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SPHERICAL TRANSFORMS AND RADON TRANSFORMS IN MOEBIUS GEOMETRY

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Abstract

We study spherical transforms on euclidean spaces through a geometric view on the action of differential operators acting onto spheres. We achieve interrelations between Radon transforms and spherical transforms. We attain inversion formulas, especially a one-radius inversion formula. The results are conformal invariant. Moreover we get two-radius and one-radius-germ support results. Finally we derive interrelations and inversion formulas for Radon transforms, spherical transforms and horospherical transforms in hyperbolic spaces.
Key Words: Spherical transform, Radon transform, inversion formula, support theorem, Moebius geometry, horospherical transform, hyperbolic geometry.
AMS Subject Classification: 53C65, 44A12, 53A30.

1 Introduction

Radon transforms go back to P. Funk (1916) and J. Radon (1917), cf. [5], [16], [9], [6]. Spherical transforms, or spherical means, in connection with plane waves and the Darboux equation go back to F. John (1935), cf. [12]. Both transformations come together from a conformal point of view, cf. [7]. The classical techniques are harmonic analysis, PDEs, integral equations, geometric means. The following studies are based on a geometric view on the action of differential operators of first order acting onto spheres and planes (3) (4). To make the geometric ideas clear we first of all concentrate on the 3-dimensional euclidean space. We attain interrelations between the spherical transform and the Radon transform, and interrelations for the spherical transform respectively (Theorem 1, Theorem 2). Moreover we obtain an inversion formula (Theorem 3) and a one-radius inversion formula (Theorem 4) for the spherical transform.
The basic results Theorem 1 (7) and Theorem 2 (13), written in euclidean terms, are conformal invariant and Möbius invariant respectively (Theorem 5). The Möbius pitch are degenerate pencils of spheres and pencils of spheres with limit points (Poncelet pencils) respectively. (Actual we fail in playing with pencils of spheres without limit points.)
Section 4 plays in hyperbolic spaces. We use conformal models of the hyperbolic space in order to apply the euclidean situations. We derive similar results concerning the geodesic Radon transform, the spherical transform and the horospherical transform respectively.
Section 5 contains support results similar to support theorem for the classical Radon transform [8] and for the spherical transform [21] respectively. We get two-radius and one-radius support theorems (Theorem 8, Theorem 9).
In section 6 we point out some first aspects of a reduction scheme in higher dimensions.

A similar geometric view on Radon transforms is treated in [19].

Let $G^{k,n}$ be the space of k-dimensional affine subspaces of the n-dimensional euclidean space E^n. The k-plane Radon transform R^k maps $C_c^\infty(E^n)$ into $C_c^\infty(G^{k,n})$, namely

$$(R^k f)(\eta) = \int_\eta f(x)\, dx \quad , \; \eta \in G^{k,n} \; , \tag{1}$$

(C_c^∞ = space of C^∞-functions with compact support; dx = euclidean volume density on the k-plane η).
Let $M^{k,n}$ be the space of k-dimensional spheres in E^n. The k-spherical transform S^k maps $C_c^\infty(E^n)$ into $C^\infty(M^{k,n})$, namely

$$(S^k f)(\xi) = \int\limits_\xi f(x)\, dx \quad , \, \xi \in M^{k,n} \, , \tag{2}$$

(dx = euclidean volume density on the k-sphere ξ).
Let D_X, $X \in V^n$ (V^n = euclidean vector space associated to E^n), denote the differential operator of first order acting on $C^\infty(M^{k,n})$ through parallel translating the spheres in E^n, i.e.

$$(D_X F)(\xi) := \frac{d}{dt}(F(\xi + tX)) \mid_{t=0} \tag{3}$$

($\xi \in M^{k,n}, F \in C^\infty(M^{k,n})$).
Let D_r denote the differential operator of first order acting on $C^\infty(M^{k,n})$ through bubbling the spheres, i.e.

$$(D_r F)(\xi) := \frac{d}{dt}(F(\xi(m,t))) \mid_{t=r} \tag{4}$$

($\xi = \xi(m,r)$ = sphere with center m and radius r).

One of the key questions is how to invert these transformations, i.e. how to reconstruct the point function f from the knowledge of the Radon transform $R^k f$ or the spherical transform $S^k f$ respectively.
For the Radon transform on the 3-dimensional euclidean space this can be done through a dual integration by means of a differential operator of second order acting onto planes, namely

$$f(x) = \frac{1}{4\pi^2} \int_{G_x^{2,3}} D^2_{N(\eta)}(R^2 f)(\eta)\, d\eta \, , \tag{5}$$

$x \in E^3$ fix, $G_x^{2,3}$ = space of 2-planes in E^3 through x = unit sphere with center x, $N(\eta)$ = unit normal vector of η, $d\eta$ = volume density of the unit sphere, D_N see (3).
The integration on the right hand side of (1) runs over all points x in the plane η, whereas the integration on the right hand side of (5) runs over all planes η through the point x. This is the play of the classical duality between points and planes in space. In the following we shall meet similar and more general situations, and in the same way we call the corresponding integrations dual integrations.
The inversion of the spherical transform is trivial, provided that $S^2 f$ is available at spheres of arbitrary small radius,

$$f(x) = \frac{1}{4\pi} \lim_{r\to\infty} \frac{1}{r^2} S^2 f(\xi(x,r)) \, , \tag{6}$$

$x \in E^3$.

2 Spherical transform and Radon transform on the euclidean space

Theorem 1 *Let η_1, η_2 be parallel 2−planes in E^3 at distance $2r$. Let a, b be real constants. Then*

$$\begin{aligned} 2\pi\Big((a+b)\cdot R^2 f(\eta_2) + (-a+b)\cdot R^2 f(\eta_1)\Big) = \\ = \int\limits_\mu \Big(\frac{a}{r}\cdot D_N + \frac{b}{r}\cdot D_r - \frac{b}{r^2}\Big)(S^2 f)(\xi)\, dm \, . \end{aligned} \tag{7}$$

Herein the dual integration on the right hand side runs over the totality of $2-$spheres $\xi = \xi(m, r)$ of radius r tangent to both η_1 and η_2, parametrized through their centers $m \in \mu$; $dm =$ euclidean volume density on μ, the mid-plane with respect to η_1 and η_2; $N =$ normal unit vector of η_1 pointing towards η_2.

Proof: The definition of the differential operators (3), (4) implies

$$D_N(S^2 f)(\xi) = \int_\xi df_{|y}(N)dy \tag{8}$$

$$D_r(S^2 f)(\xi) = -\int_\xi df_{|y}(e_1)dy + \frac{2}{r}\int_\xi f(y)dy \tag{9}$$

($e_1 =$ interior normal unit vector of ξ at y). (The vector fields N in (8) and e_1 in (9) respectively are unique up to divergence-free tangent vector fields along ξ.)
For $\xi = \xi(m, r), m \in \mu$, let $x_1 = \eta_1 \cap \xi$, resp. $x_2 = \eta_2 \cap \xi$ denote the points of contact. We take orthonormal moving frames $ye_1e_2e_3$, $y\bar{e}_1\bar{e}_2\bar{e}_3, y \in \xi$, with the following adaptations: $e_1 =$ interior normal unit vector of ξ at y, $e_2 =$ tangent unit vector at y of the oriented great circle of ξ from y to x_1, $\bar{e}_1 = N$, $\bar{e}_2 =$ tangent unit vector at y of the oriented perpendicular line from y to x_1x_2. Then

$$\begin{aligned} \bar{e}_1 &= -\frac{1}{\sin\alpha}\cdot e_2 + \frac{\cos\alpha}{\sin\alpha}\cdot \bar{e}_2 \\ e_1 &= -\frac{\cos\alpha}{\sin\alpha}\cdot e_2 + \frac{1}{\sin\alpha}\cdot \bar{e}_2 \end{aligned} \tag{10}$$

($\alpha = \angle(x_1my)$).
Hence (8), (9) yield

$$(a\cdot D_N + b\cdot D_r)(S^2 f)(\xi) = \int_\xi \left(-\frac{a}{\sin\alpha} + \frac{b\cos\alpha}{\sin\alpha}\right) df_{|y}(e_2)dy + \\ + \frac{2b}{r}\int_\xi f(y)dy \tag{11}$$

mod integrand terms $df_{|y}(\bar{e}_2)$ disappearing through the dual integration because of symmetry.
Now, using polar coordinates on ξ centered at x_1, i.e. $dy = r\sin\alpha(rd\alpha)d\varphi$ to rewrite the integrand of the first integral on the right hand side of (11), i.e. $r(-a + b\cos\alpha)df_{|y}(e_2)(rd\alpha)d\varphi$, and integrating by parts with respect to $(rd\alpha)$, this integral becomes

$$2\pi r\left((a+b)f(x_2) + (-a+b)f(x_1)\right) - \frac{b}{r}\cdot S^2(f)(\xi). \tag{12}$$

Finally, we bring (12) back into (11) and we carry-out the dual integration, taking into account $dx_1 = dx_2 = dm$. Thus we reach (7). $\square$

Remark 1. Read from the right to the left hand side (7) is pointing a way from the spherical transform to the Radon transform. For the opposite way from the Radon transform to the spherical transform, e.g. through inversion of Volterra integral equations of first kind (Abel type), cf. [9] proof of theorem 2.6..

Theorem 2 *Let η_1, η_2 be concentric 2-spheres in E^3 with center o and radii $r_1, r_2 (r_1 < r_2)$. Let a, b be real constants. Then*

$$2\pi\left((a+b)\left(\frac{r_1+r_2}{2r_2}\right)\cdot S^2 f(\eta_2) + (-a+b)\left(\frac{r_1+r_2}{2r_1}\right)\cdot S^2 f(\eta_1)\right) = \\ = \int_\mu \left(\frac{a}{r}\cdot D_{N(m)} + \frac{b}{r}\cdot D_r + \frac{2a}{r(r_1+r_2)} - \frac{b}{r^2}\right)(S^2 f)(\xi)\,dm\,. \tag{13}$$

Herein the dual integration runs over the totality of 2-spheres $\xi = \xi(m,r)$ of radius $r = \frac{r_2-r_1}{2}$ tangent to both η_1 and η_2, ξ lying outside η_1 and inside η_2, and parametrized through their centers $m \in \mu$; dm = euclidean volume density on the 2-sphere $\mu = \mu(o, \frac{r_1+r_2}{2})$; $N = N(m)$ = exterior normal unit vector of μ at m.

Proof: For $\xi = \xi(m,r), m \in \mu$, let $x_1 = \eta_1 \cap \xi$, resp. $x_2 = \eta_2 \cap \xi$ denote the points of contact. We take orthonormal moving frames $ye_1e_2e_3$, $y\bar{e}_1\bar{e}_2\bar{e}_3, y \in \xi$, with the following adaptations: e_1 = interior normal unit vector of ξ at y, e_2 = tangent unit vector at y of the oriented great circle of ξ from y to x_1, $\bar{e}_1$ = tangent unit vector at y of the oriented line from o to y, $\bar{e}_2$ = tangent unit vector at y of the oriented line from y to om perpendicular to oy. Then

$$\begin{aligned} N &= -\frac{\cos\beta}{\sin(\alpha+\beta)} \cdot e_2 + \frac{\cos\alpha}{\sin(\alpha+\beta)} \cdot \bar{e}_2 \\ e_1 &= -\frac{\cos(\alpha+\beta)}{\sin(\alpha+\beta)} \cdot e_2 + \frac{1}{\sin(\alpha+\beta)} \cdot \bar{e}_2 \end{aligned} \tag{14}$$

where $\alpha = \angle(x_1my)$, $\beta = \angle(x_1oy)$.
Hence (8), (9) yield

$$\begin{aligned} &(a \cdot D_{N(m)} + b \cdot D_r)(S^2 f)(\xi) = \\ &= \int_\xi \left(-a\frac{\cos\beta}{\sin(\alpha+\beta)} + b\frac{\cos(\alpha+\beta)}{\sin(\alpha+\beta)}\right) df_{|y}(e_2)dy + \frac{2b}{r}\int_\xi f(y)dy \end{aligned} \tag{15}$$

mod integrand terms $df_{|y}(\bar{e}_2)$ disappearing through the dual integration because of symmetry. Now, using polar coordinates on ξ centered at x_1, i.e. $dy = r\sin\alpha(rd\alpha)d\varphi$ to rewrite the integrand of the first integral on the right hand side of (15), i.e.

$$r\frac{\sin\alpha}{\sin(\alpha+\beta)}\left(-a\cos\beta + b\cos(\alpha+\beta)\right) df_{|y}(e_2)(rd\alpha)d\varphi,$$

and integrating by parts with respect to $(rd\alpha)$, this integral becomes

$$\begin{aligned} &2\pi r\left(\frac{2r_2}{r_1+r_2}(a+b)f(x_2) + \frac{2r_1}{r_1+r_2}(-a+b)f(x_1)\right) - \\ &\qquad - \left(\frac{2a}{r_1+r_2} + \frac{b}{r}\right) S^2(f)(\xi). \end{aligned} \tag{16}$$

(Some explanation: e.g. $d(br\frac{\sin\alpha\cdot\cos(\alpha+\beta)}{\sin(\alpha+\beta)})_{|y}(e_2) = d(b\frac{(r_1+r)^2-r^2-\rho^2}{2(r_1+r)})_{|y}(e_2) = -\frac{b\rho}{r_1+r} \cdot d\rho_{|y}(e_2)$ $= -\frac{b\rho}{r_1+r} \cdot d\rho_{|y}(-\sin(\alpha+\beta)\cdot\bar{e}_1 + \cos(\alpha+\beta)\cdot\bar{e}_2) = \frac{b\rho}{r_1+r}\cdot\sin(\alpha+\beta) = b\sin\alpha$, using some trigonometric formulary in triangle omy, ρ = euclidean distance between o and y.)
We bring (16) back into (15) and we carry-out the dual integration, taking into account $dx_1 = (\frac{2r_1}{r_1+r_2})^2 dm$, $dx_2 = (\frac{2r_2}{r_1+r_2})^2 dm$. That way we reach (13). $\square$

Remark 2. The same situation as in Theorem 2, but now using all the $2-$spheres ξ of radius $r = \frac{r_2+r_1}{2}$ tangent to both η_1 and η_2, η_1 lying inside ξ and ξ lying inside η_2, leads to (13) with r_1+r_2 replaced by r_2-r_1.

Remark 3. (13) with $b=0$ and $r_2 \to \infty$ gives a way from the Radon transform to the spherical transform.
(13) with $r_1 \to \infty$, $r_2 \to \infty$ and $r_2 - r_1 = 2r$ leads to (7).

Now, starting at (13) we let the sphere η_1 shrink to its center o. We calculate the geometric Taylor expansion of (13), and we get the following one-radius-germ inversion formula for the spherical transform.

Theorem 3 *Let $o \in E^3$, $r > 0$. Then*

$$\begin{aligned} 8\pi^2 f(o) &= -\int\limits_{\mu} \Big(\frac{1}{r^2} D^2_{(m,r)} + \frac{1}{r^3} D_{(m,r)}\Big)(S^2 f)(\xi) dm = \\ &= -\int\limits_{\mu} \Big(\frac{1}{r^2} D^2_{(m,r)} - \frac{1}{r^4}\Big)(S^2 f)(\xi) dm. \end{aligned} \tag{17}$$

Herein the dual integration runs over the totality of 2-spheres $\xi = \xi(m,r)$ of radius r through o, parametrized through their centers $m \in \mu = \mu(o,r)$; dm = euclidean volume density on the 2-sphere μ; $D_{(m,r)}$ and $D^2_{(m,r)}$ = differential operator of first and second order, see (19), (21), (22).

Proof: (13) with $\eta_1 = \eta_1(o, r_1)$, $\eta_2 = \eta_2(o, 2r)$ and $a = \frac{1}{2}$, $b = -\frac{1}{2}$ implies through $r_1 \to 0$

$$\int\limits_{\mu} D_{(m,r)}(S^2 f)(\xi) dm = -\frac{1}{r} \int\limits_{\mu} S^2 f(\xi) dm\,, \tag{18}$$

with

$$D_{(m,r)} := \frac{1}{2} D_{N(m)} - \frac{1}{2} D_r \tag{19}$$

at $\xi = \xi(m,r)$, $m \in \mu$,, $N(m)$ = exterior unit normal vector of μ at m. And further

$$\begin{aligned} 8\pi \cdot f(o) = \lim_{r_1 \to 0} \frac{1}{r_1} \Big(-\frac{1}{r^2} \int\limits_{\mu(r_1)} D_{(m(r_1),r)}(S^2 f)(\xi) dm(r_1) - \\ -\frac{1}{r^3} \int\limits_{\mu(r_1)} S^2 f(\xi) dm(r_1) \Big) \end{aligned} \tag{20}$$

$D_{(m(r_1),r)} := \frac{1}{2} D_{N(m(r_1))} - \frac{1}{2} D_r$ at $\xi = \xi(m(r_1), r - \frac{r_1}{2})$, $m(r_1) \in \mu(r_1) := \mu(o, r + \frac{r_1}{2})$. All 2-spheres ξ coming up belong to the family of 2-spheres tangent to $\eta_2 = \eta_2(o, 2r)$. Therefore, note $a = \frac{1}{2}$, $b = -\frac{1}{2}$,

$$D_{(m(r_1),r)} = \frac{d}{dt} \Big(\xi(m(r_1) + t \cdot u, r - \frac{r_1 + t}{2}) \Big)_{|t=0}$$

($u :=$ unit vector in direction from o to $m(r_1)$). We view the differential operator of first order as tangent vector in the space of spheres $M^{2,3}$ at $\xi = \xi(m(r_1), r - \frac{r_1}{2})$. Hence

$$D_{(m(r_1),r)}(S^2 f)(\xi) = \frac{d}{dt} \Big(S^2 f(\xi(m(r_1) + t \cdot u, r - \frac{r_1 + t}{2})) \Big)_{|t=0} \tag{21}$$

and

$$D^2_{(m(r_1),r)}(S^2 f)(\xi) = \frac{d^2}{dt^2} \Big(S^2 f(\xi(m(r_1) + t \cdot u, r - \frac{r_1 + t}{2})) \Big)_{|t=0}. \tag{22}$$

Now Taylor-expansion with respect to r_1 (direction u fixed) at $r_1 = 0$ gives

$$\begin{aligned} S^2 f(\xi(m(r_1), r - \frac{r_1}{2})) = S^2 f(\xi(m,r)) + \\ + r_1 \cdot D_{(m(\tau),r)}(S^2 f)(\xi(m(\tau), r - \frac{\tau}{2})) \end{aligned} \tag{23}$$

for some $0 < \tau < r_1$, and

$$\begin{aligned} & D_{(m(r_1),r)}(S^2 f)(\xi(m(r_1), r - \frac{r_1}{2})) = \\ & = D_{(m(r),r)}(S^2 f)(\xi(m(r), r)) + r_1 \cdot D^2_{(m(\sigma),r)}(S^2 f)(\xi(m(\sigma), r - \frac{\sigma}{2})) \end{aligned} \tag{24}$$

for some $0 < \sigma < r_1$.
Finally we bring together (20), (23), (24), (18) (note: $dm(r_1) = (r + \frac{r_1}{2})^2 r^{-2} dm$), and we reach (17). $\square$

Starting again at (13) we consider infinite many dual integrations in concentric shells around o. And for rapidly decreasing point functions f we get the following one-radius inversion formula for the spherical transform.

Theorem 4 *Let $o \in E^3$, $r > 0$, $f \in C^\infty(E^3)$ rapidly decreasing at ∞. Then*

$$8\pi^2 f(o) = -\sum_{l=0}^{\infty} \int\limits_{\mu(o,(2l+1)r)} \left(a_l \cdot D^2_{N(m)} + b_l \cdot D_{N(m)} + c_l\right) (S^2 f)(\xi(m,r))\, dm\ , \tag{25}$$

where

$$a_l := \frac{1}{r^2(2l+1)}\ ,$$

$$b_l := \frac{1}{2r^3(2l+1)} \left(\frac{6l+5}{(l+1)(2l+1)} - 1 + \sum_{i=1}^{l} \frac{1}{i(i+1)}\right)\ ,$$

$$c_l := \frac{1}{2r^4(2l+1)^2} \left(\frac{1}{l+1} - 1 + \sum_{i=1}^{l} \frac{1}{i(i+1)}\right)\ .$$

($N(m)$ = exterior unit normal vector of ν at m).

Proof: (13) with $a = 1$, $b = 0$ implies through $r_1 \to 0$

$$\pi\, S^2 f(\eta(o,2r)) = \int\limits_{\mu(o,r)} \left(\frac{1}{r} D_{N(m)} + \frac{1}{r^2}\right) (S^2 f)(\xi(m,r))\, dm\ , \tag{26}$$

and

$$\begin{aligned} 8\pi^2 f(o) = \lim_{r_1 \to 0} \frac{1}{r_1} \Bigg(&\frac{2\pi}{r_1 + 2r} (S^2 f)(\eta(o, r_1 + 2r)) - \\ &- \int\limits_{\mu(o,r_1+r)} \left(\frac{1}{r(r_1+r)} D_{N(m)} + \frac{1}{r(r_1+r)^2}\right) (S^2 f)(\xi(m,r))\, dm \Bigg)\ . \end{aligned} \tag{27}$$

Taylor-expansion in (27) with respect to r_1, taking into account (26), yields

$$\begin{aligned} 8\pi^2 f(o) = &- \int\limits_{\mu(o,r)} \left(\frac{1}{r^2} D^2_{N(m)} + \frac{2}{r^3} D_{N(m)}\right) (S^2 f)(\xi(m,r))\, dm + \\ &+ \frac{\pi}{r} \left(D_r\, |_\eta\, (S^2 f)\right)(\eta(o,2r)) - \frac{\pi}{2r^2} S^2 f(\eta(o,2r))\,. \end{aligned} \tag{28}$$

(13) with $a = 1$, $b = 0$, $r_2 = r_1 + 2r$ implies

$$\begin{aligned} S^2 f(\eta_1) = &\frac{r_1}{r_1 + 2r} S^2 f(\eta_2) - \\ &- \frac{1}{2\pi} \int\limits_{\mu(o,r_1+r)} \left(\frac{r_1}{r(r_1+r)} D_{N(m)} + \frac{r_1}{r(r_1+r)^2}\right) (S^2 f)(\xi(m,r))\, dm \end{aligned} \tag{29}$$

and

$$\begin{aligned} (D_r\, |_\eta\, S^2 f)(\eta_1) = &\frac{r_1}{r_1 + 2r} (D_r\, |_\eta\, S^2 f)(\eta_2) + \frac{2r}{(r_1+2r)^2} S^2 f(\eta_2) - \\ &- \frac{1}{2\pi} \int\limits_{\mu(o,r_1+r)} \Big(\frac{r_1}{r(r_1+r)} D^2_{N(m)} + \frac{3r_1 + r}{r(r_1+r)^2} D_{N(m)} + \\ &+ \frac{r_1}{r(r_1+r)^2}\Big) (S^2 f)(\xi(m,r))\, dm\,. \end{aligned} \tag{30}$$

In this second step (29), (30) with $r_1 = 2r$, $r_2 = 4r$, and (28) produce the first two summands (shells) in (25), up to an error along $\eta(o, 6r)$. In this manner, starting at (13) through successive application of (29), (30), shell by shell, we reach the one-radius inversion formula (25). (The sum converges and the error tends to zero because f is rapidly decreasing at ∞.) $\square$

Remark 4. (17) with $r \to \infty$ leads to the classical inversion formula for the Radon transform.

Remark 5. Other Inversion formulas for the spherical transform, cf. [12] Chpt. IV, [3], [2], [13].

3 Spherical transform on the Möbius space

Let us now open the euclidean sight to a conformal point of view. The Möbius space is the point space $E^3 \cup \{\infty\}$, the basic objects are spheres (i.e. spheres and planes under euclidean sight), the underlying group is the Möbius group. But we also consider parts of the euclidean space fit out with a conformally changed metric.

Theorem 5 *(7) Theorem 1 and (13) Theorem 2 are representatives, written in euclidean terms, of formulas invariant with respect to Möbius transformations of the euclidean space E^3 and invariant with respect to conformal changes of the euclidean metric respectively.*

Proof: Consider a conformal change of the euclidean metric g_e in E^3, i.e. $g = \rho^2 g_e$ ($\rho \neq 0$ at the points reached by the spheres ξ through the dual integration), without changing the sphere ensemble in the formulas (7), (13). For the function f in the conformaly changed setting we take $f_e := \rho^2 f$ in the euclidean setting. Then $S^2 f_e(\xi)$, $R^2 f_e(\xi)$ with respect to g_e are equal to $S^2 f(\xi)$, $R^2 f(\xi)$ with respect to the conformaly changed metric g. Therefore (7), (13), with differential operators and dual integration taken with respect to the euclidean metric, are valid too in the conformaly changed situation. Planes and spheres respectively, differential operators and the dual integration may have intrinsic meanings in the conformaly changed setting, e.g. see Theorem 6 and Theorem 7 in hyperbolic spaces. That way (7), (13) are euclidean representatives of conformal invariant formulas.
A specific case are Möbius transformations. (We may take the conformal change of the euclidean metric induced by a Möbius transformation without changing the spheres in the formulas, or equivalently we may consider a Möbius transformation of the sphere ensemble in $E^3 \cup \{\infty\}$ without changing the euclidean metric.) Here differential operators and dual integration have Möbius invariant meanings as follows:
In the situation of Theorem 1 (7) the spheres η_1 and η_2 are tangent, they define a degenerate pencil of spheres Σ. ξ is determined by its point of contact $x_1 = \eta_1 \cap \xi$ and $x_2 = \eta_2 \cap \xi$; hence ξ is determined by an orthogonal trajectory circle c_ξ to Σ; therefore ξ is parametrized through $m = c_\xi \cap \mu$, $\mu \in \Sigma$, $\mu =$ mid-sphere of η_1, η_2 (i.e. η_1, η_2, μ, o are in harmonic division, $o = \eta_1 \cap \eta_2$). Let H_Σ denote the subgroup of the Moebius group acting on Σ. Fix $r > 0$. For ξ let H_ξ be the subgroup of H_Σ acting on c_ξ. Let H_1 be the subgroup of H_ξ acting on $c_\xi \setminus \{o\}$ without fixed points. Choose V from the Lie algebra of H_1 with $\exp(2rV) \cdot \eta_1 = \eta_2$. Let H_2 be the subgroup of H_ξ acting on Σ fixing μ. Choose W from the Lie algebra of H_2 with $W^*_{|x_2} = rV^*_{|x_2}$ (W^*, resp. V^* are the vector fields on E^n associated to the action of $\exp(tW)$, resp. $\exp(tV)$, $t \in \mathbb{R}$, on E^n). Then $\frac{a}{r} \cdot V + \frac{b}{r} \cdot W - \frac{b}{r^2}$ describes the differential operator in (7) through Moebius invariant terms, acting on E^n, hence acting on the space of spheres, in particular at ξ. (V, W are not unique but their actions at ξ.) The dual integration runs over μ with volume density invariant with respect to the subgroup H_3 of H_Σ fixing η_1 and η_2 (H_3 is isomorphic to the isometry group of a euclidean plane), normalized as in the euclidean case.
In the situation of Theorem 2 (13) the spheres η_1 and η_2 do not intersect, they define a pencil of spheres Σ with limit points say o and ∞ (Poncelet pencil). ξ is determined by an orthogonal trajectory circle c_ξ to Σ, hence ξ is parametrized through $m = c_\xi \cap \mu$, $\mu \in \Sigma$, $\mu =$ fixed sphere with respect to the Moebius transfomation from H_Σ which changes o and ∞, as well as η_1 and η_2. Fix $r > 0$. For ξ let H_ξ be the subgroup of the Moebius group acting on c_ξ. Let H_1 be the subgroup of H_ξ fixing exactly ∞. Choose V from the Lie algebra of H_1 with $\exp(2rV) \cdot x_1 = x_2$.

Then r_1, r_2 are determined through $\exp(r_1 V)\cdot o = x_1$, $\exp(r_2 V)\cdot o = x_2$. Let H_2 be the subgroup of H_ξ fixing ∞. Choose W from the Lie algebra of H_2 with $W^*_{|x_2} = rV^*_{|x_2}$ and $W^*_{|x_1} = -rV^*_{|x_1}$. Then $\frac{a}{r}\cdot V + \frac{b}{r}\cdot W + \frac{2a}{r(r_1+r_2)} - \frac{b}{r^2}$ describes the differential operator in (13) through Moebius invariant terms. The dual integration runs over μ with volume density invariant with respect to the subgroup H_3 of H_Σ fixing o, η_1, η_2 and ∞ (H_3 is isomorphic to the isometry group of a euclidean sphere), normalized by $\mathrm{vol}(\mu) = ((r_1+r_2)/2)^2 4\pi$. $\square$

Remark 6. In the same line Theorem 3 (17) and Theorem 4 (25) are euclidean representatives of conformal invariant formulas.

Remark 7. (7) in the Möbius transformed sitution works for $f \in C_c^\infty(E^n)$ with $\eta_1\cap\eta_2\cap\mathrm{supp} f = \emptyset$.

4 Applications in the hyperbolic space

Let H^3 be the 3-dimensional hyperbolic standard space (i.e. geodesically complete, simply connected, constant curvature -1). Let R^2, S^2 and S^2_h denote the 2-plane Radon transform, the 2-spherical transform and the 2-horospherical transform respectively. They are defined analogeously to (1), (2), mapping $C_c^\infty(H^3)$ into $C^\infty(Y)$, $Y = G^{2,3}$ = space of 2-planes, i.e. 2-dimensional totally geodesic subspaces of H^3, $Y = M^{2,3}$ = space of 2-dimensional distance spheres in H^3 and $Y = M_h^{2,3}$ = space of 2-dimensional horospheres in H^3 respectively.
For $X \in T^1_x H^3$, $x \in H^3$, let $\tau_{(X,x)}(t)$ ($t \in \mathbb{R}$) denote the 1-parameter subgroup of hyperbolic isometries induced by geodesic parallel translation along the geodesic through x in direction X. Then $D_{(X,x)}$ denotes the differential operator of first order acting on $C_c^\infty(Y)$ through $\tau_{(X,x)}$, i.e.

$$(D_{(X,x)}F)(\xi) := \frac{d}{dt}(F(\tau_{(X,x)}(t)\cdot\xi))\,|_{t=0} \tag{31}$$

($\xi \in Y$, $F \in C^\infty(Y)$).

Theorem 6 ***a)*** *Let $\eta \in M_h^{2,3}$ be a horosphere in H^3. Then*

$$2\pi S^2_h f(\eta) = -\int_\eta \Big(D_{(N(x),x)} + 2\Big)(S^2_h f)(\xi)dx. \tag{32}$$

Herein the dual integration runs over the totality of horospheres $\xi = \xi(x)$ tangent to η, $\xi \neq \eta$, parametrized through their points of contact $x \in \eta$; $N(x)$ = exterior unit normal vector of η at x; dx = hyperbolic volume density on η.

b) *Let $\eta \in M^{2,3}$ be a distance sphere in H^3 of hyperbolic radius r. Then*

$$2\pi S^2 f(\eta) = -\frac{e^r}{2\sinh r}\int_\eta \Big(D_{(N(x),x)} + 2\Big)(S^2_h f)(\xi)dx. \tag{33}$$

Herein the dual integration runs over the totality of horospheres $\xi = \xi(x)$ tangent to η, η lying outside ξ, parametrized through their points of contact $x \in \eta$; $N(x)$ = exterior unit normal vector of η at x;dx = hyperbolic volume density on η.

c) *Let $o \in H^3$. Then*

$$\begin{aligned} 8\pi^2 f(o) &= -\int_{T^1_o H^3} \Big(D^2_{(u,o)} + 2D_{(u,o)}\Big)(S^2_h f)(\xi)du = \\ &= -\int_{T^1_o H^3} \Big(D^2_{(u,o)} - 4\Big)(S^2_h f)(\xi)du. \end{aligned} \tag{34}$$

Herein the dual integration runs over the totality of horospheres $\xi = \xi(o,u)$ through o, parametrized through their interior normal unit vector $u \in T^1_o H^3$ at o; du = euclidean volume density on the unit spherical in the tangent space of H^3 at o.

Proof: a): Let H^3 be realized through Poincaré's half-space model in $\mathbb{R}^3$, adapted to Theorem 1 by $\eta = \eta_1$, $\eta_2 =$ boundary 2-plane of the model. The euclidean metric tensor g_e and the hyperbolic metric tensor g_h at $y \in H^3$ are conformally related by $g_h = \frac{1}{\rho^2} g_e$, $\rho = \rho(y) =$ euclidean distance of y to η_2. We start at (7) with $a = -r, b = r$, and with euclidean terms replaced by their hyperbolic meanings, i.e. $R^2 \tilde{f}(\eta) = S_h^2 f(\eta)$, $S^2 \tilde{f}(\xi) = S_h^2 f(\xi)$, $\tilde{f} := f \frac{1}{\rho^2}$, $dx = \frac{1}{4r^2} dm$, $2r(-D_N + D_r) = -D_{(N(x),x)}$ at $\xi(x), x \in \eta$. We take into account that the hyperbolic isometry group acts transitively on the space of horospheres, therefore the special situation in the model setting describes the general hyperbolic situation. Thus we reach (32).
b): Let H^3 be realized through Poincaré's ball model, adapted to Theorem 2 by $\eta_1 = \eta_1(o, r_1) = \eta$, $r_1 < r_2 = 2$, $\eta_2 = \eta_2(o, 2) =$ boundary 2-sphere of the model. (13) with $a = -(1-\frac{r_1^2}{4})$, $b = (1-\frac{r_1^2}{4})$, $\tilde{f} \in C_c^\infty(H^3)$ writes, in euclidean terms,

$$2\pi(1 - \frac{r_1^2}{4}) \frac{(2 - r_1)(2 + r_1)}{2r_1} S^2 \tilde{f}(\eta) =$$

$$= \int_\mu (1 - \frac{r_1^2}{4})(-D_{N(m)} + D_r)(S^2 \tilde{f})(\xi) dm -$$

$$-2 \int_\mu S^2 \tilde{f}(\xi) dm$$

$\mu = \mu(o, \frac{r_1+2}{2})$.
The euclidean metric tensor g_e and the hyperbolic metric tensor g_h at $y \in H^3$ are conformally related by $g_h = (1 - \frac{\rho^2}{4})^{-2} g_e$, $\rho = \rho(y) =$ euclidean distance between o (= euclidean center of the model) and y. Therefore the euclidean radius r_1 and the hyperbolic radius r of η are related by $r_1 = 2\tanh \frac{r}{2}$. Then we replace the euclidean terms by their hyperbolic meanings, i.e. $S^2 \tilde{f}(\eta) = S^2 f(\eta)$, $S^2 \tilde{f}(\xi) = S_h^2 f(\xi)$, $\tilde{f} := (1 - \frac{\rho^2}{4})^{-2} f$, $dx = (1 - \frac{r_1^2}{4})^{-2} r_1^2 (\frac{r_1+2}{2})^{-2} dm$, $(1 - \frac{r_1^2}{4})^2(-D_{N(m)} + D_r) = -D_{(N(x),x)}$ at $\xi(x), x \in \eta$. Because the hyperbolic isometry group acts transitively on the space of hyperbolic distance spheres with radius r, we reach (33).
c): (33) through $r \to 0$, like in the proof of Theorem 3, yields (34). □

Through similar adaptations we obtain

Theorem 7 ***a)*** *Let $\eta \in M_h^{2,3}$ be a horosphere in H^3. Then*

$$2\pi S_h^2 f(\eta) = -\int_\eta \Big(D_{(N(x),x)} + 1\Big)(R^2 f)(\xi) dx. \tag{35}$$

Herein the dual integration runs over the totality of 2-planes $\xi = \xi(x)$ tangent to η, parametrized through their points of contact $x \in \eta$; $N(x) =$ exterior unit normal vector of η at x; $dx =$ hyperbolic volume density on η.

b) *Let $\eta \in M^{2,3}$ be a distance sphere in H^3 of hyperbolic radius r. Then*

$$2\pi S^2 f(\eta) = -\int_\eta \Big(\frac{1}{\tanh r} D_{(N(x),x)} + 1\Big)(R^2 f)(\xi) dx. \tag{36}$$

Herein the dual integration runs over the totality of 2-planes $\xi = \xi(x)$ tangent to η, parametrized through their points of contact $x \in \eta$; $N(x) =$ exterior unit normal vector of η at x; $dx =$ hyperbolic volume density on η.

c) *Let $o \in H^3$. Then*

$$8\pi^2 f(o) = -\int_{T_o^1 H^3} \Big(D^2_{(u,o)} + 1\Big)(R^2 f)(\xi) du. \tag{37}$$

Herein the dual integration runs twice over the totality of 2-planes $\xi = \xi(o, u)$ through o and perpendicular to u, parametrized through $u \in T_o^1 H^3$; du = euclidean volume density on the unit sphere in the tangent space of H^3 at o.

Remark 8. Similar adaptations lead e.g. to the following relations: from the spherical transform to the spherical transform like (13), to inversion formulas for the spherical transform like (17), or from orbital integrals with respect to distance surfaces of 2-planes to the spherical transform similar to (13), etc..
Similar applications happen in the spherical space.

Remark 9. Inversion formulas for the Radon transform in hyperbolic spaces, cf. also [8], [11], [19]. Inversion formulas for the horospherical transform, cf. also [6] Chpt. V, [10].

5 Support results

Theorem 8 *Let $f \in C^\infty(E^3)$ be such that for each integer $l > 0$, $\mid x - o \mid^l f(x)$ is bounded, ($o \in E^3$ fixed). Suppose there exists a closed ball $B(o, R)$ with center o and radius R, and a fixed radius r, such that $S^2 f(\xi) = 0$ for all spheres ξ of radius r with $B(o, R)$ outside ξ. Then $f(x) = 0$ for $x \in E^3 \setminus B(o, R)$.*

Proof: Let η_1 be a sphere of radius r_1 which encloses $B(o, R)$. Let η_2 denote the sphere of radius $r_1 + 2r$ concentric to η_1. Then (13) with $b = 0$ gives $S^2 f(\eta_1) = \frac{r_1}{r_1+2r} S^2 f(\eta_2)$. Going outwards through such steps shows $S^2 f(\eta_1) = \frac{r_1}{r_1+l2r} S^2 f(\eta_l)$, $l \in \mathbb{N}$, η_l = sphere of radius $r_1 + l2r$ concentric to η_1. Thus $S^2 f(\eta_1) = 0$ because f rapidly decreases. Hence $S^2 f(\eta) = 0$ for each sphere which encloses $B(o, R)$. Therefore [9] Ch.1 Lemma 2.7 yields that f vanishes outside $B(o, R)$. □

Theorem 9 *Let $f \in C^\infty(E^3)$. Suppose there exists a closed ball $B(o, R)$ with center o and radius R, and two fixed radii $r, \bar{r}$ with $r/\bar{r}$ irrational and $2r + 2\bar{r} < R$, such that $S^2 f(\xi) = 0$ for all spheres $\xi \subset B(o, R)$ of radius r and $S^2 f(\bar{\xi}) = 0$ for all spheres $\bar{\xi} \subset B(o, R)$ of radius $\bar{r}$. Then $f(x) = 0$ for $x \in B(o, R)$.*

Proof: Let η_1, η_2 be concentric spheres with center o and radii r_1, $r_2 = r_1 + 2r$ or $r_2 = r_1 + 2\bar{r}$, $0 < r_1 < r_2 < R$. Then (13) with $b = 0$ gives $S^2 f(\eta_1) = \frac{r_1}{r_2} S^2 f(\eta_2)$.
Let $S := \{n2r - m2\bar{r} \mid n, m \in \mathbb{Z}, n \geq 0\} \cap [0, R]$. Then S is dense in $[0, R]$, because $r/\bar{r}$ irrational. Moreover any two points in S can be connected in $[0, R]$ using segments each of length $2r$ or $2\bar{r}$. (cf. [14] pp. 88, resp. [21] proof of Lemma 3.3).
Therefore $S^2 f(\eta_1) = \frac{r_1}{r_2} S^2 f(\eta_2)$ for $r_1, r_2 \in S$. Taking into account a sequence $r_2 \to 0$ yields $S^2 f(\eta) = 0$ for all spheres $\eta(o, r)$, $r \in S$. Hence by continuity, $S^2 f(\eta) = 0$ for all spheres $\eta(o, r)$, $r < R$.
The same holds for small perturbations of o. Therefore the idea of the proof of Lemma 2.7 Chpt. 1 [9] works and shows $f(x) = 0$ for $x \in B(o, R)$. □.

Remark 10. Theorem 8 and Theorem 9 respectively are valid for f continuous and rapidly decreasing at ∞, and for f continuous respectively.
Note: The proofs above work through replacing f by the convolution $\phi * f$, where ϕ is a well chosen radial C^∞-function with support in a small ball $B(o, \epsilon) \subset E^n$, $\epsilon > 0$. In fact, for Theorem 8, and analogously for Theorem 9,

$$S^2(\phi * f)(\xi(m, r)) = (\chi * S^2 f)(\xi(m, r)) = \int_{E^n} \chi(m - x) \cdot S^2 f(\xi(x, r))\, dx\ ,$$

χ a radial C^∞-function with compact support in $B(o, \epsilon) \subset \mathbb{R}^3$, appropriated to ϕ.

Remark 11. Support and injectivity results for the spherical transform, cf. [12] Chpt. VI, [21], [1], [4], [15], [18], [20].

6 A reduction sheme in higher dimensions

Lemma 1 *Let η_1, η_2 be parallel hyperplanes in E^n at distance $2r$. Then for $k \geq 3$*

$$\int\limits_{\mu\times G_0^{k-2,n-1}} S^{k-2}f(\delta)d\delta =$$

$$= \frac{v^{k-3,k-1}}{(k-2)v^{2,n-k+1}} \int\limits_{\mu\times G_0^{k,n-1}} \Big(\frac{1}{r}D_r - \frac{1}{r^2}\Big)(S^k f)(\xi)d\xi. \tag{38}$$

Herein the dual integration on the right hand side runs over the totality of k-spheres $\xi = \xi(m,\sigma,r)$ of radius r tangent to both η_1 and η_2, parametrized through center $m \in \mu$ and σ = affine hull of $\xi \cap \mu$, $\sigma \in G_0^{k,n-1}$ (= Grassmann manifold of k-planes in μ through m), $d\xi = dmd\sigma$, dm = euclidean volume density on the hyperplane μ, $d\sigma$ = invariant volume density in $G_0^{k,n-1}$. The integration on the left hand side analogously runs over the totality of $(k-2)$-spheres $\delta = \delta(m,\tau,r)$ of radius r tangent to both η_1 and η_2, parametrized through center $m \in \mu$, $\tau \in G_0^{k-2,n-1}$. $v^{k-3,k-1}$, $v^{2,n-k+1}$ = volume of the Grassmann manifold $G_0^{k-3,k-1}$, $G_0^{2,n-k+1}$.

Proof: The definition of D_r implies

$$D_r(S^k f)(\xi) = \int\limits_{\xi} \frac{\cos\alpha}{\sin\alpha} \cdot df_{|y}(e_2)dy + \frac{k}{r}\int\limits_{\xi} f(y)dy \tag{39}$$

mod integrand terms $df_{|y}(\bar{e}_2)$ disappearing through dual integration because of symmetry (cf. proof of Proposition 2.1) ($\alpha = \angle(x_1 m y)$, $x_1 = \eta_1 \cap \xi$).
We use polar coordinates on ξ centered at x_1, i.e. $dy = r^{k-1}\sin^{k-1}\alpha(rd\alpha)du$, $u \in T^1_{x_1}\xi$, to rewrite the integrand of the first integral on the right hand side of (39), i.e. $r^{k-1}\sin^{k-2}\alpha\cdot\cos\alpha\cdot df_{|y}(e_2)(rd\alpha)du$. Integration by parts with respect to$(rd\alpha)$, this integral becomes

$$\int\limits_{\xi}(k-2)r^{k-2}\sin^{k-3}\alpha\cdot f(y)(rd\alpha)du - \frac{k-1}{r}S^k f(\xi). \tag{40}$$

We replace the integration with respect to u over the $(k-1)$-sphere $T^1_{x_1}\xi$ by a twofold integration, at first over $(k-3)$-greatspheres of $T^1_{x_1}\xi$ then over the totality of $(k-3)$-greatspheres of $T^1_{x_1}\xi$. Thus first integral in (40) becomes

$$\frac{(k-2)r}{v^{k-3,k-1}} \int\limits_{G_0^{k-2,k}} S^{k-2}f(\delta)d\tau \tag{41}$$

($\tau = T_{x_1}\delta \subset T_{x_1}\xi$).
Finally we bring (41), (40) back into (39), we carry-out the dual integrations, taking into account integration with respect to nested subspaces $\tau \subset \sigma \subset \mathbb{R}^{n-1}$ (see [17] (12.52)), and we reach (38). □

Through similar computations we get

Lemma 2 *Let η_1, η_2 be concentric $(n-1)-$spheres in E^n with center o and radii r_1, r_2 $(r_1 < r_2)$. Then for $k \geq 3$*

$$\int\limits_{\mu G_0^{k-2,n-1}} S^{k-2}f(\delta)d\delta = \frac{v^{k-3,k-1}}{(k-2)v^{2,n-k+1}} \int\limits_{\mu G_0^{k,n-1}} \Big(-\frac{r_1+r_2}{2r_1r_2}D_{N(m)}+$$

$$+\frac{(r_1+r_2)^2}{2r_1r_2(r_2-r_1)}D_r - \frac{k(r_2-r_1)^2+4r_1r_2}{(r_2-r_1)^2 r_1 r_2}\Big)(S^k f)(\xi)d\xi. \tag{42}$$

Herein the dual integration on the right hand side runs over the totality of k-spheres $\xi = \xi(m, \sigma, r)$ of radius $r = \frac{r_2 - r_1}{2}$ tangent to both η_1 and η_2, ξ outside η_1 and inside η_2, parametrized through center $m \in \mu$ and σ = affine hull of $\xi \cap T_m\mu$, $\sigma \in G_0^{k,n-1}(T_m\mu)$ (= Grassmann manifold of k-spaces in $T_m\mu$ through m), $d\xi = dmd\sigma$. The integration on the left hand side analogously runs over the totality of $(k-2)$-spheres $\delta = \delta(m, \tau, r)$ of radius r tangent to both η_1 and η_2, δ outside η_1 and inside η_2, parametrized through center $m \in \mu$, $\tau \in G_0^{k-2,n-1}(T_m\mu)$, $d\delta = dmd\tau$.

Proposition 1 *Let η_1, η_2 be parallel hyperplanes in E^n at distance $2r$, $n = 2k+1$. Let a, b be real constants. Then*

$$2\pi\Big((a+b)\cdot R^{n-1}f(\eta_2) + (-a+b)\cdot R^{n-1}f(\eta_1)\Big) = \\ = c(n)\int\limits_{\mu}\Big(\frac{a}{r}\cdot D_N + \frac{b}{r}\cdot D_r - \frac{b}{r^2}\Big)\Big(\frac{1}{r}D_r - \frac{1}{r^2}\Big)^{k-1}(S^{n-1}f)(\xi)\,dm\,. \tag{43}$$

Herein the dual integration runs over the totality of $(n-1)-$spheres $\xi = \xi(m, r)$ of radius r tangent to both η_1 and η_2, parametrized through their centers $m \in \mu$; N = normal unit vector of η_1 pointing towards η_2; $c(n)$ a constant depending on n.

Proof: (7) through successive application of (38) yields (43). (Note: Integrations in (38) and the differential operators D_N and D_r intertwine.) □

Remark 12. There are further reduction formulas in the style of (38) and (42), more general however much more complex. (Actual we don't overlook the totality of such reduction formulas.)

Remark 13. (13) and (17) respectively through successive application of the reduction formula (42) leads to the analogue of (43) for concentric $(n-1)-$spheres η_1, η_2 in E^n and to an inversion formula for the spherical transform in E^n, $n = 2k+1$. (Note: All terms coming up depend on o, r_1 and r, $(r_2 = r_1 + 2r)$. Let $I_{(\mu,r)}$ denote anyone of the integration operators in (13), (42), then $\partial/\partial r \circ I_{(\mu,r)} = I_{(\mu,r)} \circ \partial/\partial r$ and $\partial/\partial r S^k f(\xi) = D_r S^k f(\xi)$, $D_r|\mu \circ I_{(\mu,r)} = \partial/\partial(r_1 + r) \circ I_{(\mu,r)} = \partial/\partial r_1 \circ I_{(\mu,r)} = I_{(\mu,r)} \circ \partial/\partial r_1 + \frac{2(n-1)}{r_1+r_2} I_{(\mu,r)}$ and $\partial/\partial r_1 S^k f(\xi) = D_{N(m)} S^k f(\xi)$.)

Remark 14. Inversion formulas for the circular transform in the euclidean plane E^2: Let E^2 be enlarged through an orthogonal complement to $E^3 = E^2 \times \mathbb{R}$, let $F \in C_c^\infty(E^3)$ be defined by $F((x,t)) := \psi(t)\cdot f(x)$, $(x,t) \in E^2 \times \mathbb{R}$, $\psi \in C_c^\infty(\mathbb{R})$ with $\psi(0) = 1$. Then (7) with $\eta_1 = g_1 \times \mathbb{R}$, $\eta_2 = g_2 \times \mathbb{R}$, g_1, g_2 parallel lines in E^2 at distance $2r$ leads to a transformation from the circular transform $S^1(\xi)$ to the Radon transform $R^1(g_1)$, $R^1(g_2)$, looking like (7), with dual integration over the totality of circles $\xi = \xi(m, \rho)$ of radius $0 \le \rho \le r$ and centers m on the mid-parallel of g_1 and g_2, and additional coefficients depending on ρ appearing at the summands on the right hand side of (7).

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