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Preprint 2008/004

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ISSN **1613-8309**

Abstract

Almost Einstein manifolds satisfy a generalisation of the Einstein condition; they are Einstein on an open dense subspace and, in general, have a conformal scale singularity set Σ that is a conformal infinity for the Einstein metric. In case an almost Einstein manifold is closed and Σ is a hypersurface we call the corresponding Einstein metric *conformally closed*. Such Einstein metrics represent a subclass of conformally compact Poincaré-Einstein metrics. With respect to a special defining function of the boundary every Poincaré-Einstein metric can be expressed in normal form. Similarly, in this paper we discuss closed manifolds, which admit multiple almost Einstein structures, whose scale singularity sets intersect non-trivially. In a neighbourhood of that intersection set $\Sigma(\mathcal{S})$ we describe the underlying conformal geometry by normal form metrics and the S^l-doubling construction of [24]. The set $\Sigma(\mathcal{S})$ is a totally umbilic submanifold (of higher codimension).

Keywords: Conformal geometry; Poincaré-Einstein metrics; tractor calculus; almost Einstein structures; conformal Killing p-forms.

MSC 2000: Primary 53C25; Secondary 53A30

1 Introduction

Einstein metrics have a distinguished history in geometry and physics. An area of intense interest is the study of asymptotically hyperbolic Einstein metrics, which are also termed conformally compact Poincaré-Einstein metrics if the underlying space is compact with boundary. These geometric structures were introduced by Fefferman and Graham in [8] in connection with the ambient metric, and as a tool for studying the conformal geometry of the boundary. The relationship between that conformal structure and the geometry of the Riemannian interior has recently been studied intensively using spectral and scattering tools [9, 18, 19, 20, 30], and related formal asymptotics [1, 3, 10, 17]. This relationship is the geometric problem underlying the so-called AdS/CFT correspondence of String Theory [26, 31].

A certain class of Poincaré-Einstein metrics admit an expedient formulation in terms of tractor calculus. In fact, any standard tractor I on a conformal space (M, c), which is parallel with respect to the canonical tractor connection ∇, corresponds to a so-called almost Einstein structure σ_I (cf. [12, 21]). In general, an almost Einstein structure σ_I has a non-trivial scale singularity set $\Sigma(\sigma_I)$, and off this scale singularity the conformal structure c contains an Einstein metric, which stems from σ_I. In case the scale singularity $\Sigma(\sigma_I)$ is a hypersurface in M this Einstein metric is asymptotically hyperbolic at $\Sigma(\sigma_I)$ (on *both sides* of the hypersurface). Hence, if the underlying conformal space (M, c) is a closed space, i.e., a compact space without boundary, we say here the almost Einstein structure σ_I gives rise to a *conformally closed* Poincaré-Einstein metric on the closed space (M, c) with scale singularity $\Sigma(\sigma_I)$. In this sense *conformally closed* Poincaré-Einstein metrics are a subclass of conformally compact Poincaré-Einstein metrics.

In the work [14] we have studied conformal spaces (M, c), which admit multiple almost Einstein structures. A main result of this work is the finding that, if (M, c) is a closed conformal space with multiple almost Einstein structures, then (M, c) is either conformally equivalent to the Möbius sphere (the standard model of Riemannian conformal geometry), or else any almost Einstein structure σ on (M, c) admits a hypersurface scale singularity $\Sigma(\sigma)$. Immediately the question arises whether these hypersurface scale singularities can intersect non-trivially on (M, c). On the Möbius sphere this certainly happens. The main motivation of the current work is the geometric description of such instances, in general.

Using the existence result for solutions of a non-characteristic PDE, one can introduce so-called special defining functions for the boundary of a Poincaré-Einstein space. With respect to such a special defining function the Poincaré-Einstein metric on the interior can be presented in normal form near the boundary (cf. [16]). Similarly, we will describe in the current work normal form metrics in the conformal class c of a closed space M with multiple almost Einstein structures, whose scale singularities have a non-trivial intersection set $\Sigma(\mathcal{S})$. In short, the resulting geometric description says that (M, c) is (locally near $\Sigma(\mathcal{S})$) a so-called *collapsing sphere product* alias S^l-*doubling*. The S^l-doubling construction was introduced in [24]. This construction assigns to any given asymptotically hyperbolic space $\overline{F}^{m+1}$ and any number $l \geq 0$ in a natural way a smooth conformal manifold $D_l\overline{F}$ of dimension $m+l+1$ (without boundary). The *bulk* of this construction is simply the Riemannian product space of a round unit l-sphere S^l with the interior Einstein space F, i.e., a special Einstein product in the sense of [13]. Then the l-sphere S^l collapses at the boundary of $\overline{F}$, which gives rise to the so-called *pole* of the collapsing sphere product. Exactly at this pole the hypersurface scale singularities of certain multiple almost Einstein structures on $D_l\overline{F}$ intersect. An important tool for the prove of this geometric description and the normal form in our situation uses the classical results about essential conformal transformation groups on closed conformal spaces by Obata and Lelong-Ferrand in [27, 25].

The existence of a non-trivial intersection set $\Sigma(\mathcal{S})$ of hypersurface scale singularities for multiple almost Einstein structures on a conformal space (M^n, c) has further interesting implications. For example, it is well known that the hypersurface scale singularity $\Sigma(\sigma)$ of a single almost Einstein structure σ is totally umbilic in (M^n, c). In fact, the use of special defining functions shows that $\Sigma(\sigma)$ is a minimal hypersurface with respect to certain metrics in c on M. This feature generalises to the intersection set $\Sigma(\mathcal{S})$ for $\ell > 1$ hypersurface scale singularities: the set $\Sigma(\mathcal{S})$ is a totally umbilic submanifold of codimension ℓ, and $\Sigma(\mathcal{S})$ is also minimal in M with respect to certain metrics in c. Moreover, the conformal holonomy group of a space (M^n, c) with intersection set $\Sigma(\mathcal{S}) \neq \emptyset$ is decomposable. It was proven in [22] that a conformal space (M^n, c) with decomposable conformal holonomy is locally almost(!) everywhere conformally equivalent to a special Einstein product (cf. also [2]). In fact, the intersection set $\Sigma(\mathcal{S})$ of hypersurface scale singularities, which stem from $\ell > 1$ linearly independent ∇-parallel standard tractors (i.e. almost Einstein structures), provides in general an example for such points, where (M^n, c) is locally not conformally equivalent to a special Einstein product. This feature will be discussed in detail in [23]. Finally, note that we have constructed an explicit Ricci-flat Fefferman-Graham ambient metric for special Einstein products in [13]. In case (M^n, c) is closed with $\Sigma(\mathcal{S}) \neq \emptyset$ we can construct an explicit Ricci-flat Fefferman-Graham ambient metric for (M^n, c) as well (see [24]), i.e., we have found an explicit Fefferman-Graham ambient metric construction for certain almost Einstein spaces!

In Section 2 to 4 we review topics about conformal geometry, tractor calculus, almost Einstein structures, Poincaré-Einstein spaces, conformal Killing p-forms and special Einstein products as we will need them during the course of the paper. In Section 5 we discuss multiple almost Einstein structures (with scale singularity) on a conformal space (M, c). The existence of ℓ linearly independent ∇-parallel standard tractors I_i gives rise via the wedge product to a ∇-parallel ℓ-form tractor α, which in turn corresponds to a so-called decomposable nc-Killing $(\ell-1)$-form α_- (cf. [21]). In case all I_i's have hypersurface singularities $\Sigma(I_i)$ the zero set of α_- coincides with the intersection of the $\Sigma(I_i)$'s (cf. Theorem 5.2). Section 6 is the heart of the paper. There we derive under the assumption of closeness for (M, c) a normal form for the ℓ-form tractor α in a neighbourhood of its singular set $\Sigma(\mathcal{S}) = \bigcap_i \Sigma(I_i)$ (cf. Proposition 6.7). This normal form is the key to the geometric description of (M, c) with non-trivial intersection set $\Sigma(\mathcal{S})$. The precise statements are made in Section 7 (cf. Proposition 7.2 and Theorem 7.3). Finally, in Section 8 we discuss the extrinsic curvature properties of the singularity submanifold $\Sigma(\mathcal{S})$ in (M, c) (cf. Theorem 8.4 and 8.5). In particular, we present a tractor formulation, which implies total umbilicity in higher codimension (cf. Theorem 8.3).

2 Conformal geometry and tractor calculus

Let M^n be a smooth manifold of dimension $n \geq 3$. Recall that a *Riemannian conformal structure* on M is a smooth $\mathbb{R}_+$-ray subbundle $\mathcal{Q} \subset S^2T^*M$, whose fibre over $p \in M$ consists of conformally related positive definite metrics on T_pM. Smooth sections of $\mathcal{Q}$ are metrics on M and we denote the set of all such sections by c. Any two sections $g, \tilde{g} \in c$ are then related by $\tilde{g} = e^{2\varphi}g$ for some function $\varphi \in C^\infty(M)$, i.e., g and $\tilde{g}$ are conformally equivalent metrics on M. The principal

$\mathbb{R}_+$-bundle $\pi : \mathcal{Q} \to M$ induces for any representation $t \in \mathbb{R}_+ \mapsto t^{-w/2} \in \mathrm{End}(\mathbb{R})$, $w \in \mathbb{R}$, a natural real line bundle $\mathcal{E}[w]$ over M, which is called the conformal density bundle of weight w. If $\mathcal{V}$ is any vector bundle of tensors on M, we obtain for $w \in \mathbb{R}$ the tensor bundle $\mathcal{V}[w] := \mathcal{V} \otimes \mathcal{E}[w]$ of conformal weight w. We write $\boldsymbol{g}$ for the conformal metric on (M, c), that is the tautological section of $S^2T^*M[2] := S^2T^*M \otimes \mathcal{E}[2]$ determined by $\mathcal{Q}$. Note that every density bundle $\mathcal{E}[w]$, $w \in \mathbb{R}$, is trivialised by any metric $g \in c$, i.e., a section of $\mathcal{E}[w]$ is via g a real function on M.

In [29] the notion of *tractor bundles* in conformal geometry was introduced (see also [4]). For a given choice of metric $g \in c$, the standard tractor bundle $\mathcal{T} = \mathcal{T}M$ of the conformal space (M, c), may be identified with the direct sum

$$\mathcal{T} \ \cong_g \ \mathbb{R} \oplus TM \oplus \mathbb{R} \ ,$$

i.e., a section T in $\mathcal{T}$ consists of a triple (a, ψ, b), where a, b are real functions and ψ is a vector field on M. We also set $s_- := (1, 0, 0)$ and $s_+ := (0, 0, 1)$, which allows us to write $T = as_- + \psi + bs_+$ with respect to g. Under a conformal rescaling of g to $\tilde{g} = e^{2\varphi} \cdot g$ (by a smooth function φ) the triple (a, ψ, b) transforms by the rule

$$(\tilde{a}, \tilde{\psi}, \tilde{b}) \ = \ (\ e^{\varphi}a \ , \ e^{-\varphi} \cdot (\psi + a \cdot grad^g(\varphi)) \ , \ e^{-\varphi} \cdot (b - d\varphi(\psi) - \frac{a}{2}|grad^g\varphi|^2_g) \ ) \ , \tag{1}$$

i.e., $\tilde{g}$ gives rise to a different identification of the tractor bundle $\mathcal{T}$ with the direct sum $\mathbb{R} \oplus TM \oplus \mathbb{R}$. Transforming a triple (a, ψ, b) twice in a row with respect to some rescaling functions φ_1, φ_2 produces the same result as transforming (a, ψ, b) once with respect to $\varphi_1 + \varphi_2$. This explains the naturalness of the tractor bundle $\mathcal{T}$ for the conformal structure c on M. In fact, the transformation rule (1) shows that $\mathcal{T}$ invariantly admits a composition structure

$$\mathcal{T} \ = \ \mathcal{E}[1] \ \subset\!\!\!\!+ \ \ TM[-1] \ \subset\!\!\!\!+ \ \ \mathcal{E}[-1] \ ;$$

$\mathcal{E}[-1]$ may be naturally identified with a subbundle of $\mathcal{T}$ and $TM[-1]$ is a subbundle of the quotient bundle $\mathcal{T}/\mathcal{E}[-1]$. We denote by Π the natural projection from $\mathcal{T}$ to $\mathcal{E}[1]$.

The tractor bundle $\mathcal{T}$ of (M, c) carries an invariant metric $\langle \cdot, \cdot \rangle_{\mathcal{T}}$ of signature $(1, n+1)$ and a canonical invariant connection ∇, which preserves this tractor metric $\langle \cdot, \cdot \rangle_{\mathcal{T}}$. The metric $\langle \cdot, \cdot \rangle_{\mathcal{T}}$ is given for any $T = (a, \psi, b)$, $\hat{T} = (\hat{a}, \hat{\psi}, \hat{b}) \in \mathcal{T}$ with respect to $g \in c$ by

$$\langle T, \hat{T} \rangle_{\mathcal{T}} \ = \ a\hat{b} + \hat{a}b \ + \ g(\psi, \hat{\psi}) \ . \tag{2}$$

The tractor connection ∇ satisfies

$$\nabla_X \begin{pmatrix} a \\ \psi \\ b \end{pmatrix} = \begin{pmatrix} X(a) - g(X, \psi) \\ \nabla^g_X \psi + b \cdot X - a \cdot \mathsf{P}^g(X) \\ X(b) + \mathsf{P}^g(X, \psi) \end{pmatrix} \tag{3}$$

for any $X \in TM$, where ∇^g denotes the Levi-Civita connection of g on TM, and

$$\mathsf{P}^g = \frac{1}{n-2} \left(\frac{scal^g}{2(n-1)} - Ric^g \right)$$

is the Schouten tensor in terms of the Ricci tensor Ric^g and the scalar curvature $scal^g$ of g. With $\mathsf{P}^g(X)$ we denote the vector in TM, which is dual to $\mathsf{P}^g(X, \cdot)$ via g.

More generally, we have the *p-form tractor bundles* on a conformal space (M^n, c) of dimension $n \geq 3$. For the definition of these bundles, let $\mathcal{T}^* = \mathcal{T}^*M$ denote the dual vector bundle to the standard tractor bundle $\mathcal{T}$ on (M, c). The dual tractor bundle $\mathcal{T}^*$ is canonically identified with $\mathcal{T}$ via the tractor metric $\langle \cdot, \cdot \rangle_{\mathcal{T}}$. Then we obtain from $\mathcal{T}^*$ via the exterior product $\wedge$ the p-form tractor bundles $\Lambda^p\mathcal{T}^*$, $p = 0, \cdots, n+2$. The p-form tractor bundles $\Lambda^p\mathcal{T}^*$ admit the invariant composition structure

$$\Lambda^{p-1}T^*M[p] \hookleftarrow \left(\Lambda^pT^*M[p] \oplus \Lambda^{p-2}T^*M[p-2]\right) \hookleftarrow \Lambda^{p-1}T^*M[p-2] \qquad (4)$$

with natural projection

$$\Pi : \Lambda^p\mathcal{T}^* \ \rightarrow \ \Lambda^{p-1}T^*M[p],$$

i.e., for any section $\alpha \in \Gamma(\Lambda^p\mathcal{T}^*)$ the projection $\Pi(\alpha)$ is a $(p-1)$-form on (M, c) of conformal weight p.

The composition structure (4) splits with respect to any $g \in c$ into the direct sum

$$\Lambda^{p-1}T^*M \ \oplus \ \Lambda^pT^*M \ \oplus \ \Lambda^{p-2}T^*M \ \oplus \ \Lambda^{p-1}T^*M \ ,$$

and thus $\Pi(\alpha)$ is identified via g with a $(p-1)$-form $\alpha_- \in \Omega^{p-1}(M)$. We write $\alpha_- = \Pi_g(\alpha)$ to indicate the scale dependence of α_- on $g \in c$. If we denote by $s_-^\flat$ the dual of $s_- \in \mathcal{T}$ and by $s_+^\flat$ the dual of $s_+ \in \mathcal{T}$, then any p-form tractor α can be written with respect to g as

$$\alpha \ = \ s_-^\flat \wedge \alpha_- \ + \ \alpha_0 \ + \ s_-^\flat \wedge s_+^\flat \wedge \alpha_\mp \ + \ s_+^\flat \wedge \alpha_+ \qquad (5)$$

with uniquely determined differential forms $\alpha_- \in \Omega^{p-1}(M)$, $\alpha_0 \in \Omega^p(M)$, $\alpha_\mp \in \Omega^{p-2}(M)$ and $\alpha_+ \in \Omega^{p-1}(M)$ (cf. [21]). This makes the splitting of (4) explicit. Alternatively, we write the quadruple $(\alpha_-, \alpha_0, \alpha_\mp, \alpha_+)$ as a diamond

$$\alpha \ \cong_g \ \begin{pmatrix} & \alpha_- & \\ \alpha_0 & & \alpha_\mp \\ & \alpha_+ & \end{pmatrix} .$$

The transformation of the quadruple $(\alpha_-, \alpha_0, \alpha_\mp, \alpha_+)$ for conformally related metrics $g, \tilde{g} \in c$ can be deduced from (1). Note that, since $\Pi(\alpha)$ has conformal weight p, the $(p-1)$-form $\alpha_- = \Pi_g(\alpha)$ rescales for $\tilde{g} = e^{2\varphi}g$ by $\tilde{\alpha}_- = \Pi_{\tilde{g}}(\alpha) = e^{p\varphi} \cdot \alpha_-$. Also note that the tractor connection (3) extends naturally to covariant derivatives ∇ on all p-form tractor bundles $\Lambda^p\mathcal{T}^*$. Explicit formulae for ∇ acting on $\Lambda^p\mathcal{T}^*$ with respect to a metric $g \in c$ can be deduced from (3) (cf. e.g. [21, 15]).

3 Almost Einstein structures and Poincaré-Einstein metrics

A standard tractor $T \in \Gamma(\mathcal{T})$ with $\nabla T = 0$ on (M, c) is called ∇-parallel. The corresponding density $\sigma = \Pi(T) \in \Gamma(\mathcal{E}[1])$ is an *almost Einstein structure* on (M, c). We explain this notion and recall some basic facts about almost Einstein structures (cf. e.g. [12, 21]). Moreover, we explain a relation of almost Einstein structures with hypersurface singularity and Poincaré-Einstein metrics.

3.1 Almost Einstein structures

As discussed in [4], there exists an invariant second order differential operator $\mathcal{D} : \Gamma(\mathcal{E}[1]) \rightarrow \Gamma(\mathcal{T})$ for densities of weight 1 over a conformal space (M^n, c), $n \geq 3$. This differential operator $\mathcal{D}$ is acting with respect to a metric $g \in c$ on real functions $s \in C^\infty(M)$ by

$$\mathcal{D}^g s \ = \ \begin{pmatrix} s \\ grad^g(s) \\ \Box^g s \end{pmatrix} , \qquad (6)$$

where $\Box^g := -\frac{1}{n}(\Delta^g - tr_g \mathsf{P}^g)$ with Laplacian $\Delta^g s = tr_g \nabla^g ds$. For an invariant construction of $\mathcal{D}$ see [11]. It is a matter of fact that for densities $\sigma \in \Gamma(\mathcal{E}[1])$ the equation

$$\nabla \mathcal{D}\sigma \;=\; 0 \tag{7}$$

is equivalent to

$$\text{trace-free part of}\,(\nabla^g ds \;-\; s \cdot \mathsf{P}^g) \;=\; 0\;, \tag{8}$$

where s is the function which corresponds via a choice of metric $g \in c$ to the density σ. In turn, it is also true that if a tractor T satisfies $\nabla T = 0$ then the component $s = \Pi_g(T) = \langle T, s_+\rangle_{\mathcal{T}} \in C^\infty(M)$ of T with respect to $g \in c$ satisfies (8) and $\mathcal{D}^g s = T$.

It is well known that a solution σ of (7) without zeros on (M, c) has the property that $\tilde{g} = \sigma^{-2}\boldsymbol{g}$ is an Einstein metric on M in the conformal class c, i.e., $Ric^{\tilde{g}} = \frac{scal^{\tilde{g}}}{n} \cdot \tilde{g}$. On the other hand, if a metric $s^{-2} \cdot g$ in the conformal class $c = [g]$ on M is Einstein then s satisfies (8) with respect to g. However, in general one has to expect that a solution of (7), resp., (8), admits a non-trivial zero set on M. In this case the existence of an Einstein metric in c, which exists globally on M is not guaranteed. Note that, if a solution s of (8) vanishes identically on some open subset of M, then it follows directly from expression (6) for the parallel standard tractor $\mathcal{D}^g s$ that $s \equiv 0$ is the trivial solution on M. This argument proves that the zero set of a non-trivial solution s of (8) is always singular on M, i.e., the complement of the zero set of s is dense in M.

Definition 3.1 *Let (M^n, c), $n \geq 3$, be a conformal space with standard tractor bundle $\mathcal{T}$.*

1. *We call (M^n, c) an almost Einstein space if a ∇-parallel standard tractor $I \neq 0$ exists.*
2. *If $I \neq 0$ in $\Gamma(\mathcal{T})$ is ∇-parallel, then we call $\sigma = \Pi(I)$ an almost Einstein structure of (M, c). Accordingly, if $g \in c$ is a choice of metric, then we call a non-trivial solution $s \in C^\infty(M)$ of (8) an almost Einstein structure of (M, g).*
3. *If $I \neq 0$ in $\Gamma(\mathcal{T})$ is ∇-parallel, then we denote the singular zero set of the almost Einstein structure $\sigma = \Pi(I)$ by $\Sigma(\sigma)$ (resp. $\Sigma(I)$). We call $\Sigma(\sigma)$ the scale singularity set of σ (resp. I).*

On an almost Einstein manifold (M, c, I) (or (M, c, σ) with $\sigma = \Pi(I)$) we shall write $S(I)$ (or $S(\sigma)$) as a shorthand for $-\langle I, I\rangle_{\mathcal{T}}$. This may be viewed as scalar curvature quantity for the structure, since off the singularity set $\Sigma(I)$ we have $S(I) = \frac{scal^g}{n(n-1)}$ for the metric $g = \sigma^{-2}\boldsymbol{g}$. The following result is a direct consequence of (2) and (3).

Theorem 3.2 *[12] Let (M, c, I) be an almost Einstein space of Riemannian signature with $\sigma = \Pi(I)$ and $S(\sigma) = -\langle I, I\rangle_{\mathcal{T}}$. If $S(\sigma) > 0$ then $\Sigma(\sigma)$ is empty and $(M, \sigma^{-2}\boldsymbol{g})$ is Einstein with positive scalar curvature; if $S(\sigma) = 0$ then $\Sigma(\sigma)$ is either empty or consists of isolated points and $(M \setminus \Sigma(\sigma), \sigma^{-2}\boldsymbol{g})$ is Ricci-flat; if $S(\sigma) < 0$ then the scale singularity set $\Sigma(\sigma)$ is either empty or else is a smooth hypersurface, and $(M \setminus \Sigma(\sigma), \sigma^{-2}\boldsymbol{g})$ is Einstein of negative scalar curvature.*

3.2 The relation to Poincaré-Einstein spaces

In the present article we are interested in the case of (multiple) almost Einstein structures σ with hypersurface singularity. This case is closely related to the geometry of asymptotically hyperbolic metrics and Poincaré-Einstein spaces as follows. Let (M, c) be a Riemannian conformal space admitting an almost Einstein structure $\sigma \in \Gamma(\mathcal{E}[1])$ with *scalar curvature* $S(\sigma) = -1$ and $\Sigma(\sigma) \neq \emptyset$. If $g \in c$ is a metric on M, then the corresponding solution $s = \Pi_g(\mathcal{D}\sigma)$ of (8) vanishes exactly on $\Sigma(\sigma)$ and we have $ds \neq 0$ for any $p \in \Sigma(\sigma)$. Hence the real function s has positive and negative values on M. We set $\overline{M_+}(s) := \{x \in M | s(x) \geq 0\}$. By construction, the space $\overline{M_+}(s)$ is a smooth manifold with boundary $\Sigma(\sigma)$, for which s serves as a defining function. The interior $M_+(s)$ of $\overline{M_+}(s)$ is the open subset of M, where s is positive.

Since $\langle \mathcal{D}\sigma, \mathcal{D}\sigma\rangle_{\mathcal{T}} = 1$, we have $|grad^g s|_g = 1$ on the hypersurface $\Sigma(\sigma)$. This shows that the metric $g_+ = s^{-2}g$ is *asymptotically hyperbolic*, i.e., the sectional curvature of g_+ is asymptotically constant -1 at each boundary point of $\Sigma(\sigma)$ in $\overline{M_+}(s)$. The metric g_+ is also Einstein with

$$Ric^{g_+} \;=\; -(n-1)g_+$$

on the interior $M_+(s)$. Thus $(\overline{M_+}(s), g_+)$ is a *Poincaré-Einstein space* in the usual sense (cf. e.g. [8, 16]). The boundary $\Sigma(\sigma)$ with induced conformal structure $[g|_{T\Sigma(\sigma)}]$ is called a conformal infinity of the interior space $(M_+(s), g_+)$. In case $\overline{M_+}(s)$ is compact the Poincaré-Einstein metric g_+ on $M_+(s)$ is called *conformally compact* (since the metric $s^2 g_+$ is obviously smooth on the compact space $\overline{M_+}(s)$ up to the boundary $\Sigma(\sigma)$). In this situation the interior $(M_+(s), g_+)$ is geodesically complete.

Definition 3.3 *Let σ be an almost Einstein structure with $S(\sigma) = -1$ and $\Sigma(\sigma) \neq \emptyset$ on a closed (= compact without boundary) conformal space (M, c). Then we call $g_+ = \sigma^{-2}\boldsymbol{g}$ on $M \setminus \Sigma(\sigma)$ a conformally closed Poincaré-Einstein metric.*

Note that the construction as described above can also be applied to the almost Einstein structure $-s$ on (M, g). In general, the spaces $M_+(s)$ and $M_+(-s)$ equipped with the metric $g_+ = s^{-2}g$ are not isometric, even not locally near the hypersurface $\Sigma(\sigma)$. In case the two spaces are isometric the Poincaré-Einstein metric g_+ is said to be *even* (see Definition 3.6).

It is also well known that if σ is an almost Einstein structure on (M, c) with $S(\sigma) < 0$ and $\Sigma(\sigma) \neq \emptyset$, then the scale singularity set $\Sigma(\sigma)$ is a *totally umbilic* smooth hypersurface in (M, c) (see Section 8). In fact, the scale singularity set $\Sigma(\sigma)$ satisfies an even stronger assertion.

Lemma 3.4 *[16] Let (M, c, I) be an almost Einstein space with $S(I) = -1$ and $\Sigma(I) \neq \emptyset$ and let $g \in c$ be an arbitrary smooth metric on M. Then there exists a unique smooth function ω on some open neighbourhood U_ω of $\Sigma(I)$ in M with $\omega|_{\Sigma(I)} \equiv 1$ such that*

$$I \;\cong_{\tilde g}\; \begin{pmatrix} \tilde s \\ grad^{\tilde g}\tilde s \\ 0 \end{pmatrix} \tag{9}$$

with respect to the metric $\tilde g := e^{2\omega}g$ on U_ω.

Proof. We set $s = \Pi_g(I)$ and $\tilde s = \Pi_{\tilde g}(I)$ with respect to $\tilde g = e^{2\omega}g$. Then we have $\tilde s = e^\omega s$ and $d\tilde s = e^\omega(ds + s d\omega)$. The condition (9) for $\tilde s$ on some neighbourhood of $\Sigma(I)$ is equivalent to $|grad^{\tilde g}\tilde s|_{\tilde g} \equiv 1$. The latter condition in turn is equivalent to

$$2g(grad^g s, grad^g\omega) + s|grad^g\omega|_g^2 \;=\; \frac{1 - |grad^g s|_g^2}{s} \;=\; \Box^g s\ .$$

This is a non-characteristic first oder PDE for ω, so there exists a solution in some neighbourhood U_ω of $\Sigma(I)$ in M with arbitrarily prescribed ω on $\Sigma(I)$. A solution ω of this PDE with boundary condition $\omega|_{\Sigma(I)} \equiv 1$ is unique. □

Note that I restricts to a *normal standard tractor* on the hypersurface $\Sigma(I)$ in (M, c) in the sense of [4]. In general, the normal standard tractor of a hypersurface Σ is given (up to a sign) with respect to a metric on the ambient space by $N_\Sigma = (0, n, H)$, where n is a unit normal vector field on Σ in the ambient space, and H denotes the mean curvature of Σ in direction of n. This shows that with respect to a metric $\tilde g = e^{2\omega}g$ as in Lemma 3.4 for any boundary data, the mean curvature $H_{\tilde g}$ of $\Sigma(I)$ vanishes identically. We conclude that $\Sigma(I)$ is a minimal and totally geodesic smooth hypersurface of $(U_\omega, \tilde g)$ (cf. Section 8).

A metric of the form $\tilde g = e^{2\omega}g$ as in Lemma 3.4 has another special feature. It can be written in a certain normal form. To derive this normal form, we consider the almost Einstein structure

$\tilde{s}$ with respect to $\tilde{g}$ as a coordinate function on U_ω. For convenience, let us assume that $\Sigma(I)$ is closed. Then there exists some $\varepsilon > 0$ such that $\Sigma(I) \times (-\varepsilon, \varepsilon)$ is uniquely identified with a neighbourhood U_ε of $\Sigma(I)$ in U_ω via the flow of the gradient $grad^{\tilde{g}}\tilde{s}$.

Proposition 3.5 *[16] Let (M, c, I) be an almost Einstein space with $S(I) = -1$ and $\Sigma(I) \neq \emptyset$.*

1. *A metric $\tilde{g}$ as in Lemma 3.4 restricted to a neighbourhood U_ε of $\Sigma(I)$ is isometric to $d\tilde{s}^2 + g_{\tilde{s}}$ on $\Sigma(I) \times (-\varepsilon, \varepsilon)$, where $g_{\tilde{s}}$ is some smooth family of metrics on $\Sigma(I)$.*
2. *The asymptotically hyperbolic Einstein metric $g_+ = \tilde{s}^{-2} \cdot \tilde{g}$ on $U_\varepsilon \setminus \Sigma(I)$ is then given in normal form by $\tilde{s}^{-2} \cdot \left(d\tilde{s}^2 + g_{\tilde{s}}\right)$ on $\Sigma(I) \times \{(-\varepsilon, 0) \cup (0, \varepsilon)\}$.*

The proof of the normal form (1) and (2) of Proposition 3.5 works the same as for conformally compact Poincaré-Einstein spaces with boundary in [16]. Only a slight modification is necessary, since we are dealing here with a hypersurface $\Sigma(I)$ in a space M without boundary. We omit the details of this modified proof. Also note that the assumption of closeness for $\Sigma(I)$ is only made in order to have a global parameter ε at hand. In the non-compact case the normal forms of Proposition 3.5 are locally valid when $\tilde{g}$ is restricted to an appropriate neighbourhood of an arbitrary point of the hypersurface $\Sigma(I)$.

The almost Einstein structure $\tilde{s} = \Pi_{\tilde{g}}(I)$ on a neighbourhood U_ω as given in Lemma 3.4 is called a *special defining function* for the hypersurface $\Sigma(I)$. The corresponding Poincaré-Einstein metric g_+ on $M \setminus \Sigma(I)$ is called *even* if the map $(x, \tilde{s}) \in \Sigma(I) \times (-\varepsilon, \varepsilon) \mapsto (x, -\tilde{s}) \in \Sigma(I) \times (-\varepsilon, \varepsilon)$ is an isometry for the normal form metric (1) $d\tilde{s}^2 + g_{\tilde{s}}$ of Proposition 3.5, i.e., we simply have $g_{\tilde{s}} = g_{-\tilde{s}}$ for the family of metrics on $\Sigma(I)$. This definition of evenness is in the sense of [10]. The definition does not depend on the special defining function $\tilde{s}$, resp., the boundary data for the solution ω of Lemma 3.4 (which determine $\tilde{s}$).

Definition 3.6 *Let (M, c, I) be an almost Einstein space with $S(I) = -1$ and $\Sigma(I) \neq \emptyset$. We call $\sigma = \Pi(I)$ an even almost Einstein structure on (M, c) if the corresponding Poincaré-Einstein metric $g_+ = \sigma^{-2}\boldsymbol{g}$ on $M \setminus \Sigma(I)$ is even in the sense of [10].*

Alternatively, on a closed space (M, c) one can say an almost Einstein structure $\sigma = \Pi(I)$ with hypersurface singularity $\Sigma(I) \neq \emptyset$ is even if and only if there exists a conformal diffeomorphism Φ of (M, c), which satisfies $\Phi^*\sigma = -\sigma$.

The main intention of the present article is to generalise the results of Lemma 3.4 and Proposition 3.5 to the case when (M, c) is a closed conformal space with multiple almost Einstein structures and intersecting hypersurface singularities. We will develop the corresponding normal form problem in the following sections (cf. Proposition 6.7). It will turn out that multiple almost Einstein structures (resp. conformally closed Poincaré-Einstein metrics) with intersecting hypersurface singularities are typically even.

4 Normal conformal Killing p-forms and special Einstein products

The equation $\nabla\alpha = 0$ for a $(p+1)$-form tractor $\alpha \in \Gamma(\Lambda^{p+1}\mathcal{T}^*)$ on (M, c) is eligible for any $p = 0, \ldots, n+1$. If this equation is satisfied, we call α a ∇-parallel $(p+1)$-form tractor. The corresponding p-form $\varrho := \Pi(\alpha)$ of conformal weight $p+1$ is a so-called *normal conformal Killing p-form* on (M, c) (cf. [21]). For $p = 0$ this is an almost Einstein structure as introduced in the previous section. For $0 < p < n$ and *simple* α the corresponding structure is locally up to singularities that of a *special Einstein product* (cf. [22, 2, 13]).

4.1 Nc-Killing p-forms

Let (M^n,c) be a conformal space of dimension $n\geq 3$. The $(p+1)$-form tractor bundle $\Lambda^{p+1}\mathcal{T}^*$ is equipped with the induced tractor connection and $\nabla\alpha=0$ is a conformally covariant PDE for $\alpha\in\Gamma(\Lambda^{p+1}\mathcal{T}^*)$. Similar to (6), a certain conformally covariant second order differential operator

$$\mathcal{D}:\ \Omega^p(M)[p+1]\ \rightarrow\ \Gamma(\Lambda^{p+1}\mathcal{T}^*)$$

was introduced in [7], which acts naturally on p-forms of conformal weight $p+1$. The operator $\mathcal{D}$ satisfies $\Pi\circ\mathcal{D}=id$ on $\Omega^p(M)[p+1]$, and moreover, $\mathcal{D}\circ\Pi$ is the identity on ∇-parallel $(p+1)$-form tractors. The operator $\mathcal{D}$ was computed in [21] with respect to a metric $g\in c$ when acting on nc-Killing p-forms $\alpha_-=\Pi_g(\alpha)$. As result we have

$$\mathcal{D}^g\alpha_-\quad=\quad\begin{pmatrix} & \alpha_- & \\ \frac{1}{p+1}d\alpha_- & & \frac{1}{n-p+1}d^*\alpha_- \\ & \Box_p\alpha_- & \end{pmatrix}, \tag{10}$$

where d^* denotes the codifferential with respect to g and

$$\Box_p\ :=\ \frac{1}{n-2p}\left(\frac{1}{p+1}d^*d+\frac{1}{n-p+1}dd^*+tr_g\mathsf{P}^g\right)\qquad\text{for } n\neq 2p\ , \tag{11}$$

which is $\Box_p=\frac{-1}{n-2p}(\Delta^g-tr_g\mathsf{P}^g)$ with Laplacian $\Delta^g:=tr_g(\nabla^g)^2$. For $n=2p$ the operator $\Box_{n/2}$ can be expressed with respect to a (local) g-orthonormal frame $\{v_1,\dots,v_n\}$ of TM by

$$\Box_{n/2}\alpha_-\ :=\ \frac{1}{n-p}\left(\ \frac{1}{p+1}d^*d\ +\ \sum_{i=1}^{n}v_i\ \lrcorner\ (\mathsf{P}(v_i,\cdot)\wedge\alpha_-)\ \right). \tag{12}$$

The immediate consequence of (10) and a formula for the induced connection ∇ from (3) is that the tractor equation $\nabla\alpha=0$ is equivalent with respect to an arbitrary $g\in c$ to the system

$$\nabla^g_X\alpha_- -\frac{1}{p+1}X\ \lrcorner\ d\alpha_- +\frac{1}{n-p+1}X^\flat\wedge d^*\alpha_-\ =\ 0 \tag{13}$$

$$-\mathsf{P}(X)^\flat\wedge\alpha_- +\frac{1}{p+1}\nabla^g_X d\alpha_- +X^\flat\wedge\Box_p\alpha_-\ =\ 0 \tag{14}$$

$$\mathsf{P}(X)\ \lrcorner\ \alpha_- +\frac{1}{n-p+1}\nabla^g_X d^*\alpha_- +X\ \lrcorner\ \Box_p\alpha_-\ =\ 0 \tag{15}$$

$$\frac{1}{p+1}\mathsf{P}(X)\ \lrcorner\ d\alpha_- +\frac{1}{n-p+1}\mathsf{P}(X)^\flat\wedge d^*\alpha_- +\nabla^g_X\Box_p\alpha_-\ =\ 0 \tag{16}$$

of partial differential equations for $\alpha_-=\Pi_g(\alpha)$, where $X\in TM$ is arbitrary with dual 1-form $X^\flat=g(X,\cdot)$. By construction, this set of equations is conformally covariant. In fact, the first equation (13) alone is conformally covariant and simply says that $\alpha_-=\Pi_g(\alpha)$ is a so-called *conformal Killing p-form* on (M,g), resp., $\varrho=\Pi(\alpha)$ is a conformal Killing p-form of weight $p+1$ on (M,c) (cf. [28]). If, in addition, a conformal Killing p-form α_- satisfies the equations (14) to (16), then we call α_- a *normal conformal Killing p-form*, resp., in short we say α_- is a *nc-Killing p-form* (cf. [21]). It follows immediately from (13) - (16) that the zero set $\Sigma(\alpha_-)$ of a non-trivial nc-Killing p-form α_- is singular. This also shows that the natural projection $\Pi:\Gamma(\Lambda^{p+1}\mathcal{T}^*)\rightarrow\Omega^p(M)[p+1]$ restricts to an isomorphism of the space of ∇-parallel $(p+1)$-form tractors and the space of nc-Killing p-forms of weight $p+1$ on (M,c).

If (M,c) is an oriented conformal space we can introduce a Hodge $*$-operator acting on $(p+1)$-form tractors. The $*$-operator is uniquely defined by the relation

$$\alpha\wedge*\alpha=\langle\alpha,\alpha\rangle_{\mathcal{T}}\cdot vol(\mathcal{T})$$

for any $\alpha \in \Lambda^{p+1}\mathcal{T}^*$, where $vol(\mathcal{T})$ denotes the volume form tractor due to the given orientation and $\langle\cdot,\cdot\rangle_{\mathcal{T}}$ is the induced tractor metric on $(p+1)$-form tractors (cf. [21]). Obviously, since the Hodge $*$-operator is ∇-parallel, a $(p+1)$-form tractor is ∇-parallel if and only if the corresponding $(n-p+1)$-form tractor $*\alpha$ is ∇-parallel. With the conventions of [21] we have $\Pi_g(*\alpha) = (-1)^p * \Pi_g(\alpha)$, i.e., if α_- is a nc-Killing p-form then the usual Hodge dual $*\alpha_-$ is a nc-Killing $(n-p)$-form as well. Note that locally on (M,c) we can always introduce some orientation. We will do this occasionally in order to have a Hodge dual at hand for local computations on p-forms.

With respect to $g \in c$ the content of the curvature Ω^∇ of the tractor connection ∇ consists of the Weyl tensor W^g and the Cotton tensor C^g. The Weyl tensor W^g is the traceless part of the Riemannian curvature tensor, and $C^g(X,Y) := (\nabla^g_X \mathsf{P}^g)(Y) - (\nabla^g_Y \mathsf{P}^g)(X)$ for $X, Y \in TM$. We also set $C^g(X,Y,Z) := g(C^g(X,Y),Z)$ for $Z \in TM$. The tractor curvature Ω^∇ acts trivially on any ∇-parallel $(p+1)$-form tractor α. This implies directly the following curvature conditions for the existence of a nc-Killing p-form $\alpha_- = \Pi_g(\alpha)$ on (M,g):

$$W^g(X,Y) \bullet \alpha_- \;=\; 0 \tag{17}$$

$$W^g(X,Y) \bullet d\alpha_- \;=\; (p+1)C^g(X,Y)^\flat \wedge \alpha_- \tag{18}$$

$$W^g(X,Y) \bullet d^*\alpha_- \;=\; -(n-p+1)C^g(X,Y) \lrcorner\, \alpha_- \tag{19}$$

$$W^g(X,Y) \bullet \Box_p\alpha_- \;=\; -\begin{pmatrix} \frac{1}{p+1}C^g(X,Y) \lrcorner\, d\alpha_- \\ +\ \frac{1}{n-p+1}C^g(X,Y)^\flat \wedge d^*\alpha_- \end{pmatrix}, \tag{20}$$

where $\bullet$ denotes the induced action of $End(TM)$ on differential forms ΛT^*M. Moreover, we define the Bach tensor by

$$B^g(X,Y) \;:=\; \sum_{i=1}^n (\nabla^g_{v_i} C^g)(Y, v_i, X) - \sum_{i=1}^n W^g(\mathsf{P}^g(v_i), X, Y, v_i)$$

with respect to some local g-orthonormal frame $\{v_1,\ldots,v_n\}$. Taking the divergence on both sides of the equations (17) - (20) results in

$$(n-4)\cdot C^g_X \bullet \alpha_- \;=\; 0 \tag{21}$$

$$(n-4)\cdot C^g_X \bullet d\alpha_- \;=\; -(p+1)\cdot B^g(X)^\flat \wedge \alpha_- \tag{22}$$

$$(n-4)\cdot C^g_X \bullet d^*\alpha_- \;=\; (n-p+1)\cdot B^g(X) \lrcorner\, \alpha_- \tag{23}$$

$$(n-4)\cdot C^g_X \bullet \Box_p\alpha_- \;=\; \begin{pmatrix} \frac{1}{p+1}B^g(X) \lrcorner\, \alpha_- \\ +\ \frac{1}{n-p+1}B^g(X)^\flat \wedge \alpha_- \end{pmatrix} \tag{24}$$

where $C^g_X := C^g(\cdot,\cdot,X)$ for $X \in TM$.

4.2 Special Einstein products

Let $\alpha \in \Lambda^{p+1}_x\mathcal{T}^*$ be some $(p+1)$-form tractor at a point x of (M,c). We say α is a *simple* (or *decomposable*) tractor if α is a non-vanishing $\wedge$-product $\alpha_1 \wedge \cdots \wedge \alpha_{p+1}$ of 1-form tractors α_i, $i = 1,\ldots,p+1$. As one can easily see from (5), if α is simply so are the corresponding non-trivial exterior forms $\alpha_-, \alpha_0, \alpha_\mp$ and α_+ with respect to any $g \in c$. In particular, if α is a simple ∇-parallel $(p+1)$-form tractor on (M,c), then the corresponding nc-Killing p-form $\alpha_- \in \Omega^p(M)$ is simple off its zero set $\Sigma(\alpha_-)$ for any $g \in c$. If we assume in addition that the singularity set $\Sigma(\alpha_-)$ is empty, then we have a non-vanishing length function $e^{-\varphi} := |\alpha_-|_g$ with respect to $g \in c$. In this case we can rescale the metric g by $\tilde{g} = e^{2\varphi}g$ such that the rescaled nc-Killing p-form

$\tilde{\alpha}_- = \Pi_{\tilde{g}}(\alpha) = e^{(p+1)\varphi}\alpha_-$ has constant norm 1. It was observed in [22] that simple ∇-parallel $(p+1)$-form tractors without singularities occur typically on special Einstein products.

Definition 4.1 *Let (M_1, g_1) and (M_2, g_2) be Einstein spaces of dimension $n_1 > 0$ and $n_2 > 0$, respectively, such that $n_1 + n_2 \geq 3$ and*

$$n_1(n_1-1)scal^{g_2} \;=\; -n_2(n_2-1)scal^{g_1}\ . \tag{25}$$

Then we call the Riemannian product space $(M, g) := (M_1 \times M_2, g_1 \times g_2)$ a special Einstein product.

Note that Definition 4.1 is also valid for $n_1 \leq 2$ or $n_2 \leq 2$. Here any 1-dimensional space is considered to be Einstein. A 2-dimensional Einstein space has constant sectional curvature.

If $(M, g_1 \times g_2)$ is a special Einstein product of oriented spaces, then the (pull-backs of the) volume forms $vol(g_1)$ and $vol(g_2)$ are nc-Killing forms of degree n_1 and n_2, respectively, on M with conformal structure $c = [g_1 \times g_2]$. Moreover, one can easily see from (10) that $\mathcal{D}(vol(g_1)) \in \Gamma(\Lambda^{n_1+1}\mathcal{T}^*)$ and $\mathcal{D}(vol(g_2)) \in \Gamma(\Lambda^{n_2+1}\mathcal{T}^*)$ are simple ∇-parallel form tractors. On the other hand, we have the following result.

Theorem 4.2 *[22] Let α be a simple ∇-parallel $(p+1)$-form tractor with $0 < p < n$ such that $\Pi(\alpha)$ does nowhere vanish on (M, c). Then let $\tilde{g} \in c$ be the metric, for which the p-form $\tilde{\alpha}_- = \Pi_{\tilde{g}}(\alpha)$ has constant norm 1.*

1. *The p-form $\tilde{\alpha}_-$ is simple and parallel with respect to the Levi-Civita connection $\nabla^{\tilde{g}}$ of $\tilde{g}$.*
2. *The metric $\tilde{g}$ is locally isometric to a special Einstein product $g_1 \times g_2$.*

Note that, in Theorem 4.2, at least one of the scalar curvatures $scal^{g_1}$ or $scal^{g_2}$ is non-zero if $\langle\alpha, \alpha\rangle_{\mathcal{T}} \neq 0$. In case α is null $g_1 \times g_2$ is a product of Ricci-flat metrics, i.e., the product itself is Ricci-flat. If M in Theorem 4.2 is a simply connected closed space, then $(M, \tilde{g})$ is globally a special Einstein product of two simply connected closed spaces.

5 Multiple almost Einstein structures

Let (M^n, c) be a Riemannian conformal space of dimension $n \geq 3$. We use the following standard notions in regard to (M, c). The set of all smooth conformal transformations of (M^n, c) is denoted by $Aut(M, c)$, i.e., $Aut(M, c)$ consists of all diffeomorphism Φ of M with $\Phi^*\boldsymbol{g} = \boldsymbol{g}$. The set $Aut(M, c)$ is in a natural way a finite dimensional Lie group, which acts smoothly on (M, c). The corresponding Lie algebra is denoted by $\mathfrak{aut}(M, c)$. Moreover, we have the set

$$\mathfrak{inf}(M, c) \;:=\; \{\ V \in \mathfrak{X}(M) |\ L_V\boldsymbol{g} = 0\ \}$$

of conformal Killing vector fields on (M, c). The set $\mathfrak{inf}(M, c)$ equipped with the commutator $[\cdot, \cdot]$ of vector fields is a finite dimensional Lie algebra. In general, $\mathfrak{aut}(M, c)$ is properly contained in $\mathfrak{inf}(M, c)$, since not any conformal Killing vector has to be complete on M.

Furthermore, let $\mathcal{T}$ be the standard tractor bundle of (M, c). Then we denote by $\mathcal{P}(\mathcal{T}) \subset \Gamma(\mathcal{T})$ the set of ∇-parallel standard tractors. The set $\mathcal{P}(\mathcal{T})$ is a vector space of finite dimension $\mathcal{N} := dim(\mathcal{P}(\mathcal{T})) \leq n+2$. Since the tractor connection ∇ preserves the tractor metric $\langle\cdot, \cdot\rangle_{\mathcal{T}}$, a symmetric bilinear form $\langle\cdot, \cdot\rangle_{\mathcal{P}}$ is naturally induced on $\mathcal{P}(\mathcal{T})$. In fact, any tractor $T \in \mathcal{P}(\mathcal{T})$ is uniquely determined by its value T_x at a single point $x \in M$, and the definition $\langle T, T\rangle_{\mathcal{P}} := \langle T_x, T_x\rangle_{\mathcal{T}}$ is independent of $x \in M$. This shows that with respect to any (orthonormal) basis of $\mathcal{T}_x$, $x \in M$, the space $(\mathcal{P}(\mathcal{T}), \langle\cdot, \cdot\rangle_{\mathcal{P}})$ is identified with a subspace of the $(n+2)$-dimensional *Minkowski space* $\mathbb{R}^{1,n+1}$. The symmetric bilinear form $\langle\cdot, \cdot\rangle_{\mathcal{P}}$ on $\mathcal{P}(\mathcal{T})$ admits a unique signature $(\mathcal{N}_+, \mathcal{N}_-, \mathcal{N}_0)$, where $\mathcal{N}_+$ denotes the maximal possible dimension of a positive definite subspace in $\mathcal{P}(\mathcal{T})$. Accordingly, $\mathcal{N}_-, \mathcal{N}_0$ are defined as the maximal dimensions for negative definite and totally null subspaces in $\mathcal{P}(\mathcal{T})$, respectively. Note that for the case of Riemannian conformal geometry the tractor metric $\langle\cdot, \cdot\rangle_{\mathcal{T}}$ admits Lorentzian signature. Hence $\mathcal{N}_-, \mathcal{N}_0$ are either 0 or 1. If $\mathcal{N}_- = 1$ then $\mathcal{N}_0 = 0$.

5.1 The general case

Now let $\mathcal{S}$ be a linear subspace of dimension $\ell \leq \mathcal{N}$ in $\mathcal{P}(\mathcal{T})$. Then we can choose a basis $I(\mathcal{S}) := \{I_1, \ldots, I_\ell\}$ of linearly independent tractors of $\mathcal{S}$ in $\mathcal{P}(\mathcal{T})$. We set

$$\alpha_{I(\mathcal{S})} \ := \ I_1^\flat \wedge \cdots \wedge I_\ell^\flat \ ,$$

where $I^\flat := \langle I, \cdot \rangle_{\mathcal{T}}$ denotes the dual of a standard tractor I. This wedge product of dual 1-form tractors is a ℓ-form tractor. Since the $I_j^\flat$'s are ∇-parallel, the ℓ-form tractor $\alpha_{I(\mathcal{S})}$ is ∇-parallel as well. Hence $\Pi(\alpha_{I(\mathcal{S})})$ is a nc-Killing l-form with $l := \ell - 1$ of conformal weight ℓ (for $0 < \ell < n+2$). In particular, any pair $\{I_1, I_2\}$ of linearly independent tractors in $\mathcal{P}(\mathcal{T})$ gives rise to a nc-Killing 1-form $\beta_{I_1,I_2} := \Pi(I_1^\flat \wedge I_2^\flat)$ on (M, c). Such a 1-form β_{I_1,I_2} of conformal weight 2 is dual via the conformal metric $\boldsymbol{g}$ to a uniquely determined conformal Killing vector field, which we denote by V_{I_1,I_2}. We define

$$\mathfrak{inf}_{\mathcal{S}}(M, c) \ := \ span\{ \ V \in \mathfrak{X}(M) | \ V = V_{I_1,I_2} \text{ for some } I_1, I_2 \in \mathcal{S} \ \} \ .$$

Lemma 5.1 *The subset $\mathfrak{inf}_{\mathcal{S}}(M, c)$ of $\mathfrak{inf}(M, c)$ with commutator $[\cdot, \cdot]$ is a Lie subalgebra of dimension $\frac{\ell(\ell-1)}{2}$ for any subspace $\mathcal{S} \subset \mathcal{P}(\mathcal{T})$.*

Proof. Let $I_1, \ldots, I_\ell$ be a basis of $\mathcal{S}$. We set $\mathcal{P}^2(\mathcal{S}^*) := span\{I_i^\flat \wedge I_j^\flat | \, i, j = 1, \ldots, \ell\}$. The projection Π restricted to $\mathcal{P}^2(\mathcal{S}^*)$ admits no kernel. Thus the vector space $\mathfrak{inf}_{\mathcal{S}}(M, c)$ is spanned by the $\frac{\ell(\ell-1)}{2}$ linearly independent vector fields $\{ \, V_{I_i,I_j} | \, i \neq j, i, j \in \{1, \ldots, \ell\} \, \}$.

Next we notice that the 2-form tractor bundle $\Lambda^2\mathcal{T}^*$ is canonically identified with the adjoint tractor bundle $\mathcal{A}$ via the tractor metric $\langle \cdot, \cdot \rangle_{\mathcal{T}}$. The adjoint tractor bundle $\mathcal{A}$ admits a natural fibrewise bracket operation $\{\cdot, \cdot\} : \mathcal{A} \otimes \mathcal{A} \to \mathcal{A}$ (cf. e.g. [5]). Via the identification $\Lambda^2\mathcal{T}^* \cong \mathcal{A}$ the bracket $\{\cdot, \cdot\}$ induces a bracket $\{\cdot, \cdot\}_{\mathcal{P}}$ on $\mathcal{P}^2(\mathcal{S}^*)$. One can easily check that for any $i_1, i_2, i_3, i_4 \in \{1, \ldots, \ell\}$ the bracket $\{I_{i_1}^\flat \wedge I_{i_2}^\flat, I_{i_3}^\flat \wedge I_{i_4}^\flat\}_{\mathcal{P}}$ is again a linear combination of generating elements of $\mathcal{P}^2(\mathcal{S}^*)$.

Now we observe that from Proposition 3.6 of [5] the relation $\Pi(\{A, B\}_{\mathcal{P}}) = -[\Pi(A), \Pi(B)]$ follows for any $A, B \in \mathcal{P}^2(\mathcal{S}^*)$. Hence the Lie bracket $[\cdot, \cdot]$ on $\mathfrak{inf}_{\mathcal{S}}(M, c)$ is closed. □

Note that the map $A \in \mathcal{P}^2(\mathcal{S}^*) \mapsto -\Pi(A) \in \mathfrak{inf}(M, c)$ admits no kernel. Hence $(\mathcal{P}^2(\mathcal{S}^*), \{\cdot, \cdot\}_{\mathcal{P}})$ and $(\mathfrak{inf}_{\mathcal{S}}(M, c), [\cdot, \cdot])$ are isomorphic as Lie algebras. Since any fibre $(\mathcal{A}_x, \{\cdot, \cdot\}_x)$, $x \in M$, of the adjoint tractor bundle is isomorphic to the Lie algebra $\mathfrak{so}(1, n+1)$ of the *Möbius group* in dimension n, the above proof also shows that $\mathfrak{inf}_{\mathcal{S}}(M, c)$ is (naturally up to conjugation) a Lie subalgebra of $\mathfrak{so}(1, n + 1)$. It is not difficult to identify $\mathfrak{inf}_{\mathcal{S}}(M, c) \cong \mathcal{P}^2(\mathcal{S}^*)$ explicitly as Lie subalgebra of $\mathfrak{so}(1, n + 1)$. In fact, if $\mathcal{S}$ is non-degenerate in $\mathcal{P}(\mathcal{T})$ with respect to $\langle \cdot, \cdot \rangle_{\mathcal{P}}$, then $\mathcal{P}^2(\mathcal{S}^*)$ is isomorphic to the orthogonal Lie algebra $\mathfrak{so}(\mathcal{S})$, that is $\mathfrak{so}(a_-, a_+) \subset \mathfrak{so}(1, n + 1)$, where (a_-, a_+) is the signature of $\mathcal{S}$ in $\mathcal{P}(\mathcal{T})$. Note that $\mathfrak{so}(\mathcal{S}) \subset \mathfrak{so}(1, n + 1)$ is the Lie algebra of a compact connected subgroup of the Möbius group for $a_- = 0$. In case $\mathcal{S}$ is degenerate, the Lie algebra $\mathcal{P}^2(\mathcal{S}^*)$ is isomorphic to the annihilator $\mathfrak{p} \subset \mathfrak{so}(1, a_+) \subset \mathfrak{so}(1, n + 1)$ of a null vector in $\mathbb{R}^{1,a_+}$.

As we discussed in Section 3, any $I \in \mathcal{P}(\mathcal{T})$ corresponds to an almost Einstein structure $\sigma = \Pi(I)$ on (M, c). With respect to a metric $g \in c$ on M the almost Einstein structure σ is represented by a solution $s := \Pi_g(I)$ of (8). The dual tractor $I^\flat$ is then given as a triple by $(s, ds, \Box^g s)$. More generally, if $\mathcal{S} \subset \mathcal{P}(\mathcal{T})$ is a subspace with basis $I(\mathcal{S}) := \{I_1, \ldots, I_\ell\}$, $\ell \leq \mathcal{N}$, we denote the corresponding almost Einstein structures by σ_i, resp., s_i with respect to $g \in c$. The four components of the ℓ-form tractor $\alpha_{I(\mathcal{S})}$ with respect to some $g \in c$ are then given by

$$\begin{aligned}
\alpha_- &= \Sigma_{j=1}^{\ell} (-1)^{j+1} s_j \cdot ds_1 \wedge \cdots \widehat{ds_j} \cdots \wedge ds_\ell \\
\alpha_0 &= ds_1 \wedge \cdots \wedge ds_\ell \\
\alpha_\mp &= \Sigma_{i<j} (-1)^{i+j+1} (s_i \Box^g s_j - s_j \Box^g s_i) \cdot ds_1 \wedge \cdots \widehat{ds_i} \cdots \widehat{ds_j} \cdots \wedge ds_\ell \\
\alpha_+ &= \Sigma_{j=1}^{\ell} (-1)^{j+1} \Box^g s_j \cdot ds_1 \wedge \cdots \widehat{ds_j} \cdots \wedge ds_\ell \ ,
\end{aligned} \qquad (26)$$

where the hat $\widehat{\ }$ denotes omission of the underlying term.

The ℓ-form tractor $\alpha_{I(\mathcal{S})}$ is by construction ∇-parallel and simple. From Section 3 we know that all non-trivial components of $(\alpha_-, \alpha_0, \alpha_\mp, \alpha_+)$ are simple. In particular, the nc-Killing l-form $\alpha_- = \Pi(\alpha_{I(\mathcal{S})})$, $l = \ell - 1$, is simple off its zero set. Then, by Theorem 4.2, the conformal structure c on M is locally off the singularity set represented by a special Einstein product. However, it might well be that α_- does admit zeros on a space (M, c). This is suggested by the fact that a single almost Einstein structure does admit zeros, in general. We set

$$\Sigma(\mathcal{S}) \ := \ \{\ x \in M \mid \Pi(\alpha_{I(\mathcal{S})})(x) = 0\ \}\ .$$

Obviously, this definition of $\Sigma(\mathcal{S})$ depends only on the subspace $\mathcal{S} \subset \mathcal{P}(\mathcal{T})$, but not on the choice of basis $\{I_1, \ldots, I_\ell\}$ which defines $\alpha_{I(\mathcal{S})}$. In fact, note that $\alpha_{I(\mathcal{S})}$ can be seen as a constant multiple of a volume form tractor for $\mathcal{S}$. Their singularity sets are identical.

Theorem 5.2 *Let $\mathcal{S} \subset \mathcal{P}(\mathcal{T})$ be a linear subspace of dimension ℓ with basis $\{I_1, \ldots, I_\ell\}$ on a Riemannian conformal space (M^n, c) of dimension $n \geq 3$.*

1. *If $\mathcal{S}$ is Euclidean then $\Sigma(\mathcal{S}) = \bigcap_{i=1}^{\ell} \Sigma(\sigma_i)$. The singularity set $\Sigma(\mathcal{S})$ is either empty or else a smooth submanifold of codimension ℓ in M. The exterior differential of $\Pi_g(\alpha_{I(\mathcal{S})})$ does not vanish in some neighbourhood U_g of $\Sigma(\mathcal{S})$ with respect to any $g \in c$.*
2. *If $\mathcal{S}$ is degenerate then $\Sigma(\mathcal{S}) = \bigcap_{i=1}^{\ell} \Sigma(\sigma_i)$. The singularity set $\Sigma(\mathcal{S})$ is either empty or else consists of isolated points in M.*

Proof. From (26) we see immediately that $\bigcap_{i=1}^{\ell} \Sigma(\sigma_i) \subset \Sigma(\mathcal{S})$. On the other hand, let $x_o \in M \setminus \bigcap_{i=1}^{\ell} \Sigma(\sigma_i)$ be an arbitrary point. Then there exists some $k \in \{1, \ldots, \ell\}$ such that $\sigma_k \neq 0$ on some neighbourhood U of x_o. After changing the enumeration we can assume $k = 1$. We set $g = \sigma_1^{-2}\boldsymbol{g}$ on U, which is an Einstein metric. With respect to g we have $s_1 = 1$ and $ds_1 = 0$ on U. This shows $\alpha_- = ds_2 \wedge \cdots \wedge ds_\ell$ on U. If $\alpha_-(x_o) = 0$, then we see from (26) that $d\alpha_-(x_o) = 0$ and $\Box^g \alpha_-(x_o) = 0$ as well. Obviously, this implies $\langle \alpha_{I(\mathcal{S})}, \alpha_{I(\mathcal{S})} \rangle_{\mathcal{T}} < 0$, i.e., such a x_o does not exist if $\mathcal{S}$ is Euclidean or degenerate in $\mathcal{P}(\mathcal{T})$. We conclude $\Sigma(\mathcal{S}) = \bigcap_{i=1}^{\ell} \Sigma(\sigma_i)$ in these cases.

Now let us assume that $\mathcal{S}$ is Euclidean with $\Sigma(\mathcal{S}) \neq \emptyset$ and $\{I_1, \ldots, I_\ell\}$ is an orthonormal basis. Then we have $ds_i \neq 0$ for all $i \in \{1, \ldots, \ell\}$ at any $x_o \in \Sigma(\mathcal{S})$ for any $g \in c$. In fact, since the I_i's are assumed to be pairwise orthogonal standard tractors, the ds_i's are pairwise orthogonal at any $x_o \in \Sigma(\mathcal{S})$ as well. This proves that $\alpha_0 = \frac{1}{\ell} d\alpha_-$ does not vanish at any $x_o \in \Sigma(\mathcal{S})$ for any $g \in c$, which also shows that the intersection $\bigcap_{i=1}^{\ell} \Sigma(\sigma_i)$ is a submanifold of codimension ℓ in M.

In case $\mathcal{S}$ is degenerate there exists a Ricci-flat scale σ in the span of the σ_i's on (M, c). We know from Theorem 3.2 that the singularity set of the Ricci-flat scale σ is either empty or consists of isolated points. This proves that $\Sigma(\mathcal{S}) \subset \Sigma(\sigma)$ is either empty or consists of isolated points. $\Box$

The proof also shows that if $\mathcal{S}$ is not indefinite with $dim(\mathcal{S}) = n+1$ then $\Sigma(\mathcal{S}) = \emptyset$. Note that, if the subspace $\mathcal{S} \subset \mathcal{P}(\mathcal{T})$ is indefinite, then we have $\Sigma(\mathcal{S}) \not\subset \bigcap_{i=1}^{\ell} \Sigma(\sigma_i)$, in general. We find such examples on the *Möbius sphere*, which we want to discuss next. Note that the Möbius sphere is the only closed Riemannian conformal space, which admits an indefinite subspace $\mathcal{S} \subset \mathcal{P}(\mathcal{T})$ of dimension $\ell > 1$ (cf. [14]).

5.2 The model space

So let us consider the Minkowski space $\mathbb{R}^{1,n+1}$ of dimension $n+2$, $n \geq 3$, with metric $(\cdot,\cdot)_{1,n+1}$. In $\mathbb{R}^{1,n+1}$ we have the null quadric $L = \{x \in \mathbb{R}^{1,n+1} | (x, x)_{1,n+1} = 0\}$. Let L_+ denote the forward part of $L \setminus \{0\}$. The projectivation $\mathbb{P}L \cong S^n$ of the *null cone* L is a model for the Möbius sphere, the standard model of Riemannian conformal geometry. In fact, the Minkowski metric $(\cdot,\cdot)_{1,n+1}$ induces in a natural way a conformal class of metrics on $\mathbb{P}L$, which are all conformally flat. The $\mathbb{R}_+$-bundle $\pi : L_+ \to \mathbb{P}L$ is naturally isomorphic to the bundle $\mathcal{Q}$ of conformally related metrics on $\mathbb{P}L$, and the Minkowski space $\mathbb{R}^{1,n+1}$ serves as a Fefferman-Graham ambient metric space for the Möbius sphere $\mathbb{P}L$ (cf. [8, 10]). As explained in [6], the tractor bundle $\mathcal{T}$ of $\mathbb{P}L$ is naturally identified with $(T\mathbb{R}^{1,n+1}|_{L_+})/\sim$, where the equivalence relation $\sim$ is as follows: for $p, q \in L_+$ we have $T_p \sim T_q$ if and only if p and q lie in the same null ray of L_+ and T_p, T_q are parallel vectors on $\mathbb{R}^{1,n+1}$ according to the usual parallelism of $\mathbb{R}^{1,n+1}$ as affine space. Then the tractor metric and connection are induced by $(\cdot,\cdot)_{1,n+1}$ and the Levi-Civita connection on $\mathbb{R}^{1,n+1}$, respectively. It follows that there is 1-to-1-correspondence of parallel vector fields on $\mathbb{R}^{1,n+1}$ and parallel standard tractors in $\Gamma(\mathcal{T})$ on $\mathbb{P}L$. In particular, the space $\mathcal{P}(\mathcal{T})$ of $\mathbb{P}L$ has dimension $n+2$.

The almost Einstein structures on $\mathbb{P}L$ can be viewed in the following way. Let P be an arbitrary vector in $\mathbb{R}^{1,n+1}$. Then we have the orthogonal hyperplanes $P^\perp := \{x \in \mathbb{R}^{1,n+1} | (x, P)_{1,n+1} = 0\}$ and $P^\perp(\pm 1) := \{x \in \mathbb{R}^{1,n+1} | (x, P)_{1,n+1} = \pm 1\}$ to P in $\mathbb{R}^{1,n+1}$. We define G to be the intersection $(P^\perp(1) \cap L_+) \cup (P^\perp(-1) \cap L_+)$ of the latter hyperplanes with the forward null cone L_+, and by G_π we denote the image of G in $\mathbb{P}L$ under π. The image G_π is $\mathbb{P}L$ minus the projective null cone of $P^\perp$. The intersection G determines uniquely a section γ of $\pi : L_+ \to \mathbb{P}L$ over G_π, i.e., a metric g_γ on G_π. In fact, this metric g_γ is isometric to the restriction of the Minkowski metric $(\cdot,\cdot)_{1,n+1}$ to the submanifold G of $\mathbb{R}^{1,n+1}$ (which is identified via π with G_π). The metric g_γ is Einstein with constant sectional curvature $-(P, P)_{1,n+1}$. Note that if P is a timelike vector, the projective null cone of $P^\perp$ is empty, i.e., g_γ is a globally defined round metric on $\mathbb{P}L$. If P is spacelike, the image G_π is $\mathbb{P}L$ minus the sphere, which is the projective null cone of $P^\perp$. In this case g_γ is a hyperbolic metric on G_π. And, if P is null, then G_π is $\mathbb{P}L$ minus the null ray $\{\mathbb{R}P\}$. The metric g_γ on G_π is Euclidean.

Now let $\mathcal{S}'$ be a linear subspace of $\mathbb{R}^{1,n+1}$ with basis $\{P_1, \dots, P_\ell\}$. Every basis vector P_i, $i = 1, \dots, \ell$, determines a parallel standard tractor I_i and a corresponding almost Einstein structure σ_i on the Möbius sphere $\mathbb{P}L$. In particular, $\mathcal{S}'$ is naturally identified with a subspace $\mathcal{S}$ of $\mathcal{P}(\mathcal{T})$. Also, the ℓ-form $P_1^\flat \wedge \cdots \wedge P_\ell^\flat$ on $\mathbb{R}^{1,n+1}$ corresponds to the parallel ℓ-form tractor $\alpha = I_1^\flat \wedge \cdots \wedge I_\ell^\flat$ with singularity set $\Sigma(\alpha) := \{q \in \mathbb{P}L | \, \Pi(\alpha)(q) = 0\}$ on $\mathbb{P}L$. Obviously, if $\mathcal{S}'$ is Euclidean, then we have $\Sigma(\alpha) = \mathbb{P}(L \cap \mathcal{S}'^\perp) = \mathbb{P}(L \cap \bigcap_i P_i^\perp) = \bigcap_i \Sigma(\sigma_i)$. If $\mathcal{S}'$ is degenerate, then we have $\Sigma(\alpha) = \mathbb{P}(L \cap \mathcal{S}'^\perp) = \mathbb{P}(L \cap \mathcal{S}') = \{\mathbb{R}P\}$, where $\mathbb{R}P$ is the unique real null line in $\mathcal{S}'$. This explains the statements of Theorem 5.2 for the case of the Möbius sphere. On the other hand, if $\mathcal{S}'$ is indefinite we can choose a basis $\{A_1, \dots, A_{n-\ell}\}$ of the orthogonal complement $\mathcal{S}'^\perp$, which gives rise to parallel standard tractors $B_1, \dots, B_{n-\ell}$ on $\mathbb{P}L$. Then we also have the parallel $(n-\ell)$-form tractor $\alpha_\perp := B_1^\flat \wedge \cdots \wedge B_{n-\ell}^\flat$, which is (up to a constant multiple) Hodge dual to α on $\mathbb{P}L$. We observe that the singularity sets $\Sigma(\alpha)$ and $\Sigma(\alpha_\perp)$ are identical. This shows that $\Sigma(\alpha) = \mathbb{P}(L \cap \mathcal{S}') \neq \emptyset$ if $\ell > 1$, although $\bigcap_i \Sigma(\sigma_i) = \emptyset$.

Finally, note that any conformal Killing vector in $\mathfrak{inf}_{\mathcal{S}}(\mathbb{P}L)$, $\mathcal{S}' \cong \mathcal{S}$, is induced in a natural way by a wedge product of a pair of vectors in $\mathcal{S}'$. Let $V \in \mathfrak{inf}_{\mathcal{S}}(\mathbb{P}L)$ be such a vector field induced by $Q_1, Q_2 \in \mathcal{S}'$ and let P be another parallel vector on $\mathbb{R}^{1,n+1}$, which induces an Einstein metric g_γ on the subspace G_π of $\mathbb{P}L$ as described above. Then it is straightforward to see that the vector field V restricted to G_π is Killing with respect to g_γ if P is orthogonal to Q_1 and $Q_2 \in \mathcal{S}'$. In fact, $\mathfrak{inf}_{\mathcal{S}}(\mathbb{P}L)$ is a Lie algebra of Killing vectors with respect to g_γ on G_π if and only if (G_π, g_γ) comes from an orthogonal vector P to $\mathcal{S}'$. In particular, if $\mathcal{S}'$ is an Euclidean subspace of $\mathbb{R}^{1,n+1}$, then $\mathfrak{inf}_{\mathcal{S}}(\mathbb{P}L)$ is a Lie algebra of Killing vectors for any round metric on $\mathbb{P}L$, which is induced by a timelike vector $P \in \mathcal{S}'^\perp$.

6 A local geometric description

We assume the following setting throughout this section. Let (M^n, c) be a Riemannian conformal space of dimension $n \geq 3$ with standard tractor bundle $\mathcal{T}$. The space of ∇-parallel standard tractors on (M, c) is denoted by $\mathcal{P}(\mathcal{T})$. Then let $\mathcal{S} \subset \mathcal{P}(\mathcal{T})$ be an Euclidean subspace of dimension $\ell \geq 2$ with a fixed orthonormal basis $I(\mathcal{S}) := \{I_1, \dots, I_\ell\}$. The corresponding almost Einstein structures are denoted by $\sigma_j := \Pi(I_j)$, $j = 1, \cdots, \ell$, where every σ_j has negative scalar curvature $S(\sigma_j) = -1$. We set $\alpha_{I(\mathcal{S})} := I_1^\flat \wedge \cdots \wedge I_\ell^\flat$, which is a ∇-parallel ℓ-form tractor. The zero set of the corresponding nc-Killing l-form $\Pi(\alpha_{I(\mathcal{S})})$, $l := \ell - 1$, is denoted by $\Sigma(\mathcal{S})$. We assume $\Sigma(\mathcal{S}) \neq \emptyset$, which implies $\ell < n+1$. Then, as discussed in Theorem 5.2, $\Sigma(\mathcal{S})$ is a smooth submanifold of codimension ℓ in M, which coincides with the intersection $\bigcap_{j=1}^{\ell} \Sigma(\sigma_j)$ of the scale singularities of the σ_j's. Moreover, importantly, let us assume in addition that M is a closed space. We aim to study the local geometry of (M,c) in a neighbourhood of (a point of) the singularity set $\Sigma(\mathcal{S})$ for $\ell \geq 2$. For $\ell = 1$ this problem is solved by Proposition 3.5. (Note that by Theorem 1.2 of [14] any almost Einstein structure on a closed Riemannian conformal space (M,c) has to have a hypersurface scale singularity, unless (M,c) is conformally equivalent to the Möbius sphere.)

6.1 The cocloseness

The following general result for closed spaces will be used.

Theorem 6.1 *[27, 25] The automorphism group $Aut(M,c)$ of a closed conformal space (M^n,c) of dimension n is essential if and only if (M,c) is conformally equivalent to the round sphere (S^n, g_{rd}) of dimension n.*

In other words, Theorem 6.1 says that whenever (M^n,c) is not conformally equivalent to the round sphere (S^n, g_{rd}), then there exists a metric $g_* \in c$ such that $Aut(M,c)$ acts as a group of isometries on the Riemannian manifold (M, g_*). Note that a conformal transformation group $Aut(M,c)$ is *essential* if and only if it is non-compact. This result has an important consequence for our discussion here.

Corollary 6.2 *Let (M^n,c) be a closed Riemannian conformal space of dimension $n \geq 3$ and let $\mathcal{S}$ be an Euclidean subspace of $(\mathcal{P}(\mathcal{T}), \langle\cdot,\cdot\rangle_{\mathcal{P}})$. Then there exists a metric $g_* \in c$ on M such that $\mathfrak{inf}_{\mathcal{S}}(M,c)$ is a Lie subalgebra of $\mathfrak{iso}(M,g_*)$, the Lie algebra of Killing vector fields of the Riemannian manifold (M,g_*).*

Proof. Since M is assumed to be closed, every conformal vector field $V \in \mathfrak{inf}(M,c)$ on M is complete, i.e., the flow Φ_V of V is a 1-parameter group of conformal transformations on (M,c). Consequently, $\mathfrak{inf}_{\mathcal{S}}(M,c)$ is the Lie algebra of some connected subgroup $Inf_{\mathcal{S}}(M,c)$ of $Aut(M,c)$. In case (M,c) is not conformally equivalent to the standard sphere there exists a metric $g_* \in c$ such that $Aut(M,c)$ acts by isometries. In particular, $Inf_{\mathcal{S}}(M,c)$ acts by isometries. This implies that any $V \in \mathfrak{inf}_{\mathcal{S}}(M,c)$ is a Killing vector field with respect to this metric g_* on M.

Recall from Section 5 that $\mathfrak{inf}_{\mathcal{S}}(M,c)$ is in any case the Lie algebra of a compact connected subgroup of the Möbius group. In fact, we have $\mathfrak{inf}_{\mathcal{S}}(M,c) = \mathfrak{so}(l) \subset \mathfrak{so}(1,n+1)$, $l := \ell - 1$. In case (M^n,c) is conformally equivalent to the standard sphere S^n, the automorphism group $Aut(M,c)$ is the Möbius group. Hence $Inf_{\mathcal{S}}(M,c)$ is the compact connected subgroup of the Möbius group, whose Lie algebra is $\mathfrak{inf}_{\mathcal{S}}(M,c) \subset \mathfrak{so}(1,n+1)$. Thus we can conclude again that $\mathfrak{inf}_{\mathcal{S}}(M,c)$ is a Lie algebra of Killing vector fields for some metric $g_* \in c$. (In fact, the explicit discussion of the Möbius sphere $\mathbb{P}L$ at the end of Section 5 explains as well that $\mathfrak{inf}_{\mathcal{S}}(S^n,[g_{rd}])$ is a Lie algebra of Killing vector fields for certain round metrics on S^n.) □

In particular, Corollary 6.2 implies that for any pair $I_i, I_j \in I(\mathcal{S})$ the corresponding nc-Killing 1-form $\beta_{I_i,I_j} = \Pi_{g_*}(I_i^\flat \wedge I_j^\flat)$, which is dual to the vector field V_{I_i,I_j}, is coclosed with respect to g_* , i.e., $d^*\beta_{I_i,I_j} = 0$. Then expression (26) for $d^*\beta_{I_i,I_j}$ shows that the cocloseness of all the β_{I_i,I_j}'s is equivalent to

$$s_i \Box^{g_*} s_j - s_j \Box^{g_*} s_i = 0 \qquad \text{for all } \; i,j \in \{1,\ldots,\ell\}\;, \tag{27}$$

where $s_i := \Pi_{g_*}(I_i)$ This condition and again (26) for the l-form $\alpha_- = \Pi_{g_*}(\alpha_{I(\mathcal{S})})$ show that $d^*\alpha_- = 0$ on (M,g_*).

Lemma 6.3 *Let (M^n,c) be a closed Riemannian conformal space of dimension $n \geq 3$ and let $\mathcal{S} \subset (\mathcal{P}(\mathcal{T}), \langle\cdot,\cdot\rangle_{\mathcal{P}})$ be an Euclidean subspace with orthonormal basis $I(\mathcal{S}) = \{I_1,\ldots,I_\ell\}$. Then there exists a metric $g_* \in c$ such that the nc-Killing l-form $\alpha_- = \Pi_{g_*}(\alpha_{I(\mathcal{S})})$ is coclosed.*

So far we observe from our discussion that the diamond for $\alpha_{I(\mathcal{S})}$ with respect to the metric $g_* \in c$ of Lemma 6.3 takes the form

$$\alpha_{I(\mathcal{S})} \cong_{g_*} \begin{pmatrix} & \alpha_- & \\ \frac{1}{\ell} d\alpha_- & & 0 \\ & \Box^{g_*}\alpha_- & \end{pmatrix} .$$

In the following, we will explore further properties of those metrics $g_* \in c$, for which $\Pi_{g_*}(\alpha_{I(\mathcal{S})})$ is coclosed. Our ultimate ambition is to show that there exist $\breve{g} \in c$ on M such that both $d^*\Pi_{\breve{g}}(\alpha_{I(\mathcal{S})})$ and $\Box^{\breve{g}}\Pi_{\breve{g}}(\alpha_{I(\mathcal{S})})$ vanish simultaneously.

6.2 Deriving the normal form

So let us fix an arbitrary metric $g_* \in c$ on M such that the l-form $\alpha_- = \Pi_{g_*}(\alpha_{I(\mathcal{S})})$ is coclosed.

Lemma 6.4 *1. $\Box^{g_*}\alpha_- = f_* \cdot \alpha_-$ for some smooth function f_* on M.*

2. $d^*d\alpha_- = f_+ \cdot \alpha_-$ for some real function f_+ on M.

Proof. (1) Let $s_i = \Pi_{g_*}(I_i)$ be the almost Einstein structure to I_i with respect to g_* for some arbitrary $i \in \{1, \dots, \ell\}$. Then let $g = e^{2\varphi}g_*$ be a metric such that $|de^{\varphi}s_i|_g \equiv 1$, whose existence is guaranteed by Lemma 3.4 on some neighbourhood of $\Sigma(I_i)$. We have

$$\frac{\Box^{g_*}s_i}{s_i} = \frac{1 - |ds_i|^2_{g_*}}{2s_i^2} = \frac{1 - |de^{\varphi}s_i|^2_g + e^{-2\varphi}s_i^2|de^{\varphi}|^2_{g_*}}{2s_i^2} = \frac{|d\varphi|^2_{g_*}}{2}$$

on that neighbourhood. This shows that the quotient $f_* := \frac{\Box^{g_*}s_i}{s_i}$ is everywhere smooth on M. In particular, from (27) we can conclude that $I_i^\flat = (s_i, ds_i, f_* s_i)$ with respect to g_* for all $i \in \{1, \dots, \ell\}$. Obviously, (26) implies $\alpha_+ = \Box^{g_*}\alpha_- = f_*\alpha_-$.

(2) Expression (11) for $n \neq 2l$ immediately implies $d^*d\alpha_- = f_+ \cdot \alpha_-$ with $f_+ = (l+1)((n - 2l)f_* - tr\mathsf{P}^{g_*})$.

For $n = 2l$ we have the following argument. Let $Ann(\alpha_-) := \{X \in TM |\, X \lrcorner\ \alpha_- = 0\}$ denote the annihilator of the l-form α_- with orthogonal complement $Ann_{\perp_{g_*}}(\alpha_-)$ in TM. It follows from (15) that $\mathsf{P}^{g_*}(X, Y) = 0$ for any $X \in Ann(\alpha_-)$, $Y \in Ann_{\perp_{g_*}}(\alpha_-)$. This shows that for any $x \in M \setminus \Sigma(\mathcal{S})$ we can find an orthonormal basis $v := \{v_1, \dots, v_n\}$ of (T_xM, g_*) such that $\{v_1, \dots, v_{n-l}\}$ spans $Ann(\alpha_-)$ and such that $\mathsf{P}^{g_*}_x$ is diagonal with respect to v, i.e., the v_i's are eigenvectors of $\mathsf{P}^{g_*}_x : T_xM \to T_xM$.

Now, from (12) we have the expression

$$d^*d\alpha_- = (l+1) \cdot \Big(((n-l)f_* - tr\mathsf{P}^{g_*})\alpha_- + \Sigma_{i=1}^n \mathsf{P}(v_i)^\flat \wedge (v_i \lrcorner\ \alpha_-) \Big) .$$

Since $\alpha_-(x)$ is simple at $x \in M \setminus \Sigma(\mathcal{S})$, it is clear that the expression $\Sigma_{i=1}^n \mathsf{P}(v_i)^\flat \wedge (v_i \lrcorner\ \alpha_-)(x)$ is just some multiple of $\alpha_-(x)$ for any $x \in M \setminus \Sigma(\mathcal{S})$, i.e., $d^*d\alpha_- = f_+\alpha_-$ for some smooth function f_+ on $M \setminus \Sigma(\mathcal{S})$. For $x \in \Sigma(\mathcal{S})$ we have $d^*d\alpha_- = \alpha_- = 0$. $\Box$

Recall from Theorem 5.2 that there exists an open neighbourhood U_{g_*} of $\Sigma(\mathcal{S})$ in M where $\frac{1}{\ell}d\alpha_- = ds_1 \wedge \dots \wedge ds_\ell$ does not vanish. Obviously, the annihilator $Ann(d\alpha_-) := \{X \in TU_{g_*} |\, X \lrcorner\ d\alpha_- = 0\}$ is an integrable smooth distribution of rank $n - \ell$ in TU_{g_*}, i.e., there exist smooth integral leaves $Int(d\alpha_-)$ in U_{g_*} of codimension ℓ with tangent space $Ann(d\alpha_-)$. (Note that in the extremal case we have $\ell = n$, which says that the integral leaves of $Ann(d\alpha_-)$ are all single points.)

Lemma 6.5 *The g_*-orthogonal complement $Ann_{\perp_{g_*}}(d\alpha_-)$ of the annihilator $Ann(d\alpha_-)$ in TU_{g_*} is an integrable distribution of rank ℓ.*

Proof. Without loss of generality we can assume that the neighbourhood U_{g_*} is orientable, i.e., we can define a Hodge $*$-operator with respect to g_* for p-forms on U_{g_*} (cf. Section 4). We set $\beta := *d\alpha_-$. The g_*-orthogonal complement of $Ann(d\alpha_-)$ is the annihilator $Ann(\beta)$, which is a smooth distribution of rank ℓ in TU_{g_*}. We want to show that $Ann(\beta)$ is integrable, i.e., $[X,Y] \lrcorner\, \beta = 0$ for any vector field $X, Y \in \Gamma(Ann(\beta))$ on U_{g_*}. The latter condition is equivalent to

$$d\beta(X,Y,\cdot) = 0 \qquad \text{for any} \;\; X,Y \in Ann(\beta)\ . \tag{28}$$

In fact, since $\Sigma(\mathcal{S})$ is singular in U_{g_*} and $Ann(\beta)$ is smooth of constant rank, it is enough to check (28) for any $X, Y \in Ann(\beta)$ on $U_{g_*} \setminus \Sigma(\mathcal{S})$.

For this purpose recall that $d\alpha_- = \gamma \wedge \alpha_-$ for some 1-form γ on $U_{g_*} \setminus \Sigma(\mathcal{S})$, i.e., we have $Ann(d\alpha_-) \subset Ann(\alpha_-)$ and $Ann(*\alpha_-) \subset Ann(\beta)$. In fact, $Ann(*\alpha_-)$ has codimension 1 in $Ann(\beta)$ on $U_{g_*} \setminus \Sigma(\mathcal{S})$. This shows via the identity $d\beta = (-1)^{n+p^2+1} f_+ * \alpha_-$, obtained from Lemma 6.4, that $d\beta(X,Y,\cdot) = 0$ for any $X,Y \in Ann(\beta)$ on $U_{g_*} \setminus \Sigma(\mathcal{S})$. □

Now let $q \in \Sigma(\mathcal{S})$ be an arbitrary singular point and let U_q be some appropriate neighbourhood of q in U_{g_*}. Then we can choose functions $(x_1,\dots,x_{n-\ell})$ on U_q, which serve as coordinates on the integral leaves of $Ann(d\alpha_-)$, with $x_1 = \cdots = x_{n-\ell} = 0$ at $q \in U_q$. On the other hand, the functions $(s_1,\dots,s_\ell)$ serve as coordinates on the integral leaves of the orthogonal distribution $Ann_{\perp_{g_*}}(d\alpha_-)$, and the merged set $(s_1,\dots,s_\ell,x_1,\dots,x_{n-\ell})$ of functions forms a smooth coordinate system on $U_q \subset M$. We set $r := \sqrt{\Sigma_{i=1}^{\ell} s_i^2}$ and $w := \sqrt{\Sigma_{i=1}^{n-\ell} x_i^2}$. Obviously, there exist reals $\tilde r, \tilde w > 0$ such that the product $B^{\ell}_{\tilde r} \times B^{n-\ell}_{\tilde w}$ of balls is a neighbourhood of q in U_q. For convenience, we assume in the following $U_q = B^{\ell}_{\tilde r} \times B^{n-\ell}_{\tilde w}$ for some small $\tilde r, \tilde w > 0$ with respect to the given coordinates. Lemma 6.5 shows that the metric g_* takes with respect to such coordinates on U_q around $q \in \Sigma(\mathcal{S})$ the form

$$g_* \;=\; g_{*1}(s_1,\dots,s_\ell,x_1,\dots,x_{n-\ell}) \;+\; g_{*2}(s_1,\dots,s_\ell,x_1,\dots,x_{n-\ell})\ ,$$

where g_{*2} is a smooth metric on the integral leaves of $Ann(d\alpha_-)$ and g_{*1} is a smooth metric on the orthogonal leaves (for fixed $(s_1,\dots,s_\ell)$ and $(x_1,\dots,x_{n-\ell})$), respectively. Note that if $\ell = n$ we have no x_i-coordinates, but the functions $(s_1,\dots,s_\ell)$ alone serve as coordinates on U_q. The tensor g_{*2} is absent in this case.

Lemma 6.6 *(1) The metric g_{*1} takes for any $q \in \Sigma(\mathcal{S})$ on an integral leaf of $Ann_{\perp_{g_*}}(d\alpha)$ in some ball neighbourhood $U_q = B^{\ell}_{\tilde r} \times B^{n-\ell}_{\tilde w} \subset M$ the form*

$$g_{*1} \;=\; \frac{dr^2}{1-2f_* r^2} \;+\; r^2 g_{rd}\ ,$$

where $r = \sqrt{\Sigma_{i=1}^{\ell} s_i^2}$ and g_{rd} is the round metric on the unit sphere S^l of dimension $l = \ell - 1$.

(2) The function f_* depends on $(x_1,\dots,x_{n-\ell})$ and the radial coordinate r (but not independently on the coordinate functions $(s_1,\dots,s_\ell)$).

(3) The tensor g_{*2} depends on $(x_1,\dots,x_{n-\ell})$ and the radial coordinate r (but not independently on the coordinate functions $(s_1,\dots,s_\ell)$).

Proof. (1) From $\langle I_i, I_j\rangle_{\mathcal{T}} = \delta_{ij}$ and $I_i \cong_{g_*} (s_i, grad^{g_*} s_i, f_* s_i)$ we obtain $g^{ij}_{*1} := g_*(grad^{g_*} s_i, grad^{g_*} s_j) = \delta^{ij} - 2f_* s_i s_j$ for any $i,j \in \{1,\dots,\ell\}$. The inverse of the $(\ell\times\ell)$-matrix (g^{ij}_{*1}) is given by $(\delta_{ij} + \frac{2f_*}{1-2f_* r^2} s_i s_j)$. These are the coefficients of g_{*1} with respect to the $ds_i \otimes ds_j$'s on U_q. Also note $dr^2(\partial s_i, \partial s_j) = \frac{1}{r^2} s_i s_j$ for any $i,j = 1,\dots,\ell$. With $\langle\cdot,\cdot\rangle_\ell = dr^2 + r^2 g_{rd}$ (the Euclidean metric) we obtain $g_{*1} = \langle\cdot,\cdot\rangle_\ell + \frac{2f_* r^2}{1-2f_* r^2} dr^2 = \frac{dr^2}{1-2f_* r^2} \;+\; r^2 g_{rd}$.

(2) The l-form α_- is nc-Killing on (M, g_*). Note that the restriction of α_- to the integral leaves $Int_{\perp_{g_*}}(d\alpha_-) = \{x_i = const.\}$ of $Ann_{\perp_{g_*}}(d\alpha_-)$ is a non-trivial l-form, which we denote by α_- again. Since $X \lrcorner\, \alpha_- = 0$ and by (13) $\nabla^{g_*}_X \alpha_- = 0$ for any $X \in Ann(d\alpha_-)$, we have $\nabla^{g_{*1}} \alpha_- =$

$\nabla^{g_*}\alpha_-|_{Int_{\perp_{g_*}}(d\alpha_-)}$ and $d^*\alpha_- = d^*\alpha_-|_{Int_{\perp_{g_*}}(d\alpha_-)} = 0$ with respect to g_{*1} on $Int_{\perp_{g_*}}(d\alpha_-)$. This shows that α_- on $Int_{\perp_{g_*}}(d\alpha_-)$ satisfies

$$\nabla^{g_{*1}}\alpha_- \;=\; \frac{1}{l+1}d\alpha_- \;,$$

i.e., α_- is a Killing l-form on any integral leaf $\{x_i = const.\}$ of $Ann_{\perp_{g_*}}(d\alpha_-)$.

Furthermore, note that $\alpha_- \wedge rdr = \frac{r^2}{\ell}d\alpha_-$, where $|\alpha_-|_{g_{*1}} = r$, $|rdr|_{g_{*1}} = r\sqrt{1-2f_*r^2}$ and $|\frac{1}{\ell}d\alpha_-|_{g_{*1}} = \sqrt{1-2f_*r^2}$. This shows that the 1-form $\frac{rdr}{\sqrt{1-2f_*r^2}}$ is (locally) Hodge $*$-dual to α_- on the leaves $Int_{\perp_{g_*}}(d\alpha_-)$ with respect to g_{*1}. Then, since α_- is conformal Killing, the dual vector field $V = r\sqrt{1-2f_*r^2}\cdot\partial r$ of $\frac{rdr}{\sqrt{1-2f_*r^2}}$ with respect to g_{*1} is a conformal vector field on any leaf $Int_{\perp_{g_*}}(d\alpha_-) = \{x_i = const.\}$, i.e., $L_V g_{*1} = \lambda g_{*1}$ for some function λ on U_q.

Now let W be any vector field on $U_q \setminus \Sigma(\mathcal{S})$ with $W(r) = 0$, i.e., W is tangential to the cylinders $\{r = const.\}$ in $U_q \setminus \Sigma(\mathcal{S})$ and, in particular, g_{*1}-orthogonal to V. Then we compute

$$0 \;=\; \lambda g_{*1}(V,W) \;=\; (L_V g_{*1})(V,W) \;=\; W\left(r\sqrt{1-2f_*r^2}\right)\cdot g_{*1}(V,\partial r)\;.$$

This proves $W(f_*) = 0$ for any tangent vector field W to $\{r = const.\}$, i.e., the function f_* on U_q depends only on $(x_1,\dots,x_{n-\ell})$ and the radial coordinate r.

(3) From $|\alpha_-|_{g_*} = r$ we see that $\tilde\alpha_- = r^{-\ell}\alpha_- = \Pi_{\tilde g}(\alpha_{I(\mathcal{S})})$ has constant norm with respect to $\tilde g = r^{-2}g_*$ on $U_q \setminus \Sigma(\mathcal{S})$. According to Theorem 4.2 this implies that $\tilde\alpha_-$ is $\nabla^{\tilde g}$-parallel and $\tilde g = r^{-2}g_*$ is a Riemannian product space. In fact, note that $Ann(\tilde\alpha_-)$ is spanned by $\{\partial r, \partial x_1,\dots,\partial x_{n-\ell}\}$ on $U_q \setminus \Sigma(\mathcal{S})$ and its integral leaves have coordinates $(r,x_1,\dots,x_{n-\ell})$. Since g_* is radial symmetric on the leaves $\{x_1,\dots,x_{n-\ell} = const.\}$, it follows that the spheres $\{r, x_i = const.\}$ in $U_q \setminus \Sigma(\mathcal{S})$ are the integral leaves of $Ann_{\perp_{\tilde g}}(\tilde\alpha_-)$. Obviously, the rescaled metric $\tilde g$ splits by

$$g_{rd} \quad\times\quad \left(\frac{r^{-2}dr^2}{1-2f_*r^2} + r^{-2}g_{*2}\right) \tag{29}$$

into a Riemannian product metric with respect to the orthogonal integral leaves of $Ann_{\perp_{\tilde g}}(\tilde\alpha_-)$ and $Ann(\tilde\alpha_-)$ on $U_q \setminus \Sigma(\mathcal{S})$. This means $\tilde g_2 := \frac{r^{-2}dr^2}{1-2f_*r^2} + r^{-2}g_{*2}$ on $Int(\tilde\alpha_-)$ does not depend on the coordinates of spheres $Int_{\perp_{\tilde g}}(\tilde\alpha_-) = \{r, x_i = const.\}$, i.e., $\tilde g_2$ depends solely on the coordinates $(r, x_1,\dots,x_{n-\ell})$. □

In summary, Lemma 6.6 shows that any metric $g_* \in c$ on M, for which $\alpha_- = \Pi_{g_*}(\alpha_{I(\mathcal{S})})$ is coclosed, has locally on some ball neighbourhood $U_q = B_{\tilde r} \times B_{\tilde w}$ of an arbitrary $q \in \Sigma(\mathcal{S})$ a coordinate expression of the form

$$g_* \;=\; \frac{dr^2}{1-2\tilde f_*(r,x_i)r^2} \;+\; r^2 g_{rd} \;+\; \tilde g_{*2}(r,x_i)\;. \tag{30}$$

with $\tilde f_*(r,x_1,\dots,x_{n-\ell}) := f_*(s_1,\dots,s_\ell,x_1,\dots,x_{n-\ell})$ and $\tilde g_{*2}(r,x_1,\dots,x_{n-\ell}) := g_{*2}(s_1,\dots,s_\ell,x_1,\dots,x_{n-\ell})$ for $r = \sqrt{\Sigma_i s_i^2}$. Moreover, it follows from the discussion of Section 4 that (29) is a special Einstein product on $S^{\ell-1}\times((0,\tilde r)\times B_{\tilde w})$, i.e., (30) is conformally equivalent to a special Einstein product off the singularity $\{r = 0\}$.

Proposition 6.7 *Let (M^n, c) be a closed Riemannian conformal space of dimension $n \geq 3$ and let $\mathcal{S} \subset (\mathcal{P}(\mathcal{T}), \langle\cdot,\cdot\rangle_{\mathcal{P}})$ be an Euclidean subspace with orthonormal basis $I(\mathcal{S}) = \{I_1,\dots,I_\ell\}$, $\ell \geq 2$, $\Sigma(\mathcal{S}) \neq \emptyset$, and corresponding ℓ-form tractor $\alpha_{I(S)}$. Then for any $g \in c$ there exists a unique metric $\breve g \in c$ on some open neighbourhood $\breve U_{\mathcal{S}}$ of $\Sigma(\mathcal{S})$ in M with the following properties:*

1. *$\breve g = g$ on $\Sigma(\mathcal{S})$*

2. $\Box^{\check{g}} s_i = 0$ *for any* $s_i := \Pi_{\check{g}}(I_i)$, $i = 1, \ldots, \ell$, *on* $\check{U}_{\mathcal{S}}$, *i.e.,*

$$I_i^\flat \;\cong_{\check{g}}\; \begin{pmatrix} s_i \\ ds_i \\ 0 \end{pmatrix} \qquad \textit{for all} \;\; i = 1, \ldots, \ell,$$

3. $d^*\alpha_- = 0$ *and* $\Box^{\check{g}}\alpha_- = 0$ *for* $\alpha_- = \Pi_{\check{g}}(\alpha_{I(\mathcal{S})})$ *on* $\check{U}_{\mathcal{S}}$, *i.e.,*

$$\alpha_{I(\mathcal{S})} \;\cong_{\check{g}}\; \begin{pmatrix} & \alpha_- & \\ \frac{1}{\ell}d\alpha_- & & 0 \\ & 0 & \end{pmatrix} .$$

4. *For any* $q \in \Sigma(\mathcal{S})$ *there exists a neighbourhood* $\check{U}_q \subset \check{U}_{\mathcal{S}}$ *and a family of smooth metrics* $\check{g}_t$, $t \in [0, \varepsilon)$, *on* $\Sigma(\mathcal{S}) \cap \check{U}_q$ *such that* $g_+ := t^{-2}(dt^2 + \check{g}_t)$ *is an even Poincaré-Einstein metric on the interior of* $[0, \varepsilon) \times (\Sigma(\mathcal{S}) \cap \check{U}_q)$, *and* $\check{g}$ *on* $\check{U}_q$ *is isometric to*

$$dr^2 \;+\; r^2 g_{rd} \;+\; \check{g}_r \;, \tag{31}$$

where $r := \sqrt{\Sigma_{i=1}^{\ell} s_i^2}$ *and* g_{rd} *is the round metric on the* $(\ell-1)$*-sphere* $S^{\ell-1}$. *For* $\ell = n$, *the normal form is* $dr^2 + r^2 g_{rd}$.

Proof. Let $g \in c$ be an arbitrary choice of metric on M, and let $g_* \in c$ be a metric as guaranteed by Lemma 6.3, for which $\alpha_-^* = \Pi_{g_*}(\alpha_{I(\mathcal{S})})$ is coclosed. Then there exists a unique function $\omega_0 \in C^\infty(\Sigma(\mathcal{S}))$ such that $g = e^{2\omega_0} g_*$ on $\Sigma(\mathcal{S})$. We first construct for any $q \in \Sigma(\mathcal{S})$ a metric $\check{g}_q \in c$ on some neighbourhood $\check{U}_q$ such that the statements (1) to (4) of Proposition 6.7 are satisfied on $(\check{U}_q, \check{g}_q)$. In a final step we will show that statements (1) to (3) are globally true on the union $\check{U}_{\mathcal{S}} := \bigcup_{q \in \Sigma(\mathcal{S})} \check{U}_q$, which is a neighbourhood of $\Sigma(\mathcal{S})$ in M.

So let $q \in \Sigma(\mathcal{S})$ be an arbitrary point and let $U_q = B^\ell_{\tilde{r}} \times B^{n-\ell}_{\tilde{w}}$ be a ball neighbourhood with coordinates $(s_1^*, \ldots, s_\ell^*, x_1^*, \ldots, x_{n-\ell}^*)$, where $s_i^* := \Pi_{g_*}(I_i)$, $i = 1, \ldots, \ell$. The metric g_* on U_q is given by (30). We can use (30) through restriction to define the metric $h_* := \frac{dt^2}{1 - 2\tilde{f}_*(t, x_i^*)t^2} + \tilde{g}_{*2}(t, x_i^*)$ on some open $U \subset \mathbb{R}^{n-\ell+1}$ with coordinates $(t, x_1^*, \ldots, x_{n-\ell}^*)$, $t \in (-\tilde{r}, \tilde{r})$. By construction, this is a smooth Riemannian metric h_*, which is even in the sense that $h_*(t, x^*) = h_*(-t, x^*)$ for any $t \in (-\tilde{r}, \tilde{r})$. In fcact, since (29) is a special Einstein product, the coordinate t is an almost Einstein structure on (U, h_*) with scale singularity $\{t = 0\}$. (We have $S(I) = -1$ for the corresponding parallel tractor I.) Thus we can apply Lemma 3.4 to obtain a function ω on a neighbourhood of $\{t = 0\}$ in U with boundary condition $\omega|_{\{t=0\}} = \omega_0$ such that the differential $d\check{t}$ of $\check{t} := e^\omega t$ has norm 1 with respect to the rescaled metric $\check{h} := e^{2\omega} h_*$. (For the boundary condition $\omega|_{\{t=0\}} = \omega_0$ note that by construction the subset $\{t = 0\}$ of U can be considered in a natural way as a subset of $\Sigma(\mathcal{S})$.) Since the solution ω is uniquely determined by the boundary condition ω_0, the function ω has to be even in the coordinate t, i.e., $\omega(t, x^*) = \omega(-t, x^*)$.

Next we use the solution ω to define a function ω_l on a neighbourhood of $q = (0, \ldots, 0, x_o^*) \in \Sigma(\mathcal{S})$ in $U_q \subset M$ as follows. We can assume that the solution ω is given on a neighbourhood of $q' = (0, x_o^*) \in U$ of the form $(-\varepsilon, \varepsilon) \times U'$, where $\varepsilon > 0$ and U' has coordinates $(x_1^*, \ldots, x_{n-\ell}^*)$. For such a domain we define the radial symmetric function

$$\begin{array}{cccc} \omega_l : & B^\ell_\varepsilon \times U' & \to & \mathbb{R}, \\ & (s_1^*, \ldots, s_\ell^*, x^*) & \mapsto & \omega(r^*, x^*) \;, \end{array}$$

where $r^* := \sqrt{\Sigma_{i=1}^{\ell} s_i^{*2}}$ and B^ℓ_ε is a ℓ-dimensional ball of radius ε with respect to the coordinates $(s_1^*, \ldots, s_\ell^*)$. Since ω is an even function, we know from Lemma 3.3 of [24] that the function ω_l is smooth on the subset $B^\ell_\varepsilon \times U'$ of U_q in M^n.

Now we rescale the metric g_* on $\check{U}_q := B^\ell_\varepsilon \times U'$ by $\check{g}_q := e^{2\omega_l} g_*$. We claim that this metric $\check{g}_q$ on $\check{U}_q$ satisfies the desired properties of Proposition 6.7. Certainly, by construction we have (1) $\check{g}_q = g$ on $\Sigma(\mathcal{S}) \cap \check{U}_q$. Furthermore, since ω_l is radial symmetric with respect to $r^* = \sqrt{\Sigma_i s_i^*}$ on $\check{U}_q$, we obtain from (30) $grad^{g_*}\omega_l = (1 - 2\tilde{f}_*(r^*){r^*}^2)(\partial_{r^*}\omega_l)\partial_{r^*}$. The transformation rule $d^*\check{\alpha}_- = e^{(-l+1)\omega_l}(\, d^*\alpha_-^* + (n+1)grad^{g_*}\omega_l \lrcorner \alpha_-^*)$ then shows that $d^*\check{\alpha}_- = 0$. In particular, from Lemma 6.4 we have $\Box^{\check{g}_q}\check{\alpha}_- = \check{f}\check{\alpha}_-$ for some smooth function $\check{f}$ on $\check{U}_q$. Hence, in order to prove the assertions (2) and (3) on $\check{U}_q$, we only need to show $\check{f} = 0$.

In fact, we have $\check{f} = 0$ for the following reason. We set $s_i := \Pi_{\check{g}_q}(I_i)$, $i = 1, \ldots, \ell$, and $r := \sqrt{\Sigma_{i=1}^\ell s_i^2}$. The metric $\check{g}_q$ restricted to the submanifold $A := \{s_2 = \cdots = s_\ell = 0\}$ in $\check{U}_q$ is by construction isometric to $\check{h}$. Now, if we set $t = s_1$, then we have $dt = dr$ and $|dt|_{\check{h}} = |dr|_{\check{g}_q}$ on A. However, by construction of the solution ω we also have $|dt|_{\check{h}} = 1$, which shows $\check{f} = 0$ on A. Since the function $\check{f}$ is radial symmetric, this is only possible if $\check{f} = 0$ everywhere on $\check{U}_q$.

With $\check{f} = 0$ and (30) we have found for $\check{g}_q$ the local coordinate expression

$$\check{g}_q \; = \; dr^2 \; + \; r^2 g_{rd} \; + \; \tilde{g}_2(r, x_1, \ldots x_{n-\ell})$$

on a neighbourhood $\check{U}_q$ of $q \in \Sigma(\mathcal{S})$. We set $\check{g}_t := \check{g}_q(s_1, \ldots, s_\ell, x_1, \ldots, x_{n-\ell})$ for $t = \sqrt{\Sigma_{i=1}^\ell s_i^2}$, which defines a smooth family of metrics on $U' = \Sigma(\mathcal{S}) \cap \check{U}_q$. By construction of the solution ω, we know that the metric $dt^2 + \check{g}_t$ on $(-\varepsilon, \varepsilon) \times U'$ is even in the coordinate t. Moreover, the coordinate t is an almost Einstein structure, whose corresponding parallel tractor I has scalar curvature -1, i.e., $g_+ := t^{-2}(dt^2 + \check{g}_t)$ is an even Poincaré-Einstein metric on (the interior of) $[0, \varepsilon) \times U'$.

Finally, we have to show that (1) to (3) are globally true on some neighbourhood of $\Sigma(\mathcal{S})$ in M. For this purpose we choose for any $q \in \Sigma(\mathcal{S})$ a local solution $(\check{U}_q, \check{g}_q)$ as constructed above. Then we simply set $\check{U}_{\mathcal{S}} := \bigcup_{q \in \Sigma(\mathcal{S})} \check{U}_q$. Since the metric $\check{g}_q$ is uniquely determined on every neighbourhood $\check{U}_q$, $q \in \Sigma(\mathcal{S})$, by the properties (1) to (3), it is clear that for any $q, q' \in \Sigma(\mathcal{S})$ with $\check{U}_q \cap \check{U}_{q'} \neq \emptyset$ the metrics $\check{g}_q$ and $\check{g}_{q'}$ coincide on the overlap $\check{U}_q \cap \check{U}_{q'}$. This shows that the $\check{g}_q$'s define a smooth metric $\check{g}$ on the neighbourhood $\check{U}_{\mathcal{S}}$ of $\Sigma(\mathcal{S})$ in M. Obviously, the metric $\check{g}$ satisfies (1) to (3) by construction, and $\check{g}$ is uniquely determined by these properties. Statement (4) is true for any $q \in \Sigma(\mathcal{S})$ on the open neighbourhood $\check{U}_q$ of q in $\check{U}_{\mathcal{S}}$. □

6.3 Remarks about Proposition 6.7

The existence of the metric $\check{g}$ is always guaranteed on an open neighbourhood $\check{U}_{\mathcal{S}}$ of the singularity set $\Sigma(\mathcal{S})$ in M. In fact, since M is closed, there does exist a $\check{r}_o > 0$ such that the tube $\check{U}_{\check{r}_o} := \{x \in M |\, r(x) < \check{r}_o\}$ is contained in $\check{U}_{\mathcal{S}}$. Therefore, it is often convenient to use some tube neighbourhood $\check{U}_t$, $t > 0$, of the singularity set $\Sigma(\mathcal{S})$. However, note that in general a tube neighbourhood of $\Sigma(\mathcal{S})$ need not be homeomorphic to a product of $\Sigma(\mathcal{S})$ with some ball $B^\ell_{\check{r}_o}$ of radius $\check{r}_o$ in $\mathbb{R}^\ell$. This is the reason why statement (4) about the normal form (31) of Proposition 6.7 is only formulated locally in a neighbourhood of a single point $q \in \Sigma(\mathcal{S})$. The local neighbourhood can always be assumed to be homeomorphic to a product $B^\ell_{\check{r}_o} \times U'$, where U' is some neighbourhood of q in $\Sigma(\mathcal{S})$.

Proposition 6.7 is the generalisation of Lemma 3.4 and the normal form of Proposition 3.5 to closed Riemannian conformal spaces admitting multiple almost Einstein structures with intersecting scale singularities. However, while the normal form for $\ell = 1$ of Proposition 3.5 might well be based on an uneven Poincaré-Einstein metric, the Poincaré-Einstein metric involved in the normal form (31) of Proposition 6.7 for $\ell > 1$ has to be even in the sense of [10] (cf. Definition 3.6). This follows from the above proof. This behaviour can also be explained by the action of the conformal transformation group $Inf_{\mathcal{S}}(M, c)$ with Lie algebra $\mathfrak{inf}_{\mathcal{S}}(M, c)$ on the closed space (M, c). In fact, note that by (31) the metric $\check{g}$ restricted to the integral leaves of $Ann_{\perp_{\check{g}}}(d\alpha_-)$ on a tube $\check{U}_{\check{r}_o}$ with appropriate radius $\check{r}_o > 0$ is the flat metric $dr^2 + r^2 g_{rd}$, and there the conformal Killing vector fields $V_{i,j}$, $i, j \in \{1, \ldots, \ell\}$ are given by $s_i \partial s_j - s_j \partial s_i$. Thus the $V_{i,j}$'s are complete on the tube $\check{U}_{\check{r}_o}$, and the action of $Inf_{\mathcal{S}}(M, c)$ restricted to $\check{U}_{\check{r}_o}$ is well defined and isometric with respect to $\check{g}$. For example, if we set $i = 1$, $j = 2$, then the flow $\Phi_{V_{1,2}}$ of $V_{1,2}$ only rotates the (s_1, s_2)-plane in $\check{U}_{\check{r}_o}$, whereas the other coordinates remain constant. In particular, the rotation $\Phi_{V_{1,2}}(\pi)$ by 180^o

is an isometry of $(\check{U}_{\check{r}_o}, \check{g})$ with $\Phi^*_{V_{1,2}}(\pi)(\sigma_i) = -\sigma_i$, $i = 1, 2$. The restriction of $\Phi_{V_{1,2}}(\pi)$ to the $(n - \ell + 1)$-dimensional submanifold $A := \{s_2 = \ldots = s_\ell = 0\}$ in $\check{U}_{\check{r}_o}$ is an isometric reflection at $\Sigma(\mathcal{S})$, which maps s_1 to $-s_1$. Note that $\check{g}$ restricted to A is locally isometric to $dt^2 + \check{g}_t$ (as in (4) of Proposition 6.7). Thus the existence of the reflection $\Phi_{V_{1,2}}(\pi)$ on A forces by definition the evenness of the underlying Poincaré-Einstein metric $g_+ = t^{-2}(dt^2 + \check{g}_t)$ of the normal form (31).

As example, let us consider the round unit sphere S^n in $\mathbb{R}^{n+1}$ with standard coordinates $(x_1, \ldots, x_{n+1})$. We denote the restriction of the coordinate function x_i to S^n by s_i for any $i = 1, \ldots, n+1$. The functions s_i, $i = 1, \ldots, n+1$, are almost Einstein structures on S^n with respect to the round metric g_{rd} (induced by the embedding $S^n \subset \mathbb{R}^{n+1}$). The corresponding parallel tractors $I_i := \mathcal{D}^{g_{rd}} s_i$, $i = 1, \ldots, n+1$, are pairwise orthogonal with $S(I_i) = -1$, and $\Sigma(I_i)$ is an equator on S^n for any $i = 1, \ldots, n+1$. Now let $\mathcal{J}_\ell \subset \{1, \cdots, n+1\}$ be a collection of ℓ indices with $2 \leq \ell \leq n$. After a renumeration we can simply assume $\mathcal{J}_\ell = \{1, \ldots, \ell\}$. We denote by $\mathcal{S}$ the span of the I_i, $i \in \mathcal{J}_\ell$, with basis $I(\mathcal{S}) = \{I_1, \ldots, I_\ell\}$. Note that $\Sigma(\mathcal{S})$ is the intersection of ℓ (orthogonal) equators on S^n, i.e., $\Sigma(\mathcal{S})$ is a sphere of dimension $n - \ell$ in S^n. We call $\Sigma(\mathcal{S})$ a *pole* of codimension ℓ in S^n. It is straightforward to check that all the vector fields $V_{i,j}$, $i \neq j \in \mathcal{J}_\ell$ are Killing with respect to g_{rd} on S^n. In particular, the nc-Killing l-form $\alpha_- = \Pi_{g_{rd}}(\alpha_{I(\mathcal{S})})$ is coclosed on (S^n, g_{rd}). However, the length $|ds_i|_{g_{rd}}$ is not constant 1 in any neighbourhood of $\Sigma(I_i)$ for any $i \in \mathcal{J}_\ell$.

Now, Proposition 6.7 guarantees a conformal factor ω_l on a (tube) neighbourhood $\check{U}_{\mathcal{S}}$ of the pole $\Sigma(\mathcal{S})$ such that for the rescaled metric $\check{g} = e^{2\omega_l} g_{rd}$ simultaneously every almost Einstein structure $\check{s}_i = \Pi_{\check{g}}(I_i)$, $i \in \mathcal{J}_\ell$, has a differential $d\check{s}_i$ of constant length 1. The tube neighbourhood $\check{U}_{\mathcal{S}}$ can be chosen as $S^n \smallsetminus \{\Sigma_{i=1}^\ell s_i^2 = 1\}$. We set $\check{r} := \sqrt{\Sigma_{i=1}^\ell \check{s}_i^2}$. Then the rescaled metric $\check{g}$ on $\check{U}_{\mathcal{S}}$ takes the form

$$d\check{r}^2 \ + \ \check{r}^2 g^l_{rd} \ + \ (1 - (\check{r}/2)^2)^2 g^{n-l-1}_{rd} \ .$$

Note that $d\check{r}^2 + (1 - (\check{r}/2)^2)^2 g_{rd}$ is the hyperbolic metric written with respect to a special defining function $\check{r}$ for the boundary (cf. e.g. [16]).

7 S^l-doubling and main result

In [24] we have invented the *S^l-doubling of an even asymptotically hyperbolic space* $(\overline{F}, g_+)$ with boundary N. We will see in this section that the S^l-doubling is (locally) the underlying model for closed spaces admitting multiple almost Einstein structures with intersecting scale singularities.

We briefly recall the S^l-doubling construction for $l \geq 0$. A detailed explanation can be found in [24]. Let $\overline{F}^{m+1}$ be a smooth manifold of dimension $m + 1 \geq 1$ with boundary N, and let g_+ be an *even asymptotically hyperbolic metric* on the interior $F = \overline{F} \smallsetminus N$, i.e., the sectional curvature of g_+ is asymptotically constant 1 at each boundary point. Obviously, the product $S^l \times \overline{F}$ of the l-dimensional standard sphere S^l with $\overline{F}$ has boundary $S^l \times N$. Now let

$$\Lambda : S^l \times \overline{F} \to D_l \overline{F}$$

be the map, which identifies the sphere S^l at (each point of) the boundary N to a single point. It was shown in [24] that the resulting quotient space $D_l\overline{F}$ with final topology is a manifold without boundary. Moreover, if the boundary N of the smooth manifold $\overline{F}$ is equipped with an *even structure*, then $D_l\overline{F}$ is in a naturally way a smooth manifold (without boundary). In fact, in our case the even structure on N is uniquely determined by the even AH metric g_+ on the interior F. Thus we have established the natural construction of a smooth manifold $D_l\overline{F}$ of dimension $m_l := m + l + 1$ from the even asymptotically hyperbolic space $(\overline{F}, g_+)$.

Furthermore, we denote the image $\Lambda(S^l \times N)$ of identified points in $D_l\overline{F}$ by N_p. The set N_p is a smooth submanifold of codimension $l + 1$ in $D_l\overline{F}$. We call N_p the *pole* of $D_l\overline{F}$, and $D_l\overline{F} \smallsetminus N_p$ is the *bulk* of $D_l\overline{F}$, which is by construction diffeomorphic to the product space $S^l \times F$. The product $S^l \times F$ admits the conformal structure $[g_{rd} \times g_+]$, which is the conformal class of the product metric $g_{rd} \times g_+$. It is straightforward to show that this conformal structure on the bulk $S^l \times F$

extends smoothly to $D_l\overline{F}$. We denote the resulting conformal structure on $D_l\overline{F}$ by $c_l[g_+]$, and we call $(D_l\overline{F}, c_l[g_+])$ the S^l-*doubling* (alias *collapsing l-sphere product*) of $(\overline{F}, g_+)$.

Certain metrics g in $c_l[g_+]$ on $D_l\overline{F}$ can be presented in normal form locally around the pole N_p. For this, let $r \geq 0$ be a special defining function on a neighbourhood U of N in $\overline{F}$ with respect to g_+, i.e., we have $N = \{r = 0\}$ and $|dr|_{\bar{g}} = 1$ on U with respect to the metric $\bar{g} = r^2 g_+$. Via the flow of the gradient $grad^{\bar{g}}(r)$ of the special defining function r with respect to $\bar{g}$ we obtain a natural identification of a collar neighbourhood U_ε of N in $\overline{F}$ with $[0,\varepsilon) \times N$ for some small $\varepsilon > 0$. The metric g_+ is given on $[0,\varepsilon) \times N$ with respect to r in normal form by $r^{-2}(dr^2 + g_r)$, where g_r is a smooth 1-parameter family of metrics on the boundary N (cf. e.g. [16] and Proposition 3.5). Now, the pull-back of the special defining function r to the product $S^l \times \overline{F}$ induces via Λ a function r_l on the neighbourhood D_lU of the pole N_p in $D_l\overline{F}$. In fact, the function r_l vanishes exactly on the pole N_p, and the set $D_lU_\varepsilon = \{x \in D_lU : r_l(x) < \varepsilon\}$ is an ε-tube neighbourhood of N_p in $D_l\overline{F}$. The tube neighbourhood D_lU_ε is via r_l (resp. r) uniquely identified with $B^{l+1}_\varepsilon \times N$ such that r_l represents the radial coordinate of the ball B^{l+1}_ε in $\mathbb{R}^{l+1}$. It was shown in [24] that the metric g_l of the form

$$(dr_l^2 + r_l^2 \cdot g_{rd}) \ + \ g_{r_l} \ , \tag{32}$$

defined on $B^{l+1}_\varepsilon \times N$, represents the conformal class $c_l[g_+]$ of the S^l-doubling $D_l\overline{F}$ around the pole N_p. Note that $dr_l^2 + r_l^2 \cdot g_{rd}$ is the flat metric on the factor B^{l+1}_ε, and g_{r_l} is a metric on the factor N, attached to any $x \in B^{l+1}_\varepsilon \times N$ with radial coordinate r_l. In fact, the normal form (32) characterises the S^l-doubling construction exclusively.

Theorem 7.1 *[24] Let g_t, $t \in [0,\varepsilon)$, be any family of metrics on a manifold N such that $g_l := (dr^2 + r^2 g_{rd}) + g_r$ is a smooth metric on the product space $B^{l+1}_\varepsilon \times N$ with radial coordinate r. Then g_l represents the conformal class $c_l[g_+]$ of the S^l-doubling of $g_+ := t^{-2}(dt^2 + g_t)$ defined on the interior of $\overline{F} = [0,\varepsilon) \times N$. In particular, g_+ is a smooth even asymptotically hyperbolic metric on $(0,\varepsilon) \times N$.*

Obviously, we can apply Theorem 7.1 to the normal form (31) derived in statement (4) of Proposition 6.7. The immediate consequence is the following local result.

Proposition 7.2 *Let (M^n, c), $n \geq 3$, be a closed Riemannian conformal space admitting an Euclidean subspace $\mathcal{S} \subset \mathcal{P}(\mathcal{T})$ of dimension $\ell > 1$ with $\Sigma(\mathcal{S}) \neq \emptyset$. Then there exists for any $p \in \Sigma(\mathcal{S})$ a neighbourhood U_q of q in M such that (U_q, c) is conformally equivalent to the S^l-doubling $(D_l\overline{F}, c_l[g_+])$, $l := \ell - 1$, of some Poincaré-Einstein space $(\overline{F}, g_+)$ of dimension $n - l$.*

Note that $\Sigma(\mathcal{S}) \cap U_q$ in the situation of Proposition 7.2 coincides with the pole N_p of the S^l-doubling $(D_l\overline{F}, c_l[g_+])$. We also have a global result.

Theorem 7.3 *Let (M^n, c) and $\mathcal{S} \subset \mathcal{P}(\mathcal{T})$ as in Proposition 7.2. If $M \setminus \Sigma(\mathcal{S})$ is simply connected, then (M^n, c) is conformally equivalent to the S^l-doubling $(D_l\overline{F}, c_l[g_+])$ of some simply connected, conformally compact Poincaré-Einstein space $(\overline{F}, g_+)$ of dimension $n - l$ such that the singularity set $\Sigma(\mathcal{S})$ corresponds to the pole N_p.*

Proof. Let $I(\mathcal{S}) = \{I_1, \ldots, I_\ell\}$ be an orthonormal basis of $\mathcal{S}$ with corresponding ℓ-form tractor $\alpha_{I(\mathcal{S})}$. The nc-Killing l-form $\Pi(\alpha_{I(\mathcal{S})})$ has no zeros on $M \setminus \Sigma(\mathcal{S})$. Hence there exists a unique metric $\tilde{g} \in c$ on $M \setminus \Sigma(\mathcal{S})$ such that $\tilde{\alpha}_- = \Pi_{\tilde{g}}(\alpha_{I(\mathcal{S})})$ has constant length 1 with respect to $\tilde{g}$. Note that any curve γ in $M \setminus \Sigma(\mathcal{S})$ has infinite length with respect to $\tilde{g}$ if γ converges to $\Sigma(\mathcal{S})$, i.e., $\tilde{g}$ is geodesically complete. Then, since $M \setminus \Sigma(\mathcal{S})$ is simply connected, it follows from Theorem 4.2 that the Riemannian manifold $(M \setminus \Sigma(\mathcal{S}), \tilde{g})$ is isometric to a special Einstein product $(M_1 \times M_2, g_1 \times g_2)$. Since we know by Proposition 7.2 that (M, c) is locally around any point of $\Sigma(\mathcal{S})$ conformally equivalent to an S^ℓ-doubling, it is clear that one of the factors, (M_1, g_1) or (M_2, g_2), is a round S^l-sphere. We can assume $(M_1, g_1) = (S^l, g_{rd})$. (Then (M_2, g_2) has negative scalar curvature $-(n-l)(n-l-1)$). Now let $p \in M_1 \cong S^l$ be a fixed point. We obtain an embedding $\iota : x \in M_2 \mapsto (p, x) \in M \setminus \Sigma(\mathcal{S})$. The closure of the image of ι in M is $\iota(M_2) \cup \Sigma(\mathcal{S})$. We set $\overline{F} := M_2 \cup \Sigma(\mathcal{S})$, which is compact and admits via ι the smooth structure of an $(n-l)$-dimensional manifold with boundary $N := \Sigma(\mathcal{S})$, interior M_2 and metric $g_+ := g_2$. Note that M_2 is simply connected.

We show now that g_+ is an AH Einstein metric. In particular, $(\overline{F}, g_+)$ will be a conformally compact Poincaré-Einstein space, whose S^l-doubling $(D_l\overline{F}, c_l[g_+])$ is conformally equivalent to (M^n, c). For this, let $\check{g}$ be a metric on a neighbourhood of $\check{U}_{\mathcal{S}}$ of $\mathcal{S}$ in M as guaranteed by Proposition 6.7. We set $\check{r} := \Sigma_{i=1}^{\ell}\check{s}_i^2$ for $\check{s}_i := \Pi_{\check{g}}(I_i)$. We can assume $\check{U}_{\mathcal{S}} = \check{U}_{\check{r}_o}$ for some $\check{r}_o > 0$. By construction, the function $\check{r}$ is constant on the spheres $M_1 \times \{q\}$ for any $q \in M_2$, and the restriction of $\check{r}$ to $\check{U}_{\mathcal{S}} \cap (\iota(M_2) \cup \Sigma(\mathcal{S}))$ induces via the pull-back with ι a defining function r on a collar of the boundary $N = \Sigma(\mathcal{S})$ in $\overline{F}$. In particular, we see that $\check{U}_{\mathcal{S}}$ is via $\check{r}$ diffeomorphic to $B^{\ell}_{\check{r}_o} \times \Sigma(\mathcal{S})$! Moreover, with (4) of Proposition 6.7 we see that g_+ is AH Einstein and r is a special defining function for g_+ on the interior M_2 of $\overline{F}$. Then from the local result of Proposition 7.2 we conclude that $\check{U}_{\mathcal{S}}$ is (globally!) the collapsing l-sphere product of a collar of N in $\overline{F}$ such that $\Sigma(\mathcal{S})$ and N_p coincide. However, then it is also clear that (M^n, c) is globally equivalent to the S^l-doubling of $(\overline{F}, g_+)$ such that $\Sigma(\mathcal{S})$ and N_p correspond. □

Alternatively, we can describe $(\overline{F}, g_+)$ of Theorem 7.3 as the orbit space under the action of $Inf_{\mathcal{S}}(M, c)$ on (M, c) (cf. Section 7 of [24]).

8 The scale singularity $\Sigma(\mathcal{S})$ as minimal submanifold

We show here that the scale singularity $\Sigma(\mathcal{S})$ for $\mathcal{S} \subset \mathcal{P}(\mathcal{T})$ Euclidean is a *totally umbilic* submanifold of (M, c). In fact, for certain metrics in c the singularity set $\Sigma(\mathcal{S})$ is a *minimal* submanifold. This discussion includes a tractor formulation, which implies total umbilicity in higher codimension.

Let (M^n, g) be an arbitrary Riemannian space and let N^m, $dim(N) = m$, be a submanifold of M^n with codimension $s := n - m$. The restriction of the tangent bundle TM to the submanifold N admits a natural g-orthogonal decomposition into the tangential part TN with projection pr and the bundle $T^{\perp}N$ of normal vectors on N in M with projection $pr_{\perp}$. The restriction $g_N := g|_{TN}$ is a Riemannian metric on N, and the Levi-Civita connection ∇^{g_N} of g_N is the tangential part of ∇^g (restricted to tangent vector fields on N).

Definition 8.1 *The second fundamental form of a submanifold N^m in (M, g) is given by the normal part*

$$II^g(X, Y) := pr_{\perp} \circ \nabla^g_X Y, \quad X, Y \in \mathfrak{X}(N),$$

of the Levi-Civita connection of g on M. The mean curvature of N in (M, g) is the trace $H^g := \frac{1}{m} tr_g II^g$. If $m > 1$ and II^g has only a trace part, i.e., $II = H^g \otimes g_N$ on N, then we call N a totally umbilic submanifold of (M, g).

If $\tilde{g} = e^{2\varphi}g$, $\varphi \in C^{\infty}(M)$, is any conformally equivalent metric to g on M, we have

$$\nabla^{\tilde{g}}_X Y = \nabla^g_X Y + d\varphi(X)Y + d\varphi(Y)X - g(X, Y) \cdot grad^g \varphi$$

for $X, Y \in \mathfrak{X}(M)$. This relation and the fact that the decomposition $TN \oplus T^{\perp}N$ is conformally invariant imply the transformation rules

$$II^{\tilde{g}} = II^g - pr_{\perp}(grad^g \varphi) \otimes g_N \quad \text{and} \quad H^{\tilde{g}} = e^{-2\varphi}(H^g - pr_{\perp}(grad^g \varphi)) \,. \tag{33}$$

We see that N is totally umbilic in (M, g) if and only if N is totally umbilic in $(M, \tilde{g})$, i.e., the notion of total umbilicity of N as submanifold of the conformal space $(M, [g])$ is well defined. In particular, it is clear that if N is *totally geodesic* in (M, g) (i.e. $II^g = 0$), then N is totally umbilic in $(M, [g])$.

Our next aim is to formulate a tractor condition for a submanifold (of higher codimension) in a conformal space, which implies total umbilicity. This tractor formulation is in the spirit of [4] concerning hypersurfaces. Let (M, c) be a Riemannian conformal space with standard tractor bundle $\mathcal{T}$ and let N^m be a submanifold of codimension s in (M,c). For convenience, we assume here that the normal bundle $T^\perp N$ is orientable. (Locally on N, this is always the case without any further assumption!) Then with respect to $g \in c$ on M we can choose a volume form $vol(g|_{T^\perp N})$ of $T^\perp N$. We set

$$I_N \;:=\; \begin{pmatrix} 0 \\ vol(g|_{T^\perp N}) \qquad 0 \\ H^g \lrcorner\ vol(g|_{T^\perp N}) \end{pmatrix} . \tag{34}$$

Via the identification $\Lambda^s\mathcal{T}^* \cong_g \Lambda^{s-1}T^*M[s] \oplus (\Lambda^s T^*M[s] \oplus \Lambda^{s-2}T^*M[s-2]) \oplus \Lambda^{s-1}T^*M[s-2]$ we understand I_N as a section of the s-form tractor bundle $\Lambda^s\mathcal{T}^*$ of M restricted to N. In fact, the transformation rule for s-form tractors (induced by (1)) and the transformation (33) of the mean curvature show the independence of the definition of I_N from the choice of metric $g \in c$ on M (up to a choice of orientation on $T^\perp N$).

Alternatively, the s-form tractor I_N can be presented as a simple wedge product of 1-form tractors. For this, let $\{n_1,\ldots,n_s\}$ be an oriented g-orthonormal (local) frame of the normal bundle $T^\perp N$. Then we set

$$N_i := \begin{pmatrix} 0 \\ n_i \\ \nu_i \end{pmatrix}, \qquad i = 1,\ldots,s,$$

where $\nu_i := g(H, n_i)$ is the mean curvature of N in normal direction n_i. The normal standard tractor frame $\{N_1,\ldots,N_s\}$ on N in (M,c) is independent of the choice of metric as (33) shows. If α_i denotes the dual 1-form tractor to N_i, $i = 1,\ldots,s$, via the tractor metric $\langle\cdot,\cdot\rangle_{\mathcal{T}}$, we have $I_N = \alpha_1 \wedge \cdots \wedge \alpha_s$.

Definition 8.2 *Let N^m, $m \geq 1$, be a submanifold of codimension $s > 0$ in a Riemannian conformal space (M,c) (with oriented $T^\perp N$).*

1. *We call I_N as defined in (34) the normal s-form tractor of N in (M,c).*
2. *We say N is a strongly umbilic submanifold of (M,c) if I_N is ∇-parallel along N, i.e.,*

$$\nabla_X I_N = 0 \qquad \forall X \in TN\ .$$

Note that Definition 8.2 (1) for codimension $s = 1$ is equivalent to the classical definition of the normal tractor of a hypersurface in [4]. A hypersurface N in (M,c) is totally umbilic if and only if the corresponding normal tractor I_N is ∇-parallel along N. For higher codimension $s > 1$ we have at least the following (weaker) result.

Theorem 8.3 *Any strongly umbilic submanifold N^m, $m > 1$, of codimension $s > 0$ in a Riemannian conformal space (M,c) is totally umbilic.*

Proof. We compute with respect to an arbitrary metric $g \in c$, and an (oriented) g-orthonormal local frame $\{n_1,\ldots,n_s\}$ of $T^\perp N$. The corresponding normal 1-form tractors are then given by $\alpha_i = (0, \eta_i, \nu_i)$, $i = 1,\ldots,s$, with $\eta_i := g(n_i,\cdot)$. Covariant differentiation with respect to the tractor connection ∇ gives

$$\nabla_X \alpha_i = \begin{pmatrix} 0 \\ \nabla_X \eta_i + \nu_i g(X,\cdot) \\ X(\nu_i) + \mathsf{P}^g(X, n_i) \end{pmatrix}$$

for $X \in TN$ and $i = 1, \ldots, s$. We set $\kappa(X) := \Sigma_{i=1}^{s}(X(\nu_i) + \mathsf{P}^g(X, n_i))n_i$ and $\beta_i := -g(II_o(\cdot,\cdot), n_i)$ for $i = 1, \ldots, s$, where II_o denotes the trace-free part of the second fundamental form II. Then we obtain with $I_N = \alpha_1 \wedge \cdots \wedge \alpha_s$:

$$\nabla_X I_N \;=\; \begin{pmatrix} 0 \\ \Sigma_{i=1}^{s}\eta_1 \wedge \cdots \wedge \eta_{i-1} \wedge \beta_i(X,\cdot) \wedge \eta_{i+1} \wedge \cdots \wedge \eta_s \qquad 0 \\ \kappa(X) \lrcorner\, vol(g|_{T^\perp N}) \end{pmatrix}.$$

The condition $\nabla_X I_N = 0$ for all $X \in TN$ implies $\beta_i = 0$ for all $i = 1, \ldots, s$, i.e., II_o vanishes and N is totally umbilic. □

We can apply Theorem 8.3 to the situation of multiple almost Einstein structures with intersecting scale singularities.

Theorem 8.4 *Let (M^n, c), $n \geq 3$, be a Riemannian conformal space admitting an Euclidean subspace $\mathcal{S} \subset \mathcal{P}(\mathcal{T})$ of dimension $\ell > 0$ with $\Sigma(\mathcal{S}) \neq \emptyset$.*

1. *The scale singularity $\Sigma(\mathcal{S})$ is a totally umbilic submanifold of codimension ℓ in M (for $n - \ell > 1$).*
2. *For any $g \in c$ there exists a metric $\breve{g} \in c$ on M with $\breve{g}|_{T\Sigma(\mathcal{S})} = g|_{T\Sigma(\mathcal{S})}$ such that $\Sigma(\mathcal{S})$ is totally geodesic and minimal in $(M, \breve{g})$.*

Proof. (1) Let $I(\mathcal{S}) := \{I_1, \ldots, I_\ell\}$ be an orthonormal basis of $\mathcal{S}$. We set $n_i := grad^g(s_i)$ for $s_i = \Pi_g(I_i)$. Each scale singularity $\Sigma(I_i)$, $i = 1, \cdots, \ell$, is totally umbilic in M, and I_i is given on $\Sigma(I_i)$ with respect to some $g \in c$ by $(0, n_i, \Box^g s_i)$. A straightforward calculations shows

$$H_i^g \;=\; \frac{1}{n}\Delta^g s_i$$

for the mean curvature of $\Sigma(I_i)$, $i = 1, \cdots, \ell$, in (M, g) (cf. [12]). In particular, we have $g(n_i, \nabla_X^g X) = \frac{1}{n}\Delta^g s_i \cdot |X|_g^2$ for any tangent vector $X \in T\Sigma(I_i)$ and any $i = 1, \cdots, \ell$. Now let $\{e_1, \ldots, e_{n-\ell}\}$ be some orthonormal frame of the tangent bundle $T\Sigma(\mathcal{S})$ of the scale singularity $\Sigma(\mathcal{S})$. Then we have

$$H_{I(\mathcal{S})}^g \;=\; \frac{1}{n-\ell} \cdot \Sigma_{i=1}^{n-\ell}\Sigma_{j=1}^{\ell} g(n_j, \nabla_{e_i}^g e_i)n_j = \frac{1}{n}\Sigma_{j=1}^{\ell}\Delta^g s_j \cdot n_j$$

for the mean curvature of $\Sigma(\mathcal{S})$ in M.

On the other hand, for $\alpha_{I(\mathcal{S})} = I_1^\flat \wedge \cdots \wedge I_\ell^\flat$ we have

$$\alpha_{I(\mathcal{S})} \;\cong_g\; \begin{pmatrix} 0 \\ \eta_1 \wedge \cdots \wedge \eta_\ell \qquad 0 \\ \left(\frac{1}{n}\Sigma_{j=1}^{\ell}\Delta^g s_j \cdot n_j\right) \lrcorner\, \eta_1 \wedge \cdots \wedge \eta_\ell \end{pmatrix},$$

on $\Sigma(\mathcal{S})$, where $\eta_j := g(n_j, \cdot)$ for $j = 1, \cdots, \ell$. Note that $vol(g|_{T^\perp\Sigma(\mathcal{S})}) = \eta_1 \wedge \cdots \wedge \eta_\ell$. This proves $I_{\Sigma(\mathcal{S})} = \alpha_{I(\mathcal{S})}$ on $\Sigma(\mathcal{S})$. Since $\alpha_{I(\mathcal{S})}$ is ∇-parallel, it follows that $I_{\Sigma(\mathcal{S})}$ is ∇-parallel along $\Sigma(\mathcal{S})$. With Theorem 8.3 we conclude that $\Sigma(\mathcal{S})$ is totally umbilic in M for $dim\Sigma(\mathcal{S}) = n - \ell > 1$.

(2) First, note that a submanifold is by definition minimal if the mean curvature vanishes identically, i.e., a submanifold is totally geodesic if and only if it is totally umbilic and minimal.

If M is closed then we can use the metric $\breve{g} \in c$ on M of Proposition 6.7. Obviously, we have

$$I_{\Sigma(\mathcal{S})} \;=\; \alpha_{I(\mathcal{S})} \;\cong_{\breve{g}}\; \begin{pmatrix} 0 \\ vol(g|_{T^\perp\Sigma(\mathcal{S})}) \qquad 0 \\ 0 \end{pmatrix}$$

on $\Sigma(\mathcal{S})$, which proves that $H^g_{I(\mathcal{S})}$ vanishes identically. For arbitrary M we will present a general prove after the proof of Theorem 8.5 below. □

Finally, we observe that the scale singularity $\Sigma(\mathcal{S})$ in (M,c) is not only totally umbilic, but also satisfies additional extrinsic curvature properties. Here we denote by $W^g|_{\Sigma(\mathcal{S})}$ and $C^g|_{\Sigma(\mathcal{S})}$ the restriction to $\Sigma(\mathcal{S})$ of the Weyl and Cotten tensor on M with respect to $g \in c$, respectively. The (intrinsic) Weyl and Cotten tensor of $\Sigma(\mathcal{S})$ are denoted by $W^{\Sigma(\mathcal{S})}$, resp., $C^{\Sigma(\mathcal{S})}$.

Theorem 8.5 *Let $\Sigma(\mathcal{S})$ be a scale singularity in (M,c) as in Theorem 8.4 of dimension $dim(\Sigma(\mathcal{S})) \geq 3$ with normal bundle $T^\perp\Sigma(\mathcal{S})$. Let $g \in c$ be an arbitrary metric. Then*

1. *$n \lrcorner W^g = 0$ for any normal vector $n \in T^\perp\Sigma(\mathcal{S})$.*
2. *If $dim(M) \geq 5$, then $n \lrcorner C^g = 0$ for any $n \in T^\perp\Sigma(\mathcal{S})$.*
3. *$W^{\Sigma(\mathcal{S})} = W^g|_{\Sigma(\mathcal{S})}$ and $C^{\Sigma(\mathcal{S})} = C^g|_{\Sigma(\mathcal{S})}$.*

Proof. (1) Let $I(\mathcal{S}) := \{I_1, \ldots, I_\ell\}$ be an orthonormal basis of $\mathcal{S}$, and let $n_i := grad^g(s_i)$ for $s_i = \Pi_g(I_i)$. Then $n_i \lrcorner W^g = 0$ follows immediately from (18) for $\alpha_- := s_i$, $i = 1, \ldots, \ell$. Since $\{n_1, \ldots, n_\ell\}$ is a basis of $T^\perp\Sigma(\mathcal{S})$, the first statement follows. In the same manner statement (2) follows from (20) and (22) if $dim(M) \geq 5$.

(3) First, let us assume $\ell = 1$, i.e., we have a ∇-parallel tractor I_1 on M with hypersurface singularity $\Sigma(I_1)$. In this case statement (3) follows directly from Theorem 4.5 of [12]. In fact, it was shown in [12] that the restriction of the subbundle $I_1^\perp \subset \mathcal{T}$ to the hypersurface $\Sigma(I_1)$ is naturally identified with the tractor bundle $\mathcal{T}^{\Sigma(I_1)}$ of $\Sigma(I_1)$. Moreover, the restriction of the tractor connection ∇ to $I_1^\perp \to \Sigma(I_1)$ gives rise to the canonical tractor connection $\nabla^{\Sigma(I_1)}$ on $\mathcal{T}^{\Sigma(I_1)} \to \Sigma(I_1)$. It follows with $\Omega^\nabla I_1 = 0$ that the tractor curvature Ω^∇ of $\mathcal{T}$ restricts naturally to the tractor curvature of $\nabla^{\Sigma(I_1)}$ on $\mathcal{T}^{\Sigma(I_1)}$. In particular, we obtain $W^{\Sigma(I_1)} = W^g|_{\Sigma(I_1)}$ and $C^{\Sigma(I_1)} = C^g|_{\Sigma(I_1)}$.

Now let $\ell > 1$ be arbitrary. Note that above argument from [12] also shows that any I_i, $i = 2, \ldots, \ell$, restricts to a $\nabla^{\Sigma(I_1)}$-parallel tractor on $\Sigma(I_1)$. The scale singularity of the restriction of I_i is $\Sigma(I_1) \cap \Sigma(I_i)$ for $i = 2, \ldots, \ell$. In particular, $\Sigma(I_1) \cap \Sigma(I_2)$ is a totally umbilic hypersurface in $\Sigma(I_1)$, and Theorem 4.5 of [12] shows again that the Weyl tensor W and the Cotton tensor C of $\Sigma(I_1)$ restrict to W and C of $\Sigma(I_1) \cap \Sigma(I_2)$, i.e., $W^{\Sigma(I_1)\cap\Sigma(I_2)} = W^g|_{\Sigma(I_1)\cap\Sigma(I_2)}$ and $C^{\Sigma(I_1)\cap\Sigma(I_2)} = C^g|_{\Sigma(I_1)\cap\Sigma(I_2)}$. This argument applies iteratively in ℓ steps, altogether. This finally proves $W^{\Sigma(\mathcal{S})} = W^g|_{\Sigma(\mathcal{S})}$ and $C^{\Sigma(\mathcal{S})} = C^g|_{\Sigma(\mathcal{S})}$. □

Proof of Theorem 8.4 (2). Let M be arbitrary. We set $\Sigma^0(\mathcal{S}) := M$ and $\Sigma^j(\mathcal{S}) := \bigcap_{i=1}^j \Sigma(I_i)$ for $j = 1, \ldots, \ell$. Then, with the argument from the above proof it is clear that the trace-free part II_o of $\Sigma^{k+1}(\mathcal{S})$ in $\Sigma^k(\mathcal{S})$ vanishes for any $k = 0, \ldots, \ell-1$. Lemma 3.4 shows that for any metric $g(\ell)$ in the conformal class c restricted to $\Sigma(\mathcal{S})$ there is a metric $g(\ell-1)$ on a neighbourhood of $\Sigma(\mathcal{S})$ in $\Sigma^{\ell-1}(\mathcal{S})$ such that $\Sigma(\mathcal{S})$ is minimal in $\Sigma^{\ell-1}(\mathcal{S})$ with respect to $g(\ell-1)$. Now, if we use $g(k)$ as boundary condition on (a neighbourhood of $\Sigma^{k+1}(\mathcal{S})$ in) $\Sigma^k(\mathcal{S})$, then we obtain by iterated application of Lemma 3.4 for each $k = \ell-1, \ldots, 1$ a metric $g(k-1)$ on a neighbourhood of $\Sigma^k(\mathcal{S})$ in $\Sigma^{k-1}(\mathcal{S})$, for which $\Sigma^k(\mathcal{S})$ is minimal in $\Sigma^{k-1}(\mathcal{S})$.

The ultimate metric $g(0)$ is defined on a neighbourhood of $\Sigma(\mathcal{S})$ in M. We can extend $g(0)$ smoothly to a metric $\breve{g} \in c$ on M. Then we have $\breve{g}|_{T\Sigma(\mathcal{S})} = g(\ell)$, where $g(\ell)$ is the initial and arbitrary choice in c on $\Sigma(\mathcal{S})$. We set $s_i := \Pi_{\breve{g}}(I_i)$. We have by construction $ds_j(\nabla^{\breve{g}}_X X) = 0$ for any $j = 1, \ldots, \ell$ and any X tangent to $\Sigma(\mathcal{S})$. This shows that $\Sigma(\mathcal{S})$ is minimal in $(M, \breve{g})$. □

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