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Abstract

In this paper, we study convergence of two different iterative regularization methods for nonlinear ill-posed problems in Banach spaces. One of them is a Landweber type iteration the other one the iteratively regularized Gauss-Newton method with an a posteriori chosen regularization parameter in each step. We show that a discrepancy principle as a stopping rule renders these iteration schemes regularization methods, i.e., we prove their convergence as the noise level tends to zero. The theoretical findings are illustrated by two parameter identification problems for elliptic PDEs.

1 Introduction

This article is concerned with iterative solutions of nonlinear ill-posed operator equations in Banach spaces. Hence we consider an equation

$$F(x) = y \tag{1}$$

where $F : \mathcal{D}(F) \subseteq X \to Y$ is a nonlinear operator between Banach spaces X and Y. Instead of exact data y we assume that only noisy data y^δ with noise level δ are given such that

$$\left\| y^\delta - y \right\| \leq \delta\,. \tag{2}$$

Equation (1) is ill-posed in the sense that the solution of (1) does not depend continuously on the data and thus a direct inversion of noise-contaminated data y^δ would not lead to a meaningful solution. Hence a stable solution of (1) requires regularization techniques which are continuous approximations to F^{-1}. Iterative methods are widely used as regularizations of nonlinear problems.

Operator equations like (1) are thoroughly studied in the case of Hilbert spaces X and Y. Thereby the Landweber method and the Gauss - Newton method are very popular iterative solvers; convergence and stability of both of them have been well investigated. A convergence analysis of the Landweber iteration is found in Hanke, Neubauer and Scherzer [12]. The article [2] deals with the convergence of the iteratively regularized Gauss - Newton method. Convergence with rates of this iteration has been proven in [14]. Other iterative techniques that have been studied to solve (1) in Hilbert spaces are the Levenberg - Marquardt scheme [11], the method of conjugate gradients [10] and inexact Newton regularizations [19]. Overviews of iterative regularization methods for inverse problems in Hilbert spaces are also found in the books [17], [20] and [8]. A book which is entirely dedicated to iterative solvers for nonlinear operator equations is [16].

Linear ill-posed problems in Banach spaces is a growing and very lively area of research. Over the last few years a lot of theoretical and practical results have been formulated. We only name here a few. In [21] the authors presented a nonlinear extension of the Landweber method to Banach spaces using duality mappings. The iterative minimization of Tikhonov functionals in Banach spaces was outlined in [3] and convergence was proven. The article [22] deals with the solution of convex split feasibility problems in Banach spaces by cyclic projections. Convex feasibility problems in connection with Bregman projection methods are also investigated in [1]. Convergence rates results for Tikhonov regularization in Banach spaces have been formulated in [13]. A general treatise of quantitative aspects of regularizations for ill-posed problems in Banach spaces is [18].

So far, to the authors best knowledge, iterative solvers for *nonlinear* ill-posed problems in Banach spaces have not been formulated. To this end we extend in this article the well-known Landweber method and the iteratively regularized Gauss - Newton method (IRGNM) to that case, prove their convergence and demonstrate their applicability to two parameter identification problems for elliptic PDEs.

We give a brief overview of the article. Section 2 provides the mathematical setup, the iterative methods are formulated and all mathematical ingredients which are necessary for the following investigations are briefly summarized. In Section 3 we show that the Landweber type iteration converges if the step size is chosen appropriately. The convergence of the IRGNM under a certain additional condition for the regularization parameter is proven in Section 4. In Section 5 finally we present parameter identification problems for two elliptic boundary value problems and prove that these problems actually fulfill the conditions that guarantee convergence of both methods.

2 Mathematical setup

In subsection 2.1 we introduce a Landweber type method and the iteratively regularized Gauss - Newton - method (IRGNM) to solve (1). The concept of duality mappings in Banach spaces that is involved in the formulation of the Landweber type method is shortly summarized in subsection 2.2 along with essential results on Bregman distances which are needful for the convergence theory. Here, we refer to the corresponding literature for a detailed outline. The essential assumptions on the forward operator F which are supposed to be valid through the whole article are formulated in subsection 2.3.

2.1 Iterative methods

In analogy to the Landweber method in Hilbert spaces we will study the generalization of the nonlinear method from [21] to solve (1)

$$\begin{aligned} J_p(x_{k+1}^\delta) &= J_p(x_k^\delta) - \mu_k F'(x_k^\delta)^* j_r(F(x_k^\delta) - y^\delta)\,, \\ x_{k+1}^\delta &= J_q^*(J_p(x_{k+1}^\delta))\,, \qquad k = 0, 1, \ldots \end{aligned} \tag{3}$$

Here $F'(x)$ is the Fréchet derivative of F at x which has to exist for using (3). The nonlinear operators J_p, j_r and J_q^* are duality mappings from X, Y and X^* to their duals, respectively. The concept of duality mappings is concisely explained in subsection 2.2. The stepsize μ_k has to be chosen in an appropriate manner to guarantuee convergence of the method, see (15).
The second method which is considered in this article can be seen as a generalization of the iteratively regularized Gauss - Newton method (IRGNM)

$$x_{k+1}^\delta \in \operatorname{argmin}_{x \in \mathcal{D}(F)} \; \left\| T_k(x - x_k^\delta) + R_k \right\|^r + \alpha_k \left\| x - x_0 \right\|^p \,, \quad k = 0, 1, \ldots \tag{4}$$

where we abbreviate

$$T_k = F'(x_k^\delta)\,, \quad R_k = F(x_k^\delta) - y^\delta\,,$$

$p, r \in (1, \infty)$, and x_0 is some a priori guess. In (4) $F'(x)$ denotes some linearization of F at $x \in X$ satisfying a tangential cone condition according to (9), which does not necessarily imply Fréchet differentiability of F. Note that in case of Hilbert spaces X, Y with $p = r = 2$ this iteration coincides with the known IRGNM from, e.g., [2], [14]

$$x_{k+1}^\delta = x_k^\delta - (T_k^* T_k + \alpha_k I)^{-1} \Big(T_k^* R_k + \alpha_k (x_k^\delta - x_0) \Big)\,. \tag{5}$$

For the solution of the convex minimization problem in (4) we refer, e.g., to [3].

The stopping index $k_* = k_*(\delta)$ of the iterations will in both cases be determined by a discrepancy type principle

$$k_*(\delta) = \min\{k \in \mathbb{N} \;:\; \left\| F(x_k^\delta) - y^\delta \right\| \le C_{dp}\delta\}\,, \tag{6}$$

2.2 Duality mappings and Bregman distances

Let X be a real Banach space with dual X^*. For $p > 1$ the subdifferential mapping $J_p := \partial f_p : X \to 2^{X^*}$ of the convex functional $x \mapsto \frac{1}{p}\|x\|^p$ is called the duality mapping of X with gauge function $t \mapsto t^{p-1}$. It is an in general nonlinear, set-valued mapping characterized by

$$x^* \in J_p(x) \quad \Leftrightarrow \quad \langle x^* \,,\, x\rangle = \|x\|^p \quad \text{and} \quad \|x^*\| = \|x\|^{p-1}\,,$$

where we write $\langle x^* \,,\, x\rangle = \langle x \,,\, x^*\rangle = x^*(x)$ for the application of $x^* \in X^*$ on $x \in X$. For $q > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$ we denote by J_q^* the duality mapping of the dual X^* with gauge function $t \mapsto t^{q-1}$. Throughout this paper X is supposed to be uniformly smooth and uniformly convex, hence it is reflexive and the dual X^* has the same properties. For an overview of the precise definitions of smoothness and convexity and the interplay with duality mappings we refer to [21], a comprehensive treatise can be found in [7]. Here it suffices to know that under our assumptions the duality mappings J_p and J_q^* are both single-valued, uniformly continuous on bounded sets and bijective with $(J_p)^{-1} = J_q^*$. We will also consider the case of X being p-convex which is equivalent to the dual being q-smooth, i.e. there exists a constant $C_q > 0$ such that the following inequality holds for all $x^*, y^* \in X^*$, see [23]:

$$\|x^* - y^*\|^q \leq \|x^*\|^q - q\,\langle J_q^*(x^*)\,,\, y^*\rangle + C_q\|y^*\|^q\,. \tag{7}$$

To analyse the convergence of the Landweber type method we employ the Bregman distance $\Delta_p(x,y)$ between $x,y \in X$, defined as

$$\Delta_p(x,y) = \frac{1}{q}\|x\|^p - \langle J_p(x)\,,\, y\rangle + \frac{1}{p}\|y\|^p\,.$$

In Hilbert spaces we have $\Delta_2(x,y) = \frac{1}{2}\|x-y\|^2$. This notion of distance goes back to Bregman [4] and has succesfully been used in investigations of problems in Banach space settings, see e.g. [1, 5, 13, 22]. In general Δ_p is not a metric but it has some distance-like properties, especially we have

$$\Delta_p(x,y) \geq 0 \quad \text{and} \quad \Delta_p(x,y) = 0 \Leftrightarrow x = y\,.$$

Furthermore in a p-convex space X there exists some constant $c_p > 0$ such that for all $x,y \in X$ we have

$$\Delta_p(x,y) \geq c_p\,\|x-y\|^p\,. \tag{8}$$

In the following Y is allowed to be an arbitrary Banach space and we write j_r for a single-valued selection of the possibly multi-valued duality mapping of Y with gauge function $t \mapsto t^{r-1}$, $r > 1$. Possible further restrictions on X and Y will be indicated in the respective theorems.

2.3 Assumptions on the forward operator

The main assumption that we postulate for the forward operator to hold is the tangential cone condition

$$\|F(x) - F(\bar{x}) - F'(x)(x-\bar{x})\| \leq c_{tc}\,\|F(x) - F(\bar{x})\| \qquad \forall x,\bar{x} \in \mathcal{B} \tag{9}$$

for some $0 < c_{tc} < 1$, where

$$\mathcal{B} = \begin{cases} \mathcal{B}_\rho^\Delta(x^\dagger) & \text{in case of (3)} \\ \mathcal{D}(F) \cap \mathcal{B}_\rho(x_0) & \text{in case of (4)} \end{cases}$$

Here, $\mathcal{B}_\rho(x_0)$ denotes the closed ball of radius $\rho > 0$ around x_0 (possibly also $\rho = \infty$ and $\mathcal{B}_\rho(x_0) = \mathcal{D}(F)$), and $\mathcal{B}_\rho^\Delta(x^\dagger) = \{x \in X \,|\, \Delta_p(x,x^\dagger) \leq \rho\}$ is a ball with respect to the Bregman distance around some solution $x^\dagger$ of (1). Additionally we assume

- continuity of F and of F' as well as

  $$\mathcal{B}_\rho^\Delta(x^\dagger) \subseteq \mathcal{D}(F) \tag{10}$$

  in case of (3);

- (weak) sequential closedness in the sense that either

  $$\begin{gathered}(x_n \rightharpoonup x \;\wedge\; F(x_n) \to f) \\ \Rightarrow\; (x \in \mathcal{D}(F) \;\wedge\; F(x) = f)\end{gathered} \tag{11}$$

  or

  $$\begin{gathered}(J_p(x_n - x_0) \rightharpoonup x^* \;\wedge\; F(x_n) \to f) \\ \Rightarrow\; (x := J_q^*(x^*) + x_0 \in \mathcal{D}(F) \;\wedge\; F(x) = f)\end{gathered} \tag{12}$$

  for all $(x_n)_{n\in\mathbb{N}} \subseteq X$ in case of (4). Note that by $J_q^* = J_p^{-1}$ we have $J_p(x - x_0) = x^*$.

Note that nonemptyness of the interior (with respect to the norm) of $\mathcal{D}(F)$ is sufficient for (10); in a p-convex X this is an immediate consequence of (8), and in the general uniformly convex case this follows e.g. from the proof of Theorem 2.12 (e) in [21].

Remark 1. *We point out that so far, convergence of the IRGNM has been studied under somewhat stronger conditions on F even in the Hilbert space setting, compare [14, 16].*

The tangential cone condition (9) allows to show existence of an x_0-minimum-norm solution as in the Hilbert space situation:

Proposition 1. *(Proposition 2.1 in [16] and Lemma 2.10 in [21])*
Let (9) hold and $\mathcal{D}(F) \cap \mathcal{B}_\rho(x_0) = \mathcal{B}_\rho(x_0)$ (i.e., $\mathcal{D}(F)$ has nonempty interior).

(i) Then for all $x \in \mathcal{B}_\rho(x_0)$

$$M_x := \{\tilde{x} \in \mathcal{B}_\rho(x_0) \,:\, F(\tilde{x}) = F(x)\} = \Big(x + \mathcal{N}(F'(x))\Big) \cap \mathcal{B}_\rho(x_0)$$

and

$$\mathcal{N}(F'(x)) = \mathcal{N}(F'(\tilde{x})) \text{ for all } \tilde{x} \in M_x .$$

Moreover,

$$\mathcal{N}(F'(x)) \supseteq \{t(\tilde{x} - x) \,:\, \tilde{x} \in M_x, t \in \mathbb{R}\} ,$$

where instead of $\supseteq$ equality holds if $x \in \mathrm{int}(\mathcal{B}_\rho(x_0))$.

(ii) If $F(x) = y$ is solvable in $\mathcal{B}_\rho(x_0)$, then an x_0-minimum-norm solution $x^\dagger$ exists and is unique.
For $x^\dagger \in \mathrm{int}(\mathcal{B}_\rho(x_0))$ we have

$$J_p(x^\dagger) \in \overline{\mathcal{R}(F'(x^\dagger)^*)} \tag{13}$$

and if for some $\tilde{x} \in \mathcal{B}_\rho(x_0)$

$$J_p(\tilde{x}) \in \overline{\mathcal{R}(F'(x^\dagger)^*)} \text{ and } \tilde{x} - x^\dagger \in \mathcal{N}(F'(x^\dagger))$$

holds, then $\tilde{x} = x^\dagger$.

Proof. Part (i) follows analogously to part (i) of the proof of Proposition 2.1 in [16] which remains valid in Banach spaces without any modification. Part (ii) can be seen exactly as the respective assertion in the linear case as stated and proved in Lemma 2.10 of [21] up to the following small modification in the proof of (13) due to the restriction to a neighborhood of x_0:
For any $z \in \mathcal{N}(F'(x^\dagger))$ there exists an $\epsilon > 0$ such that

$$x^\dagger \pm \epsilon z \in \Big(x^\dagger + \mathcal{N}(F'(x^\dagger))\Big) \cap \mathcal{B}_\rho(x_0) = \{\tilde{x} \in \mathcal{B}_\rho(x_0) \,:\, F(\tilde{x}) = y\} .$$

Hence, by Theorem 2.5 in [21]

$$\langle J_p(x^\dagger), x^\dagger \rangle \leq \langle J_p(x^\dagger), x^\dagger \pm \epsilon z \rangle ,$$

i.e. $\langle J_p(x^\dagger), z \rangle = 0$. □

3 Convergence of the Landweber type iteration

Proposition 2. *Assume that X^* is q-smooth, that a solution $x^\dagger \in \mathcal{B}_\rho(x_0)$ to (1) exists, that F satisfies (9) with c_{tc} sufficiently small, that F and F' are continuous and that (10) holds. Let C_{dp} be chosen sufficiently large so that*

$$c_1 := c_{tc} + \frac{1 + c_{tc}}{C_{dp}} < 1 . \tag{14}$$

Then, with the choice

$$\mu_k := \frac{(1-c_1)^{p-1}}{C_q^{p-1}} \frac{\|R_k\|^{p-r}}{\max\{1, \|T_k\|^p\}} \geq 0 \tag{15}$$

with C_q being the constant in (7), monotonicity of the Bregman distances

$$\Delta_p(x_{k+1}^\delta, x^\dagger) - \Delta_p(x_k^\delta, x^\dagger) \leq -\frac{(1-c_1)^p}{p\, C_q^{p-1}} \frac{\|R_k\|^p}{\max\{1, \|T_k\|^p\}} \tag{16}$$

as well as $x_{k+1} \in \mathcal{D}(F)$ holds for all $k \leq k_*(\delta) - 1$.

Proof. Following the lines of the proof of the first part of Theorem 3.3 in [21], and using

$$\begin{aligned} \left\|F(x_k^\delta) - y^\delta - T_k(x_k^\delta - x^\dagger)\right\| &\leq c_{tc} \left\|F(x_k^\delta) - y\right\| + \delta \\ &\leq c_{tc} \left\|F(x_k^\delta) - y^\delta\right\| + (1 + c_{tc})\delta \end{aligned}$$

and (6) we have

$$\begin{aligned} &\Delta_p(x_{k+1}^\delta, x^\dagger) - \Delta_p(x_k^\delta, x^\dagger) \\ &= \frac{1}{q}\Big(\left\|x_{k+1}^\delta\right\|^p - \left\|x_k^\delta\right\|^p\Big) - \langle J_p(x_{k+1}^\delta) - J_p(x_k^\delta), x^\dagger\rangle \\ &= \Delta_p(x_{k+1}^\delta, x_k^\delta) - \mu_k \langle j_r(F(x_k^\delta) - y^\delta), T_k(x_k^\delta - x^\dagger)\rangle \\ &= \Delta_p(x_{k+1}^\delta, x_k^\delta) \\ &\quad -\mu_k\Big(\left\|F(x_k^\delta) - y^\delta\right\|^r - \langle j_r(F(x_k^\delta) - y^\delta), F(x_k^\delta) - y^\delta - T_k(x_k^\delta - x^\dagger)\rangle\Big) \\ &\leq \Delta_p(x_{k+1}^\delta, x_k^\delta) - \mu_k\Big(1 - \underbrace{\Big((c_{tc} + \frac{1 + c_{tc}}{C_{dp}}\Big)}_{=c_1}\Big) \left\|F(x_k^\delta) - y^\delta\right\|^r , \end{aligned}$$

where by inequality (7) we estimate

$$\begin{aligned} \Delta_p(x_{k+1}^\delta, x_k^\delta) &= \frac{1}{q}\Big(\left\|J_p(x_k^\delta) - \mu_k T_k^* j_r(F(x_k^\delta) - y^\delta)\right\|^q - \left\|J_p(x_k^\delta)\right\|^q\Big) \\ &\quad +\mu_k \langle T_k^* j_r(F(x_k^\delta - y^\delta), x_k^\delta\rangle \\ &\leq \frac{C_q}{q}\, \mu_k^q \, \max\{1, \|T_k\|^q\} \, \|R_k\|^{q\,(r-1)} . \end{aligned}$$

Hence we arrive at

$$\begin{aligned} &\Delta_p(x_{k+1}^\delta, x^\dagger) - \Delta_p(x_k^\delta, x^\dagger) \\ &\quad\leq -\mu_k\,(1 - c_1) \left\|F(x_k^\delta) - y^\delta\right\|^r + \frac{C_q}{q}\, \mu_k^q \, \max\{1, \|T_k\|^q\} \, \|R_k\|^{q\,(r-1)} . \end{aligned}$$

and with the choice of μ_k (15) assertion (16) is proven. □

Adapting the proof of the second part of Theorem 3.3 in [21] to the nonlinear case, the convergence result Theorem 2.4 in [16] can be generalized to the Banach space setting:

Theorem 1. *Let the assumptions of Proposition 2 be satisfied. Then the Landweber iterates x_k according to (3) applied to exact data y converge to a solution of $F(x) = y$. If $\overline{\mathcal{R}(F'(x))} \subseteq \overline{\mathcal{R}(F'(x^\dagger))}$ for all $x \in \mathcal{B}_\rho(x_0)$, then x_k converges to $x^\dagger$ as $k \to \infty$.*

Proof. The only point where the nonlinearity has to be taken into account is

$$\begin{aligned} |\langle J_p(x_{k_n}) - J_p(x_{k_l}), x_{k_n} - x\rangle &= \left|\sum_{k=k_l}^{k_n-1} \mu_k \langle T_k^* j_r(F(x_k) - y), x_{k_n} - x\rangle\right| \\ &\leq \sum_{k=k_l}^{k_n-1} \mu_k \, \|j_r(F(x_k) - y)\| \, \|T_k(x_{k_n} - x)\| , \end{aligned}$$

where we can estimate

$$\begin{aligned}
&\|T_k(x_{k_n}-x)\| \\
&\quad\le \|T_k(x_{k_n}-x_k)\| + \|T_k(x_k-x)\| \\
&\quad\le \|F(x_{k_n})-F(x_k)\| + \|F(x_k)-y\| \\
&\qquad + \|F(x_{k_n})-F(x_k)-T_k(x_{k_n}-x_k)\| + \|F(x_k)-F(x)-T_k(x_k-x)\| \\
&\quad\le (1+c_{tc})(\|F(x_{k_n})-F(x_k)\| + \|F(x_k)-y\|) \\
&\quad\le 3(1+c_{tc})\,\|F(x_k)-y\|\ .
\end{aligned}$$

□

For the sake of simplicity we restricted ourselves here to the case of a q-smooth dual. Let us mention that the same results can be proven (but in more technical way) if we only require uniform smoothness by adapting a similar proof technique and parameter choice as in [21, 3].

Theorem 2. *Let the assumptions of Theorem 1 hold with additionally Y being uniformly smooth and let $k_*(\delta)$ be chosen according to the stopping rule (6), (14). Then the Landweber iterates $x^\delta_{k_*(\delta)}$ according to (3) converge to a solution of (1) as $\delta \to 0$. If $\overline{\mathcal{R}(F'(x))} \subseteq \overline{\mathcal{R}(F'(x^\dagger))}$ for all $x \in \mathcal{B}_\rho(x_0)$, then $x^\delta_{k_*(\delta)}$ converges to $x^\dagger$ as $\delta \to 0$.*

Proof. By the uniform smoothness of Y the duality mapping j_r is also single-valued and uniformly continuous on bounded sets (cf. Theorem 2.3 (c) in [21])). Hence, for a fixed iteration index k, by continuity of F, F', J_p, J^*_q and j_r , the coefficient μ_k and hence the iterate x^δ_k continuously depend on the data y^δ.

Let $(\delta_n)_{n\in\mathbb{N}}$ be an arbitrary null sequence and $(k_n := k_*(\delta_n))_{n\in\mathbb{N}}$ the corresponding sequence of stopping indices.

The case of $(k_n)_{n\in\mathbb{N}}$ having a finite accumulation point can be treaten as in the proof of Theorem 2.6 of [16] without any changes also in the Banach space case.

As a matter of fact, this also holds true for the case $k_n \to \infty$ as $n \to \infty$, although at a first glance it looks as if the triangle inequality would be required which we do not have for the Bregman distance: Let x be a solution to (1). For arbitrary $\epsilon > 0$, by Theorem 1 we can find k such that $\Delta(x_k, x) < \frac{\epsilon}{2}$ and, by Theorem 2.12 (c) in [21], there exists n_0 such that for all $n \ge n_0$ we have $k_n \ge k$ and $|\Delta(x^{\delta_n}_k, x) - \Delta(x_k, x)| < \frac{\epsilon}{2}$. Hence, by Proposition 2

$$\Delta(x^{\delta_n}_{k_n}, x) \le \Delta(x^{\delta_n}_k, x) \le \Delta(x_k, x) + |\Delta(x^{\delta_n}_k, x) - \Delta(x_k, x)| < \epsilon\,.$$

□

4 Convergence of the IRGNM

Making use of the variational characterization (4), we provide a convergence proof with an a posteriori (instead of the so far usual a priori) choice of α_k in each step. Namley α_k is chosen as a solution to

$$\left\|T_k(x^\delta_{k+1}(\alpha) - x^\delta_k) + R_k\right\| = \theta\,\|R_k\|$$

for some $\theta \in (0, 1)$, or rather in a relaxed form such that

$$\underline{\theta}\,\|R_k\| \le \left\|T_k(x^\delta_{k+1}(\alpha) - x^\delta_k) + R_k\right\| \le \overline{\theta}\,\|R_k\| \tag{17}$$

for some $0 < \underline{\theta} \le \overline{\theta} < 1$, (cf. [11]), which corresponds to an inexact Newton method, and more precisely, to a discrepancy principle with artificial noise level $\theta\,\|R_k\|$. Here,

$$x^\delta_{k+1}(\alpha) \in \mathrm{argmin}_{x\in\mathcal{D}(F)}\ \left\|T_k(x - x^\delta_k) + R_k\right\|^r + \alpha\,\|x - x_0\|^p\ .$$

In the convergence proof we make use of the following lemma that contains a general analytical assertion.

Lemma 1. *Let X be a reflexive and strictly convex Banach space let $D \subseteq X$ be nonempty. Moreover, assume that either*

(a) $A : D \subseteq X \to Y$ is weakly closed and Y reflexive

or

(b) $A : D \subseteq X \to Y$ is weak-to-weak continuous and D weakly closed.

Denote for $\alpha > 0$

$$x(\alpha) = \text{argmin}_{x \in D} \Phi_\alpha(x) \,.$$

where $\phi_\alpha(x) := \|Ax\|_Y^r + \alpha \|x - x_0\|_X^p$. Then $\hat{\psi} : \alpha \mapsto \|Ax(\alpha)\|_Y^r$ is a continuous function on $(0, \infty)$. Furthermore the mapping $\alpha \mapsto x(\alpha)$ is continuous in case X is uniformly convex.

Proof. From (a) or (b) existence and from strict convexity of X uniqueness of $x(\alpha)$ follows by standard arguments.

We at first prove monotonicity of the mappings $\hat{\psi}$ and $\alpha \mapsto \|x(\alpha) - x_0\|_X$ in the sense that

$$\alpha_1 \le \alpha_2 \quad \Rightarrow \quad \begin{cases} \|x(\alpha_1) - x_0\|_X & \ge & \|x(\alpha_2) - x_0\|_X \\ \hat{\psi}(\alpha_1) & \le & \hat{\psi}(\alpha_2) \end{cases} \,. \tag{18}$$

Monotonicity of $\alpha \mapsto \|x(\alpha) - x_0\|_X$ follows from

$$\begin{aligned} \Phi_{\alpha_1}(x(\alpha_1)) &\le \Phi_{\alpha_1}(x(\alpha_2)) \\ &= \Phi_{\alpha_2}(x(\alpha_2)) + (\alpha_1 - \alpha_2) \|x(\alpha_2) - x_0\|_X^p \\ &\le \Phi_{\alpha_2}(x(\alpha_1)) + (\alpha_1 - \alpha_2) \|x(\alpha_2) - x_0\|_X^p \end{aligned} \tag{19}$$

which implies $(\alpha_1 - \alpha_2)\big(\|x(\alpha_2) - x_0\|_X^p - \|x(\alpha_1) - x_0\|_X^p \big) \ge 0$. Monotonicity of $\hat{\psi}$ follows from (19) and the monotonicity of $\alpha \mapsto \|x(\alpha - x_0)\|_X$.

To show continuity, let $\alpha > 0$, $\alpha_n \to \alpha$, which implies $\underline{\alpha} \le \alpha_n \le \overline{\alpha}$ for some $\underline{\alpha}, \overline{\alpha} > 0$. For all $n \in \mathbb{N}$ we have, by minimality of $x(\alpha_n)$

$$\Phi_{\alpha_n}(x(\alpha_n)) \le \Phi_{\alpha_n}(x(\alpha)) \le C \,.$$

Hence, $\|x(\alpha_n)\|_X$, $\|Ax(\alpha_n)\|_Y$ are uniformly bounded by C, $C/\underline{\alpha}$, respectively, and there exists a subequence α_{n_k} such that $x(\alpha_{n_k})$ converges weakly to some $x^* \in D$.

In case (a), by reflexivity of Y, a subsequence of $Ax(\alpha_{n_k})$ (denoted again by $Ax(\alpha_{n_k})$) converges weakly to some $y^* \in Y$ and by weak closedness $x^* \in D$, $Ax^* = y^*$ holds.

In case (b), by weak closedness of D we have $x^* \in D$, and by weak continuity of A, $Ax(\alpha_{n_k})$ converges weakly to Ax^*.

By the weak lower semicontinuity of the norms we get

$$\begin{aligned} \Phi_\alpha(x^*) &\le \liminf_{k \to \infty} \Phi_{\alpha_{n_k}}(x(\alpha_{n_k})) \le \limsup_{k \to \infty} \Phi_{\alpha_{n_k}}(x(\alpha_{n_k})) \\ &\le \limsup_{k \to \infty} \Phi_{\alpha_{n_k}}(x(\alpha)) = \Phi_\alpha(x(\alpha)) \,, \end{aligned}$$

where we have used minimality of the $x(\alpha_{n_k})$ in the third inequality. Since in a strictly convex X the minimizer of Φ_α is unique, we must have $x^* = x(\alpha)$ and thus it also follows that

$$\lim_{k \to \infty} \Phi_{\alpha_{n_k}}(x(\alpha_{n_k})) = \Phi_\alpha(x(\alpha)) \,. \tag{20}$$

In case $\alpha_{n_k} \ge \alpha$ for all k we get by (18)

$$\begin{aligned} \|x(\alpha) - x_0\|_X^p &\le \liminf_{k \to \infty} \|x(\alpha_{n_k}) - x_0\|_X^p \\ &\le \limsup_{k \to \infty} \|x(\alpha_{n_k}) - x_0\|_X^p \le \|x(\alpha) - x_0\|_X^p \,. \end{aligned}$$

Hence $\lim_{k\to\infty} \|x(\alpha_{n_k}) - x_0\|_X^p = \|x(\alpha) - x_0\|_X^p$ and from (20) we then further deduce $\lim_{k\to\infty} \hat{\psi}(\alpha_{n_k}) = \hat{\psi}(\alpha)$.

In case $\alpha_{n_k} \le \alpha$ for all k we at first similarly conclude by the monotonicity of $\hat{\psi}$ (18) that $\lim_{k\to\infty} \hat{\psi}(\alpha_{n_k}) = \hat{\psi}(\alpha)$ and then again with (20) that $\lim_{k\to\infty} \|x(\alpha_{n_k}) - x_0\|_X^p = \|x(\alpha) - x_0\|_X^p$. Subsequence arguments finally yield continuity of $\alpha \mapsto \hat{\psi}(\alpha)$ and $\alpha \mapsto \|x(\alpha) - x_0\|_X$. The latter together with the weak convergence of $x(\alpha_n)$ to $x(\alpha)$ implies strong convergence in a uniformly convex X. □

We now formulate the main convergence theorem of this section.

Theorem 3. *Assume that a solution $x^\dagger \in \mathcal{B}_\rho(x_0)$ to (1) exists, and that F satisfies (9) with c_{tc} sufficiently small as well as (11) or (12), let*

$$c_{tc} < \underline{\theta} < \overline{\theta} < 1 \,,$$

and let C_{dp} be chosen sufficiently large so that

$$c_{tc} + \frac{1 + c_{tc}}{C_{dp}} \le \underline{\theta} \text{ and } c_{tc} < \frac{1 - \overline{\theta}}{2} \,. \tag{21}$$

Moreover, assume that either

(a) $F'(x) : X \to Y$ is weakly closed for all $x \in \mathcal{D}(F)$ and Y reflexive

or

(b) $\mathcal{D}(F)$ weakly closed.

Then for all $k \le k_*(\delta) - 1$ with $k_*(\delta)$ according to (6), the iterates

$$x_{k+1}^\delta := \begin{cases} x_{k+1}^\delta = x_{k+1}^\delta(\alpha_k)\,,\ \alpha_k \text{ as in (17)} & \text{if } \left\|T_k(x_0 - x_k^\delta) + R_k\right\| \ge \overline{\theta}\, \|R_k\| \\ x_0 & \text{else} \end{cases}$$

are well-defined.

Moreover there exists a weakly convergent subsequence of

$$\begin{cases} x_{k_*(\delta)}^\delta & \text{if (11) holds} \\ J_p(x_{k_*(\delta)}^\delta - x_0) & \text{if (12) holds} \end{cases}$$

and along every such weakly convergent subsequence $x_{k_*(\delta)}^\delta$ converges strongly to a solution of (1) as $\delta \to 0$. If the solution $x^\dagger$ to (1) is unique, then $x_{k_*(\delta)}^\delta$ converges strongly to $x^\dagger$ as $\delta \to 0$.

Proof. Well-definedness of α_k in case $\left\|T_k(x_0 - x_k^\delta) + R_k\right\| \ge \overline{\theta}\, \|R_k\|$ can be seen as follows: By minimality of $x_{k+1}^\delta(\alpha)$ we have, for

$$\psi(\alpha) = \left\|T_k(x_{k+1}^\delta(\alpha) - x_k^\delta) + R_k\right\|$$

that

$$\psi(\alpha)^r + \alpha \left\|x_{k+1}^\delta(\alpha) - x_0\right\|^p \le \left\|T_k(x^\dagger - x_k^\delta) + R_k\right\|^r + \alpha \left\|x^\dagger - x_0\right\|^p \,,$$

hence

$$\limsup_{\alpha\to 0} \psi(\alpha) \le \left\|T_k(x^\dagger - x_k^\delta) + R_k\right\|^r \le (c_{tc} + \frac{1 + c_{tc}}{C_{dp}})\, \|R_k\|$$

by (2), (6), (9). On the other hand, again minimality of $x_{k+1}^\delta(\alpha)$ yields

$$\psi(\alpha)^r + \alpha \left\|x_{k+1}^\delta(\alpha) - x_0\right\|^p \le \left\|T_k(x_0 - x_k^\delta) + R_k\right\|^r \,,$$

so that

$$\left\|x_{k+1}^\delta(\alpha) - x_0\right\|^p \le \frac{1}{\alpha} \left\|T_k(x_0 - x_k^\delta) + R_k\right\|^r \to 0 \text{ as } \alpha \to \infty$$

so by continuity of T_k and the norms, there exists an $\bar{\alpha} > 0$ such that

$$\psi(\bar{\alpha}) > \underline{\theta}/\overline{\theta} \lim_{\alpha\to\infty} \psi(\alpha) = \underline{\theta}/\overline{\theta} \left\|T_k(x_0 - x_k^\delta) + R_k\right\| \ge \underline{\theta} \left\|R_k\right\| .$$

To conclude existence of an α_k satisfying (17), it remains to show continuity of ψ, which we do by using the fact that the uniformly convex Banach space X is reflexive and strictly convex, and setting $Ax = T_k(x - x_k^\delta) + R_k$, $D = \mathcal{D}(F)$ in lemma 1.
In case α_k can be chosen according to (17), by (4) we have for any solution $x^\dagger \in \mathcal{B}_\rho(x_0)$ of (1)

$$\begin{aligned} &\left\|T_k(x_{k+1}^\delta - x_k^\delta) + R_k\right\|^r + \alpha_k \left\|x_{k+1}^\delta - x_0\right\|^p \\ &\le \quad \left\|T_k(x^\dagger - x_k^\delta) + R_k\right\|^r + \alpha_k \left\|x^\dagger - x_0\right\|^p , \end{aligned} \tag{22}$$

which together with (2), (17), (6), (9) (see (17)), yields

$$\underline{\theta}^r \left\|R_k\right\|^r + \alpha_k \left\|x_{k+1}^\delta - x_0\right\|^p \le (c_{tc} + \frac{1+c_{tc}}{C_{dp}})^r \left\|R_k\right\|^r + \alpha_k \left\|x^\dagger - x_0\right\|^p \tag{23}$$

for all $k \le k_*(\delta) - 1$, provided $x_k \in \mathcal{B}_\rho(x_0)$. By (21) this implies

$$\left\|x_{k+1}^\delta - x_0\right\| \le \left\|x^\dagger - x_0\right\| , \tag{24}$$

which trivially holds in the alternative case $\left\|T_k(x_0 - x_k^\delta) + R_k\right\| > \overline{\theta} \left\|R_k\right\|$, in which we set $x_{k+1}^\delta = x_0$.

Estimate (24) allows us to conclude that $x_k^\delta \in \mathcal{B}_\rho(x_0)$ for all $k \le k_*(\delta)$ by an inductive argument.

Moreover, for all $k \le k_*(\delta) - 1$

$$\begin{aligned} &\left\|F(x_{k+1}^\delta) - y^\delta\right\| \\ &\le \quad \left\|T_k(x_{k+1}^\delta - x_k^\delta) + R_k\right\| + \left\|F(x_{k+1}^\delta) - F(x_k^\delta) - T_k(x_{k+1}^\delta - x_k^\delta)\right\| \\ &\le \quad \overline{\theta} \left\|F(x_k^\delta) - y^\delta\right\| + c_{tc} \left\|F(x_{k+1}^\delta) - F(x_k^\delta)\right\| \end{aligned}$$

hence, by the triangle inequality

$$\left\|F(x_{k+1}^\delta) - y^\delta\right\| \le \frac{\overline{\theta} + c_{tc}}{1 - c_{tc}} \left\|F(x_k^\delta) - y^\delta\right\| ,$$

which by (21) implies $k_*(\delta) < \infty$. Setting $k = k_*(\delta) - 1$ in (24), we arrive at

$$\left\|x_{k_*(\delta)}^\delta - x_0\right\| \le \left\|x^\dagger - x_0\right\| . \tag{25}$$

Hence there exist weakly convergent subsequences
$(x^l)_{l\in\mathbb{N}} := (x_{k_*(\delta_l)}^{\delta_l})_{l\in\mathbb{N}}$ and $(J_p(x^l - x_0))_{l\in\mathbb{N}} := (J_p(x_{k_*(\delta_l)}^{\delta_l} - x_0))_{l\in\mathbb{N}}$. The weak limit $\bar{x}$ of any weakly convergent subsequence $(x^l)_{l\in\mathbb{N}}$ (or $\bar{x} := J_q^*(\bar{x}^*) + x_0$ with the weak limit $\bar{x}^*$ of $(J_p(x^l - x_0))_{l\in\mathbb{N}}$) by

$$\left\|F(x^l) - y\right\| \le (C_{dp} + 1)\delta_l \to 0 \text{ as } l \to \infty$$

and the (weak) sequential closedness of F (11) (or (12)) defines a solution $\bar{x}$ of (1). Hence we can insert $\bar{x}$ in place of $x^\dagger$ in (25) to obtain, in case of (11),

$$\left\|\bar{x} - x_0\right\| \le \liminf_{l\to\infty} \left\|x^l - x_0\right\| \le \limsup_{l\to\infty} \left\|x^l - x_0\right\| \le \left\|\bar{x} - x_0\right\|$$

by the weak lower semicontinuity of the norm, i.e. convergence of $\left\|x^l - x_0\right\|$ to $\left\|\bar{x} - x_0\right\|$. Since X is uniformly convex and x_l weakly converges to $\bar{x}$, this yields norm convergence of x_l to $\bar{x}$. In case of $(J_p(x^l - x_0))_{l\in\mathbb{N}}$ weakly converging to $\bar{x}^*$ and (12), convergence in the Bregman distance can be established by the argument

$$
\begin{aligned}
&0 \le \Delta_p(x^l - x_0, \bar{x} - x_0) \\
&= \frac{1}{q}\Big(\underbrace{\|x^l - x_0\|^p}_{\le \|\bar{x}-x_0\|^p} - \|\bar{x} - x_0\|^p\Big) + \underbrace{\langle x^* - J_p(x^l - x_0), \bar{x} - x_0\rangle}_{\to 0 \text{ as } l\to\infty},
\end{aligned}
$$

which by Theorem 2.12 (d) in [21] implies strong convergence.

In case of uniqueness of $x^\dagger$, a subsequence-subsequence argument yields overall convergence. □

5 Examples

In this section, we consider two examples of parameter identification model problems, which have been used several times in the literature to illustrate convergence conditions (see, e.g., [9], [12], [19], [11], [15]). Since we wish to considerably expand the possibilities for choosing preimage and image space as compared to the Hilbert space case, we put some effort in exploiting the range of exponents in the underlying L^p spaces. A motivation for this is e.g. the fact that using an image space with large exponent and a preimage space with small exponent corresponds to making the degree of ill-posedness as small as possible. The latter – i.e. using p smaller than two in the definition of the preimage space – additionally favours sparse solutions, which has recently become a quite important trend in many inverse problems applications.

In the first example, we consider identification of the space-dependent coefficient c in the elliptic boundary value problem

$$
\begin{aligned}
-\Delta u + cu &= f \quad \text{in } \Omega \qquad (26)\\
u &= 0 \quad \text{on } \partial\Omega \qquad (27)
\end{aligned}
$$

from measurements of u in Ω (note that inhomogeneous Dirichlet boundary conditions can be easily incorporated into the right hand side f if necessary). Here $\Omega \subseteq \mathbb{R}^d$, $d \in \{1, 2, 3\}$ is assumed to be a smooth bounded domain. The forward operator

$$F : \mathcal{D}(F) \subseteq X = L^P(\Omega) \to Y = L^R(\Omega) \tag{28}$$

can be written as

$$F(c) = L(c)^{-1} f \tag{29}$$

with

$$
\begin{aligned}
L(c) : H^2(\Omega) \cap H_0^1(\Omega) &\to L^2(\Omega) \\
u &\to -\Delta u + cu \qquad (30)
\end{aligned}
$$

With (30), the derivative can formally be written as

$$F'(c)h = -L(c)^{-1}[h \cdot F(c)] .$$

To achieve smoothness and uniform convexity of X, we assume

$$P \in (1, \infty) . \tag{31}$$

Depending on whether we need (10) (for Landweber) or weak sequantial closedness (for IRGNM) we define $\mathcal{D}(F)$ in two different ways:

In the first case, i.e., for Landweber iteration, we assume that

$$P \ge \frac{d}{2} \tag{32}$$

and, in order to achieve a nonempty interior of the domain (10), similarly to [12], set

$$\mathcal{D}(F) = \{c \in L^P(\Omega) \mid \|c - \hat{c}\|_{L^P(\Omega)} \le \beta \text{ for some } \hat{c} \in L^\infty(\Omega) \text{ with } \hat{c} \ge 0 \text{ a.e.}\}\,, \tag{33}$$

where $\beta < \min\{1/\,\|id\|_{H_0^1(\Omega)\to L^{2P/(P-1)}(\Omega)}\,, 1/\,\|id\|_{W^{2,k}\cap H_0^1(\Omega)\to L^{Pk/(P-k)}(\Omega)}\}$, for some

$$\begin{aligned} & k \in [\tilde{a}, \tilde{b}] \cap (1,\infty) \text{ with} \\ \tilde{a} &= \max\{2d/(d+2), dR/(d+2R)\}\,, \\ \tilde{b} &= \min\{P, 2d/\max\{0,d-2\}, R, PR/(P+R)\} \\ &= \min\{2d/\max\{0,d-2\}, PR/(P+R)\} \\ & \text{and } (k < P \ \wedge\ R < \infty) \text{ or } k > d/2\,. \end{aligned} \tag{34}$$

It is readily checked that under the already made assumptions (31), (32) existence of such a k is equivalent to

$$R > \frac{P}{P-1}\,, \quad R \ge \frac{2dP}{dP + 2P - 2d}\,. \tag{35}$$

Well-definedness and continuity of F, F' follows from the following auxiliary result.

Lemma 2. *Let (31), (32) hold. The operator $L(c)^{-1} : L^k(\Omega) \to W^{2,k}(\Omega) \cap H_0^1(\Omega)$, where $W^{2,k} \cap H_0^1(\Omega)$ is equipped with the norm*

$$\|v\|_{W^{2,k}(\Omega)\cap H_0^1(\Omega)} = \|\Delta v\|_{L^k(\Omega)} + \|v\|_{H_0^1(\Omega)}\,,$$

is bounded for any

$$\begin{aligned} & k \in [a, b] \cap (1,\infty) \quad \textit{with} \\ a &= 2d/(d+2)\,, \\ b &= \min\{P, 2d/\max\{0,d-2\}\}\,, \\ & \textit{and } k < P \textit{ or } k > d/2\,. \end{aligned} \tag{36}$$

Proof. In the following, we will make use of Sobolev's embedding theorem several times, therefore we recall that the embedding $id : W^{s,p}(\Omega) \to W^{t,q}(\Omega)$ for some bounded C^1-smooth domain $\Omega \subseteq \mathbb{R}^d$ is continuous for $0 \le t \le s$, $p, q \in [1,\infty]$ (note that by the boundedness of Ω we need not stipulate $p \le q$) if

$$q < \infty \ \wedge\ s - \frac{d}{p} \ge t - \frac{d}{q}\,,$$

or

$$q = \infty \ \wedge\ s - \frac{d}{p} > t\,.$$

Rewriting the PDE (26) as

$$-\Delta u + \hat{c}u = f - (c - \hat{c})u\,, \tag{37}$$

we get by testing with u and integrating by parts

$$\begin{aligned} \|u\|^2_{H_0^1(\Omega)} &\le \int_\Omega \left(|\nabla u|^2 + \hat{c}u^2\right)\,dx = \int_\Omega \left(fu - (c-\hat{c})u^2\right)\,dx \\ &\le \|f\|_{L^k(\Omega)}\,\|u\|_{L^{k/(k-1)}(\Omega)} + \|c - \hat{c}\|_{L^P(\Omega)}\,\|u\|^2_{L^{2P/(P-1)}(\Omega)}\,, \end{aligned}$$

hence by

$$\left(2P/(P-1) < \infty \ \wedge\ 1 - \frac{d}{2} \ge 0 - \frac{d}{2P/(P-1)}\right) \quad \Leftrightarrow \quad \left(P > 1 \ \wedge\ P \ge \frac{d}{2}\right),$$

$$\left(k/(k-1) < \infty \ \wedge \ 1 - \frac{d}{2} \geq 0 - \frac{d}{k/(k-1)}\right) \quad \Leftrightarrow \quad \left(k > 1 \ \wedge \ k \geq \frac{2d}{d+2}\right),$$

and Poincaré's inequality

$$\|u\|_{H_0^1(\Omega)} \leq \frac{1}{1 - \|id\|_{H_0^1(\Omega)\to L^{2P/(P-1)}(\Omega)}\,\beta} \|id\|_{H_0^1(\Omega)\to L^{k/(k-1)}(\Omega)} \|f\|_{L^k(\Omega)}$$

Moreover, from (37) we get by Hölder's inequality with

$$k \leq P$$

and

$$\left(k < \infty \ \wedge \ 1 - \frac{d}{2} \geq 0 - \frac{d}{k}\right) \quad \Leftrightarrow \quad \left(k < \infty \ \wedge \ k \leq \frac{2d}{\max\{0, d-2\}}\right),$$

$$\left(Pk/(P-k) < \infty \ \wedge \ 2 - \frac{d}{k} \geq 0 - \frac{d}{Pk/(P-k)}\right) \quad \Leftrightarrow \quad \left(k < P \ \wedge \ P \geq \frac{d}{2}\right),$$

$$\left(Pk/(P-k) = \infty \ \wedge \ 2 - \frac{d}{k} > 0\right) \quad \Leftrightarrow \quad \left(k = P \ \wedge \ k > \frac{d}{2}\right),$$

with $C_1 := \|id\|_{W^{2,k}\cap H_0^1(\Omega)\to L^{Pk/(P-k)}(\Omega)}$,

$$\begin{aligned} \|\Delta u\|_{L^k(\Omega)} &\leq \|f\|_{L^k(\Omega)} + \|\hat{c}\|_{L^\infty(\Omega)} \|u\|_{L^k(\Omega)} + \|c - \hat{c}\|_{L^P(\Omega)} \|u\|_{L^{Pk/(P-k)}(\Omega)} \\ &\leq \|f\|_{L^k(\Omega)} + \|\hat{c}\|_{L^\infty(\Omega)} \|id\|_{H_0^1(\Omega)\to L^k(\Omega)} \|u\|_{H_0^1(\Omega)} \\ &\quad + \|c - \hat{c}\|_{L^P(\Omega)} C_1(\|\Delta u\|_{L^k(\Omega)} + \|u\|_{H_0^1(\Omega)}), \end{aligned}$$

hence

$$\|\Delta u\|_{L^k(\Omega)} \leq \frac{1}{1-\beta C_1}\left(\|f\|_{L^k(\Omega)} + \left\{\|\hat{c}\|_{L^\infty(\Omega)} \|id\|_{H_0^1(\Omega)\to L^k(\Omega)} + \beta C_1\right\} \|u\|_{H_0^1(\Omega)}\right)$$

□

To show the tangential cone condition with the choice (33), we consider k according to Lemma 2, denote $C_2 = \|id\|_{W^{2,k}(\Omega)\cap H_0^1(\Omega)\to L^R(\Omega)}$, and estimate

$$\begin{aligned} &\|F(c+h) - F(c) - F'(c)[h]\|_{L^R(\Omega)} \\ &= \left\|L(c)^{-1}[h\cdot(F(c+h) - F(c))]\right\|_{L^R(\Omega)} \\ &\leq C_2 \left\|L(c)^{-1}\right\|_{L^k(\Omega)\to W^{2,k}(\Omega)\cap H_0^1(\Omega)} \|h\cdot(F(c+h) - F(c))\|_{L^k(\Omega)} \\ &\leq C_2 \left\|L(c)^{-1}\right\|_{L^k(\Omega)\to W^{2,k}(\Omega)\cap H_0^1(\Omega)} \|F(c+h) - F(c)\|_{L^R(\Omega)} \|h\|_{L^{Rk/(R-k)}(\Omega)} \\ &\leq C_2 \left\|L(c)^{-1}\right\|_{L^k(\Omega)\to W^{2,k}(\Omega)\cap H_0^1(\Omega)} \|id\|_{L^P(\Omega)\to L^{Rk/(R-k)}(\Omega)} \cdot \\ &\qquad \cdot \|F(c+h) - F(c)\|_{L^R(\Omega)} \|h\|_{L^P(\Omega)} \,. \end{aligned}$$

by

$$\left(R < \infty \ \wedge \ 2 - \frac{d}{k} \geq 0 - \frac{d}{R}\right) \quad \Leftrightarrow \quad \left(R < \infty \ \wedge \ k \geq \frac{dR}{d+2R}\right)$$

$$\left(R = \infty \ \wedge \ 2 - \frac{d}{k} > 0\right) \quad \Leftrightarrow \quad \left(R = \infty \ \wedge \ k > \frac{d}{2}\right)$$

$$\left(k < R \ \wedge \ \frac{Rk}{R-k} \leq P\right) \quad \Leftrightarrow \quad \left(k < R \ \wedge \ k \leq \frac{PR}{P+R}\right)$$

(note that by $P < \infty$ and $k \leq P$ the case $k = R$ cannot occur), which together with (36) is equivalent to (34).

Consequently we have

Corollary 1. *Let F be defined by (28), (29), (33), with P, R satisfying (31), (32), (35). Then the assumptions (9), (10), continuity of F, F' of Theorem 1 for the Landweber iteration (3) are satisfied so local convergence with exact data holds.*
If additionally $R \in (1, \infty)$ holds, then we can also conclude local convergence with noisy data according to Theorem 2.

In the second case, i.e., for the IRGNM, we use

$$\mathcal{D}(F) = \{c \in L^\infty(\Omega) \mid \bar{\gamma} \geq c \geq 0 \text{ a.e.}\} \tag{38}$$

for some $\bar{\gamma} > 0$, see, e.g, [15], for which the weak sequential closedness (11) can be seen as follows: Any sequence $(c_n)_{n\in\mathbb{N}} \subseteq \mathcal{D}(F)$, (with $c_n \rightharpoonup c$ and $u_n := F(c_n) \to u$) has a weakly L^∞-subsequence c_{n_k} whose limit lies in $\mathcal{D}(F)$ and by a density argument has to coincide with c, i.e., $\mathcal{D}(F)$ is weakly closed. To show that $F(c) = u$, i.e., that u weakly solves (26), (27), we use the fact that u_n is uniformly bounded in $H^2(\Omega) \cap H_0^1(\Omega)$ and therefore has a weak $H^2(\Omega) \cap H_0^1(\Omega)$ accumulation point which again by a density argument coincides with u and lies in $H^2(\Omega) \cap H_0^1(\Omega)$. Let $(c_{n_k})_{k\in\mathbb{N}}$, $(u_{n_k})_{k\in\mathbb{N}}$ be subsequences that converge weakly in $L^P(\Omega)$ and (by uniform boundedness of u_n in H^2 and compactness) strongly in $L^{6P/(5P-6)}(\Omega)$ respectively, such that additionally u_{n_k} converges weakly in H_0^1, and consider the limit $k \to \infty$ in the following identity

$$\begin{aligned}
\forall w \in H_0^1(\Omega): \quad \int_\Omega fw\,dx &= \int_\Omega \nabla u_{n_k} \nabla w\,dx + \int_\Omega cuw\,dx \\
&\quad + \int_\Omega c_{n_k}(u_{n_k} - u)w\,dx + \int_\Omega (c_{n_k} - c)uw\,dx \\
&\to \int_\Omega \nabla u \nabla w\,dx + \int_\Omega cuw\,dx
\end{aligned}$$

where we have used uniform $L^P(\Omega)$ boundedness of c_{n_k} and the mentioned strong convergence of u_{n_k} as well as Hölder's inequality for the term $\int_\Omega c_{n_k}(u_{n_k} - u)w\,dx$. Note that this weak sequential closedness would remain valid also with (33), and that we did not even make use of convergence of u_n in $L^R(\Omega)$.

Since for the choice (38), $L(c)^{-1}$ is bounded as an operator from $L^2(\Omega)$ to $H^2(\Omega) \cap H_0^1(\Omega)$, F also satisfies the tangential cone condition provided

$$R \geq 2\,, \quad \frac{2R}{R-2} \leq P \tag{39}$$

since

$$\begin{aligned}
&\|F(c+h) - F(c) - F'(c)[h]\|_{L^R(\Omega)} \\
=\ &\left\|L(c)^{-1}[h \cdot (F(c+h) - F(c))]\right\|_{L^R(\Omega)} \\
\leq\ &\|id\|_{H^2(\Omega)\to L^R(\Omega)} \left\|L(c)^{-1}\right\|_{L^2(\Omega)\to H^2(\Omega)} \|h \cdot (F(c+h) - F(c))\|_{L^2(\Omega)} \\
\leq\ &\|id\|_{H^2(\Omega)\to L^R(\Omega)} \left\|L(c)^{-1}\right\|_{L^2(\Omega)\to H^2(\Omega)} \|F(c+h) - F(c)\|_{L^R} \|h\|_{L^{2R/(R-2)}} \\
\leq\ &\|id\|_{H^2(\Omega)\to L^R(\Omega)} \left\|L(c)^{-1}\right\|_{L^2(\Omega)\to H^2(\Omega)} \|id\|_{L^P(\Omega)\to L^{2R/(R-2)}(\Omega)} \cdot \\
&\cdot \|F(c+h) - F(c)\|_{L^R} \|h\|_{L^P}\,.
\end{aligned}$$

Corollary 2. *Let F be defined by (28), (29), (38), with P, R satisfying (31), (39). Then the assumptions (9), (11), $\mathcal{D}(F)$ weakly closed, of Theorem 3 for the IRGNM (5) are satisfied, so local convergence with noisy data according to Theorem 3 holds.*

The second example is concerned with the identification of the space-dependent coefficient a in

$$-\nabla(a\nabla u) = f \quad \text{in } \Omega \tag{40}$$

$$u = 0 \quad \text{on } \partial\Omega \tag{41}$$

Again, $\Omega \subseteq \mathbb{R}^d$, $d \in \{1, 2, 3\}$ is assumed to be a smooth bounded domain. Here we have to make sure that a is bounded away from zero by an appropriate definition of $\mathcal{D}(F)$ in

$$F : \mathcal{D}(F) \subseteq X \to Y = L^R(\Omega)\,, \tag{42}$$

$$F(a) = L(a)^{-1} f\,, \tag{43}$$

with

$$\begin{aligned} L(a) : H^2(\Omega) \cap H_0^1(\Omega) \quad &\to \quad L^2(\Omega) \\ u \quad &\to \quad -\nabla(a \nabla u) \end{aligned} \tag{44}$$

to maintain ellipticity.

Following [11] we choose

$$X = W^{1,Q}(\Omega), \quad \mathcal{D}(F) = \{a \in X \mid a \geq \underline{\alpha}\} \tag{45}$$

with $\underline{\alpha} > 0$ and

$$Q > d\,, \tag{46}$$

which indeed implies that $F(a)$ maps into $H^2(\Omega) \cap H_0^1(\Omega) \subseteq L^R(\Omega)$ for any $R \in [1, \infty]$, that $\mathcal{D}(F)$ has nonempty interior with respect to the norm in X (and hence with respect to the Bregman distance) and that the weak sequential closedness of $\mathcal{D}(F)$, i.e., $a_n \rightharpoonup a \;\Rightarrow\; a \in \mathcal{D}(F)$, (which is also the first part of (11),) holds. To show that also $u_n = F(a_n) \to u \;\Rightarrow\; F(a) = u$, consider the identity

$$\begin{aligned} \forall w \in H_0^1(\Omega) : \quad \int_\Omega f w \, dx \quad &= \quad \int_\Omega a_{n_k} \nabla u_{n_k} \nabla w \, dx \\ &= \quad \int_\Omega a \nabla u \nabla w \, dx + \int_\Omega a_{n_k} \nabla (u_{n_k} - u) \nabla w \, dx \\ &\quad + \int_\Omega (a_{n_k} - a) \nabla u \nabla w \, dx \\ &\to \quad \int_\Omega a \nabla u \nabla w \, dx \end{aligned}$$

along a subsequence a_{n_k}, u_{n_k} such that (by uniform boundedness of u_n in $H^2(\Omega)$ and compactness) $\nabla(u_{n_k} - u) \to 0$ strongly in $L^2(\Omega)$, and $a_{n_k} - a \rightharpoonup 0$ weakly in L^∞, where we use uniform boundedness of a_{n_k} in $L^\infty(\Omega)$, and the fact that $\nabla u \nabla w$ lies in the dual of $L^\infty(\Omega)$.

Under the assumption

$$Q \in (1, \infty)\,, \tag{47}$$

X is uniformly smooth and uniformly convex, cf., e.g., [23].

The tangential cone condition can be seen as follows: Using the function space $V = \overline{\mathcal{D}(\Omega)}^{\|\cdot\|_V}$ with

$$\|v\|_V = \|\Delta v\|_{L^{R/(R-1)}(\Omega)} + \|\nabla v\|_{L^{QR/(QR-Q-R)}(\Omega)}\,,$$

we estimate

$$\begin{aligned} &\|F(a+h) - F(a) - F'(a)[h]\|_{L^R(\Omega)} \\ &\quad = \quad \left\| L(a)^{-1} [\nabla (h \nabla (F(a+h) - F(a)))] \right\|_{L^R(\Omega)} \\ &\quad \leq \quad \left\| L(a)^{-1} \right\|_{V^* \to L^R(\Omega)} \|\nabla (h \nabla (F(a+h) - F(a)))\|_{V^*}\,, \end{aligned}$$

where for all $v \in V$

$$\int_\Omega \nabla (h \nabla (F(a+h) - F(a)))\, v \, dx$$

$$
\begin{aligned}
&= \int_\Omega \nabla(h\nabla v)\,(F(a+h)-F(a))\,dx \\
&\leq \|F(a+h)-F(a)\|_{L^R(\Omega)} \left(\|h\Delta v\|_{L^{R/(R-1)}(\Omega)} + \|\nabla h\,\nabla v\|_{L^{R/(R-1)}(\Omega)}\right) \\
&\leq \|F(a+h)-F(a)\|_{L^R(\Omega)} \left(\|h\|_{L^\infty(\Omega)}\,\|\Delta v\|_{L^{R/(R-1)}(\Omega)} + \right. \\
&\quad \left. + \|\nabla h\|_{L^Q(\Omega)}\,\|\nabla v\|_{L^{QR/(QR-Q-R)}(\Omega)}\right) \\
&\leq \|F(a+h)-F(a)\|_{L^R(\Omega)} \left(\|id\|_{W^{1,Q}(\Omega)\to L^\infty(\Omega)} + 1\right)\|h\|_{W^{1,Q}(\Omega)}\,\|v\|_V\ ,
\end{aligned}
$$

where we have used the assumption (46) and additionally

$$
Q \geq \frac{R}{R-1}. \tag{48}
$$

It remains to show that $L(a)^{-1} : V^* \to L^R(\Omega)$ is bounded:

Lemma 3. *Let (46), (48),*

$$
R \leq \frac{2d}{\max\{0, d-2\}} \quad \textit{and} \quad (R < \infty \ \vee\ d < 2) \tag{49}
$$

and either

$$
\begin{array}{ll}
(i) & \dfrac{\|\nabla a\|_{L^Q}}{\underline{\alpha}}\,\|id\|_{W^{2,R/(R-1)}(\Omega)\cap H_0^1(\Omega)\to W_0^{1,QR/(QR-Q-R)}(\Omega)} < 1 \\
or & \\
(ii) & \dfrac{QR}{QR-Q-R} \leq 2\,.
\end{array} \tag{50}
$$

Then the operator $L(a)^{-1} : V^* \to L^R(\Omega)$ is bounded.

Proof. By density of $\mathcal{D}(\Omega)$ in $L^{R/(R-1)}(\Omega)$ we have

$$
\begin{aligned}
\left\|L(a)^{-1}\right\|_{V^*\to L^R(\Omega)} &= \sup_{0\neq v^*\in V^*} \frac{\left\|L(a)^{-1}[v^*]\right\|_{L^R(\Omega)}}{\|v^*\|_{Y^*}} \\
&= \sup_{0\neq v^*\in V^*}\ \sup_{0\neq y\in\mathcal{D}(\Omega)} \frac{\int_\Omega L(a)^{-1}[v^*]\,y}{\|v^*\|_{Y^*}\,\|y\|_{L^{R/(R-1)}(\Omega)}} \\
&= \sup_{0\neq v^*\in V^*}\ \sup_{0\neq y\in\mathcal{D}(\Omega)} \frac{\int_\Omega v^*\,L(a)^{-1}[y]}{\|v^*\|_{V^*}\,\|y\|_{L^{R/(R-1)}(\Omega)}} \\
&= \left\|L(a)^{-1}\right\|_{L^{R/(R-1)}(\Omega)\to V}
\end{aligned}
$$

For any $y \in L^{R/(R-1)}(\Omega)$ we get for the solution $w = L(a)^{-1}y$ by testing $-\nabla(a\nabla w) = y$ with w and using Hölder's inequality

$$
\begin{aligned}
\|\nabla w\|^2_{L^2(\Omega)} &\leq \frac{1}{\underline{\alpha}}\,\|y\|_{L^{R/(R-1)}(\Omega)}\,\|w\|_{L^R(\Omega)} \\
&\leq \frac{1}{\underline{\alpha}}\,\|y\|_{L^{R/(R-1)}(\Omega)}\,\|id\|_{V\to L^R(\Omega)}\,\|w\|_V
\end{aligned} \tag{51}
$$

by

$$
\left(R < \infty \ \wedge\ 2 - \frac{d}{R/(R-1)} \geq 0 - \frac{d}{R}\right) \quad \Leftrightarrow \quad \left(R < \infty \ \wedge\ R \leq \frac{2d}{\max\{0, d-2\}}\right),
$$

$$
\left(R = \infty \ \wedge\ 2 - \frac{d}{R/(R-1)} > 0\right) \quad \Leftrightarrow \quad (R = \infty \ \wedge\ d < 2)\ ,
$$

(note that by $Q > d$ there holds $1 - \frac{d}{QR/(QR-Q-R)} = 1 - \frac{d}{R/(R-1-Q/R)} < 2 - \frac{d}{R/(R-1)}$, so indeed the Laplace term in the definition of the V norm gives the stronger embedding result).

Moreover, by $-a\Delta w = y + \nabla a \nabla w$

$$\|\Delta w\|_{L^{R/(R-1)}(\Omega)} \leq \frac{1}{\underline{\alpha}} \left(\|y\|_{L^{R/(R-1)}(\Omega)} + \|\nabla a\|_{L^Q(\Omega)} \|\nabla w\|_{L^{QR/(QR-Q-R)}(\Omega)} \right) . \tag{52}$$

So in case (i) with

$$\|v\|_{W^{2,R/(R-1)}(\Omega)\cap H_0^1(\Omega)} = \|\Delta v\|_{L^{R/(R-1)}(\Omega)} + \|v\|_{H_0^1(\Omega)} ,$$
$$C_1 = \|id\|_{W^{2,R/(R-1)}(\Omega)\cap H_0^1(\Omega)\to W_0^{1,QR/(QR-Q-R)}(\Omega)} ,$$

where we have used

$$2 - \frac{d}{R/(R-1)} > 1 - \frac{d}{QR/(QR-Q-R)} \quad \Leftarrow \quad Q > d ,$$

(hence by the strict inequality also the case $QR/(QR-Q-R) = \infty$ in Sobolev's embedding theorem is covered,) we can estimate

$$\begin{aligned}
&\|\nabla w\|^2_{L^2(\Omega)} \\
&\leq \frac{1}{\underline{\alpha}} \|y\|_{L^{R/(R-1)}(\Omega)} \|id\|_{V\to L^R(\Omega)} \left((1+C_1) \|\Delta w\|_{L^{R/(R-1)}(\Omega)} + C_1 \|\nabla w\|_{L^2(\Omega)} \right) \\
&\leq \frac{1}{2\epsilon} \frac{1}{\underline{\alpha}^2} \|y\|^2_{L^{R/(R-1)}(\Omega)} \|id\|^2_{V\to L^R(\Omega)} + \\
&+\epsilon \left((1+C_1)^2 \|\Delta w\|^2_{L^{R/(R-1)}(\Omega)} + C_1^2 \|\nabla w\|^2_{L^2(\Omega)} \right)
\end{aligned}$$

hence

$$\begin{aligned}
&\|\nabla w\|^2_{L^2(\Omega)} \\
&\leq \frac{1}{1-C_1^2\epsilon} \left(\frac{1}{2\epsilon} \frac{1}{\underline{\alpha}^2} \|y\|^2_{L^{R/(R-1)}(\Omega)} \|id\|^2_{V\to L^R(\Omega)} + \epsilon(1+C_1)^2 \|\Delta w\|^2_{L^{R/(R-1)}(\Omega)} \right) \\
&\leq \frac{1}{1-C_1^2\epsilon} \left(\frac{1}{\sqrt{2\epsilon}} \frac{1}{\underline{\alpha}} \|y\|_{L^{R/(R-1)}(\Omega)} \|id\|_{V\to L^R(\Omega)} + \sqrt{\epsilon}(1+C_1) \|\Delta w\|_{L^{R/(R-1)}(\Omega)} \right)^2
\end{aligned} \tag{53}$$

Moreover, by (52)

$$\|\Delta w\|_{L^{R/(R-1)}(\Omega)} \qquad \leq \frac{1}{\underline{\alpha}} \left(\|y\|_{L^{R/(R-1)}(\Omega)} + \|\nabla a\|_{L^Q(\Omega)} C_1 \|w\|_{W^{2,R/(R-1)}(\Omega)\cap H_0^1(\Omega)} \right)$$

Hence,

$$\begin{aligned}
&\|\Delta w\|_{L^{R/(R-1)}(\Omega)} \\
&\leq \frac{1}{\underline{\alpha} - \|\nabla a\|_{L^Q(\Omega)} C_1} \left(\|y\|_{L^{R/(R-1)}(\Omega)} + \|\nabla a\|_{L^Q(\Omega)} C_1 \|\nabla w\|_{L^2(\Omega)} \right) ,
\end{aligned} \tag{54}$$

which by inserting (53) with ϵ sufficiently small so that

$$\frac{1}{\underline{\alpha} - \|\nabla a\|_{L^Q(\Omega)} C_1} \|\nabla a\|_{L^Q(\Omega)} C_1 \frac{1}{\sqrt{1-C_1^2\epsilon}} \sqrt{\epsilon}(1+C_1) < 1$$

gives

$$\|\Delta w\|_{L^{R/(R-1)}(\Omega)} \leq C^1 \|y\|_{L^{R/(R-1)}(\Omega)}$$

for some constant $C^1 = C^1(\underline{\alpha}, \|\nabla a\|_{L^Q(\Omega)})$. Re-inserting this into (53) gives

$$\|\nabla w\|_{L^2(\Omega)} \leq C^2 \, \|y\|_{L^{R/(R-1)}(\Omega)}$$

for some constant $C^2 = C^2(\underline{\alpha}, \|\nabla a\|_{L^Q(\Omega)})$, hence

$$\|w\|_V \leq (C^1(1+C_1) + C^2 C_1) \, \|y\|_{L^{R/(R-1)}(\Omega)} \, .$$

In case (ii) with

$$\tilde{C}_1 = \|id\|_{L^2(\Omega)\to L^{QR/(QR-Q-R)}(\Omega)} \, ,$$

the estimates simplify to

$$\begin{aligned}
&\|\nabla w\|^2_{L^2(\Omega)} \\
&\leq \frac{1}{\underline{\alpha}} \, \|y\|_{L^{R/(R-1)}(\Omega)} \, \|id\|_{V\to L^R(\Omega)} \left(\|\Delta w\|_{L^{R/(R-1)}(\Omega)} + \tilde{C}_1 \, \|\nabla w\|_{L^2(\Omega)} \right) \\
&\leq \frac{1}{2\epsilon} \frac{1}{\underline{\alpha}^2} \, \|y\|^2_{L^{R/(R-1)}(\Omega)} \, \|id\|^2_{V\to L^R(\Omega)} + \epsilon \left(\|\Delta w\|^2_{L^{R/(R-1)}(\Omega)} + \tilde{C}_1^2 \, \|\nabla w\|^2_{L^2(\Omega)} \right)
\end{aligned}$$

hence

$$\begin{aligned}
&\|\nabla w\|^2_{L^2(\Omega)} \\
&\leq \frac{1}{1-\tilde{C}_1^2\epsilon} \left(\frac{1}{2\epsilon} \frac{1}{\underline{\alpha}^2} \, \|y\|^2_{L^{R/(R-1)}(\Omega)} \, \|id\|^2_{V\to L^R(\Omega)} + \epsilon \, \|\Delta w\|^2_{L^{R/(R-1)}(\Omega)} \right) \\
&\leq \frac{1}{1-\tilde{C}_1^2\epsilon} \left(\frac{1}{\sqrt{2\epsilon}} \frac{1}{\underline{\alpha}} \, \|y\|_{L^{R/(R-1)}(\Omega)} \, \|id\|_{V\to L^R(\Omega)} + \sqrt{\epsilon} \, \|\Delta w\|_{L^{R/(R-1)}(\Omega)} \right)^2
\end{aligned} \tag{55}$$

and

$$\|\Delta w\|_{L^{R/(R-1)}(\Omega)} \leq \frac{1}{\underline{\alpha}} \left(\|y\|_{L^{R/(R-1)}(\Omega)} + \|\nabla a\|_{L^Q(\Omega)} \, \tilde{C}_1 \, \|\nabla w\|_{L^2(\Omega)} \right) . \tag{56}$$

The rest follows analogously to case (i). □

Corollary 3. *Let F be defined by (42), (43), (45), with Q, R satisfying (46), (47), (48), (49). Then the assumptions (9), (11), $\mathcal{D}(F)$ weakly closed, of Theorem 3 for the IRGNM (5) are satisfied, so local convergence with noisy data according to Theorem 3 holds.*
Moreover, the assumptions (9), (10), continuity of F, F' of Theorem 1 for the Landweber iteration (3) are satisfied so local convergence with exact data holds.
If additionally $R \in (1, \infty)$ holds, then we can also conclude local convergence of Landweber with noisy data according to Theorem 2.

Conclusions

We presented two iterative methods for solving nonlinear operator equations in Banach spaces. The first one was a Landweber type method, the second one was the iteratively regularized Gauss - Newton method. Provided that the nonlinearity of the forward operator obeys a tangential cone condition we could prove convergence for both methods. Furthermore we showed that both methods are stable with respect to noisy data, if the stopping index is chosen due to an appropriate discrepancy principle. The two examples for parameter identification problems of elliptic boundary value problems demonstrate that there are interesting applications for which all conditions, that are necessary for the well-definedness and convergence of the methods, are satisfied. A sequel of this article is supposed to contain numerical evaluations of the methods with the help of these applications.

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