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Abstract

We improve the Berezin-Li-Yau inequality in dimension two by adding a positive correction term to its right-hand side. It is also shown that the asymptotical behaviour of the correction term is almost optimal. This improves a previous result by Melas, [10].

1 Introduction

Let Ω be an open bounded set in $\mathbb{R}^d$ and let $-\Delta$ be the Dirichlet Laplacian on Ω. We denote by λ_j the non-decreasing sequence of eigenvalues of $-\Delta$. The main object of our interest in this paper is the lower bound

$$\sum_{j=1}^{k} \lambda_j \geq \frac{d\, C_d}{d+2}\, V^{-\frac{2}{d}}\, k^{\frac{d+2}{d}}, \qquad C_d = (2\pi)^2 \omega_d^{-2/d}, \tag{1}$$

where V stands for the volume of Ω and ω_d denotes the volume of the unit ball in $\mathbb{R}^d$. Inequality (1) was proved in [8], and is commonly known as the Li-Yau inequality. In [7] it was pointed out that (1) is in fact the Legendre transformation of an earlier result by Berezin, see [1]. Note also that the Li-Yau inequality yields an individual lower bound on λ_k in the form

$$\lambda_k \geq \frac{d\, C_d}{d+2}\, V^{-\frac{2}{d}}\, k^{\frac{2}{d}}\, . \tag{2}$$

For further estimates on λ_k see [12, 5, 6, 7].

It is important to compare the lower bound (1) with the asymptotical behaviour of the sum on the left-hand side, which reads as follows:

$$\sum_{j=1}^{k} \lambda_j \;=\; \frac{d\, C_d}{d+2}\, V^{-\frac{2}{d}}\, k^{\frac{d+2}{d}} + \tilde{C}_d\, \frac{|\partial\Omega|}{V^{1+\frac{1}{d}}}\, k^{1+\frac{1}{d}} + o\left(k^{1+\frac{1}{d}}\right) \quad \text{as } k \to \infty \tag{3}$$

with

$$\tilde{C}_d = \frac{\sqrt{\pi}\, \Gamma\left(2+\frac{d}{2}\right)^{1+\frac{1}{d}}}{(d+1)\, \Gamma\left(\frac{3}{2}+\frac{d}{2}\right) \Gamma(2)^{\frac{1}{d}}}\, .$$

The first term in the asymptotics (3) is due to Weyl, see [15]. The second term in (3) was established, under suitable conditions on Ω, in [3, 4, 11], see also [13, Chap. 1.6].

It follows from (3) that the constant in (1) cannot be improved. On the other hand, since the second asymptotical term is positive, it is natural to ask whether one might improve (1) by adding an additional positive term of lower order in k to the right-hand side. The first step towards this goal was done by Melas, [10], who showed that the inequality

$$\sum_{j=1}^{k} \lambda_j \geq \frac{d\, C_d}{d+2}\, V^{-\frac{2}{d}}\, k^{\frac{d+2}{d}} + M_d\, \frac{V}{I}\, k, \qquad I = \min_{a \in \mathbb{R}^2} \int_{\Omega} |x-a|^2\, dx \tag{4}$$

holds true with a factor M_d which depends only on the dimension. Note however, that the additional term in the Melas bound does not have the order in k predicted by the second term in (3). Moreover, the coefficient of the second term in (3) reflects explicitly the effect of the boundary of Ω, whereas such a dependence is not seen in the coefficient V/I of (4).

Our aim is to improve (1) and (4) by adding a positive contribution which reflects the nature of the second term in the asymptotic (3). Recently, one of the authors, see [14], proved an analogous improved estimate on the quantity

$$\sum_{k} (\Lambda - \lambda_k)_+^{\sigma}, \qquad \sigma \geq 3/2$$

with a remainder term which agrees, up to a constant, with the corresponding second term in the asymptotics of $\sum_k (\Lambda - \lambda_k)_+^\sigma$ as $\Lambda \to \infty$. The proof given in [14] relies on sharp Lieb-Thirring inequalities for operator valued potentials and works only for $\sigma \geq 3/2$. Since the estimates treated in present paper concern the value $\sigma = 1$, the method of [14] cannot be carried over to this case. We will therefore develop a different approach. We wish to point out [2] (and references therein), where an improvement of (1) has been established for the discrete Laplacian. In particular, the authors make use of a more detailed analysis of the approximation of characteristic functions by low energy eigenvalues of the discrete Laplace operator. This is also the key element of this paper.

The main idea of our strategy is explained in section 2.3. It is closely related to a modified proof of inequality (1), which we briefly describe in section 2.1, see also [9, Chap. 12]. The main results which represent improved Li-Yau inequalities in case $d = 2$ are formulated in section 3. Since our proof includes many technical results concerning the geometry of the boundary of Ω, we will first give its exposition for polygons, section 4. Finally, in section 5 we extend the proof to general domains.

To keep the presentation as short and stringed as possible, we have decided to restrict ourselves to the case $d = 2$ throughout the paper.

2 Preliminaries

Following notation will be employed in the text. By $\Theta(\cdot) : \mathbb{R} \to \mathbb{R}$ we denote the Heaviside function defined by $\Theta(x) = 0$ if $x \leq 0$ and $\Theta(x) = 1$ if $x > 0$. For given $t > 0$ we denote by N_t the number of eigenvalues of the Dirichlet-Laplacian in Ω less than or equal to t. Finally, we will write $[s]$ for the integer part of a real number s.

2.1 Li-Yau bound revisited

Let ψ_j be the sequence of the normalised eigenfunctions of $-\Delta$ in Ω, i.e.

$$-\Delta\, \psi_j = \lambda_j\, \psi_j \quad \text{in } \Omega, \qquad \psi_j = 0 \quad \text{on } \partial\Omega, \qquad \int_\Omega |\psi_j|^2 = 1\,. \tag{5}$$

In order to explain the idea which will lead to an improvement of the results by Li-Yau and Melas, it is illustrative to see how to obtain inequalities (1) and (4) for $d = 2$ (the same arguments apply to higher dimensions as well). Following [1, 10] we extend the eigenfunctions ψ_j continuously by zero to the whole of $\mathbb{R}^2$ so that they remain in $H^1(\mathbb{R}^2)$. Next introduce the following functions:

$$f_j(\xi) = (2\pi)^{-1} \int_\Omega e^{-ix\cdot\xi}\, \psi_j(x)\, dx, \qquad F(\xi) := \sum_{j=1}^k |f_j(\xi)|^2. \tag{6}$$

Since $\{\psi_j\}$ is an orthonormal basis of $L^2(\Omega)$, the Parseval identity implies that

$$F(\xi) = \sum_{j=1}^k |f_j(\xi)|^2 \leq \sum_{j=1}^\infty |f_j(\xi)|^2 = (2\pi)^{-2} \int_\Omega \left|e^{-ix\cdot\xi}\right|^2 dx = (2\pi)^{-2}\, V \tag{7}$$

holds for any $\xi \in \mathbb{R}^2$. Next we denote by $F^*(|\xi|)$ the decreasing radial rearrangement of F. Using the well-known properties of the radial rearrangement we find

$$\int_{\mathbb{R}^2} F^*(|\xi|)\, d\xi = \int_{\mathbb{R}^2} F(\xi)\, d\xi = k \tag{8}$$

and

$$\sum_{j=1}^k \lambda_j = \int_{\mathbb{R}^2} |\xi|^2\, F(\xi)\, d\xi \geq \int_{\mathbb{R}^2} |\xi|^2\, F^*(|\xi|)\, d\xi. \tag{9}$$

To find a lower bound on $\sum_{j=1}^k \lambda_j$ it thus suffices to find the minimiser of the functional $\int_{\mathbb{R}^2} |\xi|^2 F^*(|\xi|)\, d\xi$ under the conditions (7) and (8).

The result of Li and Yau can be proved using the fact, [9, Chap. 12], that this functional is minimised by the function

$$\Phi_{LY}(|\xi|) = \begin{cases} (2\pi)^{-2}\, V & 0 \le |\xi| \le r_k, \\ \\ 0 & r_k < |\xi|, \end{cases} \tag{10}$$

where r_k is given by the condition

$$(2\pi)^{-1} V \int_0^{r_k} |\xi|\, d|\xi| = k \;\Rightarrow\; r_k = \sqrt{\frac{4\pi\, k}{V}}\, .$$

Inserting (10) into (9) we obtain inequality (1) for $d = 2$.

2.2 Melas' improvement revisited

Melas observed in [10] that the lower bound on the right-hand side of (9) can be improved, if one takes into account that the follwing additional regularity condition on F^* must hold

$$|(F^*)'| \le 2(2\pi)^{-2} \sqrt{V\, I} =: L\, . \tag{11}$$

It can be easily verified that, depending on the value of k, the corresponding minimiser Φ_M of the functional (9) then has the following form:

$$\text{for } k \ge \frac{V^2}{48\pi I} \qquad \Phi_M(|\xi|) = \begin{cases} (2\pi)^{-2}\, V & 0 \le |\xi| \le s_k, \\ (2\pi)^{-2}\, V - (|\xi| - s_k)\, L & s_k < |\xi| \le t_k, \\ 0 & t_k < |\xi|, \end{cases} \tag{12}$$

where the points s_k and t_k are uniquely determined by

$$2\pi \int_{\mathbb{R}_+} \Phi_M(|\xi|)\, |\xi|\, d|\xi| = k, \quad t_k = s_k + \frac{V}{4\pi^2 L}\, ,$$

see Figure 1, and

$$\text{for } k < \frac{V^2}{48\pi I} \qquad \Phi_M(|\xi|) = \left(\left(\frac{3kL^2}{\pi} \right)^{1/3} - L|\xi| \right)_+ . \tag{13}$$

Using this minimiser we obtain the lower bound

$$\sum_{j=1}^k \lambda_j \ge \frac{2\pi}{V}\, k^2 + \frac{1}{32}\, \frac{V}{I}\, k \qquad \text{if} \quad k \ge \frac{V^2}{48\pi I} \tag{14}$$

and

$$\sum_{j=1}^k \lambda_j \ge \frac{2\pi}{V}\, k^2 + \left(1 - 10 \cdot 2^{-\frac{5}{3}} 3^{-\frac{4}{3}}\right) \frac{3}{10} \left(\frac{2}{\pi}\right)^{\frac{2}{3}} L^{-\frac{2}{3}}\, k^{\frac{5}{3}} \qquad \text{if} \quad k < \frac{V^2}{48\pi I}. \tag{15}$$

Now let $a \in \mathbb{R}^2$ be such that $I = \int_\Omega |x - a|^2\, dx$ and let B_a be the disc centred in a and with the volume V. It is then straightforward to verify that

$$I \ge I(B_a) = \frac{V^2}{2\pi}\, .$$

Using this inequality and the fact that $k \ge 1$ we deduce from (14) and (15) the uniform estimate

$$\sum_{j=1}^k \lambda_j \ge \frac{2\pi}{V}\, k^2 + \frac{1}{32}\, \frac{V}{I}\, k \qquad \forall\, k \in \mathbb{N}\, . \tag{16}$$

2.3 The new correction term

Our main observation is that the crucial reservoir for improvements of (1) does not lie in the regularity of F^*, but in a more detailed analysis and improvement of the condition (7). Indeed, since

$$F(\xi) = \sum_{j=1}^{k} |f_j(\xi)|^2 = \frac{V}{4\pi^2} - \sum_{j=k+1}^{\infty} |f_j(\xi)|^2, \tag{17}$$

any estimate from below on $\sum_{j=k+1}^{\infty} |f_j(\xi)|^2$ will automatically lead to a sharper upper bound on F and therefore to an additional term in the Li-Yau inequality.

Moreover, the last term in (17) cannot go to zero arbitrarily fast as k goes to infinity. This follows from the fact that $|e^{-ix\cdot\xi}| = 1$ everywhere in Ω, which means that the Fourier coefficients $f_j(\xi)$ of $e^{-ix\cdot\xi}$ with respect to the basis $\{\psi_j\}$ cannot decay too fast in j (each ψ_j vanishes on $\partial\Omega$). In particular, the sequence $\{f_j(\xi)\}$ is not in ℓ^1. Another way to see this is to realize that the Fourier series $\sum_j f_j(\xi)\,\psi_j(\cdot)$ of continuous functions approximates, in $L^2(\mathbb{R}^2)$, the function $e^{-ix\cdot\xi}\chi_\Omega$, which has a discontinuity on $\partial\Omega$. Thus the decay properties of $\sum_{j=k+1}^{\infty} |f_j(\xi)|^2$ and consequently the additional term in Li-Yau inequality should reflect the effect of the boundary of Ω.

The main technical difficulty is to quantify this strategy into a uniform lower bound on $\sum_{j=k+1}^{\infty} |f_j(\xi)|^2$. In particular, if we can prove an estimate of the form

$$\sum_{j=k+1}^{\infty} |f_j(\xi)|^2 \geq \varepsilon\, k^{\delta} \qquad \forall \xi \in \mathbb{R}^2, \tag{18}$$

where ε and δ are positive, then the corresponding minimiser of (9) satisfying conditions (8) and (17) reads

$$\Phi(|\xi|) = \begin{cases} V/4\pi^2 - \varepsilon k^{-\delta} & 0 \leq |\xi| \leq \tau_k, \\ 0 & \tau_k < |\xi|, \end{cases} \tag{19}$$

see Figure 1. Here τ_k is defined by the condition

$$2\pi \int_{\mathbb{R}_+} \Phi(|\xi|)\, |\xi|\, d|\xi| = k.$$

A direct calculation then shows that there exists a positive coefficient $A(\varepsilon,\delta)$ such that

$$\sum_{j=1}^{k} \lambda_j \geq 2\pi \int_{\mathbb{R}_+} \Phi(|\xi|)\, |\xi|^3\, d|\xi| = \frac{2\pi}{V}\, k^2 + A(\varepsilon,\delta)\, k^{2-\delta}. \tag{20}$$

The asymptotic formula (3) implies that $\delta \geq 1/2$. For $\delta < 1$ we obtain an improvement of the Melas bound.

Let us finally mention that a similar effect of the boundary on the sum of the eigenvalues in the case of the discrete Laplace operator was already observed in [2].

3 Main results

We will state and prove the results for the case of polygons and general domains separately.

3.1 Case 1: Polygons

For a given polygon Ω we denote by p_j, $j = 1, \dots, n$ the $j-$th side of Ω. Moreover, we denote by d_j the distance between the middle third of p_j to $\partial\Omega \setminus p_j$. We can now formulate our first result.

Theorem 1 (Lower bound for polygons)**.** *Let Ω be a polygon with n sides. Let l_j be the length of the $j-th$ side of Ω. Then for any $k \in \mathbb{N}$ and any $\alpha \in [0,1]$ we have*

$$\sum_{j=1}^{k} \lambda_j \,\geq\, \frac{2\pi}{V}\, k^2 + 4\alpha\, c_3\, k^{\frac{3}{2}-\varepsilon(k)}\, V^{-\frac{3}{2}} \sum_{j=1}^{n} l_j\, \Theta\left(k - \frac{9V}{2\pi\, d_j^2}\right) + (1-\alpha)\, \frac{V}{32\, I}\, k, \tag{21}$$

where

$$\varepsilon(k) = \frac{2}{\sqrt{\log_2(2\pi k/c_1)}} \tag{22}$$

and

$$c_1 = \sqrt{\frac{3\pi}{14}}\; 10^{-11}\,, \qquad c_3 = \frac{2^{-3}}{9\sqrt{2}\,36}\,(2\pi)^{\frac{5}{4}} c_1^{1/4}\,. \tag{23}$$

3.2 Case 2: General domains

For general open domains $\Omega \subset \mathbb{R}^2$ we will have to impose certain assumptions on the regularity of $\partial\Omega$.

Assumption A. There exist C^2- smooth parts $\Gamma_j \subset \partial\Omega$ at the boundary of Ω. Let $j = 1, \dots, m$.

To be able to state the result for general domains we need some definitions. Let A_j, B_j be the end points of Γ_j and let $\{x_1^j(s), x_2^j(s)\}$ be the parametrisation of Γ_j with its length s. We define

$$\varkappa_j = \max_s\, |\varkappa_j(s)|,$$

where $\varkappa_j(s)$ denotes the curvature at the point $s \in \Gamma_j$. Moreover, let $L(\Gamma_j)$ be length of Γ_j. Now we divide Γ_j into several pieces of the same length. The tiling of Γ_j will be done in two different ways depending on the values of $\varkappa_j$ and $L(\Gamma_j)$:

(i) If

$$L(\Gamma_j) \leq \frac{3\pi}{8\,\varkappa_j}, \tag{24}$$

then we divide Γ_j into three parts of the same length and denote by d_j the distance of the middle part to $\partial\Omega_j \setminus \Gamma_j$.

(ii) If

$$L(\Gamma_j) > \frac{3\pi}{8\,\varkappa_j}, \tag{25}$$

then we divide Γ_j into $n_j = [8L(\Gamma_j)\varkappa_j/\pi]$ parts of the same length. Let a_i^j, a_{i+1}^j be the end points of the $i-$th part with $a_0^j = A_j$, $a_{n_j}^j = B_j$ and let

$$\delta_i^j = \operatorname{dist}\Big((a_i^j, a_{i+1}^j),\, \partial\Omega \setminus \{(a_{i-1}^j, a_i^j) \cup (a_i^j, a_{i+1}^j) \cup (a_{i+1}^j, a_{i+2}^j)\}\Big)$$

Then we define

$$d_j = \min_{1 \leq i \leq n-2} \delta_i^j\,.$$

Finally, we will need

$$k_j := \frac{V}{2\pi}\, \max\left\{\Lambda_3(j),\, \frac{9}{d_j^2},\, \frac{128\,\varkappa_j^2}{\pi^2}\,,\, \frac{6\varkappa_j}{d_j}\right\}\,,$$

where

$$\Lambda_3(j) := \max\left\{9 \cdot 2^{10} \max_j\, \varkappa_j^2,\, 2^{2^6}\, c_1 V^{-1}\,,\, c_1^{-1}\, 2^{22}\, 6^8\, \varkappa_j^4\, V\right\}\,.$$

Now we are in position to state the result for general domains.

Theorem 2 (Lower bound for general domains)**.** *Let Ω satisfy Assumption A. Then for any $k \in \mathbb{N}$ and any $\alpha \in [0,1]$ we have*

$$\sum_{j=1}^{k} \lambda_j \;\geq\; \frac{2\pi}{V}\, k^2 + \alpha\, c_3\, k^{\frac{3}{2}-\varepsilon(k)}\, V^{-3/2} \sum_{j=1}^{m} L(\Gamma_j)\, \Theta(k-k_j) + (1-\alpha)\, \frac{V}{32\, I}\, k. \tag{26}$$

3.3 Remarks

Remark 1. Note that the coefficient of the second term on the right hand side of (26) is very similar to the coefficient of the second term in the Weyl asymptotics (3). In particular, it reflects the expected effect of the boundary of Ω. On the other hand, this boundary term becomes visible only for k large enough. However, we would like to point out that the second term cannot be simply proportional to $\sum_j L(\Gamma_j)$. Indeed, one can make $\sum_j L(\Gamma_j)$ arbitrarily large by "folding" the boundary $\partial\Omega$ while keeping the eigenvalues λ_j with $j \leq k$ almost unchanged. This shows that the condition $k \geq k_j$ cannot be removed.

Remark 2. It would be natural to try to deduce the result for general domains from the result for polygons by approximating Ω by polygons. However, the contribution of the second term would in general disappear in such a procedure. To see this it suffices to take an open ball in $\mathbb{R}^2$ as Ω. Then the coefficients k_j would go to infinity when approximating Ω by a sequence of polygons. Therefore a different strategy will be needed in the proof of Theorem 2.

Remark 3. As for the constants in (26), notice that $\varepsilon(k) \ll 1$ for all k and that $\varepsilon(k) \to 0$ as $k \to \infty$. On the other hand, the values of k_j are in general very large. Nevertheless, the correction term on the right-hand side of (26) can be optimised according to the geometry of Ω by choosing the boundary segments Γ_j in an appropriate way.

4 Proof for polygons

The proofs of our main results rely on a careful exploitation of the ideas described in section 2.3.

Let $\lambda = \lambda_k$ and let $\mathcal{L}_k := \left\{ \sum_{i=1}^{k} c_i \psi_i : \sum_{i=1}^{k} |c_i|^2 \leq V \right\}$. Since $e^{i\xi\cdot x}$ belongs to $L^2(\Omega)$ for each $\xi \in \mathbb{R}^2$, it follows that

$$\inf_{\psi\in\mathcal{L}_k} \left\| e^{i\xi\cdot x} - \psi \right\|^2_{L^2(\Omega)} \;\leq\; \| e^{i\xi\cdot x} - \sum_{i=1}^{k} \left(e^{i\xi\cdot x},\, \psi_i \right)_{L^2(\Omega)} \psi_i \|^2_{L^2(\Omega)} = V - 4\pi^2\, F(\xi)\,, \tag{27}$$

where

$$\sum_{i=1}^{k} \left| \left(e^{i\xi\cdot x},\, \psi_i \right)_{L^2(\Omega)} \right|^2 = 4\pi^2 F(\xi) \leq V.$$

Equation (27) yields the estimate

$$F(\xi) \leq (4\pi^2)^{-1} \left(V - \inf_{\psi\in\mathcal{L}_k} \left\| e^{i\xi\cdot x} - \psi \right\|^2_{L^2(\Omega)} \right).$$

In view of the arguments given in section 2.3, to prove (21) it thus suffices to show that

$$\left\| e^{i\xi\cdot x} - \psi \right\|^2_{L^2(\Omega)} \;\geq\; \text{const}\, k^{-\frac{1}{2}-\varepsilon(k)} \qquad \forall \psi \in \mathcal{L}_k \tag{28}$$

holds for k large enough. Moreover, it is well known that $\lambda_k \sim k$ in dimension $d = 2$, which shows that (28) is equivalent to

$$\left\| e^{i\xi\cdot x} - \psi \right\|^2_{L^2(\Omega)} \;\geq\; \text{const}\, \lambda_k^{-\frac{1}{2}-\varepsilon(k)} \qquad \forall \psi \in \mathcal{L}_k\,. \tag{29}$$

The idea how to prove (29) is obvious; since $|e^{i\xi\cdot x}| = 1$ everywhere and $\psi = 0$ on $\partial\Omega$, we will estimate the left-hand side of (28) by integrating over a suitable neighbourhood of $\partial\Omega$ only. More precisely, we will make use of the contributions from integrating $|e^{i\xi\cdot x} - \psi|^2$ over squares of the size of order $\lambda^{-1/2}$ attached to the boundary of Ω, see Figure 2. To estimate these contributions from below, we will need appropriate integral upper bounds on the normal derivatives of ψ on $\partial\Omega$ in terms of λ. This will be done as the first step of the proof.

4.1 Eigenfunctions estimates

In this section we give an L^2 estimate on the derivatives the eigenfunctions ψ_i in the vicinity of $\partial\Omega$. Let

$$\omega = \left[0, \frac{1}{2\sqrt{\lambda}}\right] \times \left[-\frac{1}{4\sqrt{\lambda}}, \frac{1}{4\sqrt{\lambda}}\right]$$

and assume that λ is large enough so that the square ω can be placed inside Ω in such a way that one of its sides coincides with a part of $\partial\Omega$. We also introduce a local system of coordinates (x_1, x_2). Finally, for a given $p \in \mathbb{N}$ we define the sequence $A_n(p)$ by

$$A_n(p) = (3 + 726 \cdot 4^6\, p^4)\, A_{n-2}(p) + 150 \cdot 9^2\, p^2\, A_{n-1}(p) \tag{30}$$

where $A_0(p) = 1$ and $A_1(p) = 1$. We then have

Lemma 1. *Let ψ_i be a normalised eigenfunction of the Dirichlet Laplacian on Ω with an eigenvalue $\lambda_i \leq \lambda$. Then*

$$\left\| \frac{\partial^{p+1}\psi_i}{\partial x_1^{p+1}} \right\|^2_{L^2(\omega)} \leq\ A_p(p)\, \lambda^{p+1} \tag{31}$$

holds true for all $p \in \mathbb{N}_0$.

Proof. For $n, p \in \mathbb{N}$ we define the functions $g : [0,1] \to [0,1]$ by
$g(x) := 1 - 6x^4 + 8x^6 - 3x^8$ and $v_{n,p} : \mathbb{R} \to \mathbb{R}$ by

$$v_{n,p}(t) = \begin{cases} 1 & 0 \leq t \leq \frac{2p-n}{2p}, \\ g(2pt - 2p + n) & \frac{2p-n}{2p} \leq t \leq \frac{2p-n+1}{2p}, \\ 0 & \frac{2p-n+1}{2p} < t \end{cases}$$

with $v_{n,p}(t) = v_{n,p}(-t)$ for $t < 0$. It is easy to check that

$$|v_{n,p}(t)| \leq 1, \quad |v'_{n,p}(t)| \leq 2\alpha_1\, p, \quad |v''_{n,p}(t)| \leq 4\alpha_2\, p^2\,,$$

where $\alpha \leq 5/2$ and $\alpha_2 \leq 11$. Next we define

$$W_{n,p,\lambda}(x_1, x_2) = v_{n,p}(\sqrt{\lambda}\, x_1)\, v_{n,p}(4\sqrt{\lambda}\, x_2)$$

and note that

$$\begin{aligned} |W_{n,p,\lambda}(x_1, x_2)| \leq 1, \quad |\nabla W_{n,p,\lambda}(x_1, x_2)| &\leq 9\sqrt{\lambda}\, \alpha_1\, p \\ |\Delta W_{n,p,\lambda}(x_1, x_2)| &\leq \sqrt{2}\, 4^3\, \lambda\, \alpha_2 p^2 \end{aligned} \tag{32}$$

for all $(x_1, x_2) \in \omega$. We will prove

$$\begin{aligned} \left\| \frac{\partial^n\psi_i}{\partial x_1^n} \right\|^2_{L^2(supp\, W_{n-1,p,\lambda})} &\leq\ A_{n-1}(p)\, \lambda^n, \\ \left\| \frac{\partial^n\psi_i}{\partial x_1^{n-1}\partial x_2} \right\|^2_{L^2(supp\, W_{n-1,p,\lambda})} &\leq\ A_{n-1}(p)\, \lambda^n \end{aligned} \tag{33}$$

by induction in n for $n = 1, \dots, p$. Notice that, in view of (58), (59), the inclusion

$$\omega_n := (supp\, W_{n,p,\lambda}) \subset \Omega$$

holds true for every $p \in \mathbb{N}$ and every $n \le p$. For $n = 1$ we have

$$\left\| \frac{\partial \psi_i}{\partial x_1} \right\|^2_{L^2(\omega_0)} \le A_0(p)\,\lambda, \qquad \left\| \frac{\partial \psi_i}{\partial x_2} \right\|^2_{L^2(\omega_0)} \le A_0(p)\,\lambda.$$

Multiplying the equation $-\Delta\psi_i = \lambda_i \psi_i$ by $\frac{\partial^2 \psi_i}{\partial x_1^2}$ and integrating by parts we find out that

$$\left\| \frac{\partial^2 \psi_i}{\partial x_1^2} \right\|^2_{L^2(\omega_1)} \le A_1(p)\,\lambda^2, \qquad \left\| \frac{\partial^2 \psi_i}{\partial x_1 \partial x_2} \right\|^2_{L^2(\omega_1)} \le A_1(p)\,\lambda^2\,.$$

Hence (33) holds for $n = 1$ and $n = 2$. Now assume that (33) holds for some $n-1$ and n. We will show that it holds for $n+1$ as well. Integration by parts yields

$$\begin{aligned} \left\| \Delta \left(\frac{\partial^{n-1}\psi_i}{\partial x_1^{n-1}} W_{n-1,p,\lambda} \right) \right\|^2_{L^2(\omega_{n-1})} &= \left\| \frac{\partial^2}{\partial x_1^2} \left(\frac{\partial^{n-1}\psi_i}{\partial x_1^{n-1}} W_{n-1,p,\lambda} \right) \right\|^2_{L^2(\omega_{n-1})} + \\ \left\| \frac{\partial^2}{\partial x_2^2} \left(\frac{\partial^{n-1}\psi_i}{\partial x_1^{n-1}} W_{n-1,p,\lambda} \right) \right\|^2_{L^2(\omega_{n-1})} &+ 2 \left\| \frac{\partial^2}{\partial x_1 \partial x_2} \left(\frac{\partial^{n-1}\psi_i}{\partial x_1^{n-1}}\, W_{n-1,p,\lambda} \right) \right\|^2_{L^2(\omega_{n-1})} \end{aligned} \tag{34}$$

From the fact that $W_{n-1,p,\lambda} = 1$ on the ω_n it follows that the first and the last term on the right hand side of (34) are greater than or equal to

$$\left\| \frac{\partial^{n+1}\psi_i}{\partial x_1^{n+1}} \right\|^2_{L^2(\omega_n)} \quad \text{and} \quad \left\| \frac{\partial^{n+1}\psi_i}{\partial x_1^{n} \partial x_2} \right\|^2_{L^2(\omega_n)}$$

respectively. The second term on the right hand side of (34) is positive and since $\omega \subset supp\, W_{n,p,\lambda}$, we get

$$\left\| \frac{\partial^{n+1}\psi_i}{\partial x_1^{n+1}} \right\|^2_{L^2(\omega_n)} + \left\| \frac{\partial^{n+1}\psi_i}{\partial x_1^{n} \partial x_2} \right\|^2_{L^2(\omega_n)} \le \left\| \Delta \left(\frac{\partial^{n-1}\psi_i}{\partial x_1^{n-1}} W_{n-1,p,\lambda} \right) \right\|^2_{L^2(\omega_n)}. \tag{35}$$

Next we employ (32) and (33) to conclude that

$$\begin{aligned} &\left\| \Delta \left(\frac{\partial^{n-1}\psi_i}{\partial x_1^{n-1}} W_{n-1,p,\lambda} \right) \right\|^2_{L^2(\omega_n)} = \\ &\left\| \lambda_i \left(\frac{\partial^{n-1}\psi_i}{\partial x_1^{n-1}} \right) W_{n-1,p,\lambda} + \left(\frac{\partial^{n-1}\psi_i}{\partial x_1^{n-1}} \right) \Delta W_{n-1,p,\lambda} + 2 \left(\nabla \frac{\partial^{n-1}\psi_i}{\partial x_1^{n-1}} \right) \nabla W_{n-1,p,\lambda} \right\|^2_{L^2(\omega_n)} \\ &\le 3\lambda^{n+1} A_{n-2}(p) + 6 \cdot 4^6\, \alpha_2^2\, p^4\, \lambda^{n+1}\, A_{n-2}(p) + 24 \cdot 9^2\, \alpha_1^2\, p^2\, \lambda^{n+1}\, A_{n-1}(p) \le \lambda^{n+1}\, A_n(p)\,. \end{aligned} \tag{36}$$

□

As a consequence of this result we obtain

Corollary 1. *Let ω be as in Lemma 1. Assume that $\psi = \sum_{\lambda_i \le \lambda} c_i \psi_i$ with $\sum_{\lambda_i \le \lambda} |c_i|^2 \le V$. Then*

$$\left\| \frac{\partial^{p+1}\psi}{\partial x_1^{p+1}} \right\|^2_{L^2(\omega)} \le \frac{A_p(p) V^2(\Omega)}{4\pi}\, \lambda^{p+2}.$$

Proof. By Lemma 1 and the Cauchy-Schwarz inequality we have

$$\left\| \frac{\partial^{p+1}\psi}{\partial x_1^{p+1}} \right\|^2_{L^2(\omega)} \le \sum_{\lambda_i \le \lambda} |c_i|^2 \sum_{\lambda_i \le \lambda} \left\| \frac{\partial^{p+1}\psi_i}{\partial x_1^{p+1}} \right\|^2_{L^2(\omega)} \le N_\lambda\, V\, A_p(p)\, \lambda^{p+1}\,, \tag{37}$$

Using the lower bound on λ_i given in (53) we find out that $N_\lambda \le \frac{V}{4\pi}\,\lambda$. □

4.2 Lower bound on a square

Corollary 1 is one the two main technical results on which is based the proof of Theorems 1 and 2. The goal of this section is to prove the second one of these results, namely Proposition 2 (see page 15). We start with a couple of one dimensional estimates concerning smooth functions on an interval $[0,l]$. Unless otherwise stated, $\|\cdot\|$ denotes the L^2-norm on $[0,l]$.

Lemma 2. *Let $f \in C^{p+1}[0,l]$, $p \in \mathbb{N}$. Then*

$$\max |f^{(p)}|^2 \le \frac{3}{2}\left(\frac{1}{l}\,\|f^{(p)}\|^2 + l\,\|f^{(p+1)}\|^2\right).$$

Proof. Let $\max |f^{(p)}| = |f^{(p)}(t_0)|$ with $t_0 \in [0,l]$. For any $t \in [0,l]$ we have

$$f^{(p)}(t) = f^{(p)}(t_0) + \int_{t_0}^{t} f^{(p+1)}(\tau)\,d\tau\,.$$

Integrating with respect to t and using the Jensen inequality gives

$$\begin{aligned} l\,|f^{(p)}(t_0)|^2 &\le \frac{3}{2}\int_0^l |f^{(p)}(t)|^2\,dt + 3\int_0^l \left(\int_{t_0}^{t} f^{(p+1)}(\tau)\,d\tau\right)^2 dt \\ &\le \frac{3}{2}\,\|f^{(p)}\|^2 + 3\int_0^l t\|f^{(p+1)}\|^2\,dt = \frac{3}{2}\left(\|f^{(p)}\|^2 + l^2\|f^{(p+1)}\|^2\right). \end{aligned}$$

□

Lemma 3. *Let $f \in C^2\left[0, \frac{1}{2}\,\lambda^{-1/2}\right]$ and real-valued. Then one of the following inequalities holds true:*

$$\max|f|\,\max|f''| \le \frac{1}{4}\,\max|f'|^2 \tag{38}$$

$$\max|f'| \le 32\,\lambda^{\frac{1}{2}}\,\max|f| \tag{39}$$

Proof. Let $m_i = \max|f^{(i)}|$, $i \in \{0,1,2\}$ and let $t_0 \in \left[0, \frac{1}{2}\,\lambda^{-1/2}\right]$ be such that $m_1 = |f'(t_0)|$. Without loss of generality we assume that $t_0 < \frac{1}{4}\,\lambda^{-1/2}$, otherwise we consider the interval $[0,t_0]$ instead of $[t_0, \frac{1}{2}\,\lambda^{-1/2}]$. Assume that $f'(t_0) = m_1$. If

$$t_0 + \frac{m_1}{m_2} \le \frac{1}{2}\,\lambda^{-1/2}, \tag{40}$$

then the Taylor theorem says that

$$m_0 \ge f\left(t_0 + \frac{m_1}{m_2}\right) \ge f(t_0) + m_1\left(\frac{m_1}{m_2}\right) - \frac{m_2}{2}\left(\frac{m_1}{m_2}\right)^2 \ge -m_0 + \frac{m_1^2}{2m_2}\,,$$

which implies (38). If, on the contrary,

$$t_0 + \frac{m_1}{m_2} > \frac{1}{2}\lambda^{-1/2}, \qquad \text{then} \qquad \frac{m_1}{m_2} > \frac{1}{2}\lambda^{-1/2} - t_0 > \frac{1}{4}\lambda^{-1/2}.$$

In this case we have

$$m_0 \ge f\left(t_0 + \frac{1}{8}\lambda^{-1/2}\right) \ge f(t_0) + m_1 \frac{1}{8}\lambda^{-1/2} - \frac{m_2}{128}\lambda^{-1},$$

which implies

$$m_1 \frac{1}{8}\lambda^{-1/2} - \frac{m_1}{32\lambda^{-1/2}}\lambda^{-1} \le 2m_0.$$

From here we conclude that

$$\frac{m_1}{m_0} \le \frac{64}{3\lambda^{-1/2}} \le 32\,\lambda^{\frac{1}{2}}.$$

The proof in the case $f'(t_0) = -m_1$ is analogous. □

Proposition 1. *Let $f \in C^p\left[0, \frac{1}{2}\lambda^{-1/2}\right]$, $p \in \mathbb{N}$ and let f be real-valued. Then one of the following inequalities holds true:*

$$\max|f'| \le 4^{p+\frac{1}{2}}\lambda^{\frac{1}{2}} \max|f| \tag{41}$$

$$\max|f'| \le \left(\frac{\max|f^{(p)}|}{\max|f|}\right)^{\frac{1}{p}} 4^{p-\frac{1}{2}} \max|f|. \tag{42}$$

Proof. Let $m_i = \max|f^{(i)}|$, $i = 1, \ldots, p$. There are two possibilities. Either for all $i \le p$ holds

$$\frac{m_i}{m_{i-1}} \ge 32\,\lambda^{\frac{1}{2}}, \tag{43}$$

or there exists $i_0 \in [1, p]$, such that

$$\forall\, i < i_0 \quad \frac{m_i}{m_{i-1}} \ge 32\,\lambda^{\frac{1}{2}}, \qquad \frac{m_{i_0}}{m_{i_0-1}} < 32\,\lambda^{\frac{1}{2}}. \tag{44}$$

In the first case $\frac{m_i}{m_{i-1}} > \frac{1}{4}\frac{m_{i-1}}{m_{i-2}}$ holds for all $i \le p$, see Lemma 3. This yields

$$m_p \ge 4^{-\frac{p(p-1)}{2}} \left(\frac{m_1}{m_0}\right)^p m_0,$$

which is equivalent to (42). In the second case we have $\frac{m_i}{m_{i-1}} > \frac{1}{4}\frac{m_{i-1}}{m_{i-2}}$ for all $i \le i_0$. Combining this with (44) we conclude that

$$\frac{m_1}{m_0} \le 4^{i_0+\frac{1}{2}}\lambda^{\frac{1}{2}}.$$

□

Corollary 2. *Let $f \in C^p\left[0, \frac{1}{2}\lambda^{-1/2}\right]$, $p \in \mathbb{N}$ be a complex-valued function such that $f(0) = 0$ and $\max|f^{(p)}| \le C(p)\,\lambda^{\frac{p}{2}+1}$ for some constant $C(p)$. Then for any $\varphi_0, \varphi_1 \in \mathbb{R}$ holds*

$$\int_0^{\frac{1}{2}\lambda^{-1/2}} |f(t) - e^{i\varphi_1 t + i\varphi_0}|^2\, dt \ge \frac{\lambda^{-\frac{1}{2}}}{9} \min\left\{4^{-p-\frac{5}{2}},\ 4^{-\frac{p+3}{2}}\, 6^{\frac{1}{p}}\, C(p)^{-\frac{1}{p}}\, \lambda^{-\frac{1}{p}}\right\}. \tag{45}$$

Proof. Let $u = \mathrm{Re} f$ and $v = \mathrm{Im} f$. If $\max |f| \geq 6$, then at least one the expressions $\max |u|$, $\max |v|$ is larger than or equal to 3. Without loss of generality we assume that $\max |u| \geq 3$ and apply Proposition 1 to the function u. If u satisfies (41), then there exists an subinterval $I \subset [0, \frac{1}{2}\,\lambda^{-1/2}]$ of the length $3^{-1}4^{-p-\frac{1}{2}}\,\lambda^{-\frac{1}{2}}$ on which $|u| \geq 3/2$. This implies

$$\int_0^{\frac{1}{2}\lambda^{-1/2}} |f(t) - e^{i\varphi_1 t + i\varphi_0}|^2\, dt \geq 3^{-1}\, 4^{-p-\frac{1}{2}}\, \lambda^{-\frac{1}{2}}\,.$$

If, on the other hand, u satisfies (42), then the length of the subinterval of $[0, \frac{1}{2}\,\lambda^{-1/2}]$, on which $|u| \geq 3/2$, is at least $3^{-1}4^{-\frac{p-1}{2}}\, C(p)^{-\frac{1}{p}}\, \lambda^{-\frac{1}{2}-\frac{1}{p}}$, which gives

$$\int_0^{\frac{1}{2}\lambda^{-1/2}} |f(t) - e^{i\varphi_1 t + i\varphi_0}|^2\, dt \geq 3^{-1}\, 4^{-\frac{p-1}{2}}\, C(p)^{-\frac{1}{p}}\, \lambda^{-\frac{1}{2}-\frac{1}{p}}.$$

Assume now that $\max |f| < 6$. The latter means that $\max |u| < 6$ and $\max |v| < 6$. Since $u(0) = v(0) = 0$, there exists a subinterval of $[0, \frac{1}{2}\,\lambda^{-1/2}]$, on which $\max\{|u(t)|, |v(t)|\} \leq 1/3$, which implies $|f(t) - e^{i\varphi_1 t + i\varphi_0}|^2 \geq 1/4$. Applying Proposition 1 to the functions u, v we find out that the length of this interval is bounded from below by

$$\min\left\{3^{-2}\, 4^{-p-\frac{5}{2}}\, \lambda^{-\frac{1}{2}}\,,\, 3^{-2}\, 4^{-\frac{p+3}{2}}\, 6^{\frac{1}{p}}\, C(p)^{-\frac{1}{p}}\, \lambda^{-\frac{1}{2}-\frac{1}{p}}\right\}\,.$$

This completes the proof. $\square$

With the above auxiliary results at hand, we can finally prove the following integral estimate, which will play a central role in the proof of Theorem 1 and 2.

Proposition 2. *Let $f \in C^{p+1}[\omega]$ a complex valued function such that $f(0, x_2) = 0$ for each x_2 and*

$$\left\| \frac{\partial^{p+1} f}{\partial x_1^{p+1}} \right\|_{L^2(\omega)} \leq\, \beta_{p+1}\, \lambda^{1+\frac{p}{2}}, \qquad \left\| \frac{\partial^{p} f}{\partial x_1^{p}} \right\|_{L^2(\omega)} \leq\, \beta_{p}\, \lambda^{\frac{1}{2}+\frac{p}{2}}$$

for some positive β_p and β_{p+1}. Then the inequality

$$\left\| f - e^{i(\xi_1 x_1 + \xi_2 x_2 + \varphi)} \right\|^2_{L^2(\omega)} \geq \frac{1}{36}\, \min\left\{4^{-p-\frac{5}{2}}\, \lambda^{-1},\, 4^{-\frac{p}{2}-\frac{3}{2}}\, 6^{\frac{1}{2p}}\, (\beta_{p+1}^2 + \beta_p^2)^{-\frac{1}{2p}}\, \lambda^{-1-\frac{1}{p}}\right\} \qquad (46)$$

holds true for all $\xi_1, \xi_2, \varphi \in \mathbb{R}$.

Proof. The measure of the set

$$\left\{ x_2 \in [0, \lambda^{-1/2}]\, :\, \int_0^{\frac{1}{2\sqrt{\lambda}}} \left| \frac{\partial^i f(x_1, x_2)}{\partial x_1^i} \right|^2 dx_1\, \leq\, 8\, \beta_i^2 \lambda^{i+\frac{3}{2}},\, i \in \{p, p+1\} \right\}$$

is obviously at least $\frac{1}{4}\,\lambda^{-\frac{1}{2}}$. For such x_2 holds by Lemma 2

$$\max_{x_1} \left| \frac{\partial^p f(x_1, x_2)}{\partial x_1^p} \right| \leq \sqrt{3}\, \lambda^{1+\frac{p}{2}}\, \sqrt{\beta_{p+1}^2 + \beta_p^2}\,.$$

Corollary 2 then implies the statement. $\square$

4.3 Proof of Theorem 1

Proof of Theorem 1. Fix $\lambda > 0$. Let λ_j be the eigenvalues of the Dirichlet Laplacian on Ω and let ψ_j be the corresponding normalised eigenfunctions. For $k \in \mathbb{N}$ we define

$$F(\xi) = \sum_{j=1}^{k} |\hat{\psi}_j(\xi)|^2 ,$$

where $\hat{\psi}_j$ denotes the Fourier transform of ψ_j. Moreover, we denote by $F^*(|\xi|)$ the decreasing radial rearrangement of $F(\xi)$. Let

$$\psi(x) = \sum_{\lambda_i \le \lambda} c_i\, \psi_i(x), \quad \text{with} \quad \sum_{\mu_i \le \lambda} |c_i|^2 \le V .$$

For each $j = 1, \dots, n$ we choose on the middle part of p_j several points t_l such that $\operatorname{dist}(t_l, t_{l+1}) = \sqrt{2}\, \lambda^{-1/2}$ for all l and denote by T_l the squares with the side $\frac{1}{2}\lambda^{-1/2}$ constructed in the middle point between t_l and t_{l+1}, see Figure 2. We note that for each j the number of these squares is at least

$$N_j = \left[\frac{1}{3\sqrt{2}}\, l_j\, \lambda^{\frac{1}{2}} \right] .$$

According to Corollary 1 for each l and p we have

$$\left\| \frac{\partial^{p+1}\psi}{\partial \nu^{p+1}} \right\|^2_{L^2(T_l)} \le \frac{A_p(p) V^2}{4\pi}\, \lambda^{p+2},$$

where $\frac{\partial \psi}{\partial \nu}$ denotes the normal derivative of ψ. In view of Proposition 2 and Corollary 3 we get

$$\left\| \psi - e^{i\xi\cdot x} \right\|^2_{L^2(T_l)} \ge \frac{1}{36}\, \min\left\{ 4^{-p-\frac{5}{2}}\, \lambda^{-1},\, 4^{-\frac{p}{2}-\frac{3}{2}}\, 6^{\frac{1}{2p}}\, (\beta^2_{p+1} + \beta^2_p)^{-\frac{1}{2p}}\, \lambda^{-1-\frac{1}{p}} \right\}, \tag{47}$$

where

$$\beta^2_{p+1} = \frac{A_p(p) V^2}{4\pi} .$$

We continue by estimating the sequence $A_p(p)$. A direct inspection shows that

$$A_p(p) \le c_0\, 2^{(p+1)^2} , \quad c_0 = 7 \cdot 10^{22} . \tag{48}$$

This implies that $(\beta^2_{p+1} + \beta^2_p)^{-\frac{1}{2}} \ge 2^{-1/2}\, \sqrt{\pi}\, c_0^{-1/2}\, V^{-1}\, 2^{-(p+1)^2/2}$. Hence for

$$p = \left[\sqrt{2 \log_2(V\lambda/c_1)} \right] - 1, \quad c_1 = \sqrt{\frac{3\pi}{2}}\, c_0^{-\frac{1}{2}}$$

we obtain

$$\left\| \psi - e^{i\xi\cdot x} \right\|^2_{L^2(T_l)} \ge \frac{2^{-3}}{36}\, c_1^{-1}\, V \left(\frac{V\lambda}{c_1} \right)^{-1-\frac{2}{\sqrt{\log_2(V\lambda/c_1)}}} .$$

Taking λ large enough such that

$$\lambda^{-1/2} \le \frac{d_j}{3} .$$

we make sure that the squares T_l lie inside Ω and that they do not overlap each other. Summing this inequality for all $l = 1, \dots, N_j$ and all $j = 1, ..., n$ we thus arrive at

$$V - 4\pi^2\, F^*(|\xi|) \ge \left\| \psi - e^{i\xi\cdot x} \right\|^2_{L^2(\Omega^e)} \ge c_2\, V^{\frac{1}{2}} \left(\frac{V\lambda}{c_1} \right)^{-\frac{1}{2}-\frac{2}{\sqrt{\log_2(V\lambda/c_1)}}} \sum_{j=1}^{n} l_j\, \Theta\left(\lambda - \frac{9}{d_j^2} \right) \tag{49}$$

with $c_2 = \frac{2^{-3}}{9\sqrt{2}\,36}\, c_1^{-1/2}$. This yields the following upper bound on F^*:

$$F^*(|\xi|) \leq \; M(p,\lambda) := \frac{V}{4\pi^2}\, \left[1 - c_2\, V^{-\frac{1}{2}}\, \left(\frac{V\lambda}{c_1}\right)^{-\frac{1}{2} - \frac{2}{\sqrt{\log_2(V\lambda/c_1)}}} \, \sum_{j=1}^{n} l_j\, \Theta\left(\lambda - \frac{9}{d_j^2}\right)\right]. \tag{50}$$

Now we use the minimiser (10) with $V/4\pi^2$ replaced by $M(p,\lambda)$ to obtain

$$\sum_{j=1}^{k} \lambda_j \;\geq\; \int_{\mathbb{R}^2} F^*(|\xi|)|\xi|^2\, d\xi \;\geq\; \frac{\lambda^2 V^2}{8\pi^3\, M(p,\lambda)}\,. \tag{51}$$

Employing the definition of $M(p,\lambda)$ we then find out that

$$\sum_{j=1}^{k} \lambda_j \geq \frac{\lambda^2 V}{2\pi} + c_2\, c_1^2\, V^{-\frac{3}{2}}\, \left(\frac{V\lambda}{c_1}\right)^{\frac{3}{2} - \frac{2}{\sqrt{\log_2(V\lambda/c_1)}}} \, \sum_{j=1}^{n} l_j\, \Theta\left(\lambda - \frac{9}{d_j^2}\right). \tag{52}$$

Next we set $\lambda = \lambda_k$ and note that inequality (2) yields

$$\frac{2\pi}{V}\, k \;\leq\; \lambda_k\,. \tag{53}$$

Since the right hand side of (52) is an increasing function of λ, we can use (53) to conclude that

$$\sum_{j=1}^{k} \lambda_j \geq \frac{2\pi}{V}\, k^2 + 4\, c_3\, k^{\frac{3}{2} - \frac{2}{\sqrt{\log_2(2\pi k/c_1)}}} \, \sum_{j=1}^{n} l_j\, \Theta\left(k - \frac{9\,V}{2\pi\, d_j^2}\right)\, V^{-3/2} \tag{54}$$

where

$$c_3 = \frac{2^{-3}}{9\sqrt{2}\,36}\, (2\pi)^{\frac{5}{4}} c_1^{1/4}\,.$$

Finally, we combine inequalities (54) and (16) to get (21). □

5 Proof for general domains

From now on we suppose that Ω is a general domain satisfying assumption A. To prove a Li-Yau type inequality with the correction term we cannot directly employ the approach invented for polygons, since $\partial\Omega$ is in general nowhere straight. However, we can extend Ω by adding small "bumps" to certain parts of $\partial\Omega$, see Figure 4, in order to obtain an extended domain Ω^e whose boundary is in certain parts represented by a straight line. On these straight pieces of $\partial\Omega^e$ we will then employ the same strategy as in the case of polygons. Due to the monotonicity of eigenvalues, any lower bound on the sum of the eigenvalues on the extended domain gives also a lower bound on the sum of the eigenvalues on Ω. On the other hand, we have to make sure that the volume of Ω^e is not much bigger than V, because otherwise it could destroy the effect of the correction term in (26) by decreasing the leading term. We will again split the exposition in several steps.

5.1 Step 1: Some geometrical remarks

Here we will show that $\partial\Omega \cap \Gamma_j$ can be locally represented as a graph of a certain C^2-smooth function. Let $\Gamma = \{x_1(s), x_2(s)\}$ be a part of the boundary of Ω parametrised by its length s and such that $x_1(s), x_2(s) \in C^2(\mathbb{R}_+)$. Let

$$\varkappa_0 := \max_{\{x_1,x_2\}\in\Gamma} |\varkappa(x_1,x_2)|$$

be the maximal curvature of Γ. We consider certain points $A = \{x_1(s'), x_2(s')\} \in \Gamma$ and $B = \{x_1(s''), x_2(s'')\} \in \Gamma$ and chose a new system (u, v) such that $A = (0, 0)$ and the $u-$axes goes along the line AB.

Lemma 4. *Assume that $\varkappa_0|s' - s''| \leq \pi/4$. Then the following statements hold true.*

(i) The part of Γ connecting A and B can be written in the system of coordinates (u, v) as $v = v(u)$, $u \in [0, u_0]$, where $u_0 = |AB|$. Moreover, we have

$$\max_{u\in[0,u_0]} v(u) \leq \sqrt{2}\,\varkappa_0\, u_0^2\,. \tag{55}$$

(ii) The inequality

$$2^{-1/2}\,|s' - s''| \;\leq\; |AB| \;\leq\; |s' - s''| \tag{56}$$

holds.

Proof. Let $\{u(s), v(s)\}$ be the parametrisation of Γ in the coordinates (u, v). By assumption we have

$$\int_0^{|s'-s''|} \varkappa(s)\, ds \leq \pi/4 \tag{57}$$

This means that for any $s \in [0, |s' - s''|]$ the angle between the tangent of Γ at the point $\{u(s), v(s)\}$ and the $u-$axes is less than or equal to $\pi/4$. Assume that there exists $s_1, s_2 \in [0, |s' - s''|]$ such that $u(s_1) = u(s_2)$. Then there exists $s_3 \in [s_1, s_2]$ such that the tangent of Γ at $\{u(s_3), v(s_3)\}$ is orthogonal to the $u-$axes. The latter contradicts (57). This shows that the part of Γ between A and B can be considered as the graph of the function

$$v = v(u), \quad u \in [0, u_0], \quad v(0) = v(u_0) = 0\,.$$

This proves the first part of (i) and, in view of (57), shows that $|v'(u)| \leq 1$ on $[0, u_0]$. Next we prove inequality (56). It thus follows that

$$u_0 = |AB| \leq |s' - s''| = \int_0^{u_0} \left(1 + |v'(u)|^2\right)^{1/2}\, du \leq 2^{1/2}\, u_0\,,$$

which implies (56). To prove (55) we note that $v(u)$ is twice differentiable and therefore there exists some $u_1 \in [0, u_0]$, such that $v'(u_1) = 0$. Since $|v''(u)| = |\varkappa(u)|\,(1 + |v'(u)|^2)^{3/2} \leq 2^{3/2}\,\varkappa_0$, we obtain

$$|v'(u)| \leq \int_{u_1}^{u} |v''(u)|\, du \leq 2^{3/2}\,\varkappa_0\, u_0 \qquad \forall u \in [0, u_0].$$

The last inequality together with the fact that $v(0) = v(u_0) = 0$ finally implies

$$|v(u)| \leq \frac{1}{2}\, 2^{3/2}\,\varkappa_0\, u_0^2 = 2^{1/2}\,\varkappa_0\, u_0^2 \qquad \forall u \in [0, u_0]\,.$$

□

5.2 Step 2: Approximation of the boundary

Next we introduce a procedure that allows us to choose appropriate parts of $\partial\Omega \cap \Gamma_j$ on which we will construct the additional "bumps", see Figure 4. Let Γ_j, $j = 1 \ldots m$ be the parts of boundary defined in section 3 with the end points A_j, B_j and the partition a_i^j, $i = 0, \ldots, n_j$. We fix $j \in \{1, ..., m\}$ and take λ large enough, such that

$$\lambda^{-\frac{1}{2}} \leq \min\left\{\frac{d_j}{3}\,,\ \frac{\pi}{8\sqrt{2}\,\varkappa_j}\right\}, \quad \text{if} \quad L(\Gamma_j) > \frac{3\pi}{8\varkappa_j} \tag{58}$$

and

$$\lambda^{-\frac{1}{2}} \leq \min\left\{\frac{d_j}{3}\,,\ \frac{L(\Gamma_j)}{3\sqrt{2}}\right\}, \quad \text{if} \quad L(\Gamma_j) \leq \frac{3\pi}{8\varkappa_j}\,. \tag{59}$$

Let us consider $\Gamma_j \cap (a_i^j, a_{i+1}^j)$ with $0 < i < n_j$. On this part of the boundary we choose several disjoint arcs (b_l, b_l'), such that each of them has the length $\sqrt{2}\,\lambda^{-1/2}$ and such that

$$\sum_l s(b_l,\, b_l') \geq \frac{1}{3}\, s(a_i^j,\, a_{i+1}^j), \quad s(a_i^j,\, a_{i+1}^j) - \sum_l s(b_l,\, b_l') \leq \sqrt{2}\,\lambda^{-1/2}\,,$$

where $s(a, b)$ denotes the arc-length between a and b.

Next we pick an l and connect b_l and b_l' with a straight line and choose a local system of coordinates (y_1, y_2) so that the y_1-axis goes along the straight line from b_l to b_l' and the origin is in b_l. Notice that $s(a_{i-1}^j,\, a_{i+1}^j) = s(a_i^j,\, a_{i+2}^j) \leq \frac{\pi}{2\varkappa_j}$, which according to Lemma 4 means that in the chosen coordinate system the boundary between a_{i-1}^j and a_{i+2}^j can be written explicitly as $y_2 = f(y_1)$. Let $y_0 = \text{dist}(b_l, b_l')$. In view of Lemma 4

$$\max_{y_1} |f(y_1)| \leq \sqrt{2}\,\varkappa_j\, y_0^2 \leq 2^{\frac{3}{2}}\,\varkappa_j\,\lambda^{-1}\,.$$

Now we introduce

$$\Sigma_1 = \Big\{(y_1, y_2)\ :\ y_1 \in [0, y_0],\ y_2 = 2^{\frac{3}{2}}\,\varkappa_j\,\lambda^{-1}\Big\}$$

and

$$\Sigma_2 = \Big\{(y_1, y_2)\ :\ y_1 \in [0, y_0],\ y_2 = -2^{\frac{3}{2}}\,\varkappa_j\,\lambda^{-1}\Big\}$$

Lemma 5. *If $\lambda > 6\varkappa_j/d_j$, then*

$$\Sigma_1 \cap \partial\Omega = \Sigma_2 \cap \partial\Omega = \emptyset\,.$$

Proof. Obviously Σ_1 and Σ_2 do not cross $\partial\Omega$ between a_{i-1}^j and a_{i+2}^j. On the other hand, for each point $P = (y_1^P, y_2^P)$ holds

$$\text{dist}(P, (a_i^j,\, a_{i+1}^j))\ \leq\ 2^{3/2}\,\varkappa_j\,\lambda^{-1}\,.$$

Since $\text{dist}\Big((a_i^j,\, a_{i+1}^j),\ \partial\Omega \setminus (a_{i-1}^j,\, a_{i+2}^j)\Big) \geq d_j$, this implies

$$\text{dist}\Big(P,\ \partial\Omega \setminus (a_{i-1}^j,\, a_{i+2}^j)\Big) \geq d_j - 2^{3/2}\,\varkappa_j\,\lambda^{-1} > \frac{d_j}{2} > 0\,.$$

□

The last Lemma says that one of the sets Σ_1 and Σ_2 is inside Ω and the other one is outside Ω. Without loss of generality we assume that Σ_1 is outside Ω.

5.3 Step 3: Extended domain Ω^e.

The extended domain Ω^e differs from Ω if λ is large enough so that (58) respectively (59) is satisfied (otherwise it coincides with Ω).

To define Ω^e we proceed as follows. For a fixed $j \in \{1,\dots,m\}$, fixed $i \in \{1,...,n_j-1\}$ and fixed l, we consider the boundary between the points b_l and b_l'. If it is a straight line, we do not change it. Otherwise we replace this piece of the boundary with the segment Σ_i, where i is such that Σ_i is outside Ω, and connect the end points of Σ_1 with the boundary at certain points $\tilde{b}_l \in (b_{l-1}', b_l)$ and $\tilde{b}_l' \in (b_l', b_{l+1})$ with appropriate C^2 functions. We choose these function and the points $\tilde{b}_l$, $\tilde{b}_l'$ in such a way that the added area to Ω is less than 3 times the area of the rectangle with the corners given by b_l, b_l' and the end points of Σ_1. We then obtain a new region whose boundary, corresponding to the original piece Γ_j is again C^2-smooth and which between the original boundary points b_l and b_l' consists of a straight line. Repeating this procedure for all Γ_j, $j=1,\dots,m$, all $i \in \{1,\dots,n_j-1\}$ and all l we thus obtain a new domain Ω^e.

As a next step we construct the squares T_l of the side $\frac{1}{2}\,\lambda^{-1/2}$ between the the points b_l and b_l' centred in the middle. Note that, according to Lemma 4, $|b_l b_l'| \geq \lambda^{-1/2}/\sqrt{2}$. We have

Lemma 6. *The squares T_l do not overlap.*

Proof. First we show that every T_l does not overlap with any of the squares constructed on the part of the boundary different from the arch $(a_{i-1}^j,\, a_{i+2}^j)$. Indeed, each point of T_l has distance to (b_l, b_l') at most $\frac{1}{2}\,\lambda^{-1/2}$ and the distance between (b_l, b_l') and $\partial\Omega \setminus (a_{i-1}^j,\, a_{i+2}^j)$ is at least d_j. Since $\lambda^{-1/2} < d_j$, see (58), the result follows.

Consider now $(a_{i-1}^j,\, a_{i+2}^j)$. This part can be written as $y_2 = f(y_1)$ in the above introduced coordinate system. Consider the squares T_{l_1} and T_{l_2} with $l_1 \neq l_2$. Let y_1^1 be the y_1 coordinate of the middle point between b_{l_1} and b_{l_1}' and let y_1^2 be the y_1 coordinate of the middle point between b_{l_2} and b_{l_2}'. Since $|f'(y_1)| \leq 1$ on $(a_{i-1}^j,\, a_{i+2}^j)$, we have $|y_1^1 - y_1^2| \geq \lambda^{-1/2}$. For all points $(y_1, y_2) \in T_{l_1}$ holds $|y_1 - y_1^1| \leq \frac{1}{4}\lambda^{-1/2}$ and for all points $(y_1, y_2) \in T_{l_2}$ holds $|y_1 - y_1^2| \leq \frac{\sqrt{2}}{2}\,\lambda^{-1/2}$. Collecting these inequalities we conclude that $T_{l_1} \cap T_{l_2} = \emptyset$. $\square$

As a consequence of the last result we obtain estimates on the volume of Ω^e, which will be used in the proof of Theorem 2.

Corollary 3. *Let V^e be the volume of the extended domain Ω^e. Then*

$$V^e \;\leq\; V + 2^{\frac{3}{2}}\,\lambda^{-1} \sum_{j=1}^{m} \varkappa_j\, L(\Gamma_j)\,. \tag{60}$$

Moreover, if

$$\lambda \;\geq\; \Lambda_1 := 9 \cdot 2^{10} \max_j \varkappa_j^2\,,$$

then

$$V^e \;\leq\; 2V\,. \tag{61}$$

Proof. Inequality (60) follows directly from the construction of Ω^e, since the area of the added volume along Γ_j does not exceed $2^{\frac{3}{2}}\,\lambda^{-1}\,\varkappa_j L(\Gamma_j)$. As for the second inequality, we consider each pair b_l, $b_{l'}$ and note that for $\lambda \geq 9\cdot 2^{10}\,\varkappa_j^2$ is the area of the added volume between $\tilde{b}_l$ and $\tilde{b}_{l'}$, bounded from above by

$$12\,\varkappa_j\,\lambda^{-\frac{3}{2}} \;\leq\; \frac{1}{8}\,\lambda^{-1}\,.$$

This follows from the choice of the points b_l, see section 5.3. On the other hand, for λ chosen as above we get

$$|T_l \cap \Omega| \geq \frac{1}{2}\,|T_l| = \frac{1}{8}\,\lambda^{-1}\,.$$

Since T_l do not overlap, we obtain (61). $\square$

5.4 Proof of Theorem 2

Proof of Theorem 2. Fix $\lambda > 0$ and consider the extended domain Ω^e. Let μ_j be the eigenvalues of the Dirichlet Laplacian on Ω^e and let ϕ_j be the corresponding normalised eigenfunctions. For $k \in \mathbb{N}$ we define

$$F_e(\xi) = \sum_{j=1}^{k} |\hat{\phi}_j(\xi)|^2 \,,$$

where $\hat{\phi}_j$ denotes the Fourier transform of ϕ_j. By $F_e^*(|\xi|)$ we denote the decreasing radial rearrangement of $F_e(\xi)$. Let

$$\phi(x) = \sum_{\mu_i \le \lambda} c_i \, \phi_i(x), \quad \text{with} \quad \sum_{\mu_i \le \lambda} |c_i|^2 \le V^e$$

and let T_l be the sequence of squares constructed along Γ_j. For each j is the number of these squares at least

$$N_j = \left[\frac{1}{9\sqrt{2}} \, L(\Gamma_j) \, \lambda^{\frac{1}{2}} \right] .$$

Next we take $\lambda \ge \Lambda_1$, so that $V^e \le 2V$, see Corollary 3. According to Corollary 1 for each l and p we then have

$$\left\| \frac{\partial^{p+1} \phi}{\partial \nu^{p+1}} \right\|^2_{L^2(R_n)} \le \frac{A_p(p)(V^e)^2}{4\pi} \, \lambda^{p+2} \le \frac{A_p(p) V^2}{\pi} \, \lambda^{p+2} \,,$$

where $\frac{\partial \phi}{\partial \nu}$ denotes the normal derivative of ϕ. In view of Proposition 2 for each l holds

$$\left\| \phi - e^{i\xi \cdot x} \right\|^2_{L^2(T_l)} \ge \frac{1}{36} \min \left\{ 4^{-p-\frac{5}{2}} \, \lambda^{-1}, \, 4^{-\frac{p}{2}-\frac{3}{2}} \, 6^{\frac{1}{2p}} \, (\beta_{p+1}^2 + \beta_p^2)^{-\frac{1}{2p}} \, \lambda^{-1-\frac{1}{p}} \right\} ,$$

with

$$\beta_{p+1}^2 = \frac{A_p(p)(V^e)^2}{4\pi} \le \frac{A_p(p) V^2}{\pi} \,.$$

Now we employ the same arguments used in the proof of Theorem 1 in order to find an appropriate upper bound on F_e^*. Since $\lambda \ge \Lambda_1$ we can use Corollary 3 to arrive at

$$F_e^*(|\xi|) \le \frac{V}{4\pi^2} \left[1 + \sum_{j=1}^{m} \left(2^{3/2} V^{-1} \varkappa_j \lambda^{-1} - \frac{c_2}{2} V^{-\frac{1}{2}} \left(\frac{V\lambda}{c_1} \right)^{-\frac{1}{2} - \frac{2}{\sqrt{\log_2(V\lambda/c_1)}}} \right) L(\Gamma_j) \right] .$$

Note that for

$$\lambda \ge \Lambda_2 := 2^{2^6} \, c_1 \, V^{-1}$$

we have $\left(\frac{V\lambda}{c_1}\right)^{-\frac{1}{2} - \frac{2}{\sqrt{\log_2(V\lambda/c_1)}}} \ge \left(\frac{V\lambda}{c_1}\right)^{-\frac{3}{4}}$ and therefore

$$F_e^*(|\xi|) \le \; M_e(p, \lambda) := \frac{V}{4\pi^2} \left[1 - \frac{c_2}{4} \, V^{-\frac{1}{2}} \left(\frac{V\lambda}{c_1} \right)^{-\frac{1}{2} - \frac{2}{\sqrt{\log_2(V\lambda/c_1)}}} \sum_{j=1}^{m} L(\Gamma_j) \, \Theta(\lambda - \Lambda_3(j)) \right] .$$

where

$$\Lambda_3(j) := \max \left\{ \Lambda_1 \, , \, \Lambda_2 \, , \, c_1^{-1} \, 2^{22} \, 6^8 \, \varkappa_j^4 \, V \right\} .$$

We now use again the Li-Yau type minimiser (10) with $V/4\pi^2$ replaced by $M_e(p,\lambda)$ to obtain

$$\sum_{j=1}^{k} \lambda_j \ge \sum_{j=1}^{k} \mu_j \ge \int_{\mathbb{R}^2} F_e^*(|\xi|)|\xi|^2\, d\xi \ \ge \frac{\lambda^2 V^2}{8\pi^3\, M_e(p,\lambda)}\,.$$

As in the proof of Theorem 1 we set $\lambda = \lambda_k$ and use definition of $M_e(p,\lambda)$ together with inequalities (58),(59) and (53) to obtain

$$\sum_{j=1}^{k} \lambda_j \ge \frac{2\pi}{V}\, k^2 + c_3\, k^{\frac{3}{2}-\frac{2}{\sqrt{\log_2(2\pi k/c_1)}}} \sum_{j=1}^{m} L(\Gamma_j)\, \Theta(k-k(j))\, V^{-3/2} \tag{62}$$

where

$$k(j) := \frac{V}{2\pi}\, \max\left\{\Lambda_3(j),\, \frac{9}{d_j^2},\, \frac{128\,\kappa_j^2}{\pi^2},\, \frac{6\varkappa_j}{d_j}\right\}.$$

Finally, we combine inequalities (62) and (16) to get (26). $\square$

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