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Lattice triangulations of $\mathbb{E}^3$ and of the 3-torus

Ulrich Brehm and Wolfgang Kühnel

Abstract.[1] This paper gives answers to a few questions concerning tilings of Euclidean spaces where the tiles are topological simplices with curvilinear edges. We investigate *lattice triangulations* of Euclidean 3-space in the sense that the vertices form a lattice of rank 3 and such that the triangulation is invariant under all translations of that lattice. This is the dual concept of a primitive lattice tiling where the tiles are not assumed to be Euclidean polyhedra but only topological polyhedra. In 3-space there is a unique standard lattice triangulation by Euclidean tetrahedra (and with straight edges) but there are infinitely many non-standard lattice triangulations where the tetrahedra necessarily have certain curvilinear edges. From the view-point of Discrete Differential Geometry this tells us that there are such triangulations of 3-space which do not carry any flat discrete metric which is equivariant under the lattice. Furthermore we investigate lattice triangulations of the 3-dimensional torus as quotients by a sublattice. The standard triangulation admits such quotients with any number $n \geq 15$ of vertices. The unique one with 15 vertices is neighborly, i.e., any two vertices are joined by an edge. It turns out that for any odd $n \geq 17$ there is an n-vertex neighborly triangulation of the 3-torus as a quotient of a certain non-standard lattice triangulation. Combinatorially, one can obtain these neighborly 3-tori as slight modifications of the boundary complexes of the cyclic 4-polytopes. As a kind of combinatorial surgery, this is an interesting construction by itself.

Key words: *group action, regular tessellation, tiling, triangulated torus, neighborly triangulation, vertex-transitive triangulation, cyclic polytope, lattice*

MSC: Primary 52B70; Secondary 52C22, 05C10, 05C30, 57Q15.

1. Introduction and main results

Triangulations of spaces with a geometric structure can be regarded as special cases of face-to-face tilings where the tiles are simplices. In the case of triangulations of Euclidean spaces one would normally prefer Euclidean simplices in the sense that each d-dimensional simplex is isometric with the convex hull of certain $d+1$ points in $\mathbb{E}^d$. However, in a more general setting one may be forced to consider topological realizations of triangulations where the requirement is only that each d-simplex is homeomorphic to a Euclidean one, possibly with curvilinear edges, with non-flat triangles etc. In the theory of tilings there is an analogous distinction between convex and non-convex tiles.

We consider a triangulation of Euclidean d-space $\mathbb{E}^d$ by topological simplices such that the set of vertices forms a lattice and such that – geometrically and combinatorially – the triangulation is invariant under all translations of that lattice. We call this a *lattice triangulation*. The dual concept is that of a *primitive lattice tiling* which has been studied in the literature, compare [16], [17]. If one divides out by a sublattice then one obtains a lattice triangulation of a flat d-dimensional torus. One of the questions in this context is the following which will be answered by our Main Theorem A:

Question 1: *Is it always possible to pull the edges straight while keeping the combinatorial type of the triangulation and while keeping the property to be a lattice triangulation ? In other words: Is every lattice triangulation of Euclidean d-space combinatorially isomorphic with a lattice triangulation by Euclidean simplices with straight edges ?*

There does not seem to exist an obvious counterexample in the literature. However, we are going to show that the answer is "yes" for $d = 2$ and "no" for any $d \geq 3$. More precisely we show that there are infinitely many distinct lattice triangulations of 3-space (and higher dimensional space) where the edges cannot be pulled straight. This is in sharp contrast with the fact that in 3-space there is exactly one item with straight edges (up to affine transformations), three items in 4-space and a finite number in d-space for any fixed d. We give a construction principle and, in addition, one explicit family of non-standard items in 3-space depending on an integer parameter $k \geq 4$. There is always a straight realization of the edge graph of a given lattice triangulation since this is possible for any graph. Furthermore, locally each vertex link is a triangulated 2-sphere which admits a realization as a convex 3-polytope by Steinitz' theorem. However, globally in this case either we have forbidden overlaps of simplices, or the vertices do not form a Euclidean lattice.

[1] For an extended abstract of this paper in two parts see [5] and [23].

In the case $d = 2$ the unique standard example is the tessellation $\{3, 6\}$ by regular Euclidean triangles, six around each vertex. It is in fact the unique lattice triangulation of the plane, up to affine transformations. This is not in contradiction with the fact that for $d = 2$ many lattice tilings are known with non-convex or even quite exotic tiles, see [17].

In 3-space there is a unique standard lattice triangulation with straight edges (14 around each vertex). It coincides with the dual of the lattice tiling by truncated octahedra, and it was denoted by Type TT2 in [9]. In addition this standard lattice triangulation can be regarded as a subdivision of the lattice tiling by rhombidodecahedra. This has been used in our previous papers [24] and [21]. A.Grigis observed its uniqueness in [14] and proved a few consequences, compare Theorem 1 below. However, in general the edges of a lattice triangulation are not assumed to be straight, and there are non-standard examples as follows.

Main Theorem A (Non-standard lattice triangulations)
There are infinitely many distinct triangulations of $\mathbb{E}^3$ such that the vertices form a lattice and such that the triangulation is invariant under all translations of that lattice but in such a way that not all of the edges can be simultaneously made straight (unless one admits degenerate tetrahedra with four coplanar vertices). The number of edges emanating from a vertex can be arbitrarily large.

Consequence 1 (Primitive lattice tilings)
There are infinitely many distinct primitive lattice tilings of 3*-space. The number of facets of the (non-convex) prototile can be arbitrarily large.*

Consequence 2 (Crystallographic interpretation)
There are infinitely many distinct non-crystallographic lattice triangulations of Euclidean 3*-space in the following sense:*

If the vertices are the points of a lattice and if the triangulation is invariant under all translations then the edges cannot be straight; if all edges are straight then the vertices cannot form such a lattice with the required invariance under all of the translations.

Consequence 3 (PL curvature interpretation)
There are inifinitely many distinct triangulations of the abstract 3*-space (as a manifold) with a vertex-transitive group action of $\mathbb{Z}^3$ and with the following non-flatness property:*

If every tetrahedron is isometric with a Euclidean one and if the group $\mathbb{Z}^3$ acts by isometries then globally the resulting metric space is never isometric with the flat $\mathbb{E}^3$, and the PL curvature along the edges cannot vanish identically. In other words: No equivariant discrete metric on that triangulation can be flat. Conversely, a flat discrete metric on that triangulation cannot be preserved by the action of $\mathbb{Z}^3$.

Consequence 1 is just a dual formulation of Theorem A, in contrast with several finiteness results in this context like the one in [11] on face-transitive tilings. Consequence 2 is obvious from Theorem A. Consequence 3 follows from Theorem A in connection with Theorem 5 below. Consequence 3 can be regarded as a contribution to Discrete Differential Geometry because it involves the concepts of discrete metrics and discrete curvatures [33]. As an illustration compare Example 2 at the very end of this article. The phenomenon of vanishing PL curvature for a certain metric but non-vanishing equivariant PL curvature for any equivariant metric is in sharp contrast with the analogue for smooth Riemannian metrics: *Not every $\mathbb{Z}^3$-invariant smooth metric is flat, but certainly the flat metric on Euclidean* 3*-space is $\mathbb{Z}^3$-invariant for any $\mathbb{Z}^3$-action of a lattice.*

In dimension $d = 2$ the situation is easy to analyze: If we divide out a given lattice triangulation of the plane by a suitable sublattice of the vertex lattice then we obtain a triangulation of the ordinary 2-dimensional torus which is again invariant under a vertex transitive group. From the Euler characteristic $\chi = 0$ of the torus we deduce that every vertex is contained in precisely six edges. Such a triangulation is also called *equivelar*, compare [6] for a classification of such tori.

The universal covering is uniquely determined, at least combinatorially. Therefore the original triangulation is combinatorially equivalent to the regular tessellation $\{3,6\}$ with straight edges.

Such a type of argument systematically fails to work in any higher dimension. In fact for triangulated 3-tori the number of edges around a vertex is not bounded by any number, and infinitely many types actually occur, compare Theorem 3 below. Furthermore it is obvious that the answer to Question 1 will be "no" in any dimension $d \geq 4$ if the answer is "no" for $d = 3$. So in the sequel we can concentrate on the 3-dimensional case. The local structure of a lattice triangulation of 3-space can be studied in a suitable 3-torus as well. This leads to the following related question:

QUESTION 2: *Are there neighborly triangulations of the 3-torus with arbitrarily many vertices, in particular, such which are invariant under a vertex transitive group of Euclidean translations (in a flat torus $\mathbb{R}^3/\mathbb{Z}^3$ but possibly with curvilinear edges) ?*

A triangulation is called neighborly if any two vertices are joined by an edge. The boundary complexes of the cyclic 4-polytopes provide examples of neighborly triangulations of the 3-sphere with a dihedral symmetry group. There is a standard neighborly triangulation of the 3-torus by 15 vertices [24]. In Section 3 below we will see that the answer to Question 2 is "yes" for any given odd number $n \geq 15$ of vertices. However, it seems that even numbers cannot be attained although it is easy to construct such examples with an even number of vertices which are not neighborly. In particular the standard lattice triangulation leads to a 16-vertex triangulation of the 3-torus with eight disjoint diagonals. therefore it can be regarded as a (centrally symmetric) subcomplex of the 8-dimensional cross polytope. It was denoted by $M_1^3(16)$ in [27]. Such a triangulation is called *nearly neighborly* in [28], [29], and many other examples are presented. In Section 3 we will come back to centrally symmetric triangulations of the 3-torus.

Main Theorem B (Neighborly combinatorial 3-tori)
There is a neighborly and cyclically symmetric combinatorial 3-torus with any given odd number $n \geq 15$ of vertices. For $n \geq 17$ it cannot be obtained as a quotient of a lattice triangulation of 3-space with straight edges. Moreover, the statement of Consequence 3 above remains valid for the 3-torus if we replace the group $\mathbb{Z}^3$ by $\mathbb{Z}_n$.

The proof will be given in Section 3 below. It is remarkable that this family appears as a slight modification of the the family of the boundary complexes of cyclic 4-polytopes with the same number n of vertices. Alternatively, it can be regarded as a kind of an amalgamation of this boundary complex of a cyclic polytope with the standard n-vertex triangulation of the 3-torus which in turn is a quotient of the standard lattice triangulation.

The cubical tessellation of Euclidean 3-space is known to be the only tiling by convex polytopes such that a pure translation group acts transitively on the set of vertices and, simultaneously, on the set of tiles. In the more general setting of curvilinear polyhedra we have the following question:

QUESTION 3: *Are there other tilings of $\mathbb{E}^3$ such that a group of pure translations acts transitively on the set of natural vertices and, simultaneously, on the set of tiles ?*

In this case the prototile will not be required to be a Euclidean polyhedron. There may be curved edges. A positive answer follows from our Main Theorem A in connection with the results of [37].

Theorem (Doubly transitive lattice tilings) *There are infinitely many distinct non-standard tilings of $\mathbb{E}^3$ by topological polyhedral 3-balls such that the translations of a lattice act transitively on the set of vertices and, simultaneously, on the set of tiles.*

2. Lattice triangulations of 3-space: Basic properties

A triangulation of X will always be understood as a (possibly infinite but locally finite) simplicial complex whose carrier is homeomorphic to X. We will distinguish between the automorphism group consisting of all homeomorphisms preserving an abstract triangulation and the symmetry group consisting of all Euclidean symmetries preserving a given concrete triangulation of Euclidean space. A simplex embedded into Euclidean space may have curvilinear edges but it is – by definition – not admitted that the volume is zero. This excludes degenerate cases. When regarded as tilings, geometric realizatios of triangulations of Euclidean spaces are face-to-face tilings. It is well known that there is no regular tessellation of Euclidean 3-space (or higher-dimensional space) by simplices, i.e., no one with a flag-transitive automorphism group, compare [32]. The reason is that there is just no appropriate Schläfli symbol $\{3,3,q\}$ in Euclidean 3-space, similarly in higher dimensions. There are triangulations with a symmetry group acting transitively on the tetrahedra, classified in [9]. Vertex-transitive tilings of a special type were classified in [10]. As another substitute of regular triangulations we investigate *lattice triangulations* of 3-space (or higher-dimensional space) in the following sense:

Definition A triangulation of $\mathbb{E}^d$ is called a lattice triangulation if the set of vertices forms a Euclidean lattice of rank d and if the triangulation is invariant under all translations of that lattice. A lattice is defined as a discrete additive subgroup of $\mathbb{R}^d$. Up to affine transformations we can assume that the lattice is nothing but the subgroup $\mathbb{Z}^d \subset \mathbb{R}^d$, so that the vertices can be identified with the points having integer coordinates. The star of a vertex v in a given triangulation is defined to be the set of all d-dimensional simplices containing v. The link of a vertex v in a given triangulation is defined to be the set of all $(d-1)$-dimensional simplices Δ such that the join $\Delta * \{v\}$ is a d-dimensional simplex of the triangulation. Since the translations act transitively on the set of vertices, each vertex link is isomorphic with the link of the origin $(0,\ldots,0) \in \mathbb{Z}^d$ which we call the basic vertex link, similarly we can talk about the basic vertex star. Throughout this paper we will assume that all triangulations are combinatorial in the sense that the link of any k-dimensional simplex is a $(d-k-1)$-dimensional sphere, so that in particular the basic vertex link is a triangulated $(d-1)$-sphere. A combinatorial manifold is a manifold together with such a combinatorial triangulation. Note that the set of vertices in the basic vertex link (but not necessarily the set of simplices) is invariant under the central involution $x \mapsto -x$. It follows that the number of vertices in the basic vertex link is always even. It is also clear that the linear part of the abstract automorphism group is a subgroup of the unimodular affine linear group $AGL(d,\mathbb{Z})$ acting on $\mathbb{Z}^d$. The symmetry of a lattice triangulation can be measured by the purely linear part of the automorphism group in $GL(d,\mathbb{Z})$ which is contained in the automorphism group of the basic vertex link.

In the case $d=2$ the standard situation is the following: The basic link is the convex hull of the union of the two squares with vertices

$$(0,0),(0,1),(1,0),(1,1) \text{ and } (0,0),(0,-1),(-1,0),(-1,-1)$$

where we subdivide each square by the main diagonal $(0,0), \pm(1,1)$. This is a non-regular hexagon, and any other lattice triangulation is equivalent to this one by an affine transformation. The basic link and its translate by the vector $(1,1)$ overlap along a subdivided square, like two regular hexagons can overlap along a rhombus. The matrix $\left(\begin{smallmatrix} 0 & -1 \\ 1 & -1 \end{smallmatrix}\right) \in SL(2,\mathbb{Z})$ of order 3 acts on this lattice triangulation in the same way as the rotation by $2\pi/3$ acts on the regular tessellation $\{3,6\}$. The central symmetry $x \mapsto -x$ is obvious. In higher dimensions there is a similar standard triangulation which, however, does not seem to have a standard name or a standard notation. For $d=3$ it is unique by Theorem 1 below.

Proposition 1 (Standard lattice triangulation)
There is a standard lattice triangulation *of* $\mathbb{E}^d$ *with straight edges by subdividing the cubical tessellation* $\{4,3^{d-1}\}$ *in the standard way. This means that in each cube we are introducing the main*

diagonal from $(x_1, \ldots, x_d)$ *to* $(x_1 + 1, x_2 + 1, \ldots, x_d + 1)$ *with* $d!$ *top-dimensional simplices around it, compare* [31]. *In this case the vertex link is a triangulated* $(d-1)$*-sphere with* $2^{d+1} - 2$ *vertices which can be defined as a subdivided boundary complex of the dual of Coxeter's expanded simplex* $e\alpha_d$ [26]. *According to a theorem of Minkowski this is the maximum number: Any parallelohedron in d-space has at most* $2^{d+1} - 2$ *facets. The matrix*

$$\begin{pmatrix} 0 & 0 & 0 & \cdots & 0 & -1 \\ 1 & 0 & 0 & \cdots & 0 & -1 \\ 0 & 1 & 0 & \cdots & 0 & -1 \\ 0 & 0 & 1 & \cdots & 0 & -1 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 & -1 \end{pmatrix}$$

in $GL(d, \mathbb{Z})$ *of order* $d + 1$ *acts on this triangulation in the same way as the shift matrix acts on the hyperplane* $\sum_i x_i = 0$ *in* $(d+1)$*-space.*

Corollary (The 3-dimensional case)

1. *For* $d = 3$ *the basic link in the standard triangulation is combinatorially a subdivided boundary of a cube where each square is subdivided by one additional vertex at its centre. This is combinatorially equivalent to the rhombidodecahedron where each of the rhombi is subdivided by the short diagonal. It coincides with Type* TT2 *(Fig.* 16*) in* [9]. *For the uniqueness see Theorem* 1 *below.*

2. *After an affine transformation, the* 3*-dimensional standard lattice triangulation can be regarded as a subdivision of the tessellation of* 3*-space by rhombidodecahedra where one puts an extra vertex at the centre of each rhombidodecahedron. The various rhombidodecahedra are overlapping each other in the same way as the hexagons are overlapping in the planar tessellation* $\{6, 3\}$ *after subdivision of each hexagon by an additional vertex at its centre.*

3. *Alternatively, the* 3*-dimensional standard lattice triangulation can be regarded as the dual of the lattice tiling of* 3*-space by truncated (regular) octahedra. From this fact it follows that the* 3*-dimensional Euclidean symmetry group acts transitively on the set of vertices and on the set of tetrahedra as well. However, the action of pure translations is transitive only on the set of vertices, not on edges or higher-dimensional faces.*

Remark (Compact quotients)
This standard lattice triangulation admits a quotient with $n_d = 2^{d+1} - 1$ vertices (or any $n \geq n_d$) and $n_d \cdot d!$ top-dimensional simplices (or $n \cdot d!$, resp.) which is a combinatorial d-torus, see [24], [26], compare [15]. For arbitrary $n \geq n_d$ this example was denoted by $M_1^d(n)$ in [27] as part of a family with several parameters, compare Question 7 below. In [24] the rhombidodecahedral shape of the vertex link is emphasized, including a discussion and illustration of the overlapping rhombidodecahedra. It is possible that this number n_d is the minimum number of vertices for any combinatorial d-torus but this has not yet been proved. It is remarkable that in the cases $d = 4$ and $d = 8$ other lattice triangulations with straight edges have been found, admitting another quotient 4-torus with $n_4 = 31$ vertices and several 8-tori with $n_8 = 511$ vertices, see [14], [8].

Furthermore, by the central symmetry this standard lattice triangulation admits a branched quotient with 2^d vertices which is a generalized combinatorial Kummer variety (d-torus modulo the involution $x \mapsto -x$), see [21], [22]. The case $d = 3$ is also known as the most symmetric 3-dimensional pseudomanifold with 8 vertices where each vertex link is a real projective plane, see [2, $\mathcal{P}$], [28, p.73]. The case $d = 4$ (the classical Kummer variety with 16 nodes) is the unique combinatorial 4-pseudomanifold (with at most isolated singularities) with 16 vertices and with a primitive automorphism group which is not a manifold, see [7].

Corollary (Combinatorial d-tori)
Any combinatorial d-torus which is a quotient of a lattice triangulation of d-space with convex simplices has at least $n_d = 2^{d+1} - 1$ vertices with equality if and only if it is 2-neighborly. This bound is attained for the triangulations given in [26] *which are quotients of the standard lattice triangulation.*

For $d = 3$ the corollary was observed by Grigis [14], and the uniqueness of such a triangulation with $n = n_d$ vertices was shown. For $d \geq 4$ uniqueness does not hold, see [14], [8].

Proof. For a given combinatorial d-torus of this type we consider the dual tiling of the universal covering. This is a primitive lattice tiling with convex faces, and the prototile is a parallelohedron [16, Ch.32]. *Primitive* means that at each vertex precisely $d+1$ tiles meet, the minimum number. By a theorem of Voronoi [35, p.67] such a primitive prototile has exactly $2^{d+1} - 2$ facets, compare also [15]. By duality this implies that in the lattice triangulation of d-space every vertex has a link with precisely $2^{d+1} - 2$ distinct vertices. Consequently, in the d-torus we must have at least one full vertex star with precisely $2^{d+1} - 1$ vertices. □

Conjecture *Any combinatorial d-torus has at least $2^{d+1} - 1$ vertices.*

So far the conjecture is proven only for $d = 2$ where in addition the 7-vertex 2-torus is combinatorially unique. For $d = 3$ a similar uniqueness (without any additional assumptions) can be conjectured, compare Section 3 below.

We now turn to a closer examination of the 3-dimensional case. Here the link of the vertex $(0, 0, 0)$ in the standard lattice triangulation of 3-space is depicted in Figure 1. It has an automorphism group of order 48 (contained in $GL(3, \mathbb{Z})$), as the ordinary 3-cube. It can be defined as a subdivision of the cube by one additional vertex in each of the six facets, it can also be defined as the subdivision of the rhombidodecahedron by the short diagonal in each of the rhombi. One method for getting more examples is to modify this standard basis vertex link. This raises the following question:

Question 4: *How can the basic vertex link of a lattice triangulation look like ? Can it have arbitrarily many vertices ?*

We will see in Sections 3 and 4 below that in fact the basic vertex link can have arbitrarily many vertices. A special and fairly symmetric example with 18 vertices is depicted in Figure 2. It has no vertices of valence 3, and its automorphism group coincides with that of the tetrahedron. For getting started we first have to list a few elementary properties of the basic vertex star and the basic vertex link. At this point it is quite important to distinguish between an abstract triangulation and a concrete geometric realization of it.

Definition (Abstract vs. geometric triangulation)

1. For the abstract version of a lattice triangulation we require that the vertices are in bijection with $\mathbb{Z}^3$ and that the triangulation is invariant under this group acting on itself, so that the automorphism group of the triangulation contains $\mathbb{Z}^3$ as a subgroup in a natural way. For a better distinction we call such a triangulation a $\mathbb{Z}^3$-invariant triangulation of $\mathbb{E}^3$ if the carrier of the triangulation (or the union of all tetrahedra) is homeomorphic with Euclidean 3-space.
2. On the more geometric side we can talk about such a triangulation together with a homeomorphism with $\mathbb{E}^3$ such that the set of vertices is mapped onto $\mathbb{Z}^3$ and such that the group of Euclidean translations induced by $\mathbb{Z}^3$ acts on the triangulation. This means that any tetrahedron is mapped to another tetrahedron which is congruent to the first one by a Euclidean translation. This is what we call a lattice triangulation of $\mathbb{E}^3$ in a geometric sense.

In a very rough version the difference is that the *abstract basic vertex link* is an abstract triangulation such that the vertices are labeled by elements of $\mathbb{Z}^3$ whereas the geometric basic vertex

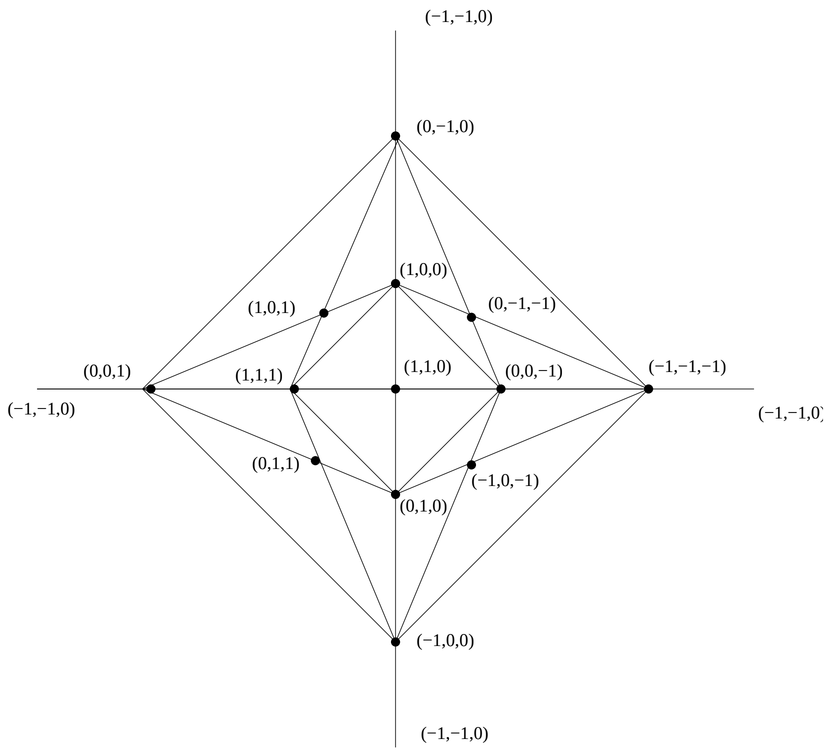

Figure 1: The unique standard basic vertex link 14_1 with 14 vertices

link is embedded into Euclidean 3-space such that the vertices actually get integer positions (the same for the *geometric basic vertex star*) together with additional strong geometric conditions as given in Lemma 1.

Lemma 1 (The geometric basic vertex star)
Any basic vertex star has the following properties:

1. *It is a simplicial* 3*-ball embedded in* 3*-space (possibly with curvilinear edges and with non-planar triangles) where all vertices are in* $\mathbb{Z}^3$*.*
2. *There are three vertices in the link which (regarded as position vectors starting from the origin) form an integer basis of the* $\mathbb{Z}$*-module* $\mathbb{Z}^3$*.*
3. *If* $\langle 0, x, y, z\rangle$ *is a tetrahedron of the basic star then so are the following tetrahedra*

$$\langle -x, 0, y-x, z-x\rangle, \langle -y, x-y, 0, z-y\rangle, \text{ and } \langle -z, x-z, y-z, 0\rangle,$$

and all four are congruent with one another under pure translations by x, y, z*.*

Proof. Condition 1 is obvious. Condition 2 follows because the contrary would imply that the vertices of the basic vertex star would generate a proper subgroup of $\mathbb{Z}^3$. This contradicts the fact that the edge graph of such a triangulation is connected. For the proof of 3 we observe that the triangulation is invariant under the translations by $-x, -y, -z$ sending the first tetrahedron to the four other one which, obviously, also belong to the star of the vertex 0. □

It is quite obvious that a basic vertex star with the properties 1, 2 and 3 in Lemma 1 above induces a unique lattice triangulation of 3-space, just by the union of all translates of the basic vertex star by integer vectors. The translates of the various vertex stars cover the 3-space in such a way that the overlaps correspond to subcomplexes of the resulting triangulation. In particular the star of

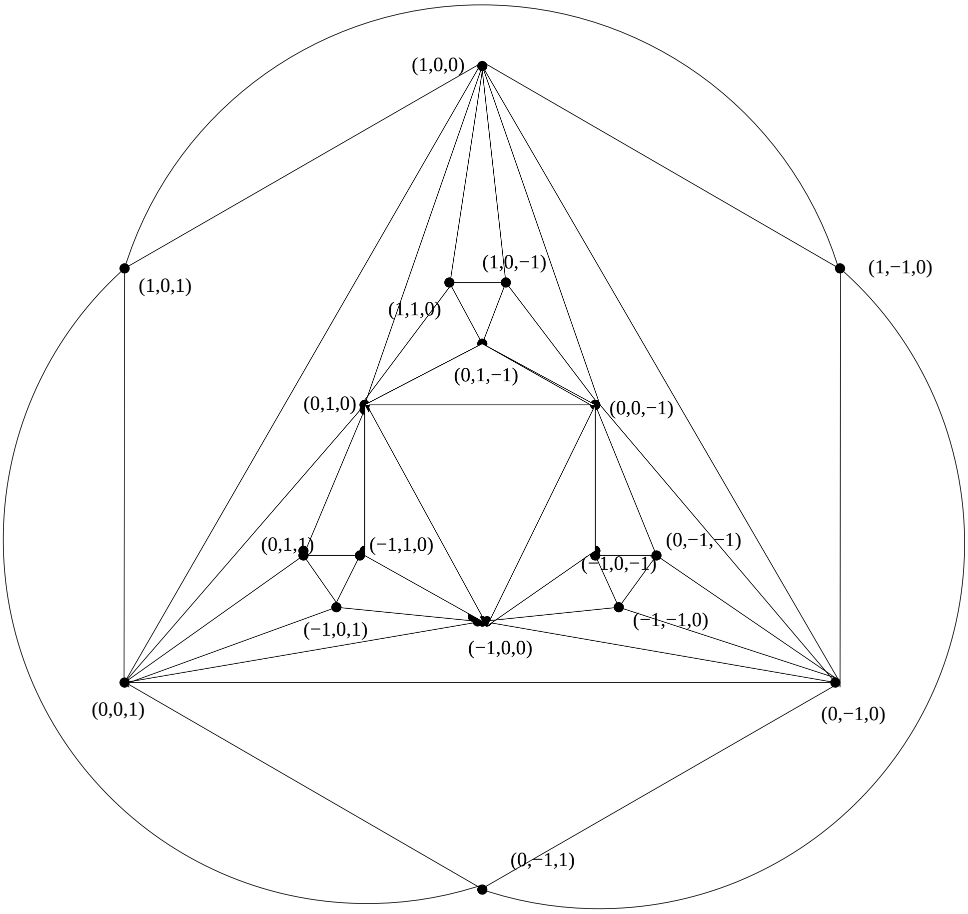

Figure 2: A special basic vertex link 18_2 with 18 vertices

the vertex 0 and the star of the vertex $x \neq 0$ intersect in the star of the edge $\langle 0\ x \rangle$ if x is a vertex of the star of 0.

On the abstract level we have to start with the situation that the basic vertex link is assumed to be an abstract triangulated 2-sphere such that the vertices have labels in $\mathbb{Z}^3$. We formulate this in a weaker version of Lemma 1 as follows

Lemma 2 (The abstract basic vertex link)
Any basic vertex link has the following properties:

1. *It is a simplicial* 2*-sphere where all vertices are elements of* $\mathbb{Z}^3$.
2. *There are three vertices in the link which (regarded as vectors) form an integer basis of the* $\mathbb{Z}$*-module* $\mathbb{Z}^3$.
3. *If* $\langle x, y, z \rangle$ *is a triangle of the basic link then so is* $\langle -x, y - x, z - x \rangle$.
4. *For any triangle* $\langle x, y, z \rangle$ *in the basic link there exists a unique vector* $0 \neq u \in \mathbb{Z}^3$ *such that* $\langle x - u, y - u, z - u \rangle$ *is another triangle in the basic link, and both are mapped into each other under the translation by* $\pm u$.

Proof. Condition 1 is obvious. The proof of 2 is the same as in the proof os Lemma 1 above. For the proof of 3 we observe that a tetrahedron $\langle 0, x, y, z \rangle$ in the $\mathbb{Z}^3$-invariant triangulation implies that $\langle -x, 0, y-x, z-x\rangle$ is another tetrahedron by the equivariance under the translation by the vector $-x$, and vice versa. Note that Condition 4 is a consequence of 1, 2 and 3 only: We start with the triangles $\langle x, y, z\rangle$ and $\langle -x, y-x, z-x\rangle$ in the basic link. Since the edge $\langle y-x, z-x\rangle$ is in precisely two triangles there is a unique triangle $\langle v, y-x, z-x\rangle$ in the basic link where $v \neq -x$. Now let $u := v + x$ then $\langle u-x, y-x, z-x\rangle$ is in the basic link and, by 3., the same holds for $\langle x-u, y-u, z-u\rangle$. □

Again it is quite obvious that an abstract basic vertex link with the properties 1, 2, 3 in Lemma 2 above induces uniquely an abstract $\mathbb{Z}^3$-invariant triangulation of some non-compact 3-manifold, just by applying the group $\mathbb{Z}^3$ to the vertex star which in turn is just the abstract cone over the vertex link. The translates of the various vertex stars cover this 3-manifold in such a way that the overlaps correspond to subcomplexes of the resulting triangulation. In particular the star of the vertex 0 and the star of the vertex $x \neq 0$ intersect in the star of the edge $\langle 0\ x\rangle$ if x is a vertex of the link of 0.

Question 5: *Is this 3-manifold always homeomorphic with Euclidean 3-space ? If yes, is the $\mathbb{Z}^3$-action conjugate to the standard one in the group of all homeomorphisms ?*

Proposition 2 (Modification of the basic vertex link)
Assume we have a lattice triangulation of Euclidean 3-space with an n-vertex basic vertex link such that one triangle in the basic vertex link is also in the link of another vertex u such that u and $-u$ are not in the basic vertex link. Then it can be modified into another lattice triangulation with an $(n+2)$-vertex basic vertex link.

Proof. The modification can be achieved by simultaneous bistellar flips. Let $\langle x, y, z\rangle$ be a triangle in the basic vertex link together with a vector u such that we have the two tetrahedra $\langle 0, x, y, z\rangle$ and $\langle x, y, z, u\rangle$. In the union of them we remove the triangle $\langle x, y, z\rangle$ and introduce the diagonal $\langle 0, u\rangle$, surrounded by three tetrahedra corresponding to the three edges $\langle x, y\rangle, \langle y, z\rangle, \langle z, x\rangle$. Then this operation is transferred by all translations to all the other vertices. In particular, the edge $\langle -u, 0\rangle$ is also introduced. As a result, the new basic vertex link contains all the vertices as before and, in addition, the vertices u and $-u$. Note that this modification can be carried out for abstract $\mathbb{Z}^3$-invariant triangulations as well as for geometric triangulations. The additional edges will have to be curvilinear unless the union of the two adjacent tetrahedra is a convex 3-polytope with 5 proper vertices. □

Illustration (The relationship between the possible basic vertex links by Proposition 2)
We can interpret one step according to Proposition 2 as an edge in a graph where the vertices represent the possible basic vertex links or rather their equivalence classes under the $GL(3,\mathbb{Z})$-action. From the unique type 14_1 there is an edge to the unique type 16_1. From there we obtain a unique type 18_1 but there is another type 18_2 (depicted in Figure 2) which can only be obtained by the inverse process from a certain type with 20 vertices. The following Figure 3 shows the graph of all possible basic vertex links with up to 22 vertices together with their automorphism groups (D_8 denotes the dihedral group of order 8). In addition there are 28 items with 24 vertices (among them one which is directly joined with the special type 22_9 but not with any of $22_1, \ldots, 22_8$), and 80 items with 26 vertices.

Question 6: *Does every abstract $\mathbb{Z}^3$-invariant triangulation of 3-space induce a lattice triangulation, possily with curvilinear edges ? In other words: Can every abstract basic vertex link in the sense of Lemma 2 be realized as the boundary of an embedded geometric basic vertex star in the sense of Lemma 1 ? Is this realization unique up to isotopy in space ?*

If the answer is "yes" then we would obtain a classification of lattice triangulations by the classification of all triangulated 2-spheres with an even number of vertices and with a labeling of these vertices by integer vectors according to Lemma 2. This is a discrete set of data in any case so that we would obtain a countably infinite classification. No finite classification can exist by our

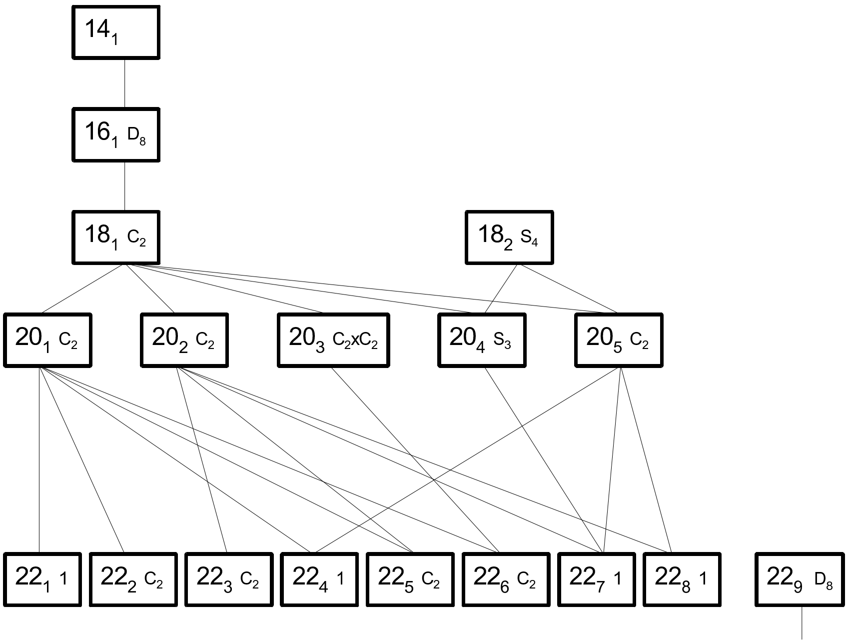

Figure 3: The basic vertex links with up to 22 vertices

Theorem 2 below. An old result by Heesch [18] states that there is no finite classifcation of lattice tilings. Our results imply that there is no finite classification of primitive lattice tilings either but there might be a countable classification, depending on an answer to Question 6. Furthermore it seems that a positive answer to Question 6 would imply a positive answer to Question 5.

Theorem 1 (Uniqueness)

1. *(Fedorov-Voronoi* [35]*) Up to affine transformations, the dual of the standard lattice triangulation is the unique lattice tiling of* 3*-space by convex polytopes which is primitive in the sense that every vertex is contained in precisely four tiles.*

2. *(Grigis* [14]*) Up to affine transformations, the standard lattice triangulation of* 3*-space is the unique lattice triangulation with straight edges.*

Proof. The claim in 1. is part of a more general theorem which states that (up to affine transformations) there are precisely five translational tilings by polytopes in 3-space, and in each case the prototile is a parallelohedron, see [35, p.161ff.], [1], [33], compare [16, Ch.32] and [33] for the case of higher dimensions. The five prototiles are the cube, the hexagonal prism, the rhombidodecahedron, the elongated dodecahedron, and the truncated octahedron. Among them, only the last one leads to a primitive tiling in the sense that only four tiles meet at a vertex. The *truncated octahedron* with $8 + 6 = 14$ facets is the prototile of the dual of the standard lattice triangulation with the standard basic vertex link having $8 + 6 = 14$ vertices, see Figure 1. This prototile is also called *orthic tetrakaidekahedron* [20]. It is of great importance in the sciences.

The claim in 2. follows by duality from 1. since there is a duality principle between tilings with convex tiles. However, in [14] an independent and direct proof was given which is not based on the dual result of 1. Our lattice triangulations were called *invariant triangulations* there. □

Remark Such a uniqueness theorem does not hold in higher dimensions $d \geq 4$. It is well known that there are three distinct primitive parallelohedra for $d = 4$ [35, p.164ff.] and 222 ones for $d = 5$ [12], [33]. Consequently, we have as many distinct duals which are lattice triangulations with straight edges and convex simplices. It is the main result of [14] that precisely two of the three lattice triangulations of 4-space with straight edges, each having a basic vertex link with 30 vertices, admit a quotient combinatorial 4-torus with 31 vertices.

Theorem 2 (Existence)
There are infinitely many distinct lattice triangulations of 3*-space where the edges cannot be made straight by Theorem* 1 *above. The basic vertex link can have any even number* $n-1 \geq 14$ *of vertices.*

In a dual formulation, there are infinitely many distinct primitive lattice tilings in 3*-space where the (non-convex) prototile can have any even number* $n-1 \geq 14$ *of facets.*

This follows from Theorem 4 below. The proof will be given in Section 4, after constructing triangulated 3-tori with n vertices which are neighborly and cyclically symmetric in the sense of a cyclic and vertex transitive automorphism group.

3. Neighborly triangulations of the 3-dimensional torus

It was the discovery of the cyclic 4-polytopes with n equidistant vertices on the trigonometric moment curve $t \mapsto (e^{it}, e^{2it})$ that led to the construction of neighborly triangulations of the 3-sphere with arbitrarily many vertices such that a vertex transitive group acts by geometric transformations. In particular, in this case the vertex link has arbitrarily many vertices. It seems to be a natural question whether there are neighborly triangulations of the 3-torus with arbitrarily many vertices and with a vertex transitive automorphism group, possibly realizable by geometric transformations preserving the local Euclidean structure. This section contains a proof of our Main Theorem B. We start with a quotient of the standard eqivariant triangulation.

The standard 15-vertex torus As mentioned in the remark after Proposition 1, the standard lattice triangulation of 3-space admits simplicial quotient 3-tori by pure translation groups with any number $n \geq 15$ of vertices and with a dihedral automorphism group of order $2n$, see [27], [24], [26]. It was denoted by $M_1^3(n)$ in [27]. For $n \leq 14$ one obtains double edges in the quotient, so this does not lead to simplicial complexes. The smallest simplicial example is the 15-vertex neighborly triangulation of the 3-torus with an automorphism group of order 120 including the cyclic shift $x \mapsto x+1$ (15) and including the multipliers $\pm1, \pm2, \pm4, \pm8$ (15), realized by Euclidean symmetries in a flat 3-torus. It was denoted by III_{15} in [25] where the authors were assuming there that it is a sporadic example not followed by a type III_{17} or others (this is actually true under the assumption of a dihedral automorphism group but not with a cyclic one). It seems that this example is even unique and has the minimum number of vertices among all combinatorial 3-tori. As a corollary of Theorem 1, this has been proved only under the assumption of lattice symmetry since in this case the vertex link has 14 vertices, see [14]. The uniqueness of the 15-vertex 3-torus in general is still open. Its 90 tetrahedra are given by the $\mathbb{Z}_{15}$-orbits of the six generating tetrahedra

$$\mathbf{0\,1\,3\,7} \quad \mathbf{0\,1\,5\,7} \quad \mathbf{0\,2\,3\,7} \quad \mathbf{0\,2\,6\,7} \quad \mathbf{0\,4\,5\,7} \quad \mathbf{0\,4\,6\,7}.$$

These six tetrahedra form a subdivided cube with main diagonal **0 7**. From the combinatorial point of view one recognizes the six permutations of the three differences 1, 2, 4 modulo 15 with a complementary difference 8, in short notation [**1 2 4**], compare [27] for these permuted differences as a construction principle in more generality. From the geometric point of view it seems to be the most natural triangulation of the 3-torus with few vertices, an appropriate analogue of the unique 7-vertex triangulation of the 2-torus. As a kind of a combinatorial Clifford torus it is a subcomplex of a cyclically symmetric simplicial 5-sphere, see the remark at the end of [4]. This can lead to an explicit triangulation of complex projective 3-space with the equilibrium 3-torus at its centre (F.H.Lutz, unpublished). The basic vertex link of the 15-vertex 3-torus is precisely the one in Figure 1 with the labeling $\varphi\colon \mathbb{Z}^3 \to \mathbb{Z}_{15}$ which is obtained from

$$\varphi(1,0,0) = \mathbf{1}, \quad \varphi(0,1,0) = \mathbf{2}, \quad \varphi(0,0,1) = \mathbf{4}$$

by linear continuation, i.e., the sum of vectors corresponds to the sum of labels modulo 15. In [8] such a labeling φ is called a *separating map* because it is a $\mathbb{Z}$-linear map which is bijective when restricted to the vertices of the star of the vertex $(0,0,0)$. This 15-vertex triangulation can be regarded as a Euclidean triangulation (with straight edges) of a flat 3-torus such that any combinatorial automorphism is realized by a Euclidean symmetry.

Proposition 3 (F.H.Lutz [29])
There is a unique neighborly combinatorial 3*-torus with* 17 *vertices and with a vertex transitive automorphism group. There is no one with* 16 *vertices and with the same properties otherwise. There is a unique one with* 15 *vertices also (namely, the standard example above).*

The construction of the 17-vertex 3-torus is the following: Its $7 \cdot 17 = 119$ tetrahedra are given by the $\mathbb{Z}_{17}$-orbits of the seven generating tetrahedra

$$\mathbf{0\,1\,3\,8} \quad \mathbf{0\,1\,5\,7} \quad \mathbf{0\,2\,3\,7} \quad \mathbf{0\,3\,7\,8} \quad \mathbf{0\,4\,5\,7} \quad \mathbf{0\,4\,6\,7} \quad \mathbf{0\,1\,7\,8}.$$

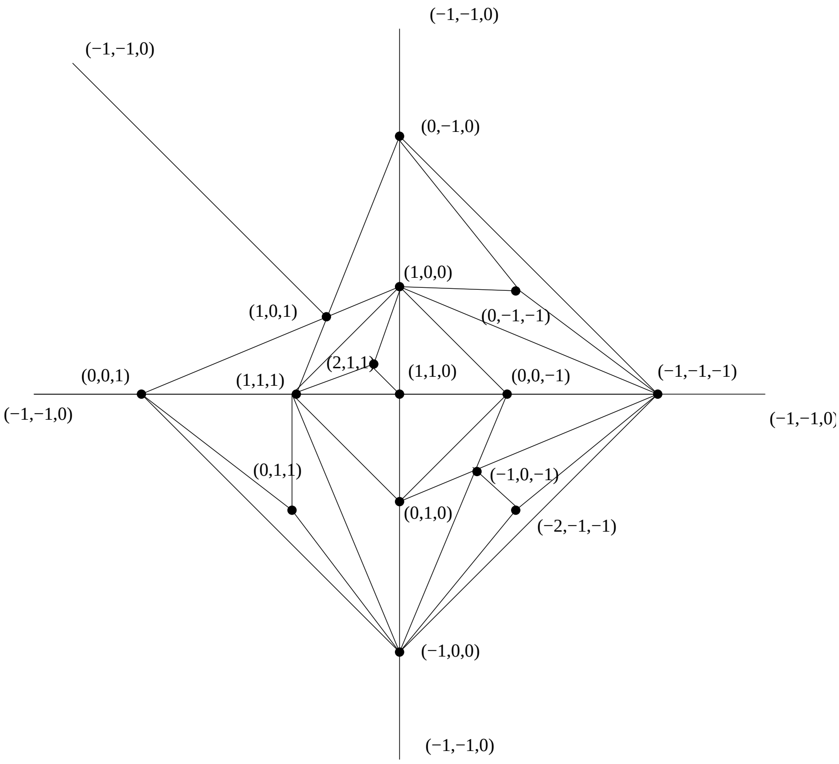

Figure 4: The unique basic vertex link 16_1 with 16 vertices

The vertex link is shown in Figure 4 with the labeling $\varphi\colon \mathbb{Z}^3 \to \mathbb{Z}_{17}$ which is obtained from

$$\varphi(1,0,0) = \mathbf{1}, \quad \varphi(0,1,0) = \mathbf{2}, \quad \varphi(0,0,1) = \mathbf{4}$$

by linear continuation, i.e., the sum of vectors corresponds to the sum of labels modulo 17, as an example: $\varphi(0,0,0) = \mathbf{0}$, $\varphi(1,1,1) = \mathbf{7}$, $\varphi(2,1,1) = \mathbf{8}$. This link has a dihedral automorphism group of order 8 but no automorphism carries over to the entire triangulation which, consequently, has an automorphism group isomorphic with $\mathbb{Z}_{17}$. Note that with these coordinates the basic link is not the boundary of a strictly convex polyhedron: The edge joining $(0,0,0)$ and $(2,1,1)$ meets the edge joining $(1,0,0)$ and $(1,1,1)$. In accordance with Theorem 1 this indicates that this 16-vertex 2-sphere is not the basic link of a lattice triangulation with straight edges, unless one admits tetrahedra with four coplanar vertices.

One can pass in a combinatorial way from the 15-vertex 3-torus above to the 17-vertex one by regarding the six generating tetrahedra as cyclic orbits of 4-tuples modulo 17 (this triangulation is not yet neighborly because the vertex link is still the same) and then by replacing the two orbits **0 1 3 7** and **0 2 6 7** by the three orbits **0 1 3 8**, **0 1 7 8**, **0 3 7 8**. This corresponds to 17 simultaneous bistellar flips (or an equivariant bistellar flip) in the union of **0 1 3 7** and **1 3 7 8** (and all its translates) by cutting out the triangle **1 3 7** and by introducing the diagonal **0 8** (and all its translates) as an edge. The uniqueness of this triangulation as well as the non-existence in the case of 16 vertices was established by a computer check by F.H.Lutz [29].

Remark: Similarly one can pass further to neighborly 3-tori with 19 and 21 vertices: Regard the seven generators above as cyclic orbits modulo 19 and replace the two orbits **0 2 3 7** and **0 1 5 7** by the three orbits **0 2 3 9**, **0 2 7 9**, **0 3 7 9**. This leads to an example with 19 vertices. Then we can regard these eight generators as cyclic orbits modulo 21 and replace the two orbits **0 1 3 8** and **0 2 7 9** by the three orbits **0 1 3 10**, **0 1 8 10**, **0 3 8 10**. This leads to an example with 21 vertices. However, it seems that one cannot continue by this type of equivariant bistellar flips in the same way. Compare Theorem 3 below for an alternative construction.

Corollary *If the neighborly 17-vertex triangulation of the 3-torus above is realized by Euclidean tetrahedra in a flat 3-torus then the cyclic automorphism group cannot be realized by isometries of the flat metric. Conversely, an equivariant discrete metric on this triangulation cannot be flat (compare Section 6).*

Proof. Because 17 is a prime number there are no fixed points of the $\mathbb{Z}_{17}$-action on the torus. Assume that the automorphism group $\mathbb{Z}_{17}$ is realized by isometries in a flat 3-torus $\mathbb{R}^3/\Gamma$. Then in the universal covering $\mathbb{R}^3$ this group generates a discrete group G of Euclidean motions without fixed points. If all elements in G are translations then there are three distinct elements in $\mathbb{Z}_{17}$ generating three linearly independent translations. So the triangulation by Euclidean tetrahedra is a lattice triangulation. Hence by Theorem 1 it coincides with the standard lattice triangulation, in contradiction with the 16-vertex basic link. If an element of G is not a translation then G must contain a screw motion with a basic angle $2\pi/17$ of the rotational part. Since this screw motion acts on the 3-torus $\mathbb{R}^3/\Gamma$ it follows that it acts also on a fundamental domain (modulo Γ). Hence this screw motion must be compatible with the lattice Γ, a contradiction. With modifications this argument can be extended to any free cyclic group action of any order, see [19]. This proves the first part. The second claim is essentially the same as in Theorem 5. □

Example 1 (A vertex link which is not possible in a neighborly 3-torus)
It is remarkable that not every basic vertex link with $n-1$ vertices can occur in a neighborly quotient 3-torus with n vertices. A counterexample is the special type depicted in Figure 2. Here it is impossible to find a $\mathbb{Z}$-linear separating map $\varphi : \mathbb{Z}^3 \to \mathbb{Z}_{19}$ which is bijective when restricted to the star of the vertex $(0,0,0)$. Assume that there is such a φ. The $\mathbb{Z}_{19}$-action and the symmetries of the link imply that without loss of generality we may assume that

$$\varphi(1,0,0) = \mathbf{1}, \quad \varphi(0,1,0) = \mathbf{a}, \quad \varphi(0,0,1) = \mathbf{b}$$

with $\mathbf{2} \le \mathbf{a} < \mathbf{b} \le \mathbf{9}$ where $\mathbb{Z}_{19} = \{\mathbf{0}, \mathbf{1}, \mathbf{2}, \ldots, \mathbf{18}\}$. However, $\mathbf{a} = \mathbf{2}$ is impossible because otherwise we would obtain $\mathbf{1} = \varphi(1,0,0) \neq \varphi(-1,1,0) = \mathbf{2} - \mathbf{1} = \mathbf{1}$, a contradiction. Similarly $\mathbf{b} = \mathbf{a} + \mathbf{1}$ is impossible because of $\varphi(1,1,0) \neq \varphi(0,0,1)$, $\mathbf{b} = \mathbf{a} + \mathbf{2}$ is impossible because it would imply $\mathbf{a} + \mathbf{1} = \varphi(1,1,0) \neq \varphi(-1,0,1) = \mathbf{b} - \mathbf{1}$. Furthermore, $\mathbf{b} = \mathbf{9}$ is impossible because of $\mathbf{9} + \mathbf{1} = -\mathbf{9}$. $\mathbf{b} = \mathbf{2a}$ is impossible because of $\varphi(0,-1,1) \neq \varphi(0,1,0)$, $\mathbf{b} = \mathbf{2a} + \mathbf{1}$ is impossible because of $\varphi(0,-1,1) \neq \varphi(1,1,0)$, $\mathbf{b} = \mathbf{2a} - \mathbf{1}$ is impossible because of $\varphi(-1,1,0) \neq \varphi(0,-1,1)$. Finally $\mathbf{a} = \mathbf{3}$ and $\mathbf{b} = \mathbf{8}$ is impossible because of $\varphi(0,1,1) \neq \varphi(0,0,-1)$, and $\mathbf{a} = \mathbf{5}$ and $\mathbf{b} = \mathbf{8}$

is impossible because of $\varphi(0,1,1) \neq \varphi(-1,-1,0)$. Therefore no such φ exists. For the same phenomenon in dimensions 4 and 8 see [14], [8].

On the other hand there is a $\mathbb{Z}$-linear separating map $\varphi : \mathbb{Z}^3 \to \mathbb{Z}_{20}$ defined by

$$\varphi(1,0,0) = \mathbf{2}, \quad \varphi(0,1,0) = \mathbf{5}, \quad \varphi(0,0,1) = \mathbf{6}$$

which shows that the link in Figure 2 is the link of a nearly neighborly 3-torus with 20 vertices and a natural $\mathbb{Z}_{20}$-action (and with ten diagonals $\langle \mathbf{x}, \mathbf{x}+\mathbf{10}\rangle$ for $\mathbf{x} = \mathbf{0}, \mathbf{1}, \ldots, \mathbf{9}$) because $\mathbf{10}$ is not in the imeage of φ.

Goal: Starting with the 15-vertex example and the 17-vertex example above, we construct two infinite series of neighborly combinatorial 3-tori by a combinatorial extrapolation as follows. The universal coverings of such will provide lattice triangulations of 3-space, see Section 4.

Theorem 3 (Neighborly triangulations of 3-tori)
For any odd number $n \geq 15$ there is a neighborly combinatorial 3-torus with n vertices and with a vertex transitive automorphism group being isomorphic with $\mathbb{Z}_n$ (for $n \geq 17$) or containing $\mathbb{Z}_n$ (for $n = 15$).

Remark: We missed this series in the previous paper [25], due to the lower speed of the computers. At that time a classification of 17-vertex triangulations admitting only the cyclic group seemed to be out of range. Therefore only the case $n = 15$ was covered.

Proof. The f-vector $f = (n, f_1, f_2, f_3)$ of such a triangulation coincides with the f-vector of any neighborly 3-manifold with n vertices. In particular it coincides with the one of the boundary complex of the cyclic 4-polytope with n vertices which is

$$f = \left(n, \tbinom{n}{2}, n(n-3), \tfrac{n}{2}(n-3)\right).$$

Hence for the tetrahedra we expect to see $\frac{n-3}{2}$ orbits of length n under the $\mathbb{Z}_n$-action.
The construction is the following: Let the vertices be the elements of $\mathbb{Z}_n$. For any $k \geq 4$ the example is given by the union of the orbits generated by the tetrahedra in Table I under the natural $\mathbb{Z}_n$-action where we have to distinguish between the two cases $n = 4k-1$ with $\frac{n-3}{2} = 2k-2$ orbits and $n = 4k+1$ with $\frac{n-3}{2} = 2k-1$ orbits:

Case 1 : $n = 4k-1$			
0	1	$k-1$	$2k-1$
0	1	$k+1$	$k+3$
0	$k-2$	$k-1$	$2k-1$
0	$k-2$	$2k-2$	$2k-1$
0	k	$k+1$	$k+3$
0	k	$k+2$	$k+3$
0	1	3	4
0	1	4	5
0	1	5	6
⋮	⋮	⋮	⋮
0	1	$k-3$	$k-2$
0	1	$k-2$	$k-1$
0	1	$k+3$	$k+4$
0	1	$k+4$	$k+5$
0	1	$k+5$	$k+6$
⋮	⋮	⋮	⋮
0	1	$2k-3$	$2k-2$
0	1	$2k-2$	$2k-1$

Case 2 : $n = 4k+1$			
0	1	$k-1$	$2k$
0	1	$k+1$	$k+3$
0	$k-2$	$k-1$	$2k-1$
0	$k-1$	$2k-1$	$2k$
0	k	$k+1$	$k+3$
0	k	$k+2$	$k+3$
0	1	3	4
0	1	4	5
0	1	5	6
⋮	⋮	⋮	⋮
0	1	$k-3$	$k-2$
0	1	$k-2$	$k-1$
0	1	$k+3$	$k+4$
0	1	$k+4$	$k+5$
0	1	$k+5$	$k+6$
⋮	⋮	⋮	⋮
0	1	$2k-2$	$2k-1$
0	1	$2k-1$	$2k$

Table I: generating tetrahedra for neighborly n-vertex 3-tori

In Case 1 one recognizes three blocks of size 6, $k-4$ and $k-4$. The first block originates from the 15-vertex 3-torus. In fact, for $k=4$ the triangulation consists of the first block only, and the two others are empty. The two other blocks are part of the boundary complex of the cyclic 4-polytope with $4k-1$ vertices (compare Gale's evenness condition). The missing orbits are the first one, the last one and the four middle ones in the standard order $0\ 1\ 2\ 3,\ 0\ 1\ 3\ 4,\ \ldots\ ,\ 0\ 1\ 2k-1\ 2k$. Therefore in some sense and for $k \geq 5$ this neighborly triangulated 3-torus with n vertices can be regarded as a slight modification of the boundary complex of the cyclic 4-polytope with n vertices: For any k only six orbits have to be exchanged. One can also think of it as a kind of an amalgamation of the 15-vertex 3-torus and the cyclic polytope with n vertices. The same type of modification was successful in constructing a particular sequence II_n of n-vertex neighborly triangulated 3-dimensional Klein bottles $S^1 \veebar S^2$ from the boundary complexes I_n of the cyclic 4-polytopes, see [25]. Here the amalgamation procedure starts with $n=10$, the case $n=9$ is the original and vertex minimal example due to D.Walkup [36], compare [22] and [3].

In Case 2 one recognizes three blocks of size 6, $k-4$ and $k-3$. The first block originates from the 17-vertex 3-torus or, indirectly, from the 15-vertex 3-torus in Case 1. In fact, for $k=4$ the triangulation consists of the first block only, together with one item $\langle 0\ 1\ 7\ 8\rangle$ in the last block. The second block is empty for $n=17$. As in Case 1, the last two blocks are part of the boundary complex of the cyclic 4-polytope with $4k+1$ vertices (compare Gale's evenness condition). Consequently, as above the construction can be regarded as a slight modification of the boundary complex of the cyclic 4-polytope.

One can pass from Case 1 to Case 2 as above from 15 to 17 vertices: Regard the orbits for $\mathbb{Z}_{4k-1}$ literally as orbits for $\mathbb{Z}_{4k+1}$ and replace the two orbits $\langle 0\ 1\ k-1\ 2k-1\rangle$ and $\langle 0\ k-2\ 2k-2\ 2k-1\rangle$ by the three orbits $\langle 0\ 1\ k-1\ 2k\rangle$, $\langle 0\ k-1\ 2k-1\ 2k\rangle$ and $\langle 0\ 1\ 2k-1\ 2k\rangle$. This corresponds to n simultaneous bistellar flips, introducing the additional edges $\langle x\ x+2k\rangle$ for $x \in \mathbb{Z}_{4k+1}$.

In either case (1 or 2), for $n \geq 17$ the first block does not admit the multiplier -1 (n). The other part contained in the boundary complex of the cyclic polytope does admit the multiplier -1 (n) but not any other. Any additional automorphism would lead to a nontrivial symmetry of the link of 0 which interchanges 1 and -1. From the structure of the link it becomes clear that for $k \geq 5$ such a global automorphism does not exist. In the case $n=17$ the automorphism group is just $\mathbb{Z}_{17}$, compare Proposition 3. Consequently, the automorphism group coincides with $\mathbb{Z}_n$ for any $n \geq 17$.

It remains to show that this construction leads to a triangulation of the 3-dimensional torus. The link of the vertex 0 is easily checked to be a triangulated 2-sphere with $n-1$ vertices. In fact it appears as a slight modification of the boundary complex of the cyclic 3-polytope with $n-1$ vertices. Therefore we have a neighborly combinatorial 3-manifold. The topology follows from considering the universal covering with a deck transformation group consisting of pure translations as in the following Theorem 4. □

Corollary (Centrally symmetric triangulations of 3-tori)
For any even number $n = 2m \geq 16$ there is a nearly neighborly and centrally symmetric combinatorial 3-torus with n vertices and with a vertex transitive automorphism group being isomorphic with $\mathbb{Z}_n$ (for $n \geq 18$) or containing $\mathbb{Z}_n$ (for $n = 16$). This simplicial complex can be regarded as a subcomplex of the m-dimensional cross polytope which contains the full 1-skeleton of the polytope and which is invariant under the central involution $X \mapsto -X$.

The construction is essentially the same as in the proof of Theorem 3. We regard the vertices as the elemente of $\mathbb{Z}_n$ and generate the triangulation by the orbits of the same generating tetrahedra as in Theorem 3 above, separately for $n=4k$ (Case 1) and $n=4k+2$ (Case 2). Since any of the tetrahedra is contained in a section of length at most $2k$ (or $2k+1$, resp.) subsequent integers mod n, the vertex link will be the same as before. The triangulation contains all possible edges except for the edges between x and $x+m$, $x = 0, \ldots, m-1$, the *diagonals*. In the cross polytope the shift $x \mapsto x + m$ (n) appears as the central involution (antipodal mapping) without fixed points. The topology of this manifold follows by the same argument as in Theorem 3 from the universal covering studied in Theorem 4. □

4. Lattice triangulations of 3-space: Existence

In this section we prove our Main Theorem A (which is also formulated as Theorem 2) by explicitly constructing an infinite family of distinct lattice triangulations of 3-space, depending on an integer parameter $k \geq 4$. These triangulations can be distinguished by their basic vertex links: Any even number $n - 1 \geq 14$ can occur as the number of vertices in the basic vertex link. The neighborly triangulations of the 3-torus from Theorem 3 then appear as quotients of the lattice triangulations of 3-space or, conversely, the lattice triangulations are the universal coverings of the torus triangulations if the positions of the vertices are chosen appropriately.

Theorem 4 (Non-standard lattice triangulations of 3-space)
The universal covering of each of the examples in Theorem 3 above with n vertices is a lattice triangulation of 3-space such that the basic vertex link has $n-1$ vertices. For any $n \geq 17$ the edges cannot simultaneously be made straight while preserving the equivariance under the lattice (unless one admits tetrahedra with four coplanar vertices).

Proof. First of all the case $n = 15$ leads to the standard lattice triangulation of 3-space, compare Proposition 1. This is our starting object. From Theorem 1 it is clear that for any other case it will be impossible to make all edges simultaneously straight. In order to construct the examples we are looking for explicit integer coordinates for the basic vertex link in the universal covering of the examples in Theorem 3. Elements of $\mathbb{Z}^3$ are denoted by row vectors (z_1, z_2, z_3). The quotient map is then induced by the $\mathbb{Z}$-linear separating map $\varphi : \mathbb{Z}^3 \to \mathbb{Z}_n$ defined by

$$\varphi(1,0,0) = \mathbf{1}, \quad \varphi(0,1,0) = \mathbf{2}, \quad \varphi(0,0,1) = \mathbf{k}$$

where the bold face symbols on the right hand side refer to the cyclic labeling of the vertices of the 3-torus by integers modulo n where $4n = 4k - 1$ or $n = 4k + 1$, respectively. Compare the two cases with $k = 4$ and $n = 15$ or $n = 17$ above. Once we have the right coordinates for the vertices, edges and faces in space, this mapping φ can be easily checked to be a bijection when restricted to the star of the vertex $(0,0,0)$. Our goal is to find these coordinates and to construct a concrete lattice triangulation of 3-space such that the triangulation in Theorem 3 is the quotient of the triangulated 3-space by the sublattice defined by $\varphi^{-1}(\mathbf{0})$. It is the same in the Corollary after Theorem 3 for $\varphi : \mathbb{Z}^3 \to \mathbb{Z}_{2m}$ where in this case φ is injective but not surjective when restricted to the star of the vertex $(0,0,0)$ since no vertex corresponds to the label $\mathbf{m} \in \mathbb{Z}_{2m}$. This leads to the diagonals in the triangulation.

By the invariance under the lattice $\mathbb{Z}^3$ it is sufficient to give coordinates for the generating tetrahedra, one in each orbit under the group. The three blocks of generating tetrahedra in the proof of Theorem 3 above (see Table I) can be realized with integer coordinates according to the following Table II:

Case 1 : $n = 4k-1$				Case 2 : $n = 4k+1$			
$(0,0,0)$	$(1,0,0)$	$(k-3,1,0)$	$(k-3,1,1)$	$(0,0,0)$	$(1,0,0)$	$(k-3,1,0)$	$(k-2,1,1)$
$(0,0,0)$	$(1,0,0)$	$(1,0,1)$	$(1,1,1)$	$(0,0,0)$	$(1,0,0)$	$(1,0,1)$	$(1,1,1)$
$(0,0,0)$	$(k-4,1,0)$	$(k-3,1,0)$	$(k-3,1,1)$	$(0,0,0)$	$(k-4,1,0)$	$(k-3,1,0)$	$(k-3,1,1)$
$(0,0,0)$	$(k-4,1,0)$	$(k-4,1,1)$	$(k-3,1,1)$	$(0,0,0)$	$(k-3,1,0)$	$(k-3,1,1)$	$(k-2,1,1)$
$(0,0,0)$	$(0,0,1)$	$(1,0,1)$	$(1,1,1)$	$(0,0,0)$	$(0,0,1)$	$(1,0,1)$	$(1,1,1)$
$(0,0,0)$	$(0,0,1)$	$(0,1,1)$	$(1,1,1)$	$(0,0,0)$	$(0,0,1)$	$(0,1,1)$	$(1,1,1)$
$(0,0,0)$	$(1,0,0)$	$(1,1,0)$	$(2,1,0)$	$(0,0,0)$	$(1,0,0)$	$(1,1,0)$	$(2,1,0)$
$(0,0,0)$	$(1,0,0)$	$(2,1,0)$	$(3,1,0)$	$(0,0,0)$	$(1,0,0)$	$(2,1,0)$	$(3,1,0)$
$(0,0,0)$	$(1,0,0)$	$(3,1,0)$	$(4,1,0)$	$(0,0,0)$	$(1,0,0)$	$(3,1,0)$	$(4,1,0)$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
$(0,0,0)$	$(1,0,0)$	$(k-5,1,0)$	$(k-4,1,0)$	$(0,0,0)$	$(1,0,0)$	$(k-5,1,0)$	$(k-4,1,0)$
$(0,0,0)$	$(1,0,0)$	$(k-4,1,0)$	$(k-3,1,0)$	$(0,0,0)$	$(1,0,0)$	$(k-4,1,0)$	$(k-3,1,0)$
$(0,0,0)$	$(1,0,0)$	$(1,1,1)$	$(2,1,1)$	$(0,0,0)$	$(1,0,0)$	$(1,1,1)$	$(2,1,1)$
$(0,0,0)$	$(1,0,0)$	$(2,1,1)$	$(3,1,1)$	$(0,0,0)$	$(1,0,0)$	$(2,1,1)$	$(3,1,1)$
$(0,0,0)$	$(1,0,0)$	$(3,1,1)$	$(4,1,1)$	$(0,0,0)$	$(1,0,0)$	$(3,1,1)$	$(4,1,1)$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
$(0,0,0)$	$(1,0,0)$	$(k-5,1,1)$	$(k-4,1,1)$	$(0,0,0)$	$(1,0,0)$	$(k-4,1,1)$	$(k-3,1,1)$
$(0,0,0)$	$(1,0,0)$	$(k-4,1,1)$	$(k-3,1,1)$	$(0,0,0)$	$(1,0,0)$	$(k-3,1,1)$	$(k-2,1,1)$

Table II: generating tetrahedra for lattice triangulations of 3-space

By translations these tetrahedra generate a basic vertex star with $4 \cdot (2k-2)$ (or $4 \cdot (2k-1)$, resp.) tetrahedra and, furthermore, the entire triangulation of 3-space by the conditions in Lemma 1 above. By induction we can pass from any k to $k+1$, see below. the starting case $k = 4, n = 15$ is nothing but the standard lattice triangulation with straight edges. Under the assumption that in the standard lattice triangulation (i.e., a basic link with 14 vertices) all edges emanating from $(0,0,0)$ are straight, then the additional edges from $(0,0,0)$ to $(m,1,1)$ and $(m,1,0)$ with $m \geq 2$ cannot be made straight (see the bistellar flips below).

One can pass from Case 1 to Case 2 by the bistellar flip procedure according to Proposition 2 above: The triangle $\langle (1,0,0), (k-3,1,0), (k-3,1,1) \rangle$ in the link of $(0,0,0)$ (see the very first line in the table above) is also contained in the tetrahedron

$$\langle (1,0,0), (k-3,1,0), (k-3,1,1), (k-2,1,1) \rangle$$

where the last vertex is not in the basic vertex link. After adding the edge

$$\langle (0,0,0), (k-2,1,1) \rangle$$

by a geometric bistellar flip (the same for all of its translates) we obtain precisely the generating tetrahedra in Case 2. Note that the various translates of these pair of tetrahedra ever overlap in interior points, so the procedure can be carried out globally. For $k = 4$ the additional edge cannot be made straight since otherwise two edges would cross. It has to be below the straight one, measured in terms of the 3rd component (the z-axis in 3-space). In the other cases the additional edge cannot be made straight either since it has to be even below the other ones.

By one further step we can pass from Case 2 for $n = 4k+1$ to Case 1 for $n = 4k+3$: The triangle $\langle (1,0,0), (k-3,1,0), (k-2,1,1) \rangle$ in the link of $(0,0,0)$ (see the very first line in the table above) is also contained in the tetrahedron

$$\langle (1,0,0), (k-3,1,0), (k-2,1,1), (k-2,1,0) \rangle$$

where the last vertex is not in the basic vertex link. After adding the edge

$$\langle (0,0,0), (k-2,1,0) \rangle$$

by a geometric bistellar flip (the same for all of its translates) we obtain precisely the generating tetrahedra in Case 1 where k is replaced by $k+1$. Again the translates of these pairs of tetraedra never overlap in interior points, so the procedure can be carried out globally. For Â§k=4Â§ the additional edge cannot be made straight since otherwise two edges would cross. It has to be above the straight one, measured in terms of the 3rd component (the z-axis in 3-space). In the other cases the additional edge cannot be made straight either since it has to be even above the other ones.

This proves Theorem 4 by induction on k. □

The method of combining a starting triangulation of the 3-torus (or rather its cyclic orbits) with a large part of a neighborly triangulation of the 3-sphere (or rather its cyclic orbits) in the proof of Theorem 3 suggests the following question:

QUESTION 7: *Are there neighborly triangulations with arbitrarily many vertices also in all the other cases of the triangulated total spaces of sphere bundles (of dimension $d \geq 3$) which were investigated in* [27] *?*

A partial answer is the following: For the case of the sphere itself in any dimension $d \geq 3$ we have the sequence of the cyclic polytopes. For the nontrivial S^2-bundle over S^1 we have the sequence II_n, $n \geq 9$ from [27]. For the product $S^1 \times S^2$ it seems we have the sequence with $n = 2k+8 \geq 10$ vertices defined by the following generating tetrahedra modulo $\mathbb{Z}_n$:

$$0\,1\,2\,5 \quad 0\,1\,2\,6 \quad 0\,1\,4\,6 \qquad 0\,2\,5\,7 \quad 0\,2\,6\,8 \quad \ldots \quad 0\,2\,k+4\,k+6.$$

The last orbit has length $\frac{n}{2} = k+4$ only. The cases $k = 1, 2, 3$ can be found in [28]. The universal covering of any of these triangulations is a triangulated version of the space $\mathbb{R} \times S^2$ which carries one of Thurston's eight geometries on 3-manifolds. Therefore this series is a non-Euclidean analogue of the triangulations in Theorem 4. For the 3-torus we have the sequence in Theorem 3 above. The first case which is not yet covered would be an extrapolation of the standard 19-vertex triangulation of the 2-sphere bundle over the 2-torus, denoted by M_2^4 in [27]. One may ask for neighborly n-vertex triangulations of the same 4-manifold with any given odd $n \geq 21$. Its universal covering would triangulate $\mathbb{R}^2 \times S^2$. In the case of $(2m-1)$-manifolds one may in addition ask whether there is a similar cyclic amalgamation of the standard example with the boundary complex of a cyclic $2m$-polytope as above. The first instance to be considered here is the standard 23-vertex triangulation of M_3^5 in [27].

Neighborly and other triangulations of 3-manifolds carrying a geometric structure were investigated in [29] and [30], in particular such with few vertices. However, infinite families of such with explicit and finite descriptions still seem to be rare.

Additional remark (Double transitivity)
It is well known that the standard cubical tessellation is the only tiling of 3-space by convex polytopes with double transitivity meaning that the translations act transitively on the set of vertices *and*, simultaneously, on the set of tiles. This follows by the same argument as in the proof of Theorem 1 above: There are only five possible prototiles of lattice tilings, and among them only one (the cube) admits a tiling where the translations act transitively also on the vertices. However, from the construction in Theorem 4 above it is not difficult to obtain many other tilings of 3-space with non-convex tiles and with precisely this double transitivity. The idea is to build a prototile by taking precisely one tetrahedron from each of the $\mathbb{Z}^3$-orbits, as described in [37]. This proves the theorem at the end of Section 1 above. More precisely, one can arrange that the prototile is an abstract polyhedral ball in a certain sense to be specified as follows.

We consider lattice tilings by topological 3-balls which are triangulated by topological tetrahedra, possibly with curved edges. The boundary of such a prototile carries the structure of a topological triangulation of the 2-sphere, given by triangles, edges and vertices. However, the intersection of two tiles is allowed to be a union of such faces. We define the natural faces (of any dimension) of a tiling as the minimal non-empty sets which can be represented as a Boolean combination (union, intersection, complement) of tiles. In particular for a tiling of 3-space we obtain the natural facets, natural edges, and natural vertices of the prototile. In particular the closure of each natural facet is a union of original triangles. The structure defined by the natural faces can be used for classifying lattice tilings.

There is no finite classification of lattice tilings of 3-space by [18]. As an extension of this result we obtain infinitely many distinct lattice tilings with double transitivity, by this and by other methods.

5. Equivariant PL curvature

Every abstract simplicial complex can be equipped with a discrete metric in the sense that a length is assigned to each edge in such a way that all possible triangle inequalities are satisfied. This induces a Euclidean metric on each simplex such that adjacent simplices fit together by an isometry. More precisely, every k-dimensional simplex becomes isometric with the convex hull of $k+1$ points in general position in k-dimensional Euclidean space. Consequently, any two points in the same simplex can be joined by a unique shortest geodesic within this simplex. We remark that degenerate k-simplices with a vanishing k-dimensional volume are not admitted here because otherwise the topology induced from the metric would not coincide with the natural topology induced from glueing together the simplices of the abstract simplicial complex.

Definition (PL curvature)
For a triangulated 2-dimensional surface with a discrete metric on it the curvature $K(v)$ at a vertex v is defined by

$$K(v) = 2\pi - \sum_i \alpha_i$$

where the α_i denote the interior angles of the triangles at v. Similarly for a 3-manifold we have the same formula

$$K(e) = 2\pi - \sum_i \beta_i$$

for the curvature along an edge e where the β_i denote the dihedral angles of the tetrahedra at e.

In higher dimensions a similar curvature is defined along the skeleton of codimension 2. In dimension 4 this leads to the Regge functional which is regarded as a discrete analogue of the Hilbert-Einstein functional in General Relativity. For a survey about ideas on the relation between discrete and smooth curvatures see [34].

Definition (equivariant PL curvature)
For a given abstract lattice triangulation of Euclidean d-space we call a discrete metric equivariant if each of the abstract translations of the corresponding lattice acts as an isometry. Such an equivariant metric induces an equivariant PL curvature as a function on the set of orbits of codimension-two faces under the lattice.

The class of equivariant discrete metrics is always non-empty since one can make every simplex into a regular one with one fixed edge length. In this case the full combinatorial automorphism group acts by isometries. By introducing several classes of edge lengths the isometry group can become smaller.

QUESTION 8: *Given an abstract lattice triangulation of d-space, can one always associate an equivariant discrete metric on it such that the PL curvature vanishes along all codimension-two faces ?*

For $d = 2$ the answer ist “yes” because such a triangulation is combinatorially equivalent to the regular tessellation $\{3, 6\}$. If all triangles are made regular then the metric is flat and we have $K(v) = 0$ for all vertices.

For $d = 3$ the answer is "no" according to the following theorem.

Theorem 5 (Vanishing equivariant PL curvature)
Assume we have an abstract lattice triangulation of Euclidean 3*-space, equipped with an equivariant discrete metric by Euclidean simplices such that the group $G \cong \mathbb{Z}^3$ of all translations of the lattice acts isometrically with respect to this metric. Assume further that the equivariant PL curvature $K(e)$ vanishes along all edges e. Then the triangulation is combinatorially unique and, moreover, the metric is affinely equivalent with the one of the standard lattice triangulation of* 3*-space, as defined in Proposition* 1*.*

Conversely it follows from Theorem 4 *that there are infinitely many distinct abstract lattice triangulations of* 3*-space where the equivariant PL curvature cannot vanish.*

The proof of Theorem 5 is obtained in the following three steps:

Step 1. From the condition $K(e) = 0$ at the edges we conclude that around each edge we have a flat metric (except possibly at the endpoints). Now we consider a neighborhood of a vertex v. In that neighborhood let $S_\varepsilon(v)$ denote the distance sphere in distance ε from v. Since outside of v the metric is flat this $S_\varepsilon(v)$ is a round sphere of radius ε, at least piecewise in each tetrahedron. If we consider all these pieces in all the tetrahedra around v then they fit together at certain vertices (resulting from the original edges) with an interior angle sum of 2π. But that implies that $S_\varepsilon(v)$ is globally isometric with a round sphere of radius ε because by assumption the vertex v is not a topological singularity. Hence the neighborhood of v is isometric with an open part of Euclidean 3-space. We remark that this step would break down for $d = 2$ when literally applied to the edges because in this case the distance sphere can be a nontrivial isometric quotient of another sphere which is globally different as a metric space. This can lead to metric cones at the vertices without topological singularities.

Step 2. It follows that the flat Euclidean space is triangulated by Euclidean tetrahedra, and that the Euclidean group acts on it by translations and transitively on the vertices. Consequently, the dual of it is a primitive lattice tiling of Euclidean 3-space by Euclidean convex polyhedra.

Step 3. By a classical theorem of Fedorov-Voronoi [35] there is precisely one primitive lattice tiling of 3-space by convex polyhedra (up to affine transformations), namely, the one where the prototile is a truncated octahedron, see also Theorem 1 above. By duality, our triangulation with the equivariant discrete metric is affinely equivalent with the standard lattice triangulation. □

The standard lattice triangulation of 3-space has only tetrahedra with one right dihedral angle. After affine transformations at least one angle will still be non-acute. It is not even trivial that there are triangulations of Euclidean 3-space at all with only acute tetrahedra [13].

Example 2 (Vanishing PL curvature vs. non-vanishing equivariant PL curvature)
The lattice triangulation of 3-space with the unique 16-vertex basic vertex link (which is nothing but the universal covering of the 17-vertex 3-torus in Proposition 3) cannot carry any equivariant discrete metric with vanishing PL curvature by Theorem 5. However it can be realized in Euclidean 3-space with Euclidean tetrahedra and, consequently, with vanishing PL curvature. In this case the isometry group does no longer act transitively on the vertices. With the integer positions of the vertices there would be crossings of edges: Each edge introduced by a bistellar flip according to the transition from $n = 15$ to $n = 17$ in the proof of Theorem 4 would intersect one of the other edges, compare the proof of Proposition 3. In order to avoid this we have to modify the positions of the vertices. We introduce the function $f(x) = \varepsilon x^2$ for a suitable real number ε with $0 < \varepsilon < \frac{1}{2}$. Then we replace the position of each of the original vertices $(x, y, z) \in \mathbb{Z}^3$ by $(x, y, z - f(x)) \in \mathbb{R}^3$, that is to say, we replace the original lattice by a kind of a "parabolic lattice". In each single "layer" $\{(x, y, z) \mid x_0 \leq x \leq x_0 + 1\}$ for an integer x_0 the transformation can be regarded as an affine transformation. This implies that the standard lattice triangulation with a 14-vertex link is still embedded with the new coordinates since each of its tetrahedra is contained in such a layer. Moreover, with these new positions the required bistellar flips can be

realized in a Euclidean way, and the additional edges between (x, y, z) and $(x+2, y+1, z+1)$ for any $(x, y, z) \in \mathbb{Z}^3$ can be introduced as straight edges. The reason is that this edge meets the triangle $\big\langle (x+1, y, z), (x+1, y+1, z), (x+1, y+1, z+1) \big\rangle$ at an interior point with barycentric coordinates $\frac{1}{2}, \varepsilon, \frac{1}{2} - \varepsilon$. This is independent of x, y, z. Therefore we obtain globally a triangulation of Euclidean 3-space by Euclidean tetrahedra which is still invariant under the translations of a 2-dimensional lattice $\mathbb{Z}^2$ acting on the (y, z)-planes. Hence the PL curvature vanishes everywhere, and combinatorially the triangulation coincides with the non-standard lattice triangulation with the 16-vertex basic vertex link, depicted in Figure 4.

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