,\mathbf{u}_0\|_{L_2(\mathbb{R}^3)} \le \widehat{\mathcal{C}}_{15}\varepsilon\|\mathbf{F}\|_{\mathfrak{G}}, \quad 0<\varepsilon\le 1,$$
$$\widehat{\mathcal{C}}_{15} = \mathcal{C}_{15} + \underline{\nu}c(\eta^0,\underline{\nu})r_0^{-1}.$$

We arrive at the following statement.

Theorem 22.3. *Suppose that conditions of Theorem* 22.1 *are satisfied. Let* $\mathbf{u}_\varepsilon$ *be the solution of the equation* (22.13)*, and let* $\mathbf{u}_0$ *be the solution of the homogenized equation* (22.14)*. Let* $a(\mathbf{x})$ *be the* (3×3)*-matrix with the columns* $\nabla\widetilde{\Phi}_j(\mathbf{x})+\mathbf{c}_j$*,* $j=1,2,3$*, where* $\widetilde{\Phi}_j$ *is the* Γ*-periodic solution of the problem* (22.5)*. Then for* $0<\varepsilon\le 1$ *we have*

$$\|(\eta^\varepsilon)^{-1}\mathrm{rot}\,\mathbf{u}_\varepsilon - a^\varepsilon\mathrm{rot}\,\mathbf{u}_0\|_{\mathfrak{G}} \le \widetilde{\mathcal{C}}_{15}\varepsilon\|\mathbf{F}\|_{\mathfrak{G}}, \tag{22.24}$$

$$\|\nu^\varepsilon\mathrm{div}\,\mathbf{u}_\varepsilon - \underline{\nu}\mathrm{div}\,\mathbf{u}_0\|_{L_2(\mathbb{R}^3)} \le \widehat{\mathcal{C}}_{15}\varepsilon\|\mathbf{F}\|_{\mathfrak{G}}.$$

The constants $\widetilde{\mathcal{C}}_{15}$ *and* $\widehat{\mathcal{C}}_{15}$ *depend only on* $\|\eta\|_{L_\infty}$*,* $\|\eta^{-1}\|_{L_\infty}$*,* $\|\nu\|_{L_\infty}$*,* $\|\nu^{-1}\|_{L_\infty}$*, and on parameters of the lattice* Γ*.*

Note that (see Remark 12.2) the weak $\mathfrak{G}$-limit of the functions $(\eta^\varepsilon)^{-1}\mathrm{rot}\,\mathbf{u}_\varepsilon$ is equal to $(\eta^0)^{-1}\mathrm{rot}\,\mathbf{u}_0$.

22.4. Special cases

The case where $g^0=\overline{g}$ is realized, if $\nu=\mathrm{const}$ and $\eta^0=\underline{\eta}$, i. e., if the columns of the matrix η^{-1} are potential vectors (representations of the form (17.28) are true). In this case, Theorems 10.9 and 11.5 are applicable. This gives the following result.

Theorem 22.4. *Suppose that conditions of Theorem* 22.1 *are satisfied. Suppose also that* $\nu=\mathrm{const}$ *and* $\eta^0=\underline{\eta}$*. Then for* $0<\varepsilon\le 1$ *we have*

$$\|(\mathcal{L}_\varepsilon+I)^{-1}-(\mathcal{L}^0+I)^{-1}\|_{\mathfrak{G}\to\mathfrak{G}^1} \le \mathcal{C}_{12}\varepsilon. \tag{22.25}$$

Besides, for $0\le s\le 1$ *and* $0<\varepsilon\le 1$ *we have*

$$\|(\mathcal{L}_\varepsilon+I)^{-1}-(\mathcal{L}^0+I)^{-1}\|_{\mathfrak{G}\to\mathfrak{G}^s} \le \widetilde{\mathcal{C}}_s\varepsilon^{2-s}.$$

The constant $\mathcal{C}_{12}$ *depends only on* $\|\eta\|_{L_\infty}$*,* $\|\eta^{-1}\|_{L_\infty}$*,* $\|\nu\|_{L_\infty}$*,* $\|\nu^{-1}\|_{L_\infty}$*, and on parameters of the lattice* Γ*, while the constant* $\widetilde{\mathcal{C}}_s$ *depends on the same parameters and on* s*.*

The case $g^0=\underline{g}$ is realized, if $\nu(\mathbf{x})$ is arbitrary and $\eta^0=\overline{\eta}$ (i. e., if the columns of the matrix $\eta(\mathbf{x})$ are solenoidal). Then we can apply Theorem 12.4, which gives the following result.

Theorem 22.5. *Suppose that conditions of Theorem* 22.1 *are satisfied. Suppose also that* $\eta^0 = \overline{\eta}$ *(i. e., the columns of the matrix* $\eta(\mathbf{x})$ *are solenoidal vectors). Let* $\mathbf{u}_\varepsilon$ *be the solution of the equation* (22.13)*, and let* $\mathbf{u}_0$ *be the solution of the equation* (22.14)*. Then for* $0 < \varepsilon \le 1$ *we have*

$$\|g^\varepsilon b(\mathbf{D})\mathbf{u}_\varepsilon - g^0 b(\mathbf{D})\mathbf{u}_0\|_{\mathfrak{G}_*} \le \mathcal{C}_{16}\varepsilon\|\mathbf{F}\|_{\mathfrak{G}},$$

which implies two estimates

$$\|(\eta^\varepsilon)^{-1}\mathrm{rot}\,\mathbf{u}_\varepsilon - (\overline{\eta})^{-1}\mathrm{rot}\,\mathbf{u}_0\|_{\mathfrak{G}} \le \mathcal{C}_{16}\varepsilon\|\mathbf{F}\|_{\mathfrak{G}}, \tag{22.26}$$

$$\|\nu^\varepsilon \mathrm{div}\,\mathbf{u}_\varepsilon - \underline{\nu}\mathrm{div}\,\mathbf{u}_0\|_{L_2(\mathbb{R}^3)} \le \mathcal{C}_{16}\varepsilon\|\mathbf{F}\|_{\mathfrak{G}}.$$

The constant $\mathcal{C}_{16}$ *depends only on* $\|\eta\|_{L_\infty}$*,* $\|\eta^{-1}\|_{L_\infty}$*,* $\|\nu\|_{L_\infty}$*,* $\|\nu^{-1}\|_{L_\infty}$*, and on parameters of the lattice* Γ*.*

22.5. Splitting of the operator $\mathcal{L}$

We put

$$J = \{\mathbf{u} \in \mathfrak{G} : \ \mathrm{div}\,\mathbf{u} = 0\}.$$

We use the orthogonal Weyl decomposition

$$\mathfrak{G} = L_2(\mathbb{R}^3; \mathbb{C}^3) = J \oplus G, \tag{22.27}$$

where

$$G = \{\mathbf{u} = \nabla\phi : \ \phi \in H^1_{\mathrm{loc}}(\mathbb{R}^3), \ \nabla\phi \in \mathfrak{G}\}.$$

Decomposition (22.27) reduces the operator (22.1):

$$\mathcal{L} = \mathcal{L}_J \oplus \mathcal{L}_G.$$

The operator $\mathcal{L}_J$ acting in the subspace J corresponds to the differential expression $\mathrm{rot}\,\eta^{-1}\mathrm{rot}$, and the operator $\mathcal{L}_G$ acting in G is given by the expression $-\nabla\nu\mathrm{div}$. The operators $\mathcal{L}_\varepsilon$ and $\mathcal{L}^0$ are also reduced by decomposition (22.27):

$$\mathcal{L}_\varepsilon = \mathcal{L}_{J,\varepsilon} \oplus \mathcal{L}_{G,\varepsilon}, \quad \mathcal{L}^0 = \mathcal{L}^0_J \oplus \mathcal{L}^0_G.$$

(For applications to the Maxwell system), we are interested mainly in the operators $\mathcal{L}_J$, $\mathcal{L}_{J,\varepsilon}$, and $\mathcal{L}^0_J$. Since they do not depend on the coefficient ν, it suffices to consider the case where $\nu = 1$.

Let $\mathcal{P}$ be the orthogonal projection of $\mathfrak{G}$ onto J. Then (see [BSu2, Subsection 2.4 of Ch. 7]) $\mathcal{P}$ (restricted to $\mathfrak{G}^s$) is also the orthogonal projection of the space $\mathfrak{G}^s = H^s(\mathbb{R}^3; \mathbb{C}^3)$ onto the subspace $J^s = \mathfrak{G}^s \cap J$, for all $s > 0$.

Restricting the operators in (22.8) onto the subspace J and multiplying them by $\mathcal{P}$ from the left, we obtain:

$$\|(\mathcal{L}_{J,\varepsilon} + I_J)^{-1} - (\mathcal{L}^0_J + I_J)^{-1} - \varepsilon\mathcal{P}\Psi^\varepsilon\Pi_\varepsilon \mathrm{rot}\,(\mathcal{L}^0_J + I_J)^{-1}\|_{J\to J^1} \le \mathcal{C}'_{10}\varepsilon, \quad 0 < \varepsilon \le 1. \tag{22.28}$$

The constant $\mathcal{C}'_{10}$ is equal to $\mathcal{C}_{10}$ with $\nu = 1$, whence $\mathcal{C}'_{10}$ depends only on $\|\eta\|_{L_\infty}$, $\|\eta^{-1}\|_{L_\infty}$, and on parameters of the lattice Γ.

Similarly, restricting the operators in (22.12) onto the subspace J and multiplying them by $\mathcal{P}$ from the left, we obtain:

$$\|(\mathcal{L}_{J,\varepsilon}+I_J)^{-1}-(\mathcal{L}_J^0+I_J)^{-1}-\varepsilon\widetilde{K}_J(\varepsilon)\|_{J\to J^s}\leq \mathcal{C}'_s\varepsilon^{2-s},\quad 0\leq s\leq 1,\ \ 0<\varepsilon\leq 1, \tag{22.29}$$

where

$$\begin{aligned}\widetilde{K}_J(\varepsilon)=\ &\mathcal{P}\Psi^\varepsilon\Pi_\varepsilon\mathrm{rot}\,(\mathcal{L}_J^0+I_J)^{-1}\\ &+(\mathcal{L}_J^0+I_J)^{-1}\Pi_\varepsilon\mathrm{rot}\,(\Psi^\varepsilon)^t-(\mathcal{L}_J^0+I_J)^{-1}\mathcal{E}(\mathbf{D})(\mathcal{L}_J^0+I_J)^{-1}.\end{aligned} \tag{22.30}$$

The constant $\mathcal{C}'_s$ is equal to $\mathcal{C}_s$ with $\nu=1$, whence $\mathcal{C}'_s$ depends only on s, on $\|\eta\|_{L_\infty}$, $\|\eta^{-1}\|_{L_\infty}$, and on parameters of the lattice Γ.

We arrive at the following result.

Theorem 22.6. *Suppose that conditions of Theorem* 22.1 *are satisfied. Let* $\mathcal{L}_{J,\varepsilon}=\mathrm{rot}\,(\eta^\varepsilon)^{-1}\mathrm{rot}$ *be the part of the operator* $\mathcal{L}_\varepsilon$ *in the solenoidal subspace* J. *Let* $\mathcal{L}_J^0=\mathrm{rot}\,(\eta^0)^{-1}\mathrm{rot}$ *be the part of the effective operator* $\mathcal{L}^0$ *in* J. *Let* $\mathcal{P}$ *be the orthogonal projection of* $\mathfrak{G}$ *onto the subspace* J. *Let* $\widetilde{K}_J(\varepsilon)$ *be the corrector defined by* (22.30). *Then estimate* (22.28) *is true, and for* $0\leq s\leq 1$ *estimate* (22.29) *is valid. The constant* $\mathcal{C}'_{10}$ *from* (22.28) *depends on* $\|\eta\|_{L_\infty}$, $\|\eta^{-1}\|_{L_\infty}$, *and on parameters of the lattice* Γ, *while the constant* $\mathcal{C}'_s$ *from* (22.29) *depends on the same parameters and also on* s.

Now we apply (22.24) with $\mathbf{F}\in J$. Then, by the splitting of the operators $\mathcal{L}_\varepsilon$ and $\mathcal{L}^0$, we have:

$$\mathbf{u}_\varepsilon=(\mathcal{L}_{J,\varepsilon}+I_J)^{-1}\mathbf{F},\quad \mathbf{u}_0=(\mathcal{L}_J^0+I_J)^{-1}\mathbf{F},\quad \mathbf{F}\in J. \tag{22.31}$$

By $\widetilde{\mathcal{C}}'_{15}$ we denote the constant $\widetilde{\mathcal{C}}_{15}$ with $\nu=1$. We arrive at the following result.

Theorem 22.7. *Suppose that conditions of Theorem* 22.6 *are satisfied. Let* $\mathbf{u}_\varepsilon$ *and* $\mathbf{u}_0$ *be the vector-valued functions defined by* (22.31). *Let* $a(\mathbf{x})$ *be the* (3×3)*-matrix with the columns* $\nabla\widetilde{\Phi}_j(\mathbf{x})+\mathbf{c}_j$, $j=1,2,3$, *where* $\widetilde{\Phi}_j$ *is the* Γ*-periodic solution of the problem* (22.5). *Then for* $0<\varepsilon\leq 1$ *we have*

$$\|(\eta^\varepsilon)^{-1}\mathrm{rot}\,\mathbf{u}_\varepsilon-a^\varepsilon\mathrm{rot}\,\mathbf{u}_0\|_{\mathfrak{G}}\leq\widetilde{\mathcal{C}}'_{15}\varepsilon\|\mathbf{F}\|_{\mathfrak{G}}.$$

The constant $\widetilde{\mathcal{C}}'_{15}$ *depends only on* $\|\eta\|_{L_\infty}$, $\|\eta^{-1}\|_{L_\infty}$, *and on parameters of the lattice* Γ.

Now we distinguish the special cases. Restricting the operators in (22.25) onto J and using the notation $\mathcal{C}'_{12}$ for the constant $\mathcal{C}_{12}$ with $\nu=1$, we arrive at the following result.

Theorem 22.8. *Suppose that conditions of Theorem* 22.6 *are satisfied. Suppose also that* $\eta^0=\underline{\eta}$. *Then for* $0<\varepsilon\leq 1$ *we have*

$$\|(\mathcal{L}_{J,\varepsilon}+I_J)^{-1}-(\mathcal{L}_J^0+I_J)^{-1}\|_{J\to J^1}\leq\mathcal{C}'_{12}\varepsilon.$$

Besides, for $0\leq s\leq 1$ *and* $0<\varepsilon\leq 1$ *we have*

$$\|(\mathcal{L}_{J,\varepsilon}+I_J)^{-1}-(\mathcal{L}_J^0+I_J)^{-1}\|_{J\to J^s}\leq\widetilde{\mathcal{C}}'_s\varepsilon^{2-s}.$$

The constant $\mathcal{C}'_{12}$ *depends on* $\|\eta\|_{L_\infty}$, $\|\eta^{-1}\|_{L_\infty}$, *and on parameters of the lattice* Γ, *while the constant* $\widetilde{\mathcal{C}}'_s$ *depends on the same parameters and on* s.

The following statement is deduced from (22.26) with $\mathbf{F}\in J$; here we use the notation $\mathcal{C}'_{16}$ for the constant $\mathcal{C}_{16}$ with $\nu=1$.

Theorem 22.9. *Suppose that conditions of Theorem* 22.6 *are satisfied. Suppose also that* $\eta^0=\overline{\eta}$. *Then for* $0<\varepsilon\leq 1$ *we have*

$$\|(\eta^\varepsilon)^{-1}\mathrm{rot}\,\mathbf{u}_\varepsilon-(\overline{\eta})^{-1}\mathrm{rot}\,\mathbf{u}_0\|_{\mathfrak{G}}\leq\mathcal{C}'_{16}\varepsilon\|\mathbf{F}\|_{\mathfrak{G}}.$$

The constant $\mathcal{C}'_{16}$ *depends on* $\|\eta\|_{L_\infty}$, $\|\eta^{-1}\|_{L_\infty}$, *and on parameters of the lattice* Γ.

§23. Comments

Along with the material of this section, it is useful to be acquainted with the similar §15 from [BSu4].

23.1

We did not consider the problem of approximations of higher order for the resolvent $(\widehat{\mathcal{A}}_\varepsilon + I)^{-1}$. The significance of such approximations for applications is scarcely big. We also mention the following. Suppose that, using approximations of higher order, we want to obtain the remainder estimates of „right“ order in the natural operator norms. Then we may expect that these approximations will contain new terms besides the standard ones, i. e., those that can be found by the twoscale expansions method of N. S. Bakhvalov (see [BaPa]). As a result, the formulas will become yet more bulky.

The fear expressed above is indirectly confirmed by the observation related to the homogenization problem for the stationary Maxwell system. In [Su1,2], for some physical fields the following was clarified. In order to find approximation in the $(L_2 \to L_2)$-norm with the error term of order ε, it is not sufficient to take the resolvent of the homogenized Maxwell system. We are forced to add terms of order $O(1)$ (that are rapidly oscillating as $\varepsilon \to 0$) to it. In the weak limit procedures, these terms make no contribution and are not noticed.

23.2

Often, in the homogenization theory, not the problem in the whole space is considered, but the problem in a fixed bounded domain $\mathcal{O}$ ($\partial\mathcal{O}$ is smooth), under an appropriate classical boundary condition. Such problem is more difficult than the problem in the whole space, since the homogenization effect itself interacts with the effects occuring in the boundary layer. Sometimes, however, it is useful first to solve the homogenization problem in $\mathbb{R}^d$, and afterwards try to satisfy the boundary conditions on $\partial\mathcal{O}$. It is this way, that was used for the proof of the $(L_2 \to H^1)$-estimates in $\mathcal{O}$ in the paper [ZhPas]. Herewith, the order of the error estimate worsens up to $\varepsilon^{1/2}$.

Some results about H^1-estimates in a bounded domain can be found in [Gr1,2].

23.3. Correctors. Smoothing operators

We emphasize once more that, a fortiori, the corrector is defined not uniquely. Herewith, the extent of this non-uniqueness depends on the choice of the (operator) norm, in which we wish to estimate the error of aproximation. This is already seen from comparence of formulas (0.8)–(0.10), and also of (0.10) and (11.5). Note also that (see [BSu4, Proposition 8.8]) the operator K_3 in the corrector (0.9) coincides with the weak L_2-derivative of the operator-valued function $(\widehat{\mathcal{A}}_\varepsilon + I)^{-1}$ with respect to ε at $\varepsilon = 0$. Therefore, K_3 is defined uniquely. At the same time, (0.10) does not contain K_3.

Usually, in the proof of estimates (0.8) or (0.10), we cannot avoid including of some smoothing operator in the corrector (our smoothing operator is the pseudodifferential operator Π_ε defined by (10.4)). The smoothing operator is also defined not uniquely. Evidently, in general it is impossible to get rid of (some) smoothing operator. However, sometimes it is possible to eliminate it, i. e., to replace it by I. We attentively looked for the sufficient conditions for such elimination. As it has already been mentioned, the sufficient conditions for elimination of Π_ε in the case of estimate (0.8) are wider than in the case of estimate (0.10). It is also interesting that, in the case of the model operator of electrodynamics (see §22), in the approximation of the resolvent in the $(L_2 \to H^1)$-norm we did not succeed to eliminate the operator Π_ε in the corrector, while in the approximation of the fluxes in the $(L_2 \to L_2)$-norm we succeeded to eliminate Π_ε. The latter fact is essential for applications to the Maxwell operator theory.

23.4. The stationary Maxwell system

From the point of view of homogenization problems, the stationary Maxwell system with the periodic characteristics of the medium is of significant interest. Up to now, approximations with the error estimates of order ε in the $(L_2 \to L_2)$-norm are not obtained for all physical fields. The homogenization problem for the Maxwell operator can be reduced to the similar problem for an appropriate elliptic second order DO. This operator admits a factorization of the form $\mathcal{X}^*\mathcal{X}$, where $\mathcal{X}$ is a homogeneous first order DO. In general, the latter operator cannot be represented in the form $\mathcal{X} = hb(\mathbf{D})f$ (see Subsection 4.1). (This representation is assumed in [BSu2,4] and also in the present paper.) However, our abstract results are applicable to the DO $\mathcal{X}^*\mathcal{X}$. On this basis, in a separate work [Su1,2], approximations with the error estimates of order ε in the $(L_2 \to L_2)$-norm were obtained, but, unfortunately, not for all physical fields.

However, a little bit earlier, the following was mentioned (see [BSu2, Ch. 7]). If one of two characteristics of the medium (for instance, the magnetic permeability μ) is equal to identity (or is constant), then the corresponding second order DO belongs to the class of operators of the form (0.5). On this way, in [BSu2], for the Maxwell operator approximation in the $(L_2 \to L_2)$-norm with the error estimate of order ε was found for one (of six!) physical field. For two more fields, similar results directly follow from the general statements of [Su1,2]; herewith, approximations contain the corrector of order zero (with respect to ε), which is rapidly oscillating as $\varepsilon \to 0$.

Finally, using the results of §22 about approximation of the fluxes, we can deduce the required approximations and estimates for the remaining three fields. Thus, for the case where $\mu = 1$, the required results about homogenization of the Maxwell operator are now obtained. Note that all six approximations do not contain smoothing operators. The results for $\mu = 1$ will be written in details elsewhere. Up to now, the homogenization problem for the Maxwell operator of general type is not studied completely.

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