SPHERICAL TRANSFORMS
AND RADON TRANSFORMS
IN MOEBIUS GEOMETRY

Eberhard Teufel

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Abstract

We study spherical transforms on euclidean spaces through a geometric view on the action of differential operators acting onto spheres. We achieve interrelations between Radon transforms and spherical transforms. We attain inversion formulas, especially a one-radius inversion formula. The results are conformal invariant. Moreover we get two-radius and one-radius-germ support results. Finally we derive interrelations and inversion formulas for Radon transforms, spherical transforms and horospherical transforms in hyperbolic spaces.

Key Words: Spherical transform, Radon transform, inversion formula, support theorem, Moebius geometry, horospherical transform, hyperbolic geometry.

AMS Subject Classification: 53C65, 44A12, 53A30.

1 Introduction

Radon transforms go back to P. FUNK (1916) and J. RADON (1917), cf. [5], [16], [9], [6]. Spherical transforms, or spherical means, in connection with plane waves and the Darboux equation go back to F. JOHN (1935), cf. [12]. Both transformations come together from a conformal point of view, cf. [7]. The classical techniques are harmonic analysis, PDEs, integral equations, geometric means. The following studies are based on a geometric view on the action of differential operators of first order acting onto spheres and planes (3) (4). To make the geometric ideas clear we first of all concentrate on the 3-dimensional euclidean space. We attain interrelations between the spherical transform and the Radon transform, and interrelations for the spherical transform respectively (Theorem 1, Theorem 2). Moreover we obtain an inversion formula (Theorem 3) and a one-radius inversion formula (Theorem 4) for the spherical transform.

The basic results Theorem 1 (7) and Theorem 2 (13), written in euclidean terms, are conformal invariant and Moebius invariant respectively (Theorem 5). The Moebius pitch are degenerate pencils of spheres and pencils of spheres with limit points (Poncelet pencils) respectively. (Actual we fail in playing with pencils of spheres without limit points.)

Section 4 plays in hyperbolic spaces. We use conformal models of the hyperbolic space in order to apply the euclidean situations. We derive similar results concerning the geodesic Radon transform, the spherical transform and the horospherical transform respectively.

Section 5 contains support results similar to support theorem for the classical Radon transform [8] and for the spherical transform [21] respectively. We get two-radius and one-radius support theorems (Theorem 8, Theorem 9).

In section 6 we point out some first aspects of a reduction scheme in higher dimensions.

A similar geometric view on Radon transforms is treated in [19].

Let $G^{k,n}$ be the space of $k$-dimensional affine subspaces of the $n$-dimensional euclidean space $E^n$. The $k$-plane Radon transform $R^k$ maps $C_c^\infty(E^n)$ into $C_c^\infty(G^{k,n})$, namely

$$ (R^k f)(\eta) = \int f(x) \, dx \quad , \quad \eta \in G^{k,n} , $$

(1)
\( C^\infty_c \) = space of \( C^\infty \)-functions with compact support; \( dx \) = euclidean volume density on the \( k \)-plane \( \eta \).

Let \( M^{k,n} \) be the space of \( k \)-dimensional spheres in \( E^n \). The \( k \)-spherical transform \( S^k \) maps \( C^\infty_c (E^n) \) into \( C^\infty (M^{k,n}) \), namely

\[
(S^k f)(\xi) = \int f(x) \, dx \quad , \quad \xi \in M^{k,n} ,
\]

(\( dx \) = euclidean volume density on the \( k \)-sphere \( \xi \)).

Let \( D_X \), \( X \in V^n \) (\( V^n \) = euclidean vector space associated to \( E^n \)), denote the differential operator of first order acting on \( C^\infty (M^{k,n}) \) through parallel translating the spheres in \( E^n \), i.e.

\[
(D_X F)(\xi) := \frac{d}{dt} (F(\xi + tX)) \mid_{t=0}
\]

(\( \xi \in M^{k,n}, F \in C^\infty (M^{k,n}) \)).

Let \( D_r \) denote the differential operator of first order acting on \( C^\infty (M^{k,n}) \) through bubbling the spheres, i.e.

\[
(D_r F)(\xi) := \frac{d}{dt} (F(\xi(m,t))) \mid_{t=r}
\]

(\( \xi = \xi(m,r) = \text{sphere with center } m \text{ and radius } r \)).

One of the key questions is how to invert these transformations, i.e. how to reconstruct the point function \( f \) from the knowledge of the Radon transform \( R^k f \) or the spherical transform \( S^k f \) respectively.

For the Radon transform on the 3-dimensional euclidean space this can be done through a dual integration by means of a differential operator of second order acting onto planes, namely

\[
f(x) = \frac{1}{4\pi^2} \int_{G^2_x} D_{N(\eta)}^2(R^2 f)(\eta) \, d\eta ,
\]

(\( x \in E^3 \) fix, \( G^2_x \) = space of 2-planes in \( E^3 \) through \( x = \text{unit sphere with center } x \), \( N(\eta) \) = unit normal vector of \( \eta \), \( d\eta \) = volume density of the unit sphere, \( D_N \) see (3)).

The integration on the right hand side of (1) runs over all points \( x \) in the plane \( \eta \), whereas the integration on the right hand side of (5) runs over all planes \( \eta \) through the point \( x \). This is the play of the classical duality between points and planes in space. In the following we shall meet similar and more general situations, and in the same way we call the corresponding integrations dual integrations.

The inversion of the spherical transform is trivial, provided that \( S^2 f \) is available at spheres of arbitrary small radius,

\[
f(x) = \frac{1}{4\pi} \lim_{r \to \infty} \frac{1}{r^2} S^2 f(\xi(x,r)) ,
\]

(\( x \in E^3 \)).

2 Spherical transform and Radon transform on the euclidean space

Theorem 1 Let \( \eta_1, \eta_2 \) be parallel 2-planes in \( E^3 \) at distance 2r. Let \( a, b \) be real constants. Then

\[
2\pi \left( (a + b) \cdot R^2f(\eta_2) + (-a + b) \cdot R^2f(\eta_1) \right) =
\]

\[
= \int_{\mu} \left( \frac{a}{r} \cdot D_N + \frac{b}{r} \cdot D_r - \frac{b}{r^2} \right) (S^2 f)(\xi) \, dm .
\]
Herein the dual integration on the right hand side runs over the totality of 2-spheres $\xi = \xi(m,r)$ of radius $r$ tangent to both $\eta_1$ and $\eta_2$, parametrized through their centers $m \in \mu$; $dm$ = euclidean volume density on $\mu$, the mid-plane with respect to $\eta_1$ and $\eta_2$; $N$ = normal unit vector of $\eta_1$ pointing towards $\eta_2$.

Proof: The definition of the differential operators (3), (4) implies
\[ D_N(S^2f)(\xi) = \int_\xi df_y(N)dy \]  
\[ D_r(S^2f)(\xi) = -\int_\xi df_y(e_1)dy + \frac{2}{r} \int f(y)dy \]  
\[ (e_1 = \text{interior normal unit vector of } \xi \text{ at } y). \]  
(The vector fields $N$ in (8) and $e_1$ in (9) respectively are unique up to divergence-free tangent vector fields along $\xi$.)

For $\xi = \xi(m,r), m \in \mu, \text{ let } x_1 = \eta_1 \cap \xi, \text{ resp. } x_2 = \eta_2 \cap \xi \text{ denote the points of contact. We take orthonormal moving frames } y e_1 e_2 e_3, y \tilde{e}_1 \tilde{e}_2 \tilde{e}_3, y \in \xi, \text{ with the following adaptations: } e_1 = \text{interior normal unit vector of } \xi \text{ at } y, e_2 = \text{tangent unit vector at } y \text{ of the oriented great circle of } \xi \text{ from } y \text{ to } x_1, \tilde{e}_1 = N, \tilde{e}_2 = \text{tangent unit vector at } y \text{ of the oriented perpendicular line from } y \text{ to } x_1 x_2. \text{ Then}
\[ e_1 = -\frac{1}{\sin \alpha} e_2 + \frac{\cos \alpha}{\sin \alpha} \tilde{e}_2 \]
\[ e_1 = -\frac{\cos \alpha}{\sin \alpha} e_2 + \frac{1}{\sin \alpha} \tilde{e}_2 \]  
\[ (\alpha = \angle(x_1 m y)). \]

Hence (8), (9) yield
\[ (a \cdot D_N + b \cdot D_r)(S^2f)(\xi) = \int_\xi \left( -\frac{a}{\sin \alpha} + \frac{b \cos \alpha}{\sin \alpha} \right) df_y(e_2)dy + \frac{2b}{r} \int f(y)dy \]  
\[ \text{mod integrand terms } df_y(\tilde{e}_2) \text{ disappearing through the dual integration because of symmetry.} \]

Now, using polar coordinates on $\xi$ centered at $x_1$, i.e. dy = r sin $\alpha$ (rda) d$\phi$ to rewrite the integrand of the first integral on the right hand side of (11), i.e. r($-a + b \cos \alpha$) df_y(e_2)(rda) d$\phi$, and integrating by parts with respect to (rda), this integral becomes
\[ 2\pi r ((a + b)f(x_2) + (-a + b)f(x_1)) - \frac{b}{r} \cdot S^2(f)(\xi). \]

Finally, we bring (12) back into (11) and we carry-out the dual integration, taking into account $dx_1 = dx_2 = dm$. Thus we reach (7). $\square$

Remark 1. Read from the right to the left hand side (7) is pointing a way from the spherical transform to the Radon transform. For the opposite way from the Radon transform to the spherical transform, e.g. through inversion of Volterra integral equations of first kind (Abel type), cf. [9] proof of theorem 2.6.

Theorem 2 Let $\eta_1, \eta_2$ be concentric 2-spheres in $E^3$ with center o and radii $r_1, r_2 (r_1 < r_2)$. Let $a, b$ be real constants. Then
\[ 2\pi \left( (a + b) \left( \frac{r_1 + r_2}{2r_2} \right) \cdot S^2f(\eta_2) + (-a + b) \left( \frac{r_1 + r_2}{2r_1} \right) \cdot S^2f(\eta_1) \right) = \right. \]
\[ \left. = \int_{\mu} \left( \frac{a}{r} \cdot D_N(m) + \frac{b}{r} \cdot D_r + \frac{2a}{r(r_1+r_2)} - \frac{b}{r^2} \right) (S^2f)(\xi) dm. \right. \]  
\[ (13) \]
Herein the dual integration runs over the totality of 2-spheres \( \xi = \xi(m,r) \) of radius \( r = \frac{\sqrt{r_1 r_2}}{2} \) tangent to both \( \eta_1 \) and \( \eta_2 \), \( \xi \) lying outside \( \eta_1 \) and inside \( \eta_2 \), and parametrized through their centers \( m \in \mu; \ dm = \text{euclidean volume density on the 2-sphere} \mu = \mu(\frac{\sqrt{r_1 r_2}}{2}); N = N(m) = \text{exterior normal unit vector of} \mu \) at \( m \).

Proof: For \( \xi = \xi(m,r), m \in \mu, \) let \( x_1 = \eta_1 \cap \xi, \) resp. \( x_2 = \eta_2 \cap \xi \) denote the points of contact. We take orthonormal moving frames \( y e_1 e_2 e_3, y e_1 e_2 e_3, y \in \xi, \) with the following adaptations: \( e_1 = \text{interior normal unit vector of} \xi \) at \( y, e_2 = \text{tangent unit vector at} \ y \) of the oriented great circle of \( \xi \) from \( y \) to \( x_1, e_1 = \text{tangent unit vector at} \ y \) of the oriented line from \( o \) to \( y, e_2 = \text{tangent unit vector at} \ y \) of the oriented line from \( y \) to \( \text{om perpendicular to} \ oy. \) Then

\[
N = -\frac{\cos \beta}{\sin(\alpha + \beta)} \cdot e_2 + \frac{\cos \alpha}{\sin(\alpha + \beta)} \cdot \bar{e}_2 \\
e_1 = -\frac{\cos(\alpha + \beta)}{\sin(\alpha + \beta)} \cdot e_2 + \frac{1}{\sin(\alpha + \beta)} \cdot \bar{e}_2
\]

(14)

where \( \alpha = \angle(x_1 my), \beta = \angle(x_1 oy). \)

Hence (8), (9) yield

\[
(a \cdot D_{N(m)} + b \cdot D_r)(S^2 f)(\xi) = \int \frac{\sin \alpha}{\sin(\alpha + \beta)} (-a \cos \beta + b \cos(\alpha + \beta)) df(y)(e_2) dy + \frac{2b}{r} \int f(y) dy
\]

(15)

mod integrand terms \( df(y)(e_2) \) disappearing through the dual integration because of symmetry.

Now, using polar coordinates on \( \xi \) centered at \( x_1, i.e. \ dy = r \sin \alpha (rd\alpha) d\varphi \) to rewrite the integrand of the first integral on the right hand side of (15), i.e.

\[
r\frac{\sin \alpha}{\sin(\alpha + \beta)} (-a \cos \beta + b \cos(\alpha + \beta)) df(y)(e_2)(rd\alpha) d\varphi,
\]

and integrating by parts with respect to \( (rd\alpha), \) this integral becomes

\[
2\pi r \left( \frac{2r_2}{r_1 + r_2} (a + b) f(x_2) + \frac{2r_1}{r_1 + r_2} (-a + b) f(x_1) \right) - \left( \frac{2a}{r_1 + r_2} + \frac{b}{r} \right) S^2(f)(\xi).
\]

(16)

(Some explanation: e.g. \( d(br \frac{\sin \alpha \cos(\alpha + \beta)}{\sin(\alpha + \beta)}) df(y)(e_2) = \frac{b}{2(\sqrt{r} + r^2)} df(y)(e_2) = \frac{b}{r_1 + r_2} \cdot d\rho y (\sin(\alpha + \beta) \cdot \bar{e}_1 + \cos(\alpha + \beta) \cdot \bar{e}_2) = \frac{b}{r_1 + r_2} \cdot \sin(\alpha + \beta) = b \sin \alpha, \) using some trigonometric formulary in triangle \( omy, \rho = \text{euclidean distance between} \ o \ and \ y. \)

We bring (16) back into (15) and we carry-out the dual integration, taking into account \( dx_1 = \left( \frac{2r_1}{r_1 + r_2} \right)^2 dm, \ dx_2 = \left( \frac{2r_2}{r_1 + r_2} \right)^2 dm. \) That way we reach (13). \) □

Remark 2. The same situation as in Theorem 2, but now using all the 2-spheres \( \xi \) of radius \( r = \frac{\sqrt{r_1 r_2}}{2} \) tangent to both \( \eta_1 \) and \( \eta_2, \eta_1 \text{ lying inside} \ \xi \) and \( \xi \text{ lying inside} \ \eta_2, \) leads to (13) with \( r_1 + r_2 \) replaced by \( r_2 - r_1. \)

Remark 3. (13) with \( b = 0 \) and \( r_2 \to \infty \) gives a way from the Radon transform to the spherical transform.

(13) with \( r_1 \to \infty, r_2 \to \infty \) and \( r_2 - r_1 = 2r \) leads to (7).

Now, starting at (13) we let the sphere \( \eta_1 \) shrink to its center \( o. \) We calculate the geometric Taylor expansion of (13), and we get the following one-radius-germ inversion formula for the spherical transform.
Theorem 3 Let \( o \in E^3, r > 0 \). Then
\[
8\pi^2 f(o) = -\int_\mu \left( \frac{1}{r^2} D_{(m,r)}^2 + \frac{1}{r^3} D_{(m,r)} \right)(S^2 f)(\xi) \, dm = -\int_\mu \left( \frac{1}{r^2} D_{(m,r)}^2 - \frac{1}{r^3} \right)(S^2 f)(\xi) \, dm.
\]
(17)

Herein the dual integration runs over the totality of 2-spheres \( \xi = \xi(m,r) \) of radius \( r \) through \( o \), parametrized through their centers \( m \in \mu(o,r) \); \( dm = \text{eucieidean volume density on the 2-sphere} \mu; D_{(m,r)} \) and \( D_{(m,r)}^2 = \text{differential operator of first and second order, see (19), (21), (22).} \)

Proof: (13) with \( \eta_1 = \eta_1(o,r_1), \eta_2 = \eta_2(o,2r) \) and \( a = \frac{1}{2}, b = -\frac{1}{2} \) implies through \( r_1 \to 0 \)
\[
\int_\mu D_{(m,r)}(S^2 f)(\xi) \, dm = -\frac{1}{r} \int_\mu S^2 f(\xi) \, dm,
\]
(18)
with
\[
D_{(m,r)} := \frac{1}{2} D_{N(m)} - \frac{1}{2} D_r
\]
(19)
at \( \xi = \xi(m,r), m \in \mu, N(m) = \text{exterior unit normal vector of} \mu \) at \( m \). And further
\[
8\pi \cdot f(o) = \lim_{r_1 \to 0} \frac{1}{r_1} \left( -\frac{1}{r^2} \int_{\mu_1} D_{(m(r_1),r)}(S^2 f)(\xi) \, dm(r_1) - \frac{1}{r^2} \int_{\mu_1} S^2 f(\xi) \, dm(r_1) \right)
\]
(20)
\[
D_{(m(r_1),r)} := \frac{1}{2} D_{N(m(r_1))} - \frac{1}{2} D_r \text{ at } \xi = \xi(m(r_1), r - \frac{r_1}{2}), m(r_1) \in \mu(r_1) := \mu(o,r + \frac{r_1}{2}). \text{ All 2-spheres} \xi \text{ coming up belong to the family of 2-spheres tangent to } \eta_2 = \eta_2(o,2r). \text{ Therefore, note } a = \frac{1}{2}, b = -\frac{1}{2},
\]
\[
D_{(m(r_1),r)} = \frac{d}{dt} \left( \xi(m(r_1) + t \cdot u, r - \frac{r_1 + t}{2}) \right)|_{t=0}
\]
(21)
\[
D_{(m(r_1),r)}(S^2 f)(\xi) = \frac{d}{dt} \left( S^2 f(\xi(m(r_1) + t \cdot u, r - \frac{r_1 + t}{2})) \right)|_{t=0}
\]
(22)
Now Taylor-expansion with respect to \( r_1 \) (direction \( u \) fixed) at \( r_1 = 0 \) gives
\[
S^2 f(\xi(m(r_1), r - \frac{r_1}{2})) = S^2 f(\xi(m,r)) + r_1 \cdot D_{(m(r_1),r)}(S^2 f)(\xi(m(r), r - \frac{r_1}{2}))
\]
(23)
for some \( 0 < \tau < r_1 \), and
\[
D_{(m(r_1),r)}(S^2 f)(\xi(m(r_1), r - \frac{r_1}{2})) = D_{(m(r),r)}(S^2 f)(\xi(m(r), r)) + r_1 \cdot D_{(m(r),r)}^2(S^2 f)(\xi(m(r), r - \frac{\tau}{2}))
\]
(24)
for some $0 < \sigma < r_1$.

Finally we bring together (20), (23), (24), (18) (note: $dm(r_1) = (r + \frac{r_1}{r})^2 r^{-2} dm$), and we reach (17). □

Starting again at (13) we consider infinite many dual integrations in concentric shells around $o$. And for rapidly decreasing point functions $f$ we get the following one-radius inversion formula for the spherical transform.

**Theorem 4** Let $o \in E^3$, $r > 0$, $f \in C^\infty(E^3)$ rapidly decreasing at $\infty$. Then

$$8\pi^2 f(o) = - \sum_{l=0}^{\infty} \int_{\mu(o, (2l+1)r)} \left(a_l \cdot D^2_{N(m)} + b_l \cdot D_{N(m)} + c_l\right) (S^2 f)(\xi(m,r)) \, dm ,$$

where

$$a_l := \frac{1}{r^2(2l+1)} ,$$

$$b_l := \frac{1}{2r^2(2l+1)} \left( \frac{6l+5}{l(l+1)(2l+1)} - 1 + \sum_{i=1}^{l} \frac{1}{i(i+1)} \right) ,$$

$$c_l := \frac{1}{2r^4(2l+1)^2} \left( \frac{l+1}{l+1} - 1 + \sum_{i=1}^{l} \frac{1}{i(i+1)} \right) .$$

($N(m) =$ exterior unit normal vector of $\nu$ at $m$).

Proof: (13) with $a = 1$, $b = 0$ implies through $r_1 \to 0$

$$\pi \cdot S^2 f(\eta(o, 2r)) = \int_{\mu(o,r)} \left( \frac{1}{r} D_{N(m)} + \frac{1}{r^2} \right) (S^2 f)(\xi(m,r)) \, dm ,$$

and

$$8\pi^2 f(o) = \lim_{r_1 \to 0} \frac{1}{r_1^2} \left( \frac{2\pi}{r_1 + 2r} (S^2 f)(\eta(o, r_1 + 2r) \right) -$$

$$- \int_{\mu(o, r_1 + r)} \left( \frac{1}{r(r_1 + r)} D_{N(m)} + \frac{1}{r^2} \right) (S^2 f)(\xi(m,r)) \, dm .$$

Taylor-expansion in (27) with respect to $r_1$, taking into account (26), yields

$$8\pi^2 f(o) = - \int_{\mu(o,r)} \left( \frac{1}{r^2} D^2_{N(m)} + \frac{2}{r^3} D_{N(m)} \right) (S^2 f)(\xi(m,r)) \, dm +$$

$$+ \frac{\pi}{r} (D_r \mid_\eta (S^2 f)(\eta(o, 2r)) - \frac{\pi}{2r^2} S^2 f(\eta(o, 2r)) .$$

(13) with $a = 1$, $b = 0$, $r_2 = r_1 + 2r$ implies

$$S^2 f(\eta_1) = \frac{r_1}{r_1 + 2r} S^2 f(\eta_2) -$$

$$- \frac{1}{2\pi} \int_{\mu(o, r_1 + r)} \left( \frac{r_1}{r(r_1 + r)} D_{N(m)} + \frac{r_1}{r^2} \right) (S^2 f)(\xi(m,r)) \, dm$$

(29)

and

$$(D_r \mid_\eta S^2 f)(\eta_1) = \frac{r_1}{r_1 + 2r} (D_r \mid_\eta S^2 f)(\eta_2) + \frac{2r}{(r_1 + 2r)^2} S^2 f(\eta_2) -$$

$$- \frac{1}{2\pi} \int_{\mu(o, r_1 + r)} \left( \frac{r_1}{r(r_1 + r)} D^2_{N(m)} + \frac{3r_1 + r}{r^2} \right) D_{N(m)} +$$

$$+ \frac{r_1}{r(r_1 + r)^2} (S^2 f)(\xi(m,r)) \, dm .$$

(30)
In this second step (29), (30) with \( r_1 = 2r, r_2 = 4r \), and (28) produce the first two summands (shells) in (25), up to an error along \( \eta(0, 6r) \). In this manner, starting at (13) through successive application of (29), (30), shell by shell, we reach the one-radius inversion formula (25). (The sum converges and the error tends to zero because \( f \) is rapidly decreasing at \( \infty \).) □

**Remark 4.** (17) with \( r \to \infty \) leads to the classical inversion formula for the Radon transform.

**Remark 5.** Other inversion formulas for the spherical transform, cf. [12] Chpt. IV, [3], [2], [13].

### 3 Spherical transform on the Möbius space

Let us now open the euclidean sight to a conformal point of view. The Möbius space is the point space \( E^3 \cup \{ \infty \} \), the basic objects are spheres (i.e. spheres and planes under euclidean sight), the underlying group is the Möbius group. But we also consider parts of the euclidean space fit out with a conformally changed metric.

**Theorem 5** (7) Theorem 1 and (13) Theorem 2 are representatives, written in euclidean terms, of formulas invariant with respect to Möbius transformations of the euclidean space \( E^3 \) and invariant with respect to conformal changes of the euclidean metric respectively.

Proof: Consider a conformal change of the euclidean metric \( g_e \) in \( E^3 \), i.e. \( g = \rho^2 g_e \) (\( \rho \neq 0 \) at the points reached by the spheres \( \xi \) through the dual integration), without changing the sphere ensemble in the formulas (7), (13). For the function \( f \) in the conformally changed setting we take \( f_e := \rho^2 f \) in the euclidean setting. Then \( S^2 f_e(\xi), R^2 f_e(\xi) \) with respect to \( g_e \), are equal to \( S^2 f(\xi) \), \( R^2 f(\xi) \) with respect to the conformaly changed metric \( g \). Therefore (7), (13), with differential operators and dual integration taken with respect to the euclidean metric, are valid too in the conformal changed situation. Planes and spheres respectively, differential operators and the dual integration may have intrinsic meanings in the conformally changed setting, e.g. see Theorem 6 and Theorem 7 in hyperbolic spaces. That way (7), (13) are euclidean representatives of conoformal invariant formulas.

A specific case are Möbius transformations. (We may take the conformal change of the euclidean metric induced by a Möbius transformation without changing the spheres in the formulas, or equivalently we may consider a Möbius transformation of the sphere ensemble in \( E^3 \cup \{ \infty \} \) without changing the euclidean metric.) Here differential operators and dual integration have Möbius invariant meanings as follows:

In the situation of Theorem 1 (7) the spheres \( \eta_1 \) and \( \eta_2 \) are tangent, they define a degenerate pencil of spheres \( \Sigma \). \( \xi \) is determined by its point of contact \( x_1 = \eta_1 \cap \xi \) and \( x_2 = \eta_2 \cap \xi \); hence \( \xi \) is determined by an orthogonal trajectory circle \( c_\xi \) to \( \Sigma \); therefore \( \xi \) is parametrized through \( m = c_\xi \cap \mu, \mu \in \Sigma, \mu = \text{mid-sphere of } \eta_1, \eta_2 \) (i.e. \( \eta_1, \eta_2, \mu, o \) are in harmonic division, \( o = \eta_1 \cap \eta_2 \)). Let \( H_\Sigma \) denote the subgroup of the Moebius group acting on \( \Sigma \). Fix \( r > 0 \). For \( \xi \) let \( H_\xi \) be the subgroup of \( H_\Sigma \) acting on \( c_\xi \). Let \( H_1 \) be the subgroup of \( H_\xi \) acting on \( c_\xi \setminus \{ o \} \) without fixed points. Choose \( V \) from the Lie algebra of \( H_1 \) with \( \exp(2rV) \cdot \eta_1 = \eta_2 \). Let \( H_2 \) be the subgroup of \( H_\xi \) acting on \( \Sigma \) fixing \( \mu \). Choose \( W \) from the Lie algebra of \( H_2 \) with \( W_{|_{x_2}} = rV_{|_{x_2}} \) (\( W^* \), resp. \( V^* \) are vector fields on \( E^n \) associated to the action of \( \exp(tW) \), resp. \( \exp(tV) \), \( t \in \mathbb{R} \), on \( E^n \)). Then \( \frac{2}{3} V + \frac{1}{3} W - \frac{1}{3} \xi \) describes the differential operator in (7) through Moebius invariant terms, acting on \( E^n \), hence acting on the space of spheres, in particular at \( \xi \). (\( V, W \) are not unique but their actions at \( \xi \)). The dual integration runs over \( \mu \) with volume density invariant with respect to the subgroup \( H_\Sigma \) of \( H_\Sigma \) fixing \( \eta_1 \) and \( \eta_2 \) (\( H_\Sigma \) is isomorphic to the isometry group of a euclidean plane), normalized as in the euclidean case. In the situation of Theorem 2 (13) the spheres \( \eta_1 \) and \( \eta_2 \) do not intersect, they define a pencil of spheres \( \Sigma \) with limit points say \( o \) and \( \infty \) (Poncelet pencil). \( \xi \) is determined by an orthogonal trajectory circle \( c_\xi \) to \( \Sigma \), hence \( \xi \) is parametrized through \( m = c_\xi \cap \mu, \mu \in \Sigma, \mu = \text{fixed sphere with respect to the Moebius transformation from } H_\Sigma \text{ changes } o \text{ and } \infty, \) as well as \( \eta_1 \) and \( \eta_2 \). Fix \( r > 0 \). For \( \xi \) let \( H_\xi \) be the subgroup of the Moebius group acting on \( c_\xi \). Let \( H_1 \) be the subgroup of \( H_\xi \) fixing exactly \( \infty \). Choose \( V \) from the Lie algebra of \( H_1 \) with \( \exp(2rV) \cdot x_1 = x_2 \).
Then $r_1, r_2$ are determined through $\exp(r_1 V) \cdot o = x_1, \exp(r_2 V) \cdot o = x_2$. Let $H_2$ be the subgroup of $H_3$ fixing $\infty$. Choose $W$ from the Lie algebra of $H_2$ with $W|_{x_1} = rV|_{x_2}$ and $W|_{x_2} = -rV|_{x_1}$.

Then $\frac{d}{dt} V + \frac{d}{dt} W + \frac{d}{dt} (rV) = \frac{d}{dt} W$ describes the differential operator in (13) through Moebius invariant terms. The dual integration runs over $\mu$ with volume density invariant with respect to the subgroup $H_3$ of $H_3$ fixing $o, \eta_1, \eta_2$ and $\infty$ ($H_3$ is isomorphic to the isometry group of a euclidean sphere), normalized by $\text{vol}(\mu) = ((r_1 + r_2)/2)^2 4\pi$. □

Remark 6. In the same line Theorem 3 (17) and Theorem 4 (25) are euclidean representatives of conformal invariant formulas.

Remark 7. (7) in the M"{o}bius transformed situation works for $f \in C_c^\infty(E^n)$ with $\eta_1 \cap \eta_2 \cap \text{supp} f = \emptyset$.

4 Applications in the hyperbolic space

Let $H^3$ be the 3-dimensional hyperbolic standard space (i.e. geodesically complete, simply connected, constant curvature $-1$). Let $R^2, S^2$ and $S^2_h$ denote the 2-plane Radon transform, the 2-spherical transform and the 2-horospherical transform respectively. They are defined analogously to (1), (2), mapping $C_c^\infty(H^3)$ into $C_c^\infty(Y), Y = G^{2,3}$ space of 2-planes, i.e. 2-dimensional totally geodesic subspaces of $H^3$, $Y = M^{2,3} = \text{space of 2-dimensional distance spheres in } H^3$ and $Y = M^{2,3}_h = \text{space of 2-dimensional horospheres in } H^3$ respectively.

For $X \in T^1_H H^3, x \in H^3$, let $\tau_{(X,x)}(t)$ ($t \in \mathbb{R}$) denote the 1-parameter subgroup of hyperbolic isometries induced by geodesic parallel translation along the geodesic through $x$ in direction $X$. Then $D_{(X,x)}$ denotes the differential operator of first order acting on $C_c^\infty(Y)$ through $\tau_{(X,x)}$, i.e.

\[ (D_{(X,x)}F) (\xi) := \frac{d}{dt} (F(\tau_{(X,x)}(t) \cdot \xi)) |_{t=0} \]

($\xi \in Y, F \in C_c^\infty(Y)$).

Theorem 6  

a) Let $\eta \in M^{2,3}_h$ be a horosphere in $H^3$. Then

\[ 2\pi S^2_h f(\eta) = - \int_{\eta} \left( D_{(N(x),x)} + 2 \right) (S^2_h f)(\xi) dx. \]

Herein the dual integration runs over the totality of horospheres $\xi = \xi(x)$ tangent to $\eta, \xi \neq \eta$, parametrized through their points of contact $x \in \eta; N(x) = \text{exterior unit normal vector of } \eta$ at $x; dx = \text{hyperbolic volume density on } \eta$.

b) Let $\eta \in M^{2,3}$ be a distance sphere in $H^3$ of hyperbolic radius $r$. Then

\[ 2\pi S^2 f(\eta) = - \frac{e^r}{2 \sinh r} \int_{\eta} \left( D_{(N(x),x)} + 2 \right) (S^2_h f)(\xi) dx. \]

Herein the dual integration runs over the totality of horospheres $\xi = \xi(x)$ tangent to $\eta, \eta$ lying outside $\xi$, parametrized through their points of contact $x \in \eta; N(x) = \text{exterior unit normal vector of } \eta$ at $x; dx = \text{hyperbolic volume density on } \eta$.

c) Let $o \in H^3$. Then

\[ 8\pi^2 f(o) = - \int_{T^2_o H^3} \left( D^2_{(u,o)} + 2D_{(u,o)} \right) (S^2_h f)(\xi) du = \]

\[ = - \int_{T^2_o H^3} \left( D^2_{(u,o)} - 4 \right) (S^2_h f)(\xi) du. \]

Herein the dual integration runs over the totality of horospheres $\xi = \xi(o,u)$ through $o, \text{parametrized through their interior normal unit vector } u \in T^1_o H^3$ at $o; du = \text{euclidean volume density on the unit sphere in the tangent space of } H^3$ at $o$.  

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Proof: a): Let $H^3$ be realized through Poincaré’s half-space model in $\mathbb{R}^3$, adapted to Theorem 1 by $\eta = \eta_1, \eta_2$ = boundary 2-plane of the model. The euclidean metric tensor $g_e$ and the hyperbolic metric tensor $g_h$ at $y \in H^3$ are conformally related by $g_h = \frac{1}{\rho^2} g_e$, $\rho = \rho(y) =$ euclidean distance of $y$ to $\eta_2$. We start at (7) with $a = -r, b = r$, and with euclidean terms replaced by their hyperbolic meanings, i.e. $R^2 f(\eta) = S_h^2 f(\eta)$, $S^2 f(\xi) = S_h^2 f(\xi)$, $\tilde{f} := f_r$, $dx = \frac{1}{\rho^2} dm$, $2r (-D_N + D_r) = -D(r(x),x) \text{ at } \xi(x), x \in \eta$. We take into account that the hyperbolic isometry group acts transitively on the space of horospheres, therefore the special situation in the model setting describes the general hyperbolic situation. Thus we reach (32).

b): Let $H^3$ be realized through Poincaré’s ball model, adapted to Theorem 2 by $\eta_1 = \eta_1(a, r_1) = \eta$, $r_1 < r_2 = 2, \eta_2 = \eta_2(2, \eta)$ = boundary 2-sphere of the model. (13) with $a = -(1 + \frac{r^2}{4}), b = (1 + \frac{r^2}{4})$, $\tilde{f} \in C^\infty_c (H^3)$ writes, in euclidean terms,

$$
2\pi (2 - r_1)(2 + r_1) S^2 f(\eta) = \\
\int (1 - \frac{r^2}{4}) (-D_N + D_\tau)(S^2 f(\xi)) dm - \\
-2 \int S^2 f(\xi) dm
$$

$\mu = \mu(o, \frac{\pi + 4}{2})$. 

The euclidean metric tensor $g_e$ and the hyperbolic metric tensor $g_h$ at $y \in H^3$ are conformally related by $g_h = (1 - \frac{r^2}{4})^{-2} g_e$, $\rho = \rho(y) =$ euclidean distance between $o = $ euclidean center of the model and $y$. Therefore the euclidean radius $r_1$ and the hyperbolic radius $r$ of $\eta$ are related by $r_1 = 2 \tanh \frac{r}{2}$. Then we replace the euclidean terms by their hyperbolic meanings, i.e. $S^2 f(\eta) = S^2 f(\eta)$, $S^2 f(\xi) = S_h^2 f(\xi)$, $\tilde{f} := (1 - \frac{r^2}{4})^{-2} f$, $dx = (1 - \frac{r^2}{4})^{-2} r^2 (1 + \frac{r^2}{2})^{-2} dm$, $(1 - \frac{r^2}{4})^2 (-D_N + D_\tau) = -D(N(x),x) \text{ at } \xi(x), x \in \eta$. Because the hyperbolic isometry group acts transitively on the space of hyperbolic distance spheres with radius $r$, we reach (33).

c): (33) through $r \to 0$, like in the proof of Theorem 3, yields (34). $\square$

Through similar adaptations we obtain

**Theorem 7**  
**a)** Let $\eta \in M^{2,3}_h$ be a horosphere in $H^3$. Then

$$
2\pi S^2 f(\eta) = - \int_\eta \left( D_N + 1 \right) (R^2 f)(\xi) dx.
$$

Herein the dual integration runs over the totality of 2-planes $\xi = \xi(x)$ tangent to $\eta$, parametrized through their points of contact $x \in \eta$; $N(x) =$ exterior unit normal vector of $\eta$ at $x$; $dx =$ hyperbolic volume density on $\eta$.

**b)** Let $\eta \in M^{2,3}$ be a distance sphere in $H^3$ of hyperbolic radius $r$. Then

$$
2\pi S^2 f(\eta) = - \int_\eta \left( \frac{1}{\tanh r} D(N(x),x) + 1 \right) (R^2 f)(\xi) dx.
$$

Herein the dual integration runs over the totality of 2-planes $\xi = \xi(x)$ tangent to $\eta$, parametrized through their points of contact $x \in \eta$; $N(x) =$ exterior unit normal vector of $\eta$ at $x$; $dx =$ hyperbolic volume density on $\eta$.

**c)** Let $o \in H^3$. Then

$$
8\pi^2 f(o) = - \int_{T^3_o H^3} \left( D^2(o) + 1 \right) (R^2 f)(\eta) du.
$$

(37)
Herein the dual integration runs twice over the totality of 2-planes $\xi = \xi(o, u)$ through $o$ and perpendicular to $u$, parametrized through $u \in T_o H^3$; $du =$ euclidean volume density on the unit sphere in the tangent space of $H^3$ at $o$.

Remark 8. Similar adaptations lead e.g. to the following relations: from the spherical transform to the spherical transform like (13), to inversion formulas for the spherical transform like (17), or from orbital integrals with respect to distance surfaces of 2-planes to the spherical transform similar to (13), etc..

Similar applications happen in the spherical space.


5 Support results

Theorem 8 Let $f \in C^\infty(E^3)$ be such that for each integer $l > 0$, $| x - o |^l f(x)$ is bounded, $(o \in E^3$ fixed). Suppose there exists a closed ball $B(o, R)$ with center $o$ and radius $R$, and a fixed radius $r$, such that $S^2 f(\xi) = 0$ for all spheres $\xi$ of radius $r$ with $B(o, R)$ outside $\xi$. Then $f(x) = 0$ for $x \in E^3 \setminus B(o, R)$.

Proof: Let $\eta_1$ be a sphere of radius $r_1$ which encloses $B(o, R)$. Let $\eta_2$ denote the sphere of radius $r_1 + 2r$ concentric to $\eta_1$. Then (13) with $b = 0$ gives $S^2 f(\eta_1) = \frac{1}{r_1 + 2r} S^2 f(\eta_2)$. Going outwards through such steps shows $S^2 f(\eta_1) = \frac{1}{r_1 + 2r} s^2 f(\eta_l)$, $l \in \mathbb{N}$, $\eta_l =$ sphere of radius $r_1 + l2r$ concentric to $\eta_1$. Thus $S_2 f(\eta_1) = 0$ because $f$ rapidly decreases. Hence $S^2 f(\eta) = 0$ for each sphere which encloses $B(o, R)$. Therefore [9] Ch.1 Lemma 2.7 yields that $f$ vanishes outside $B(o, R)$.

Theorem 9 Let $f \in C^\infty(E^3)$. Suppose there exists a closed ball $B(o, R)$ with center $o$ and radius $R$, and two fixed radii $r, \bar{r}$ with $r/\bar{r}$ irrational and $2r + 2\bar{r} < R$, such that $S^2 f(\xi) = 0$ for all spheres $\xi \subset B(o, R)$ of radius $r$ and $S^2 f(\bar{\xi}) = 0$ for all spheres $\bar{\xi} \subset B(o, R)$ of radius $\bar{r}$. Then $f(x) = 0$ for $x \in B(o, R)$.

Proof: Let $\eta_1, \eta_2$ be concentric spheres with center $o$ and radii $r_1, r_2 = r_1 + 2r$ or $r_2 = r_1 + 2\bar{r}$, $0 < r_1 < r_2 < R$. Then (13) with $b = 0$ gives $S^2 f(\eta_1) = \frac{1}{r_1 + 2r} S^2 f(\eta_2)$. Let $S := \{n2r - m2\bar{r} | n, m \in \mathbb{Z}, n \geq 0 \} \cap [0, R]$. Then $S$ is dense in $[0, R]$, because $r/\bar{r}$ irrational. Moreover any two points in $S$ can be connected in $[0, R]$ using segments each of length $2r$ or $2\bar{r}$. (cf. [14] pp. 88, resp. [21] proof of Lemma 3.3.) Therefore $S^2 f(\eta_1) = \frac{1}{r_1 + 2r} S^2 f(\eta_2)$ for $r_1, r_2 \in S$. Taking into account a sequence $r_2 \to 0$ yields $S^2 f(\eta) = 0$ for all spheres $\eta(o, r), r \in S$. Hence by continuity, $S^2 f(\eta) = 0$ for all spheres $\eta(o, r), r < R$.

The same holds for small perturbations of $o$. Therefore the idea of the proof of Lemma 2.7 Chpt. 1 [9] works and shows $f(x) = 0$ for $x \in B(o, R)$.

Remark 10. Theorem 8 and Theorem 9 respectively are valid for $f$ continuous and rapidly decreasing at $\infty$, and for $f$ continuous respectively.

Note: The proofs above work through replacing $f$ by the convolution $\phi \ast f$, where $\phi$ is a well chosen radial $C^\infty$-function with support in a small ball $B(o, \epsilon) \subset E^n, \epsilon > 0$. In fact, for Theorem 8, and analogously for Theorem 9,

$$S^2(\phi \ast f)(\xi(m, r)) = (\chi \ast S^2 f)(\xi(m, r)) = \int_{E^n} \chi(m - x) \cdot S^2 f(\xi(x, r)) dx,$$

$\chi$ a radial $C^\infty$-function with compact support in $B(o, \epsilon) \subset \mathbb{R}^3$, appropriated to $\phi$.

Remark 11. Support and injectivity results for the spherical transform, cf. [12] Chpt. VI, [21], [1], [4], [15], [18], [20].
6 A reduction scheme in higher dimensions

**Lemma 1** Let \( \eta_1, \eta_2 \) be parallel hyperplanes in \( E^n \) at distance \( 2r \). Then for \( k \geq 3 \)

\[
\int_{\mu \times G_0^{k-2,n-1}} S^{k-2} f(\delta) d\delta = \frac{v^{k-3,k-1}}{(k-2)\mu^{2,n-k+1}} \int_{\mu \times G_0^{k-2,n-1}} \left( \frac{1}{r^2} D_r - \frac{1}{r^2} \right)(S^k f)(\xi) d\xi. \tag{38}
\]

Herein the dual integration on the right hand side runs over the totality of \( k \)-spheres \( \xi = \xi(m, \sigma, r) \) of radius \( r \) tangent to both \( \eta_1 \) and \( \eta_2 \), parametrized through center \( m \in \mu \) and \( \sigma = \text{affine hall of } \xi \cap \mu, \sigma \in G_0^{k,n-1} \) (= Grassmann manifold of \( k \)-planes in \( \mu \) through \( m \)), \( d\xi = dmd\sigma, d\sigma = \text{euclidean volume density on the hyperplane } \mu, d\sigma = \text{invariant volume density in } G_0^{k,n-1} \). The integration on the left hand side analogously runs over the totality of \((k-2)\)-spheres \( \delta = \delta(m, \tau, r) \) of radius \( r \) tangent to both \( \eta_1 \) and \( \eta_2 \), parametrized through center \( m \in \mu, \tau \in G_0^{k-2,n-1} \). \( v^{k-3,k-1} \) is the volume of the Grassmann manifold \( G_0^{k-3,k-1} \), \( G_0^{k,n+1} \).

Proof: The definition of \( D_r \) implies

\[
D_r(S^k f)(\xi) = \int_{\xi} \frac{\cos \alpha}{\sin \alpha} df_y(e_2) dy + \frac{k}{r} \int_{\xi} f(y) dy \tag{39}
\]

mod integrand terms \( df_y(e_2) \) disappearing through dual integration because of symmetry (cf. proof of Proposition 2.1) \( (\alpha = \angle(x_1my), x_1 = \eta_1 \cap \xi) \).

We use polar coordinates on \( \xi \) centered at \( x_1 \), i.e. \( dy = r^{k-1} \sin^{k-1} \alpha (rda) du, u \in T_{x_1}^{1} \xi, \) to rewrite the integrand of the first integral on the right hand side of (39), i.e. \( \frac{v^{k-1} \sin^{k-2} \alpha \cdot \cos \alpha \cdot df_y(e_2)(rda)du} {r} \). Integration by parts with respect to \( r(da) \), this integral becomes

\[
\int_{\xi} (k-2) v^{k-2} \sin^{k-3} \alpha \cdot f(y)(rda) du - \frac{k-1}{r} S^k f(\xi). \tag{40}
\]

We replace the integration with respect to \( u \) over the \((k-1)\)-sphere \( T_{x_1}^{1} \xi \) by a twofold integration, at first over \((k-3)\)-greatspheres of \( T_{x_1}^{1} \xi \) then over the totality of \((k-3)\)-greatspheres of \( T_{x_1}^{1} \xi \). Thus first integral in (40) becomes

\[
\frac{(k-2)r}{v^{k-3,k-1}} \int_{G_0^{k-2,k}} S^{k-2} f(\delta) d\tau \tag{41}
\]

\( (\tau = T_{x_1} \delta \subset T_{x_1} \xi) \).

Finally we bring (41), (40) back into (39), we carry-out the dual integrations, taking into account integration with respect to nested subspaces \( \tau \subset \sigma \subset \mathbb{R}^{n-1} \) (see [17] (12.52)), and we reach (38). \( \square \)

Through similar computations we get

**Lemma 2** Let \( \eta_1, \eta_2 \) be concentric \((n-1)\)-spheres in \( E^n \) with center \( o \) and radii \( r_1, r_2 \) \((r_1 < r_2) \).

Then for \( k \geq 3 \)

\[
\int_{\mu \times G_0^{k-2,n-1}} S^{k-2} f(\delta) d\delta = \frac{v^{k-3,k-1}}{(k-2)\mu^{2,n-k+1}} \int_{\mu \times G_0^{k-2,n-1}} \left( -\frac{r_1 + r_2}{2r_1 r_2} D_{N(m)} + \frac{(r_1 + r_2)^2}{2r_1 r_2(r_2 - r_1)} D_r - \frac{k}{r_2 - r_1} D_r \right)(S^k f)(\xi) d\xi. \tag{42}
\]
Herein the dual integration on the right hand side runs over the totality of $k$-spheres $\xi = \xi(m, \sigma, r)$ of radius $r = \frac{m - m_0}{r_0-1}$ tangent to both $\eta_1$ and $\eta_2$, $\xi$ outside $\eta_1$ and inside $\eta_2$, parametrized through center $m \in \mu$ and $\sigma = \text{affine hull of } \xi \cap T_{m\mu}$, $\sigma \in C_0^{k,n-1}(T_{m\mu})$ (= Grassmann manifold of $k$-spaces in $T_{m\mu}$ through $m$), $d\sigma = dmd\sigma$. The integration on the left hand side analogously runs over the totality of $(k-2)$-spheres $\delta = \delta(m, \tau, r)$ of radius $r$ tangent to both $\eta_1$ and $\eta_2$. \vspace{1ex}

Proposition 1 Let $\eta_1, \eta_2$ be parallel hyperplanes in $E^n$ at distance $2r$, $n = 2k + 1$. Let $a, b$ be real constants. Then

\[ 2\pi \left( (a - b) \cdot R^{n-1}f(\eta_2) + (a + b) \cdot R^{n-1}f(\eta_1) \right) = c(n) \int_{\mu} \left( \frac{a}{r} D_N + \frac{b}{r} \cdot D_r - \frac{b}{r^2} \left( \frac{1}{r} D_r - \frac{1}{r^2} \right)^{-1} (S^{n-1}f)(\xi) \right) dm. \]

Herein the dual integration runs over the totality of $(n-1)-$spheres $\xi = \xi(m, r)$ of radius $r$ tangent to both $\eta_1$ and $\eta_2$, parametrized through their centers $m \in \mu$; $N = \text{normal unit vector of } \eta_1$ pointing towards $\eta_2$; $c(n)$ a constant depending on $n$.

Proof: (7) through successive application of (38) yields (43). (Note: Integrations in (38) and the differential operators $D_N$ and $D_r$ intertwine.) □

Remark 12. There are further reduction formulas in the style of (38) and (42), more general however much more complex. (Actual we don’t overlook the totality of such reduction formulas.)

Remark 13. (13) and (17) respectively through successive application of the reduction formula (42) leads to the analogue of (43) for concentric $(n-1)-$spheres $\eta_1, \eta_2$ in $E^n$ and to an inversion formula for the spherical transform in $E^n$, $n = 2k + 1$. (Note: All terms coming up depend on $o$, $r_1$ and $r$, $(r_2 = r_1 + 2r)$. Let $I_{(\mu, r)}$ denote anyone of the integration operators in (13), (42), then $\partial/\partial r \circ I_{(\mu, r)} = I_{(\mu, r)} \circ \partial/\partial r$ and $\partial/\partial r S^k f(\xi) = D_r S^k f(\xi), D_r|\mu \circ I_{(\mu, r)} = \partial/\partial (r_1 + r) \circ I_{(\mu, r)} = \partial/\partial r_1 \circ I_{(\mu, r)} = I_{(\mu, r)} \circ \partial/\partial r_1 + \frac{2(n-1)}{r_1^2} I_{(\mu, r)}$ and $\partial/\partial r_1 S^k f(\xi) = D_{N|\mu} S^k f(\xi)$.)

Remark 14. Inversion formulas for the circular transform in the euclidean plane $E^2$. Let $E^2$ be enlarged through an orthogonal complement to $E^3 = E^2 \times \mathbb{R}$, let $F \in C_0^\infty(E^3)$ be defined by $F((x, t)) := \psi(t) \cdot f(x), (x, t) \in E^2 \times \mathbb{R}$, $\psi \in C_0^\infty(\mathbb{R})$ with $\psi(0) = 1$. Then (7) with $\eta_1 = g_1 \times \mathbb{R}, \eta_2 = g_2 \times \mathbb{R}, g_1, g_2$ parallel lines in $E^2$ at distance $2r$ leads to a transformation from the circular transform $S^1(\xi)$ to the Radon transform $R^1(g_1), R^1(g_2)$, looking like (7), with dual integration over the totality of circles $\xi = \xi(m, r)$ of radius $0 \leq \rho \leq r$ and centers $m$ on the mid-parallel of $g_1$ and $g_2$, and additional coefficients depending on $\rho$ appearing at the summands on the right hand side of (7).

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Eberhard Teufel
Institut für Geomtrie und Topologie (IGT), Fachbereich Mathematik,
Universität Stuttgart, 70550 Stuttgart, Germany
e-mail: Eberhard.Teufel@mathematik.uni-stuttgart.de

Eberhard Teufel
Fachbereich Mathematik, Universität Stuttgart, Pfaffenwaldring 57, D-70550 Stuttgart, Germany
E-Mail: Eberhard.Teufel@mathematik.uni-stuttgart.de
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