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PÓLYA'S CONJECTURE IN THE PRESENCE OF A CONSTANT MAGNETIC FIELD

ABSTRACT. We consider the Dirichlet Laplacian with a constant magnetic field in a two-dimensional domain of finite measure. We determine the sharp constants in semi-classical eigenvalue estimates and show, in particular, that Pólya's conjecture is not true in the presence of a magnetic field.

1. INTRODUCTION

Let $\Omega \subset \mathbb{R}^2$ be a domain of finite measure and define the Dirichlet Laplacian H^Ω in $L_2(\Omega)$ as the Friedrichs extension of $-\Delta$ initially given on $C_0^\infty(\Omega)$. This defines a self-adjoint non-negative operator, and by Rellich's compactness theorem its spectrum is discrete and accumulates at infinity only. The spectrum of H^Ω plays an important role in many physical models (such as membrane vibration or quantum mechanics) and its determination is a classical problem in mathematical physics.

Let (λ_n) be the non-decreasing sequence of eigenvalues of $H^\Omega = -\Delta$ (taking multiplicities into account) and let $N(\lambda, H^\Omega) := \#\{n: \ \lambda_n < \lambda\}$ denote their counting function.

In 1911 H. Weyl [W] (see also [RS, Ch. XIII]) showed the asymptotic formula

$$\lambda_n = \frac{4\pi n}{|\Omega|}(1 + o(1)), \quad n \to \infty.$$

In terms of the counting function this is equivalent to

$$N(\lambda, H^\Omega) = \frac{1}{4\pi}\lambda|\Omega|(1 + o(1)), \quad \lambda \to +\infty. \tag{1.1}$$

Integrating the latter formula one finds as well the asymptotic behavior of the eigenvalue means

$$\operatorname{tr}(H^\Omega - \lambda)_-^\gamma := \sum_{n:\lambda > \lambda_n} (\lambda - \lambda_n)^\gamma = L_{\gamma,2}^{\mathrm{cl}}\lambda^{\gamma+1}|\Omega|(1 + o(1)), \quad \lambda \to +\infty, \tag{1.2}$$

where $\gamma \geq 0$ and

$$L_{\gamma,2}^{\mathrm{cl}} := (4\pi(\gamma + 1))^{-1}. \tag{1.3}$$

Note that the expression on the right hand side of (1.2) equals the classical phase space average

$$L_{\gamma,2}^{\mathrm{cl}}\lambda^{\gamma+1}|\Omega| = (2\pi)^{-2} \iint_{\Omega \times \mathbb{R}^2} (|\xi|^2 - \lambda)_-^\gamma \, dx d\xi \tag{1.4}$$

of the symbol $|\xi|^2$ of the Laplacian.

Pólya [P] found in 1961 that for *tiling domains*[1] Ω the asymptotic expression (1.1) is in fact an upper bound on the counting function, namely

$$N(\lambda, H^\Omega) \leq \frac{1}{4\pi}\lambda|\Omega|, \quad \lambda \geq 0. \tag{1.5}$$

[1]A domain $\Omega \subset \mathbb{R}^2$ is tiling if one can cover $\mathbb{R}^2$ up to a set of measure zero by pairwise disjoint congruent copies of Ω.

By (1.1) the constant in this bound is optimal. Moreover, Pólya conjectured that this bound should hold true for *arbitrary* domains Ω with the same sharp constant $\frac{1}{4\pi}$.

The fact that the counting function $N(\lambda, H^\Omega)$ can be estimated by

$$N(\lambda, H^\Omega) \leq C\lambda|\Omega|, \quad \lambda \geq 0. \tag{1.6}$$

with some constant C which does not depend on λ or the shape of the domain is due to Rozenblum [R1], Lieb [L2] and Metivier [M]. Results with sharp constants for *sums* of eigenvalues have been obtained by Berezin and by Li and Yau. Indeed, Berezin [B1] proved that

$$\mathrm{tr}(H^\Omega - \lambda)_-^\gamma \leq L_{\gamma,2}^{\mathrm{cl}}\lambda^{\gamma+1}|\Omega| \quad \text{for} \quad \gamma \geq 1. \tag{1.7}$$

In view of the Weyl asymptotics (1.2) the constant in this bound is optimal. This estimate in the case $\gamma = 1$ implies after taking the Legendre transform the celebrated result by Li and Yau [LY]

$$\sum_{j=1}^n \lambda_j \geq \frac{2\pi n^2}{|\Omega|}, \quad n \in \mathbb{N}. \tag{1.8}$$

Both (1.7) and (1.8) give rise to the best known upper bound $C \leq (2\pi)^{-1}$ on the sharp constant C in (1.6). However, Pólya's conjecture, namely that (1.5) holds for general domains, remains open. In fact, this question is unresolved even in the case where the domain is a disk.

The main goal of this paper is to *disprove* the analogous conjecture for the Dirichlet Laplacian with a constant magnetic field.

Put $D = -i\nabla$ and let A be a sufficiently regular real vector field on Ω. We consider the operator $(D - A)^2$ on $L_2(\Omega)$ with Dirichlet boundary conditions defined in the quadratic form sense. If $|\Omega|$ has finite measure, the spectrum of $(D - A)^2$ is discrete and as above, we can introduce the ordered sequence of eigenvalues and the corresponding counting function. It is well-known that the asymptotic formulae (1.1) and (1.2) remain true in the magnetic case as well. This is in accordance with the fact that the magnetic field leaves the classical phase space average unchanged,

$$(2\pi)^{-2} \iint_{\Omega \times \mathbb{R}^2} (|\xi - A(x)|^2 - \lambda)_-^\gamma \, dxd\xi = (2\pi)^{-2} \iint_{\Omega \times \mathbb{R}^2} (|\xi|^2 - \lambda)_-^\gamma \, dxd\xi.$$

Therefore, it seems reasonable to discuss Pólya-type bounds in the magnetic case as well. In fact, it turns out that the bound (1.6) extends to the magnetic case with a suitable constant C which does not depend on A, Ω and λ, see e.g. [R2].

There are also results concering magnetic estimates with *sharp* semi-classical constants. As recalled in the appendix, a result by Laptev and Weidl [LW1] implies the bound

$$\mathrm{tr}((D - A)^2 - \lambda)_-^\gamma \leq L_{\gamma,2}^{\mathrm{cl}}\lambda^{\gamma+1}|\Omega| \tag{1.9}$$

for *arbitrary* A and all $\gamma \geq 3/2$. In [ELV] this result was extended to $\gamma \geq 1$ in the special case of a homogeneous magnetic field, $A(x) = \frac{B}{2}(-x_2, x_1)^T$. The latter two results motivate the question, whether Pólya's conjecture could be true in the magnetic case.

In this note we shall show that this intuition is wrong and that the Pólya estimate (1.5) in the magnetic case can be violated even for tiling domains. More precisely, we consider a homogeneous magnetic field, $A(x) = \frac{B}{2}(-x_2, x_1)^T$, and show that for arbitrary domains Ω of finite measure the bound

$$N(\lambda, (D - A)^2) \leq \frac{1}{2\pi}\lambda|\Omega| = 2L_{0,2}^{\mathrm{cl}}\lambda|\Omega| \tag{1.10}$$

holds true. We prove that the constant in this bound is optimal and that the numerical factor 2 on the right hand side cannot be improved - not even in the tiling case. A similar phenomenon occurs for eigenvalue moments of order $\gamma \in (0, 1)$.

As a consequence of our result we see, in particular, that any attempt to prove Pólya's conjecture with a method which extends to constant magnetic fields must fail.

2. Main results

Let $\Omega \subset \mathbb{R}^2$ be a domain of finite measure. For $B > 0$ we consider the self-adjoint operator

$$H_B^\Omega := (D - BA)^2 \qquad \text{in } L_2(\Omega)$$

with Dirichlet boundary conditions, i.e., closing the form $\|(D - BA)u\|^2$ on $C_0^\infty(\Omega)$. The magnetic vector potential A is always chosen in the form

$$A(x) := \frac{1}{2}(-x_2, x_1)^T,$$

and we remark that $\operatorname{curl} BA \equiv B$. In other words, we restrict the vector potential for a constant magnetic field from $\mathbb{R}^2$ to Ω. [2]

The operator H_B^Ω has compact resolvent and we denote by $N(\lambda, H_B^\Omega)$ the number of its eigenvalues less than λ, counting multiplicities. Our first main result is

Theorem 2.1. *Let $\Omega \subset \mathbb{R}^2$ be a domain of finite measure. Then for all $B > 0$ and $\lambda > 0$*

$$N(\lambda, H_B^\Omega) \leq R_0 L_{0,2}^{\mathrm{cl}} |\Omega| \lambda \tag{2.1}$$

and

$$\operatorname{tr}(H_B^\Omega - \lambda)_-^\gamma \leq R_\gamma L_{\gamma,2}^{\mathrm{cl}} |\Omega| \lambda^{\gamma+1}, \qquad 0 < \gamma < 1, \tag{2.2}$$

where $R_0 = 2$ and $R_\gamma = 2\left(\gamma/(\gamma+1)\right)^\gamma$ for $0 < \gamma < 1$. One has $R_\gamma > 1$ and these constants can not be improved, not even if Ω is tiling. More precisely, for any $0 \leq \gamma < 1$, $\varepsilon > 0$ and $B > 0$ there exists a square Ω and $\lambda > 0$ such that

$$\operatorname{tr}(H_B^\Omega - \lambda)_-^\gamma \geq (1 - \varepsilon) R_\gamma L_{\gamma,2}^{\mathrm{cl}} |\Omega| \lambda^{\gamma+1}. \tag{2.3}$$

We emphasize that for linear and superlinear moments one has the semi-classical bound

$$\operatorname{tr}(H_B^\Omega - \lambda)_-^\gamma \leq L_{\gamma,2}^{\mathrm{cl}} |\Omega| \lambda^{\gamma+1}, \qquad \gamma \geq 1, \tag{2.4}$$

without an excess factor. The inequality (2.4) is essentially contained in [ELV] but will be rederived in Corollary 4.4 below.

Our second main result concerns tiling domains. We shall show that in this case the inequalities (2.1) and (2.2) can be strengthened if one is willing to allow the right hand side depend on B. Let us define

$$\mathfrak{B}_\gamma(B, \lambda) := (2\pi)^{-1} B \sum_{k \in \mathbb{N}_0} (\lambda - B(2k + 1))_+^\gamma. \tag{2.5}$$

For $\gamma = 0$ this is defined to be left-continuous in λ, i.e., $0_-^0 := 0$.

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[2]For simply connected domains Ω this choice of A is up to gauge invariance unique in the class of all vector potentials inducing a constant magnetic field in Ω. If Ω is not simply connected, then one has gauge invariant classes of magnetic vector potentials inducing a constant magnetic field inside Ω, but which are not restrictions of a vector potential producing a constant magnetic field on the whole of $\mathbb{R}^2$. In this paper we do not consider such vector potentials.

Theorem 2.2. *Let $\Omega \subset \mathbb{R}^2$ be a tiling domain of finite measure. Then for all $B > 0$ and $\lambda > 0$*

$$N(\lambda, H_B^\Omega) \leq \mathfrak{B}_0(B, \lambda)|\Omega| \tag{2.6}$$

and

$$\operatorname{tr}(H_B^\Omega - \lambda)_-^\gamma \leq \mathfrak{B}_\gamma(B, \lambda)|\Omega|, \qquad 0 < \gamma < 1, \tag{2.7}$$

and these estimates can not be improved. More precisely, for any $0 \leq \gamma < 1$, $\varepsilon > 0$, $B > 0$, $\lambda > 0$ there exists a square Ω such that

$$\operatorname{tr}(H_B^\Omega - \lambda)_-^\gamma \geq (1 - \varepsilon)\mathfrak{B}_\gamma(B, \lambda)|\Omega|. \tag{2.8}$$

We emphasize that for $\gamma \geq 1$ one has the bound

$$\operatorname{tr}(H_B^\Omega - \lambda)_-^\gamma \leq \mathfrak{B}_\gamma(B, \lambda)|\Omega|, \qquad \gamma \geq 1, \tag{2.9}$$

for an *arbitrary* domain $\Omega \subset \mathbb{R}^2$ of finite measure. This is again essentially contained in [ELV]. We give an independent proof in Theorem 4.1 below and show also that (2.9) is stronger than (2.4). The question whether (2.6) and (2.7) extend to not necessarily tiling domains is left open.

Remark 2.3. There are estimates intermediate between (2.1) and (2.6) with the right hand side depending on B but in a simpler way than in (2.6). For example, we mention the estimate

$$N(\lambda, H_B^\Omega) \leq \frac{1}{4\pi}(\lambda + B)|\Omega| \tag{2.10}$$

for Ω tiling. Note that this estimate is *stronger* than (2.1) since $N(\lambda, H_B^\Omega) = 0$ for $\lambda \leq B$. In particular, it coincides with the estimate (1.5) for $B = 0$.

Remark 2.4. There is an essentially equivalent way of stating the estimates (2.1) and (2.10). Namely denoting the eigenvalues of H_B^Ω by $\lambda_{B,j}^\Omega$ and passing to the limit $\lambda \to \lambda_{B,j}^\Omega+$ in these estimates we find

$$\lambda_{B,N}^\Omega \geq 2\pi|\Omega|^{-1}N$$

and, respectively,

$$\lambda_{B,N}^\Omega \geq 4\pi|\Omega|^{-1}N - B.$$

Remark 2.5. For the lower bound (2.3) we fix $B > 0$ and choose Ω and λ. Alternatively, one can fix a cube Ω and choose B and λ. This follows by a simple scaling argument.

3. THE MAGNETIC DENSITY OF STATES

3.1. **The magnetic density of states.** In this section we shall use a slightly modified notation. When $\Omega = (-L/2, L/2)^2$ we shall denote the operator H_B^Ω by $H_B^D(L)$. Recall that $\mathfrak{B}_0(B, \lambda)$ was defined in (2.5). Our goal is to prove

Proposition 3.1. *Let $B > 0$ and $\lambda > 0$. Then*

$$\lim_{L \to \infty} L^{-2}N(\lambda, H_B^D(L)) = \mathfrak{B}_0(B, \lambda). \tag{3.1}$$

Hence $\mathfrak{B}_0(B, \cdot)$ is the density of states for the Landau Hamiltonian $H_B := (D - BA)^2$ in $L_2(\mathbb{R}^2)$. This is certainly well-known, but we include the proof for the sake of completeness. This will be done in the remaining part of this section. A different proof may be found in [N]. Alternatively, one can also use the known result that

$$\lim_{L \to \infty} L^{-2}N(\lambda, H_B^D(L)) = \lim_{L \to \infty} L^{-2}\operatorname{tr}(\chi_{Q_L}\chi_{(0,\lambda)}(H_B)).$$

The RHS can be evaluated using the explicit form of the spectral projections of H_B, see the proof of Theorem 4.1.

3.2. **Explicit solution on the torus.** In this subsection we consider the case of a square, $\Omega = (-L/2, L/2)^2 =: Q_L$, and define an operator $H_B^P(L)$ in $L_2(Q_L)$ which differs from H_B^Ω by the choice of magnetic periodic boundary conditions. However, its spectrum will turn out to be explicitly computable.

To define $H_B^P(L)$ we shall fix $B, L > 0$ such that

$$(2\pi)^{-1}L^2 B \in \mathbb{N} \tag{3.2}$$

and introduce the 'magnetic translations'

$$(T_1 u)(x) := e^{-iBLx_2/2} u(x_1 + L, x_2),$$
$$(T_2 u)(x) := e^{iBLx_1/2} u(x_1, x_2 + L).$$

(The dependence on B and L is not reflected in the notation.) The assumption (3.2) implies that T_1 and T_2 commute, and hence any function u on Q_L has a unique extension to a function $\tilde{u}$ on $\mathbb{R}^2$ by means of the operators T_1, T_2. We introduce the Sobolev spaces

$$H_{per}^k(Q_L) := \{u \in H^k(Q_L): \; \tilde{u} \in H_{loc}^k(\mathbb{R}^2)\}.$$

Then the operator $H_B^P(L) := (D - BA)^2$ in $L_2(Q_L)$ with domain $H_{per}^2(Q_L)$ is self-adjoint. It is generated by the quadratic form $\|(D - BA)u\|^2$ with form domain $H_{per}^1(Q_L)$. The spectrum of this operator is described in

Proposition 3.2. *Assume* (3.2). *Then the spectrum of $H_B^P(L)$ consists of the eigenvalues $B(2k+1)$, $k \in \mathbb{N}_0$, with common multiplicity $(2\pi)^{-1}L^2 B$. In particular, for all $\lambda > 0$,*

$$N(\lambda, H_B^P(L)) = L^2 \mathfrak{B}_0(B, \lambda). \tag{3.3}$$

We recall the proof from [CV].

Proof. Consider the closed operator $Q := (D_1 - BA_1) + i(D_2 - BA_2)$ with domain $H_{per}^1(Q_L)$. Its adjoint is given by $Q^* := (D_1 - BA_1) - i(D_2 - BA_2)$ with domain $H_{per}^1(Q_L)$ and one has

$$\|(D - BA)u\|^2 = \|Qu\|^2 + B\|u\|^2 = \|Q^*u\|^2 - B\|u\|^2, \qquad u \in H_{per}^1(Q_L).$$

Hence $H_B^P(L) = Q^*Q + B$ and $QQ^* - Q^*Q = 2B$. By standard arguments using these commutation relations one computes the spectrum of $H_B^P(L)$ to consist of the eigenvalues $B(2k+1)$, $k \in \mathbb{N}_0$, with a common multiplicity, say m. To determine m we note that

$$N(\lambda, H_B^P(L)) = m\#\{k \in \mathbb{N}_0: \; B(2k+1) < \lambda\} \sim m\lambda/2B \quad \text{as} \quad \lambda \to \infty.$$

On the other hand, the Weyl-type asymptotics on the counting function holds true for the Dirichlet and the Neumann boundary conditions, and hence also for the periodic operator,

$$N(\lambda, H_B^P(L)) \sim \lambda L^2/4\pi \quad \text{as} \quad \lambda \to \infty.$$

Comparing the two asymptotics above one finds that $m = L^2 B/2\pi$. [3] $\square$

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[3]Alternatively, we may determine m using the Aharonov-Casher theorem. Indeed, the multiplicity m is the dimension of the kernel of the Pauli operator $(\sigma \cdot (D - BA))^2$ acting on the sections of a complex line bundle over the torus $(\mathbb{R}/L\mathbb{Z})^2$.

3.3. **Boundary conditions.** In this subsection we shall quantify the intuition that a change of the boundary conditions of a differential operator has only a relatively small effect on the overall eigenvalue distribution. We shall denote by $H_B^N(L)$ the operator $(D - BA)^2$ with (magnetic) Neumann boundary conditions in $\Omega = Q_L = (-L/2, L/2)^2$, that is the operator generated by the quadratic form $\|(D - BA)u\|^2$ with form domain $H^1(Q_L)$. We denote by $\|K\|_1 = \mathrm{tr}(K^*K)^{1/2}$ the trace norm of a trace class operator K.

A special case of a result by Nakamura [N] (who also allows for a variable magnetic field and an electric potential) is

Proposition 3.3. *Let $m \in \mathbb{N}$ and $B > 0$. Then there exists a constant $C_m(B) > 0$ such that for all $L \geq 1$*

$$\|(H_B^D(L) + I)^{-2m-1} - (H_B^N(L) + I)^{-2m-1}\|_1 \leq C_m(B)L. \tag{3.4}$$

3.4. **Proof of Proposition 3.1.** Throughout the proof, B will be fixed and, for the sake of simplicity, dropped from the notation. First note that since

$$N(\lambda, H^D(L')) \leq N(\lambda, H^D(L)) \leq N(\lambda, H^D(L''))$$

for $L' \leq L \leq L''$ it suffices to prove Proposition 3.1 only for $L \to \infty$ with the flux constraint (3.2), which we shall assume henceforth. One has $H^D(L) \geq H^P(L)$ and hence by the variational principle

$$N(\lambda, H^D(L)) \leq N(\lambda, H^P(L)).$$

In view of Proposition 3.2 this proves the upper bound in (3.1).

To prove the lower bound we write

$$N(\lambda, H^D(L)) = n((\lambda + 1)^{-3}, (H^D(L) + I)^{-3})$$

where $n(\kappa, K)$ denotes the number of singular values larger than κ of a compact operator K. Now by the Ky-Fan inequality [BS, Ch. 11 Sec. 1] for any $\varepsilon > 0$

$$n((\lambda + 1)^{-3}, (H^D(L) + I)^{-3})$$
$$\geq n((1 + \varepsilon)(\lambda + 1)^{-3}, (H^N(L) + I)^{-3})$$
$$\quad - n(\varepsilon(\lambda + 1)^{-3}, (H^N(L) + I)^{-3} - (H^D(L) + I)^{-3}).$$

We treat the two terms on the RHS separately. The second one can be estimated using Proposition 3.3 as follows,

$$n(\varepsilon(\lambda + 1)^{-3}, (H^N(L) + I)^{-3} - (H^D(L) + I)^{-3})$$
$$\leq n(\varepsilon(\lambda + 1)^{-3}, (H^N(L) + I)^{-3} - (H^D(L) + I)^{-3})$$
$$\leq \varepsilon^{-1}(\lambda + 1)^3 \|(H^N(L) + I)^{-3} - (H^D(L) + I)^{-3}\|_1$$
$$\leq \varepsilon^{-1}(\lambda + 1)^3 C_3(B)L.$$

On the other hand, writing $\lambda_\varepsilon := (1 + \varepsilon)^{-1/3}(\lambda + 1) - 1$ and applying Proposition 3.2 one finds that for $L^2 \in 2\pi B^{-1}\mathbb{N}$

$$n((1 + \varepsilon)(\lambda + 1)^{-3}, (H^N(L) + I)^{-3}) = N(\lambda_\varepsilon, H^N(L))$$
$$\geq N(\lambda_\varepsilon, H^P(L)) = L^2 \mathfrak{B}_0(B, \lambda_\varepsilon).$$

Noting that $\lambda_\varepsilon < \lambda$ and that $\mathfrak{B}_0(B, \lambda)$ is left-continuous in λ we see that for all sufficiently small $\varepsilon > 0$ one has

$$\mathfrak{B}_0(B, \lambda_\varepsilon) = \mathfrak{B}_0(B, \lambda).$$

Collecting all the estimates we find that as $L \to \infty$ with $L^2 \in 2\pi B^{-1}\mathbb{N}$

$$\liminf L^{-2}N(\lambda, H^D(L)) \geq \mathfrak{B}_0(B, \lambda).$$

This proves the lower bound in (3.1).

4. PROOF OF THE MAIN RESULTS

4.1. **Non-convex moments for tiling domains.** This subsection is devoted to the proof of Theorem 2.2. We assume that Ω is tiling, so we can write

$$\mathbb{R}^2 = \bigcup_{n \in \mathbb{Z}^2} \Omega_n \qquad \text{up to measure } 0$$

where $\Omega_0 = \Omega$ and all the Ω_n are disjoint and congruent to Ω. For $L > 0$ let $Q_L = (-L/2, L/2)^2$ and

$$J_L := \{n \in \mathbb{Z}^2 : \Omega_n \subset Q_L\}, \qquad \Omega^L := \text{int}\left(\text{clos}\bigcup_{n \in J_L} \Omega_n\right).$$

We note that

$$\lim_{L \to \infty} L^{-2}\#J_L = |\Omega|^{-1}. \tag{4.1}$$

Moreover, one has the operator inequalities

$$H_B^{Q_L} \leq H_B^{\Omega^L} \leq \sum_{n \in J_L} \oplus H_B^{\Omega_n}.$$

(The first inequality is, of course, understood in terms of the natural embedding $L_2(\Omega^L) \subset L_2(Q_L)$ by extension by zero.) Noting that all the $H_B^{\Omega_n}$ are unitarily equivalent we obtain from the variational principle that

$$N(\lambda, H_B^\Omega) \leq (\#J_L)^{-1}N(\lambda, H_B^{Q_L}).$$

The bound (2.6) follows now from (4.1) and Proposition 3.1 by letting L tend to infinity. This implies also the sharpness of (2.6). Indeed, by Proposition 3.1 for any $\varepsilon > 0$, $B > 0$ and $\lambda > 0$ there exists a cube Ω satisfying (2.8) for $\gamma = 0$.

To prove (2.7) we write, in the spirit of [AL],

$$\text{tr}(H_B^\Omega - \lambda)_-^\gamma = \gamma \int_0^\infty N(\lambda - \mu, H_B^\Omega)\mu^{\gamma-1}\,d\mu \tag{4.2}$$

and

$$\mathfrak{B}_\gamma(B, \lambda) = \gamma \int_0^\infty \mathfrak{B}_0(B, \lambda - \mu)\mu^{\gamma-1}\,d\mu. \tag{4.3}$$

Hence (2.7) follows from (2.6). Moreover, Proposition 3.1, the formulae (4.2), (4.3) and an easy approximation argument based on (2.6) imply that

$$\lim L^{-2}\text{tr}(H_B^D(L) - \lambda)_-^\gamma = \mathfrak{B}_\gamma(B, \lambda).$$

As before, this proves the sharpness of the estimate (2.7) and concludes the proof of Theorem 2.2.

4.2. **Convex moments for arbitary domains.** From now on we shall consider arbitrary, not necessarily tiling domains Ω. Our goal is to prove

Theorem 4.1. *Let $\Omega \subset \mathbb{R}^2$ be a domain of finite measure and let $\gamma \geq 1$. Then for all $B > 0$ and $\lambda > 0$,*

$$\operatorname{tr}(H_B^\Omega - \lambda)_-^\gamma \leq \mathfrak{B}_\gamma(B, \lambda)|\Omega|. \tag{4.4}$$

As we will explain after Corollary 4.4 this improves slightly the main result of [ELV].

Proof. In the case $\Omega = \mathbb{R}^2$ we write H_B instead of H_B^Ω. By the variational principle and the Berezin-Lieb inequality (see [B2], [L1] and also [LS], [L]), one has for any non-negative, convex function φ vanishing at infinity that

$$\operatorname{tr} \varphi(H_B^\Omega) \leq \operatorname{tr} \chi_\Omega \varphi(H_B).$$

Now, if $P_B^{(k)}$ denotes the spectral projection of H_B corresponding to the k-th Landau level,

$$\varphi(H_B) = \sum_{k \in \mathbb{N}_0} \varphi(B(2k+1)) P_B^{(k)}.$$

To evaluate the above trace we recall that the integral kernel of $P_B^{(k)}$ is constant on the diagonal (this follows from the translation invariance of the Landau Hamiltonian) and has the value

$$P_B^{(k)}(x, x) = \frac{B}{2\pi}.$$

(This is easily seen by diagonalizing H_B with the help of a harmonic oscillator, see also [F].) It follows that $\operatorname{tr} \chi_\Omega P_B^{(k)} = B|\Omega|/2\pi$. [4] This proves that

$$\operatorname{tr} \varphi(H_B^\Omega) \leq \frac{B|\Omega|}{2\pi} \sum_{k \in \mathbb{N}_0} \varphi(B(2k+1)).$$

Specializing to the case $\varphi(\mu) = (\mu - \lambda)_-^\gamma$, $\gamma \geq 1$, one obtains the estimate (4.4). $\qquad\square$

4.3. **Diamagnetic inequalities for the semi-classical symbol.** This subsection illustrates on a semi-classical level the effects that appear when passing from the 'magnetic symbol' $\mathfrak{B}_\gamma(B, \lambda)$ appearing in Theorem 2.2 to the 'non-magnetic symbol' $L_{\gamma,2}^{\mathrm{cl}} \lambda^{\gamma+1}$ appearing in Theorem 2.1. The convex case $\gamma \geq 1$ appears to be different from the non-convex case $0 < \gamma < 1$. We shall prove

Proposition 4.2. *Let $\gamma \geq 0$ and $B > 0$. Then*

$$\sup_{\lambda > 0} \frac{\mathfrak{B}_\gamma(B, \lambda)}{L_{\gamma,2}^{\mathrm{cl}} \lambda^{\gamma+1}} = \begin{cases} 2 & \text{if } \gamma = 0, \\ 2\left(\frac{\gamma}{\gamma+1}\right)^\gamma & \text{if } 0 < \gamma < 1, \\ 1 & \text{if } \gamma > 1. \end{cases}$$

Moreover, for $0 < \gamma < 1$ the supremum is attained for $\lambda = B(\gamma + 1)$ and for $\gamma = 0$ the supremum is attained in the limit $\lambda \to B+$.

We shall need the elementary

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[4]To justify this, identify the LHS as the square of the Hilbert-Schmidt norm of $\chi_\Omega P_B^{(k)}$ and use that $\int |P_B^{(k)}(x, y)|^2 \, dy = P_B^{(k)}(x, x)$ since $P_B^{(k)}$ is a projection.

Lemma 4.3. *Let $\sigma > \gamma \geq 0$ and $\mu > \lambda$. Then for all $E \geq 0$*

$$(E - \lambda)_-^\gamma \leq C(\gamma, \sigma)(\mu - \lambda)^{-\sigma+\gamma}(E - \mu)_-^\sigma$$

with $C(0, \sigma) := 1$ if $\gamma = 0$ and $C(\gamma, \sigma) := \sigma^{-\sigma}\gamma^\gamma(\sigma - \gamma)^{\sigma - \gamma}$ if $\sigma > \gamma > 0$.

For the proof of Lemma 4.3 one just has to maximize $(\lambda - E)^\gamma(\mu - \lambda)^{\sigma - \gamma}$ as function of λ on the interval (E, μ).

Proof of Proposition 4.2. By scaling, we may assume $B = 1$. First let $\gamma \geq 1$ and note that the function $\varphi(\mu) := (\lambda - \mu)_+^\gamma$ is convex. Then by the mean value property of convex functions

$$\varphi(2k + 1) \leq \frac{1}{2}\int_{2k}^{2k+2}\varphi(\mu)\,d\mu.$$

Summing over $k \in \mathbb{N}_0$ yields the assertion in the case $\gamma \geq 1$.

Now let $0 \leq \gamma < 1$. Lemma 4.3 with $\sigma = 1$ together with the inequality that we have already proved implies that for any $\mu > \lambda$

$$\mathfrak{B}_\gamma(1, \lambda) \leq C(\gamma, 1)(\mu - \lambda)^{-1+\gamma}\mathfrak{B}_1(1, \mu)$$

$$\leq C(\gamma, 1)L_{1,2}^{\mathrm{cl}}(\mu - \lambda)^{-1+\gamma}\mu^2$$

Applying the lemma again, i.e. optimizing in μ, yields the estimate

$$\mathrm{tr}(H_B^\Omega - \lambda)_-^\gamma \leq R_\gamma L_{\gamma,2}^{\mathrm{cl}}|\Omega|\lambda^2$$

where

$$R_\gamma = \frac{C(\gamma, 1)}{C(\gamma + 1, 2)}\frac{L_{1,2}^{\mathrm{cl}}}{L_{\gamma,2}^{\mathrm{cl}}} = 2\left(\frac{\gamma}{\gamma + 1}\right)^\gamma. \tag{4.5}$$

This proves the claimed upper bound on the supremum in the proposition. Choosing λ as stated shows that this upper bound is sharp. □

Combining Theorem 4.4 with Proposition 4.2 we obtain

Corollary 4.4. *Let $\Omega \subset \mathbb{R}^2$ be a domain of finite measure and let $\gamma \geq 1$. Then for all $B > 0$ and $\lambda > 0$,*

$$\mathrm{tr}(H_B^\Omega - \lambda)_-^\gamma \leq L_{\gamma,2}^{\mathrm{cl}}|\Omega|\lambda^{\gamma+1}. \tag{4.6}$$

Using an idea from [LW2] we now show that (4.6) implies the inequality

$$\sum_{j=1}^N \lambda_j(H_B^\Omega) \geq 2\pi|\Omega|^{-1}N^2 \tag{4.7}$$

from [ELV] for the eigenvalues $\lambda_j(H_B^\Omega)$ of H_B^Ω. For this, we recall the definition of the Legendre transform of a function $f : \mathbb{R}_+ \to \mathbb{R}$,

$$\tilde{f}(p) := \sup_{\lambda > 0}(p\lambda - f(\lambda)),$$

and note that the inequality $f \leq g$ for *convex* functions f, g is equivalent to the reverse inequality $\tilde{f} \geq \tilde{g}$ for their Legendre transforms. Hence an easy calculation shows that (4.6) with $\gamma = 1$ is equivalent to the inequality

$$(p - [p])\lambda_{[p]+1}(H_B^\Omega) + \sum_{j=1}^{[p]}\lambda_j(H_B^\Omega) \geq (4L_{1,2}^{\mathrm{cl}}|\Omega|)^{-1}p^2, \qquad p \geq 0,$$

where $[p]$ denotes the integer part of p. Choosing $p = N$ one obtains (4.7).

In passing we note that by the same argument the inquality (4.4) (which is stronger than (4.6)) is in the case $\gamma = 1$ equivalent to the inequality

$$(p - [p])\lambda_{[p]+1}(H_B^\Omega) + \sum_{j=1}^{[p]} \lambda_j(H_B^\Omega) \geq \frac{B^2}{2\pi}\left((\tilde{p} - [\tilde{p}])(2[\tilde{p}] + 1) + [\tilde{p}]^2\right), \qquad p \geq 0,$$

where we have set $\tilde{p} = 2\pi p/(B|\Omega|)$. Estimating the RHS from below by $B^2\tilde{p}^2/(2\pi)$ one obtains again (4.7).

4.4. **Non-convex moments for arbitrary domains.** In this subsection we shall prove Theorem 2.1. We deduce the inequalities (2.1) and (2.2) from Corollary 4.4 in the case $\gamma = 1$. The proof is analogous to that of Proposition 4.2. Indeed, Lemma 4.3 and (4.6) imply that for any $0 \leq \gamma < 1$ and for any $\mu > \lambda$,

$$\operatorname{tr}(H_B^\Omega - \lambda)_-^\gamma \leq C(\gamma, 1)(\mu - \lambda)^{-1+\gamma} \operatorname{tr}(H_B^\Omega - \mu)_-$$
$$\leq C(\gamma, 1)L_{1,2}^{cl}|\Omega|(\mu - \lambda)^{-1+\gamma}\mu^2.$$

Applying the lemma again, i.e. optimizing in μ, yields the estimate

$$\operatorname{tr}(H_B^\Omega - \lambda)_-^\gamma \leq R_\gamma L_{\gamma,2}^{cl}|\Omega|\lambda^2$$

with R_γ as in (4.5). This proves (2.1) and (2.2).

To prove sharpness of these bounds we note that if $0 < \gamma < 1$ and $\lambda_\gamma = \gamma + 1$ then

$$\mathfrak{B}_\gamma(B, B\lambda_\gamma) = R_\gamma L_{\gamma,2}^{cl}(B\lambda_\gamma)^{\gamma+1}.$$

Similarly, if $\gamma = 0$ one has

$$\lim_{\lambda \to 1+} \mathfrak{B}_0(B, B\lambda) = 2L_{0,2}^{cl}B.$$

Hence (2.8) implies that for any $\varepsilon > 0$, $0 \leq \gamma < 1$ and $B > 0$ there exists a cube Ω satisfying (2.3) with $\lambda = B\lambda_\gamma$. This concludes the proof of Theorem 2.1.

5. ADDITIONAL REMARKS

5.1. **The three-dimensional case.** The our proof of semi-classical inequalities for the two-dimensional Dirichlet problem with constant magnetic field is based on two observations. Firstly, it seems to be appropriate to estimate eigenvalue sums $\operatorname{tr}(H_B^\Omega - \lambda)_-^\gamma$ in terms of the respective average of the *magnetic* symbol $\mathfrak{B}_\gamma(B, \lambda)$. Indeed, the bound

$$\operatorname{tr}(H_B^\Omega - \lambda)_-^\gamma \leq \mathfrak{B}_\gamma(B, \lambda)|\Omega|,$$

which holds true for arbitrary Ω for $\gamma \geq 1$ and for tiling domains for $\gamma \geq 0$, is sharp, since the ratio

$$\frac{\operatorname{tr}(H_B^\Omega - \lambda)_-^\gamma}{\mathfrak{B}_\gamma(B, \lambda)|\Omega|}$$

can be made arbitrary close to 1 by a suitable choice of (large) Ω.

Secondly, the average of the *magnetic* symbol satisfies a sharp estimate by the standard non-magnetic phase space average from above

$$\mathfrak{B}_\gamma(B, \lambda) \leq L_{\gamma,2}^{cl}\lambda^{\gamma+1}.$$

for $\gamma \geq 1$ only. For $\gamma < 1$ this leads in conjunction with the asymptotic argument to the counterexamples stated above.

As we shall see in this subsection, in the *three-dimensional case* the asymptotic behavior of eigenvalue moments is still governed by the average of a suitable magnetic symbol. However, this average will not exceed the corresponding classical phase space average for all $\gamma \geq 1/2$. Therefore our approach produces counterexamples to inequalities with semi-classical constants only for $0 \leq \gamma < 1/2$. We shall discuss this below in more detail.

Let $\Omega \subset \mathbb{R}^3$ be a domain of finite measure and consider for $B > 0$ the self-adjoint operator

$$H_B^\Omega := (D - BA)^2 \qquad \text{in } L_2(\Omega)$$

with Dirichlet boundary conditions where now

$$A(x) := \frac{1}{2}(-x_2, x_1, 0)^T.$$

In the three-dimensional case the magnetic symbol is define as

$$\mathfrak{B}_\gamma^{(3)}(B, \lambda) := (2\pi)^{-1} \int_\mathbb{R} \mathfrak{B}_\gamma(B, \lambda - |\xi|^2)\, d\xi$$

$$= \frac{\Gamma(\gamma + 1)}{\Gamma(\gamma + 3/2)} \frac{B}{4\pi^{3/2}} \sum_{k \in \mathbb{N}_0} (\lambda - B(2k+1))_+^{\gamma + 1/2}.$$

Similarly as in Subsection 4.2 one proves that

$$\text{tr}(H_B^\Omega - \lambda)_-^\gamma \leq \mathfrak{B}_\gamma^{(3)}(B, \lambda)|\Omega|, \qquad \gamma \geq 1. \tag{5.1}$$

Put

$$L_{\gamma,3}^{\text{cl}} := (2\pi)^{-3} \int_{\{|\xi| < 1\}} (1 - |\xi|^2)^\gamma\, d\xi = \frac{1}{8\pi^{3/2}} \frac{\Gamma(\gamma + 1)}{\Gamma(\gamma + 5/2)}$$

By the same argument as in Proposition 4.2 one has

$$B_\gamma^{(3)}(B, \lambda) \leq L_{\gamma,3}^{\text{cl}} \lambda^{\gamma + 3/2}, \qquad \gamma \geq 1/2, \tag{5.2}$$

and hence

$$\text{tr}(H_B^\Omega - \lambda)_-^\gamma \leq L_{\gamma,3}^{\text{cl}} \lambda^{\gamma + 3/2}|\Omega|, \qquad \gamma \geq 1.$$

Again the quantity $\mathfrak{B}_0^{(3)}(B, \lambda)$ arises as the density of states. More precisely, if $Q_L := (-L/2, L/2)^3$ then a three-dimensional version of Proposition 3.3 allows to prove that

$$\lim_{L \to \infty} L^{-3} N(\lambda, H_B^{Q_L}) = \mathfrak{B}_0^{(3)}(B, \lambda). \tag{5.3}$$

This implies as in the two-dimensional case

Theorem 5.1. *Let $\Omega \subset \mathbb{R}^3$ be a tiling domain of finite measure. Then for all $B > 0$ and $\lambda > 0$*

$$N(\lambda, H_B^\Omega) \leq \mathfrak{B}_0^{(3)}(B, \lambda)|\Omega| \tag{5.4}$$

and

$$\text{tr}(H_B^\Omega - \lambda)_-^\gamma \leq \mathfrak{B}_\gamma^{(3)}(B, \lambda)|\Omega|, \qquad 0 < \gamma < 1, \tag{5.5}$$

and these estimates cannot be improved. More precisely, for any $0 \leq \gamma < 1$, $\varepsilon > 0$, $B > 0$, $\lambda > 0$ there exists a cube Ω such that

$$\text{tr}(H_B^\Omega - \lambda)_-^\gamma \geq (1 - \varepsilon)\mathfrak{B}_\gamma^{(3)}(B, \lambda)|\Omega|. \tag{5.6}$$

The estimates (5.4), (5.5) and Proposition 4.2 imply that for tiling domains Ω and for $0 \leq \gamma < 1/2$,

$$\text{tr}(H_B^\Omega - \lambda)_-^\gamma \leq R_{\gamma+1/2} L_{\gamma,3}^{\text{cl}} \lambda^{\gamma+3/2} \qquad (5.7)$$

with R_γ as in Theorem 2.1. Moreover, the asymptotics (5.3) imply that this constant can not be replaced by a smaller one. However, in contrast to the two-dimensional case we do not know whether the constant in this estimate has to be further increased if non-tiling domains are considered.

On the other hand, (5.5) and (5.2) imply that for tiling domains Ω and for $\gamma \geq 1/2$,

$$\text{tr}(H_B^\Omega - \lambda)_-^\gamma \leq L_{\gamma,3}^{\text{cl}} \lambda^{\gamma+3/2}. \qquad (5.8)$$

We do not know whether the constant in this estimate has to be increased if $1/2 \leq \gamma < 1$ and if non-tiling domains are considered.

The method of Appendix A allows to deduce from (5.1) (probably non-sharp) estimates on $\text{tr}(H_B^\Omega - \lambda)_-^\gamma$ for $0 \leq \gamma < 1$ and arbitrary Ω. We omit the details.

Another remark concerns domains with product structure.

Proposition 5.2. *Let $\omega \subset \mathbb{R}^2$ be a domain of finite measure, $I \subset \mathbb{R}$ a bounded open interval and $\Omega := \omega \times I$, and let $\gamma \geq 1/2$. Then for all $B > 0$ and $\lambda > 0$,*

$$\text{tr}(H_B^\Omega - \lambda)_-^\gamma \leq \mathfrak{B}_\gamma^{(3)}(B, \lambda) |\Omega|. \qquad (5.9)$$

It follows from (5.2) that for domains of this form and for $\gamma \geq 1/2$ one has also (5.8).

Proof. We follow Laptev's lifting idea [L]. By separation of variables we can write

$$\text{tr}(H_B^\Omega - \lambda)_-^\gamma = \sum_{n \in \mathbb{N}} \text{tr}\left(H_B^\omega + \left(\frac{\pi n}{|I|}\right)^2 - \Lambda\right)_-^\gamma.$$

Pólya's estimate on an interval states that

$$\sum_{n \in \mathbb{N}} \left(\left(\frac{\pi n}{|I|}\right)^2 - E\right)_-^\gamma \leq L_{\gamma,1}^{\text{cl}} |I| E^{\gamma+1/2}$$

where

$$L_{\gamma,1}^{\text{cl}} := \frac{1}{2\sqrt{\pi}} \frac{\Gamma(\gamma+1)}{\Gamma(\gamma+3/2)}.$$

Hence

$$\text{tr}(H_B^\Omega - \lambda)_-^\gamma \leq L_{\gamma,1}^{\text{cl}} |I| \, \text{tr}(H_B^\omega - \lambda)_-^{\gamma+1/2}.$$

Applying Theorem 4.4 and noting that

$$L_{\gamma,1}^{\text{cl}} \mathfrak{B}_{\gamma+1/2}(B, \lambda) = \mathfrak{B}_\gamma^{(3)}(B, \lambda)$$

completes the proof. $\qquad \square$

5.2. **The role of the integrated density of states.** Our reasoning in Subsection 4.1 has shown that the important idea in Pólya's proof is not the high energy limit, but the large domain limit. (In the non-magnetic case these two limits are equivalent by scaling.) The large domain limit corresponds to the passage to the density of states.

More generally, one can prove the following. For the sake of simplicity we return to the two-dimensional case. Assume that $\Omega \subset \mathbb{R}^2$ is a tiling domain and write

$$\mathbb{R}^2 = \bigcup_{n \in \mathbb{Z}^2} \Omega_n \qquad \text{up to measure } 0.$$

Here $\Omega_0 = \Omega$ and all the Ω_n are disjoint with $\Omega_n = G_n \Omega$ for G_n a composition of a translation and a rotation. Let V and A be a sufficiently regular real-valued function, respectively vectorfield on Ω and consider the self-adjoint operator $H^\Omega := (D - A)^2 + V$ with Dirichlet boundary conditions in $L_2(\Omega)$.

We extend V and A to the whole plane in such a way that $V(x) = V(G_n^{-1}x)$ and $\operatorname{curl} A(x) = \operatorname{curl} A(G_n^{-1}x)$ for $x \in \Omega_n$. This allows to define a self-adjoint operator $H := (D - A)^2 + V$ in $L_2(\mathbb{R}^2)$. Our main assumption is that this operator possesses an integrated density of states at a certain $\lambda \in \mathbb{R}$, i.e., there exists a number $n(\lambda) \geq 0$ such that

$$\lim_{L \to \infty} L^{-2} N(\lambda, H^{Q_L}) = n(\lambda). \tag{5.10}$$

Here as before, $Q_L = (-L/2, L/2)$. Under this assumption one has for this given value of λ the Pólya estimate

$$N(\lambda, H^\Omega) \leq n(\lambda)|\Omega|.$$

This is proved in the same way as Theorem 2.2.

A special case is when the G_n are translations. If the flux of $\operatorname{curl} A$ through Ω vanishes, then A can be chosen periodic and one can apply Floquet theory. In this case it is well-known that the limit (5.10) exists for any λ and defines a non-negative, increasing and left-continuous function n on $\mathbb{R}$. A more general case is that of G_n's which correspond to almost-periodic tilings. The existence of the limit (5.10) in the almost-periodic case under broad conditions on the coefficients has been proved, e.g., in [S].

APPENDIX A. THE CASE OF AN ARBITRARY MAGNETIC FIELD

In this section we consider an *arbitrary* magnetic field $A \in L_{2,\mathrm{loc}}(\overline{\Omega})$ with $\Omega \subset \mathbb{R}^d$ in any dimension $d \geq 2$ and define $H_\Omega(A) = (D - A)^2$ on Ω with Dirichlet boundary conditions. We shall prove the estimate

$$\operatorname{tr}(H_\Omega(A) - \lambda)_-^\gamma \leq \rho_{\gamma,d} L_{\gamma,d}^{\mathrm{cl}} \lambda^{\gamma + d/2} |\Omega|, \qquad 0 \leq \gamma < 3/2. \tag{A.1}$$

Here

$$\rho_{\gamma,d} := \frac{\Gamma(5/2)\,\Gamma(\gamma + d/2 + 1)}{\Gamma((5+d)/2)\,\Gamma(\gamma+1)} 3^{-3/2}(3+d)^{(3+d)/2}(2\gamma)^\gamma (2\gamma + d)^{-\gamma - d/2}$$

and

$$L_{\gamma,d}^{\mathrm{cl}} = \frac{\Gamma(\gamma+1)}{2^d \pi^{d/2} \Gamma(\gamma + \frac{d}{2} + 1)}.$$

Note that for $d = 2$ the constant $\rho_{\gamma,d}$ equals

$$\rho_{\gamma,2} := (5/3)^{3/2}(\gamma/(\gamma+1))^\gamma,$$

and it follows from our main result that this is off at most by a factor $(5/3)^{3/2}/2 \approx 1.0758$.

To prove (A.1) we recall the sharp Lieb-Thirring bound on the negative spectrum of a magnetic Schrödinger operator $H_{\mathbb{R}^d}(A, V) = (D - A)^2 - V$ in $\mathbb{R}^d$ from [LW1],

$$\operatorname{tr}(H_{\mathbb{R}^d}(A, V))_-^{3/2} \leq L_{3/2,d}^{\mathrm{cl}} \int_{\mathbb{R}^d} V(x)_+^{(3+d)/2} \, dx \, .$$

Here we extend the given magnetic vector potential A on $\overline{\Omega}$ by 0 to $\mathbb{R}^d$. Since the negative eigenvalues of $H_\Omega(A) - \mu$ are not below those of $H_{\mathbb{R}^d}(A, V)$ with $V(x) := \mu$ for $x \in \Omega$ and $V(x) := 0$ for $x \in \mathbb{R} \setminus \Omega$, we find

$$\operatorname{tr}(H_\Omega(A) - \mu)_-^{3/2} \leq \operatorname{tr}(H_{\mathbb{R}^d}(A, V))_-^{3/2} \leq L_{3/2,d}^{\mathrm{cl}} |\Omega| \mu^{(3+d)/2} \, .$$

Lemma 4.3 with $\sigma = 3/2$ shows now that for $0 \leq \gamma < 3/2$

$$\operatorname{tr}(H_\Omega(A) - \lambda)_-^\gamma \leq C(\gamma, 3/2)(\mu - \lambda)^{-3/2+\gamma} \operatorname{tr}(H_\Omega(A) - \mu)_-^{3/2}$$
$$\leq C(\gamma, 3/2) L_{3/2,d}^{\mathrm{cl}} |\Omega| (\mu - \lambda)^{-3/2+\gamma} \mu^{(3+d)/2}$$

for any $\mu > \lambda$. Again by this lemma, i.e., optimizing in μ, we get (A.1) with excess factor

$$\rho_{\gamma,d} = \frac{L_{3/2,d}^{\mathrm{cl}}}{L_{\gamma,d}^{\mathrm{cl}}} \frac{C(\gamma, 3/2)}{C(3/2 - \gamma, (3+d)/2)} = \frac{L_{3/2,d}^{\mathrm{cl}}}{L_{\gamma,d}^{\mathrm{cl}}} \frac{(3+d)^{(3+d)/2}}{3^{3/2}} \frac{(2\gamma)^\gamma}{(2\gamma + d)^{\gamma+d/2}} \, .$$

Recalling the definition of $L_{\gamma,d}^{\mathrm{cl}}$ we obtain the claimed statement.

Besides the case of a homogeneous magnetic field also the case of a δ-like magnetic field (Aharonov-Bohm field) has received particular attention. In [FH] the above value of the excess factor $\rho_{\gamma,2}$ could be slightly improved for this case, but it is still unknown whether or not this factor can be chosen one for $0 \leq \gamma < 3/2$.

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