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## 1. Introduction

It is known for a long time that there is a close connection between the geometry of hyperbolic space $\mathbb{H}^{n+1}$ of $n+1$ dimensions and the conformal geometry of the $n$-sphere $S^n$, viewed as the sphere at infinity of $\mathbb{H}^{n+1}$. This picture was generalised by Fefferman and Graham in [6] to the case of asymptotically hyperbolic Einstein geometry (alias Poincaré-Einstein geometry) with conformal infinity structure on a boundary space, in relation with conformal invariant theory and the ambient metric construction. In recent years this relationship of the Riemannian interior and its conformal boundary has been studied intensively using spectral and scattering tools [7, 16, 17, 27], and related formal asymptotics [5, 8, 15]. This relationship is also the geometric problem underlying the so-called AdS/CFT correspondence of String Theory [23, 28].

The Einstein condition for a Riemannian metric finds an expedient formulation in terms of conformal tractor calculus. In fact, the prolongation of the Einstein condition for a metric in a conformal class gives naturally rise to the so-called standard tractor bundle $\mathcal{T}$ with tractor connection $\nabla$. Then parallel sections $I$ of $\mathcal{T}$ with respect to $\nabla$ over a conformal manifold $(M, c)$ correspond uniquely to so-called *almost Einstein structures* $\sigma_I$. Almost Einstein structures slightly generalise the notion of Einstein metrics. In fact, an almost Einstein structures gives rise in a unique way to an Einstein metric on an open dense subset of the underlying manifold $M$. However, in general, an almost Einstein structure admits a scale singularity set $\Sigma \neq \emptyset$, which may be viewed as a conformal infinity for the corresponding Einstein metric on the complement $M \setminus \Sigma$. In case the scale singularity $\Sigma$ is a hypersurface in $M$ the corresponding Einstein metric is asymptotically hyperbolic at $\Sigma$. This shows a close relationship between Poincaré-Einstein metrics and almost Einstein structures with hypersurface scale singularity.

Usually a Poincaré-Einstein metric $g_+$ is defined on the interior $M$ of a smooth (compact) manifold $\overline{M}$ with boundary $N = \partial\overline{M}$. It was pointed out in [24], for example, that a manifold $\overline{M}^{n+1}$ of dimension $n+1$ with boundary can be doubled to a smooth manifold $D\overline{M}$ of dimension $n+1$ without boundary. This doubling construction takes two copies of $\overline{M}$ and glues them together at their boundaries $N$ via the identity map. In case $\overline{M}$ admits an *even structure* on the boundary $N$ the smooth structure on the doubling $D\overline{M}$ is natural. Such an even structure on the boundary is induced for example by any even AH metric $g_+$ on the interior $M$. Consequently, any smooth even Poincaré-Einstein space $(\overline{M}^{n+1}, g_+)$ gives rise in a natural way to an almost Einstein manifold $(D\overline{M}, \sigma_I)$ without boundary, but with hypersurface scale singularity $\Sigma$ (cf. Section 5.2 of [10]).

In this article we aim to describe a generalisation of the doubling construction for even asymptotically hyperbolic spaces $(\overline{M}^{n+1}, g_+)$ with boundary. We call this construction the *collapsing $\ell$-sphere product* alias *$S^\ell$-doubling* of $(\overline{M}^{n+1}, g_+)$. In brief, the $S^\ell$-doubling construction works as follows. Let $\ell \geq 0$ be a number, and let $S^\ell$ be the standard smooth $\ell$-sphere. The product $S^\ell \times \overline{M}$ is a $n_\ell$-manifold of dimension $n_\ell := n + \ell + 1$ with boundary $S^\ell \times N$. If we identify the sphere $S^\ell$ at each boundary point $q \in N$ to a single point, then we obtain in canonical way a topological manifold $D_\ell\overline{M}$ of dimension $n_\ell$ without boundary. In addition, if the boundary $N$ is even, then a natural smooth structure is induced on $D_\ell\overline{M}$. The manifold $D_\ell\overline{M}$ has no boundary by construction, but it admits a natural smooth submanifold $N_p$ of codimension $\ell + 1$, which we call the *pole* of $D_\ell\overline{M}$. The pole $N_p$ is the subset, where the sphere $S^\ell$ collapses to single points. This picture gives rise to the notion of the collapsing $\ell$-sphere product. And, if the interior $M$ is equipped with an even AH metric $g_+$, then the

Riemannian product metric $g_{rd} \times g_+$ on the bulk $S^\ell \times M$ of $D_\ell\overline{M}$, where $g_{rd}$ denotes the round standard metric on the factor $S^\ell$, collapses smoothly to a conformal structure on $D_\ell\overline{M}$. We call $D_\ell\overline{M}$ with this conformal structure the $S^\ell$-doubling of the AH metric $g_+$ on the interior of $\overline{M}$.

The $S^\ell$-doubling $D_\ell\overline{M}$ of $(\overline{M}^{n+1}, g_+)$ has some interesting properties. First, we note that the $S^\ell$-doubling of the hyperbolic space $\mathbb{H}^{n+1}$ gives rise to the conformally flat model (= Möbius sphere) $S^{n_\ell}$. In general, the $S^\ell$-doubling $D_\ell\overline{M}$ admits a natural smooth action of the orthogonal group $\mathrm{O}(\ell+1)$ by conformal transformations. The orbit space under this action is the initial AH space $(\overline{M}^{n+1}, g_+)$. If $g_+$ is a Poincaré-Einstein metric, then $D_\ell\overline{M}$ admits multiple almost Einstein structures with hypersurface singularities. These hypersurface singularities intersect exactly at the pole $N_p$. Conformal manifolds with multiple almost Einstein structures and intersecting hypersurface singularities are discussed from a general point of view in [22]. Moreover, in the Poincaré-Einstein case the conformal holonomy group of the $S^\ell$-doubling $D_\ell\overline{M}$ has a decomposable standard representation. Such a decomposable conformal holonomy representation is typical for special Einstein products (cf. [20, 2]). However, in general, the $S^\ell$-doubling is not a special Einstein product at the pole $N_p$! Furthermore, the pole $N_p$ is for any AH metric $g_+$ a totally umbilic submanifold of the $S^\ell$-doubling $D_\ell\overline{M}$. In fact, with respect to certain normal form metrics in the conformal class of $D_\ell\overline{M}$, the pole $N_p$ is minimal. Finally, we remember that the bulk $S^\ell \times M$ of $D_\ell\overline{M}$ with special Einstein product $g_{rd} \times g_+$ (in the AH Einstein case) admits a Ricci-flat ambient metric, which was explicitly constructed in [11]. This ambient metric extends smoothly to the pole $N_p$ and will be presented here in explicit form.

The course of the article is as follows. In Section 2 we recall the doubling construction for a manifold $\overline{M}$ with boundary. In particular, we define the evenness of the boundary and equip the doubling space with a natural smooth structure. In Section 3 and 4 we generalise the doubling construction by the collapsing sphere product $D_\ell\overline{M}$. For asymptotically hyperbolic metrics on the interior of $\overline{M}$ we obtain naturally a conformal structure on $D_\ell\overline{M}$. In Section 5 we show that the collapsing sphere product of the hyperbolic space $\mathbb{H}^{n+1}$ is the conformally flat model space. Section 6 recalls the basic notions of conformal tractor calculus and almost Einstein structures. In particular, the relation with Poincaré-Einstein metrics and conformal holonomy is explained. In Section 7 we discuss basic geometric properties of the $S^\ell$-doubling $D_\ell\overline{M}$. In particular, we will see in the AHE case that the pole is the intersection of scale singularities of certain almost Einstein structures. Section 8 establishes a geometric construction of a Ricci-flat ambient space for $D_\ell\overline{M}$. In the last section we discuss extrinsic curvature properties of the pole $N_p$ in $D_\ell\overline{M}$. In particular, we will verify *strong umbilicity* for the pole in $D_\ell\overline{M}$.

## 2. Even boundaries and smooth doubling

We recall here the doubling construction of a manifold with boundary (cf. e.g. [24]). The evenness of the boundary ensures the existence of a natural smooth structure on the doubling space.

Let $\overline{M}^{n+1}$ be a (topological) manifold of dimension $n+1$ with boundary $N := \partial\overline{M}$ of dimension $n \geq 0$. There exists a well known topological doubling construction for any such $\overline{M}$. We take two copies of $\overline{M}$ and glue them together via the identity map $id_N$ at their boundaries $N$. The resulting set $D\overline{M}$ is equipped with the final topology induced by the gluing. This topology is locally Euclidean and equips the set $D\overline{M}$ naturally with the structure of a topological manifold. We call the manifold $D\overline{M}$ the *(topological) doubling* of $\overline{M}$. The doubling $D\overline{M}$ admits by construction no boundary.

Recall that a smooth (differentiable) structure $\mathcal{D}$ on a manifold $\overline{M}^{n+1}$ with boundary $N$ is a complete (or maximal) atlas of charts (with and without boundary) such that the coordinate transformations on the overlaps for all these charts in $\mathcal{D}$ are smooth. Note that a coordinate transformation between two charts with boundary is called smooth if it is the restriction of a smooth diffeomorphism between some open subsets of $\mathbb{R}^{n+1}$.

Let us consider a smooth structure $\mathcal{D}$ more closely at the boundary $N$. For this purpose, we denote by $\mathbb{R}^{n+1}_+$ the closed subset of $\mathbb{R}^{n+1}$, which consists of all points $x = (x^0, \ldots, x^n)$, whose coordinate $x^0$ is non-negative. The reflection of $\mathbb{R}^{n+1}$ at the hyperplane $\{x^0 = 0\}$, which sends $(x^0, x^1 \ldots, x^n)$ to $(-x^0, x^1 \ldots, x^n)$, is denoted by $I$. Then, for any open subset $U'$ of $\mathbb{R}^{n+1}_+$, we denote by $DU'$ the union (or doubling) $U' \cup I(U')$. The doubling $DU'$ is an open subset of $\mathbb{R}^{n+1}$ and the restriction of the reflection $I$ is an involutive diffeomorphism on $DU'$. This is also true if $U'$ has a boundary, i.e., when the intersection of $U'$ with the hyperplane $\{x^0 = 0\}$ is non-empty. In general, we call a smooth diffeomorphism $\tilde{\alpha}$ between open subsets of $\mathbb{R}^{n+1}$ *even* if $\tilde{\alpha}$ commutes with the reflection $I$, i.e., $I$ is a bijection on the domain and image of $\tilde{\alpha}$ and $I \circ \tilde{\alpha} = \tilde{\alpha} \circ I$.

A chart $\varphi : U \to U'$ for a $C^\infty$-manifold $(\overline{M}, \mathcal{D})$ with boundary is a continuous embedding of an open subset $U \subset \overline{M}$ into $\mathbb{R}^{n+1}_+$ with image $U'$. Let $\psi : V \to V'$ be another chart of $(\overline{M}, \mathcal{D})$ such that $U \cap V \neq \emptyset$. We denote the corresponding coordinate transformation from $\varphi(U \cap V)$ to $\psi(U \cap V)$ by $\alpha := \psi \circ \varphi^{-1}$. By definition of $\mathcal{D}$, this coordinate transformation $\alpha$ is the restriction of some smooth diffeomorphism $\tilde{\alpha} : D(\varphi(U \cap V)) \to D(\psi(U \cap V))$. Now, if there exists a smooth even diffeomorphism $\tilde{\alpha}$, whose restriction to $\varphi(U \cap V)$ is $\alpha$, we call $\alpha : \varphi(U \cap V) \to \psi(U \cap V)$ an *even coordinate transformation* and the charts $\varphi$ and $\psi$ are *evenly related*. Obviously, a coordinate transformation on the overlap of any two charts, which does not intersect the boundary of $\overline{M}$, is even. However, in general, a coordinate transformation on a chart overlap with boundary (in $\overline{M}$) is not even!

**Definition 2.1.** *Let $(\overline{M}, \mathcal{D})$ be a smooth manifold with boundary $N$. Then, we call a complete atlas $\mathcal{D}_{ev}$ of evenly related charts in $\mathcal{D}$ an even smooth structure on $\overline{M}$. We also say in this case that $\mathcal{D}_{ev}$ defines an even structure on the boundary $N$ in $\overline{M}$.*

Note that any smooth atlas of evenly related charts determines in a unique way an even smooth structure $\mathcal{D}_{ev}$ and, of course, a corresponding smooth structure $\mathcal{D}$ on $\overline{M}$.

Now let us come back to the doubling construction for a manifold $\overline{M}$ with boundary $N$. It is a matter of fact that in general a smooth structure $\mathcal{D}$ on $\overline{M}$ does not induce in a natural way a smooth structure on the topological doubling $D\overline{M}$. In particular, there is no natural class of differentiable functions on the topological doubling $D\overline{M}$! This phenomenon is implied by the existence of smooth coordinate transformations on $\overline{M}$, which are not even, in general. However, if $\mathcal{D}_{ev}$ is an even smooth structure on the manifold $\overline{M}$, then the doubling $D\overline{M}$ is equipped with a natural smooth structure $\mathcal{D}^d_{ev}$ induced by $\mathcal{D}_{ev}$.

This natural smooth structure $\mathcal{D}^d_{ev}$ on $D\overline{M}$ can be defined as follows. There are two natural continuous embeddings of $\overline{M}$ into the doubling $D\overline{M}$. We choose one of these embeddings and denote it by $\iota_{\overline{M}}$. Note that $D\overline{M}$ is equipped with a canonical involution $I_{D\overline{M}}$. Then we can define for any open subset $U$ in $\overline{M}$ (considered as subset of $D\overline{M}$ via $\iota_{\overline{M}}$) the doubling $DU := U \cup I_{D\overline{M}}(U)$, which is open in $D\overline{M}$. Now let $\varphi : U \to U'$ be an arbitrary chart of $\mathcal{D}_{ev}$

on $\overline{M}$. The chart $\varphi$ can be doubled (as a map) in the obvious way. We set

$$\begin{array}{rccc} \varphi_d : & DU & \to & DU', \\ & u \in U & \mapsto & \varphi(u), \\ & u \in I_{D\overline{M}}(U) & \mapsto & I(\varphi(I_{D\overline{M}}(u))) \ . \end{array}$$

Lemma 3.1 below states (in a more general situation) that if $\varphi$ and $\psi$ are arbitrary charts in $\mathcal{D}_{ev}$ then the coordinate transformation between the doubled charts $\varphi_d$ and $\psi_d$ is smooth. This shows that the collection of all doubled charts $\varphi_d$ for $\varphi \in \mathcal{D}_{ev}$ establishes a smooth atlas on $D\overline{M}$. We denote the induced smooth structure on $D\overline{M}$ by $\mathcal{D}^d_{ev}$. The definition of $\mathcal{D}^d_{ev}$ does not depend on the choice of the embedding $\iota_{\overline{M}}$. In fact, $\mathcal{D}^d_{ev}$ is in a natural and unique way induced by $\mathcal{D}_{ev}$ on $\overline{M}$.

**Definition 2.2.** *Let $(\overline{M}, \mathcal{D}_{ev})$ be a smooth manifold with even boundary $N$. Then we call the space $D\overline{M}$ with smooth structure $\mathcal{D}^d_{ev}$ the (smooth) doubling of $(\overline{M}, \mathcal{D}_{ev})$.*

Note that with this construction the boundary $N$ of $\overline{M}$ is naturally embedded as a smooth submanifold of codimension 1 in $D\overline{M}$.

## 3. A generalised doubling – the collapsing sphere product

The smooth doubling of the previous section has a useful generalisation. This is the construction of the so-called *collapsing sphere product*. In order to equip a collapsing sphere product of a smooth manifold admitting a boundary with a natural smooth structure, the evenness of the boundary will be (as before for the doubling case) a sufficient precondition.

3.1. **The topology.** Let $(\overline{M}^{n+1}, \mathcal{D}_{ev})$ be a smooth manifold of dimension $n+1$ with even smooth boundary $N := \partial\overline{M}$. We denote by $M := \overline{M} \setminus \partial\overline{M}$ the interior of $\overline{M}$, which is an open submanifold without boundary. It is a matter of fact that any such manifold $\overline{M}$ admits an open neighbourhood $W$ of the boundary $N$, which is diffeomorphic to $[0,\varepsilon) \times N$ for some $\varepsilon > 0$. A corresponding diffeomorphism $\Phi$ can be chosen such that $\Phi$ restricted to the boundary $N$ is the identity. We call such a neighbourhood $W$ a *collar* of $N$.

The sphere $S^\ell$ of dimension $\ell \geq 0$ is a closed (= compact, without boundary) smooth manifold. If $\ell = 0$ then we think of the sphere as two separate points $\{-1, 1\}$, which is a smooth manifold of dimension 0. Given a sphere $S^\ell$ we can form the product space $S^\ell \times \overline{M}^{n+1}$ of dimension $n_\ell := n + \ell + 1$ with smooth product structure and (even) boundary $S^\ell \times N$. The interior of $S^\ell \times \overline{M}$ is given by $S^\ell \times M$. Note that there exists a collar of the boundary $S^\ell \times N$ in $S^\ell \times \overline{M}$, which is diffeomorphic to $S^\ell \times [0,\varepsilon) \times N$.

Let us consider for a moment just a product of the form $S^\ell \times [0,\varepsilon)$ with $\varepsilon > 0$. As a topological space we can identify the boundary $S^\ell \times \{0\}$ of that product to a single point. The resulting quotient space with final topology is homeomorphic to an Euclidean ball $B^{\ell+1}_\varepsilon$ of radius $\varepsilon$ in $\mathbb{R}^{\ell+1}$. In fact, we have an explicit map for this identification with a ball by

$$\begin{array}{rccc} i : & S^\ell \times [0,\varepsilon) & \to & B^{\ell+1}_\varepsilon, \\ & (y, r) & \mapsto & r \cdot y, \end{array}$$

where we understand $y \in S^\ell$ as a point of the unit sphere in $\mathbb{R}^{\ell+1}$.

More generally, let (as before) $\overline{M}$ be a manifold with boundary $N$ and let $\Phi$ be a diffeomorphism of some collar $W$ of $N$ onto $[0,\varepsilon)\times N$ (which restricts to the identity on $N$). Then we can identify $S^\ell \times W$ in $S^\ell \times \overline{M}$ via the map $\tilde{\Phi} := (i \times id_N)\circ(id_{S^\ell}\times\Phi)$ to a topological space, which is homeomorphic to $B^{\ell+1}_\varepsilon \times N$. In fact, this construction for the collar $W$ via the identification $\tilde{\Phi}$ can be trivially extended to an identification for $S^\ell\times\overline{M}$. We denote the resulting topological space with natural projection by

$$\Lambda : S^\ell \times \overline{M} \to D_\ell\overline{M}\ .$$

Note that the natural inclusion of $S^\ell \times M$ and the image of $B^{\ell+1}_\varepsilon \times N$ by $\tilde{\Phi}^{-1}$ cover the space $D_\ell\overline{M}$. The corresponding inverse maps serve as charts (mapping into other manifolds) for the quotient space $D_\ell\overline{M}$. This shows that $D_\ell\overline{M}$ is a topological manifold. Note that the definition of the final topology on $D_\ell\overline{M}$ does not depend on the choice of the collar $W$ with diffeomorphism $\Phi$ (and, of course, also not on the even smooth structure $\mathcal{D}_{ev}$). In fact, one can easily see that the coordinate transformation between any two charts $B^{\ell+1}_{\varepsilon_1}\times N$ and $B^{\ell+1}_{\varepsilon_2}\times N$ around the identified points in $D_\ell\overline{M}$, which are induced by homeomorphisms $\Phi_1$ and $\Phi_2$ of some collars in $\overline{M}$, is a homeomorphism. We call $D_\ell\overline{M}$ the *(topological) collapsing $\ell$-sphere product* of $\overline{M}$. If $\overline{M}$ is a compact space, then $D_\ell\overline{M}$ is closed. Note that the set of all (non-trivially) identified points in $D_\ell\overline{M}$ is homeomorphic to the boundary manifold $N$ of $\overline{M}$. We call this set the *pole* of $D_\ell\overline{M}$ and denote it by $N_p$ (or simply just $N$ again). The pole $N_p$ is closed in $D_\ell\overline{M}$. The complement $D_\ell\overline{M}\setminus N_p$ is called the *bulk* (or interior) of $D_\ell\overline{M}$. Also note that by construction $D_0\overline{M} = D\overline{M}$ as topological manifolds, i.e., the collapsing 0-sphere product is just the doubling as introduced in the previous section.

Before we come to the construction of a natural smooth structure on $D_\ell\overline{M}$, we want to remark that $D_\ell\overline{M}$ admits certain naturally induced continuous functions for any $\ell\geq 0$. For this, let $f:\overline{M}\to\mathbb{R}$ be an arbitrary continuous function. The function $f$ lifts uniquely to a function $\tilde{f}$ on $S^\ell\times\overline{M}$, which is constant along the $S^\ell$-fibres. This function $\tilde{f}$, in turn, passes down via the identification $\Lambda$ to a uniquely and well defined function $f_\ell$ on $D_\ell\overline{M}$. The function $f_\ell$ is continuous by construction.

3.2. **The smooth structure.** So far we have constructed from a (smooth) manifold $\overline{M}$ with boundary $N$ a topological manifold $D_\ell\overline{M}$ for any $\ell\geq 0$. We want to show now that the even smooth structure $\mathcal{D}_{ev}$ on $\overline{M}$ induces in a natural way a smooth structure $\mathcal{D}^\ell_{ev}$ on the collapsing $\ell$-sphere product of $\overline{M}$. The following Lemma 3.1 is the key observation for this construction.

**Lemma 3.1.** *Let $U', V'$ be open subsets of $\mathbb{R}^n$ and $\varepsilon_1,\varepsilon_2>0$. Let $\ell\geq 0$ be an arbitrary integer. If*

$$\alpha = (\alpha^0,\alpha^a):\quad [0,\varepsilon_1)\times U' \quad\to\quad [0,\varepsilon_2)\times V'$$

*is an even smooth diffeomorphism (between sets with boundaries), then the map*

$$\begin{aligned} \alpha_\ell:\ \ B^{\ell+1}_{\varepsilon_1}\times U' &\ \to\ B^{\ell+1}_{\varepsilon_2}\times V', \\ (r\cdot y, u) &\ \mapsto\ (\ \alpha^0(r,u)\cdot y\ ,\ \alpha^a(r,u)\ ), \qquad y\in S^\ell\subset\mathbb{R}^{\ell+1}, \end{aligned}$$

*is a smooth diffeomorphism between open subsets in $\mathbb{R}^{n_\ell}$.*

We need for the proof of Lemma 3.1 the following Lemmata 3.2 and 3.3.

**Lemma 3.2.** *Let $\gamma : (-\varepsilon, \varepsilon) \to \mathbb{R}$ be an even smooth function, i.e., $\gamma(x) = \gamma(-x)$ for all $x \in (-\varepsilon, \varepsilon)$. Then the function $\Gamma(t) := \gamma(\sqrt{t})$, $t \in [0, \varepsilon^2)$, is smooth on $(0, \varepsilon^2)$ and its $k$th derivative $\Gamma^{(k)}$ is continuous on $[0, \varepsilon^2)$ for all $k \geq 0$.*

Proof. By definition, the function $\Gamma(t)$ is smooth on $(0, \varepsilon^2)$. We have to show that all derivatives $\Gamma^{(k)}(t)$, $k \geq 0$, converge to some finite value for $t \to 0$.

For this, we observe that the evenness of $\gamma$ implies that all odd derivatives $\gamma^{(2k+1)}(x)$, $k \in \mathbb{Z}_{\geq 0}$, vanish for $x = 0$. We set $\gamma_s(x) := \Sigma_{i=0}^{s} \gamma^{(i)}(0) \cdot x^i$ for the Taylor serious expansion of $\gamma$ at 0 to order $s \geq 0$. Obviously, the functions $\gamma_s$ are smooth in the variable $x^2$ for all $s$. Hence the limit of $\gamma_s^{(k)}(\sqrt{t})$ for $t \to 0$ exists and is finite for any $k$. Moreover, l'Hospital's rule implies that $\big(\ \gamma(\sqrt{t}) - \gamma_s(\sqrt{t})\ \big)^{(k)} \to 0$ for $t \to 0$ when $s \geq k$. This shows that $\lim_{t \to 0} \gamma^{(k)}(\sqrt{t})$ exists and is finite for any $k$. □

Note that, by construction, we have $\gamma(x) = \Gamma(x^2)$ for all $x \in (-\varepsilon, \varepsilon)$, i.e., Lemma 3.2 says that an even smooth function in a variable $x$ can be considered as a smooth function in $x^2$.

We also remark that there is a slight generalisation of Lemma 3.2. For this, let us assume that $U'$ is a subset of $\mathbb{R}^n$ with coordinates $u = (u^1, \ldots, u^n)$ and let $\gamma : (-\varepsilon, \varepsilon) \times U' \to \mathbb{R}$ be a smooth function, which is even in the coordinate of the interval $(-\varepsilon, \varepsilon)$. Then we can define the function $\Gamma(t, u) := \gamma(\sqrt{t}, u)$. Obviously, the function $\Gamma(t, u)$ is smooth on $(0, \varepsilon^2) \times U'$. Moreover, the limits of all $\Gamma^{(k)}(t, u)$, $k \geq 0$, for $t \to 0$ are smooth in the parameters $u = (u^1, \ldots, u^n)$. This follows, since all partial derivatives of $\Gamma(t, u)$ with respect to the coordinates $(u^1, \ldots, u^n)$ are even when considered as functions in $t$. Then Lemma 3.2 applies, which says that the limits for $t \to 0$ of the partial $u$-derivatives of the $\Gamma^{(k)}(t, u)$ exist for $k \geq 0$. These limits represent the partial $u$-derivatives of the $\Gamma^{(k)}(0, u)$.

**Lemma 3.3.** *Let $\gamma : [0, \varepsilon) \times U' \to \mathbb{R}$ be the restriction of a smooth function on an open subset $(-\varepsilon, \varepsilon) \times U'$ of $\mathbb{R}^{n+1}$, which is even in the coordinate of the interval $(-\varepsilon, \varepsilon)$. Then, for any integer $\ell \geq 0$, the function*

$$\begin{array}{rccc} \gamma_\ell : & B_\varepsilon^{\ell+1} \times U' & \to & \mathbb{R}, \\ & (x, u) & \mapsto & \gamma(|x|, u), \end{array}$$

*on the $(\ell + 1)$-dimensional ball $B_\varepsilon^{\ell+1}$ of radius $\varepsilon$ in $\mathbb{R}^{\ell+1}$ times $U'$ is smooth.*

Proof. We have $\gamma_\ell(x, u) = \Gamma(|x|^2, u)$, where $\Gamma$ is defined as in Lemma 3.2 (for fixed $u$). The function $|x|^2$ is smooth on $\mathbb{R}^{\ell+1}$. By Lemma 3.2, the function $\Gamma$ is smooth as well and all partial derivatives $\Gamma^{(k)}$ are continuous on $[0, \varepsilon^2) \times U'$. Moreover, by the above remark following Lemma 3.2, we know that $\Gamma$ and its partial derivatives $\Gamma^{(k)}$ on $[0, \varepsilon^2) \times U'$ depend smoothly on the $u$-coordinates. Then, it follows immediately by application of the chain rule that any partial derivative (with respect to the $x$- and $u$-coordinates) of the superposition $\Gamma(|x|^2, u)$ is continuously differentiable. This proves the smoothness of $\gamma_\ell$. □

Proof of Lemma 3.1. Obviously, the map $\alpha_\ell$ is a homeomorphism. We have to show that $\alpha_\ell$ is smooth. The smoothness of the inverse $(\alpha_\ell)^{-1}$ follows then by the same argument, since $\alpha^{-1}$ is also even and smooth.

Let us denote the standard coordinates of $B^{\ell+1}_{\varepsilon_1}$ in $\mathbb{R}^{\ell+1}$ by $x = (x^1, \dots, x^{\ell+1})$ and let $u = (u^1, \dots, u^n)$ be the coordinates of $U' \subset \mathbb{R}^n$. Then we can write

$$\begin{array}{rccl} \alpha_\ell : & B^{\ell+1}_{\varepsilon_1} \times U' & \to & B^{\ell+1}_{\varepsilon_2} \times V', \\ & (x,u) & \mapsto & \left( \frac{\alpha^0(r,u)}{r} \cdot x \ , \ \alpha^a(r,u) \ \right), \end{array}$$

where $r := |x|$. By assumption, $\alpha$ is the restriction of an even smooth diffeomorphism, which we denote by $\tilde{\alpha} = (\tilde{\alpha}^0, \tilde{\alpha}^a)$. This means that the functions $\frac{\tilde{\alpha}^0(r,u)}{r}$ and $\tilde{\alpha}^a(r,u)$, $a = 1, \dots, n$, are all even in the variable $r$. By l'Hospital's rule again, it follows that $\frac{\tilde{\alpha}^0(r,u)}{r}$ is smooth as a function in $r$ on the interval $(-\varepsilon_1, \varepsilon_1)$. We can conclude that $\frac{\tilde{\alpha}^0(r,u)}{r}$ and $\tilde{\alpha}^a(r,u)$, $a = 1, \dots, n$, are smooth functions in $r$ and the parameters $(u^1, \dots, u^n)$ of $U'$. Then it follows by Lemma 3.3 that $\frac{\alpha^0(|x|,u)}{|x|}$ and $\alpha^a(|x|,u)$, $a = 1, \dots, n$, are smooth functions on $B^{\ell+1}_{\varepsilon_1} \times U'$, i.e., $\alpha_\ell$ is a smooth map. □

Now let us consider the collapsing $\ell$-sphere product $D_\ell\overline{M}$ of $(\overline{M}, \mathcal{D}_{ev})$ with natural projection $\Lambda : S^\ell \times \overline{M} \to D_\ell\overline{M}$. For $U \subset \overline{M}$, we set $U_\ell := \Lambda(S^\ell \times U)$. Then a coordinate chart in $\mathcal{D}_{ev}$ of the form $\varphi = (\varphi^0, \varphi^a) : U \to [0, \varepsilon) \times U'$ induces in a natural way a chart

$$(1) \qquad \begin{array}{rccl} \varphi_\ell : & U_\ell & \to & B^{\ell+1}_{\varepsilon} \times U', \\ & \overline{(s,m)} & \mapsto & (\varphi^0(m)s, \varphi^a(m)), \end{array}$$

where $\overline{(s,m)}$ denotes the class of $(s,m) \in S^\ell \times U$ via the identification $\Lambda$. It follows directly from Lemma 3.1 that the coordinate transformation between any charts $\varphi_\ell$ and $\psi_\ell$ of the form (1) is smooth. Moreover, note that all the charts of the form (1) cover an open neighbourhood of the pole $N_p$ in $D_\ell\overline{M}$ and they are all compatible with the smooth product structure on the bulk $S^\ell \times M$ of $D_\ell\overline{M}$. This shows that the charts $\varphi_\ell$ of the form (1) together with the smooth product structure on the bulk induce a smooth structure $\mathcal{D}^\ell_{ev}$ on $D_\ell\overline{M}$. Note that the construction of $\mathcal{D}^\ell_{ev}$ does not depend on the choice of charts of the special form (1). In fact, any chart $\varphi \in \mathcal{D}_{ev}$ gives rise in a natural way to a chart $\varphi_\ell$, which then belongs to $\mathcal{D}^\ell_{ev}$.

**Definition 3.4.** *Let $(\overline{M}, \mathcal{D}_{ev})$ be a smooth manifold with even boundary $N$. Then we call $(D_\ell\overline{M}, \mathcal{D}^\ell_{ev})$ the (smooth) collapsing $\ell$-sphere product of $\overline{M}$ with pole $N_p$.*

Note that $(D_0\overline{M}, \mathcal{D}^0_{ev})$ is naturally diffeomorphic to the smooth doubling $(D\overline{M}, \mathcal{D}^d_{ev})$. Also recall that for any $\ell \geq 0$ the pole $N_p$ is a closed subset in $D_\ell\overline{M}$. In fact, $N_p$ is a smooth submanifold in $(D_\ell\overline{M}, \mathcal{D}^\ell_{ev})$ of codimension $\ell + 1$. The bulk $D_\ell\overline{M} \setminus N_p$ is an open smooth submanifold of $D_\ell\overline{M}$ with respect to $\mathcal{D}^\ell_{ev}$, which is diffeomorphic to the product space $S^\ell \times M$. A further remark is expedient. Let $V^{\kappa+1}$ be a $(\kappa+1)$-dimensional subspace of $\mathbb{R}^{\ell+1}$ with $0 \leq \kappa \leq \ell$. The intersection of $V$ with the unit sphere $S^\ell$ in $\mathbb{R}^{\ell+1}$ defines an embedding of the unit $\kappa$-sphere $S^\kappa \subset V$ into $S^\ell$. This embedding induces in an obvious way a natural smooth embedding $\iota_{\kappa,\ell}$ of $D_\kappa\overline{M}$ into $D_\ell\overline{M}$ such that $\iota_{\kappa,\ell}$ is the identity map on the pole $N_p$. In particular, the doubling $(D\overline{M}, \mathcal{D}^d_{ev})$ is contained in any collapsing $\ell$-sphere product $(D_\ell\overline{M}, \mathcal{D}^\ell_{ev})$ as a smooth submanifold of codimension $\ell$.

## 4. The collapsing sphere product of asymptotically hyperbolic spaces

So far the collapsing $\ell$-sphere product is a standard construction, which generates from a smooth manifold of dimension $n+1$ with even boundary a smooth manifold of dimension $n_\ell$ without boundary. In this section we show that this construction admits a natural usage in conformal geometry. In fact, we will see that the metric product of a round $\ell$-sphere $(S^\ell, g_{rd})$ with $(M, g_+)$ collapses to a smooth conformal structure on $D_\ell\overline{M}$ if the metric $g_+$ on the interior $M$ of $\overline{M}$ is asymptotically hyperbolic and even at the boundary $N$.

4.1. **The even structure of an AH metric.** Let $(\overline{M}^{n+1}, \mathcal{D})$ be a smooth manifold of dimension $n+1 \geq 1$ with boundary $N$. A defining function $r$ for the boundary $N$ is a smooth non-negative function on $\overline{M}$ such that $r = 0$ and $dr \neq 0$ on $N$ (and otherwise $r > 0$ on the interior $M$). Now let $g_+$ be a smooth Riemannian metric on the interior $M$ of $\overline{M}$ such that $\overline{g} = r^2 g_+$ extends smoothly as a metric to $\overline{M}$ for some defining function $r$ of the boundary $N$. Then the restriction of $\overline{g}$ to $TN$ rescales conformally upon changing the defining function $r$, and so defines invariantly a smooth conformal class $c := [\overline{g}|_{TN}]$ of metrics on the boundary $N$. The conformal class (or structure) $c$ on $N$ is called a *conformal infinity* of $g_+$ on $M$. If, in addition, $\overline{M}$ is a compact space, then we say that the metric $g_+$ is *conformally compact* (cf. e.g. [13]).

In the special case when $|dr|^2_{\overline{g}} = 1$ on $N$ the sectional curvature of $g_+$ is asymptotically constant 1 at each boundary point in $N$, and the metric $g_+$ is called *asymptotically hyperbolic.* Note that this definition does not depend on the choice of the defining function $r$. In particular, a metric $g_+$ is asymptotically hyperbolic if it satisfies the normalised Einstein condition

$$Ric(g_+) \ = \ -n g_+ \ .$$

In this case we call $g_+$ an *AH Einstein metric.* Alternatively, we say that $(M, g_+)$ is a *Poincaré-Einstein space.*

Let us assume in the following that $g_+$ is an asymptotically hyperbolic metric on the interior $M$ of $\overline{M}$. A defining function $r$ of the boundary $N$ defines for any small $\varepsilon > 0$ a collar $N_\varepsilon$ of $N$ in $\overline{M}$. This collar $N_\varepsilon$ is by definition the preimage of $[0, \varepsilon)$ with respect to $r$ near the boundary. Note that the gradient $grad^{\overline{g}}(r)$ of $r$ is defined with respect to the metric $\overline{g} = r^2 g_+$ and $grad^{\overline{g}}(r)$ does not vanish on $N_\varepsilon$ for small $\varepsilon$. This feature of $r$ allows us to define uniquely an explicit diffeomorphism of $N_\varepsilon$ with $[0, \varepsilon) \times N$ via the flow of the integral curves of $grad^{\overline{g}}(r)$ on $N_\varepsilon$ (cf. [13]). In addition, if we choose a smooth chart $\varphi : U \to U'$ of the boundary $N$, then we obtain via $r$ in an obvious and unique way a chart $\varphi_r : U_\varepsilon \to [0, \varepsilon) \times U'$ of $\mathcal{D}$ on $\overline{M}$. The collar $N_\varepsilon$ can be covered by such charts $\varphi_r$, i.e., the smooth structure $\mathcal{D}$ of $\overline{M}$ at the boundary $N$ is determined by a smooth defining function $r$ and the smooth structure of $N$.

A smooth metric in the conformal class $c$ on $N$ is induced by (many) different defining functions $r$ of the boundary $N$. However, the condition $|dr|^2_{\overline{g}} \equiv 1$ on a collar $N_\varepsilon$ of $N$ in $\overline{M}$ determines $r$ uniquely for a given metric $\overline{g}|_{TN}$ in the conformal infinity $c$ on $N$. Such a defining function $r$ is smooth and exists for any smooth metric $g_0$ in $c$ (cf. e.g. Lemma 2.1 of [13]). We call this $r$ a *special defining function* for $g_0$ on the boundary $N$. Now it is straightforward to see that the asymptotically hyperbolic metric $g_+$ is given on a collar $N_\varepsilon \cong [0, \varepsilon) \times N$ with respect to a special defining function $r$ by

$$r^{-2} \left( \, dr^2 \ + \ g_r \, \right) , \qquad (2)$$

where $g_r$ is a smooth family of (smooth) metrics on the boundary $N$ with $g_0 = r^2 g_+|_{TN}$ (cf. [13]). We say that $g_+$ is given by (2) in *normal form* with respect to the special defining function $r$. In the following, we call a smooth defining function $r$ on $\overline{M}$ *generalised special* for the boundary $N$ if $r > 0$ on the interior $M$ and $|dr|^2_{\bar{g}} \equiv 1$ on a collar $N_\varepsilon$ for some $\varepsilon > 0$, i.e., the defining function $r$ is special at least on a small collar of $N$ in $\overline{M}$.

There exists also a natural notion of *evenness* for an asymptotically hyperbolic metric $g_+$. To explain this notion, let us consider (the normal form (2) of) $g_+$ on a collar $[0, \varepsilon) \times N$ with respect to some special defining function $r$. We call $g_+$ *even* if there exists a smooth metric $h$ on the manifold $(-\varepsilon, \varepsilon) \times N$ such that $h|_{[0,\varepsilon)\times N} = r^2 g_+$ and the involution $r \mapsto -r$ is an isometry of $h$. This definition of evenness for $g_+$ does not depend on the choice of the special defining function $r$ (cf. Section 4 of [8]).

Furthermore, it is also true that if $g_+$ is even then any two special defining functions $r_1$ and $r_2$ of $N$ are *evenly related* in the following sense. Let $\varphi : U \to U'$ be any chart of the smooth structure on $N$ (which is induced by $\mathcal{D}$). Then we have corresponding charts $\varphi_{r_1}$ and $\varphi_{r_2}$ in $\mathcal{D}$ on $\overline{M}$ (as explained above). It is a matter of fact that the coordinate change for these charts $\varphi_{r_1}$ and $\varphi_{r_2}$ is an even diffeomorphism of open subsets in $\mathbb{R}^{n+1}_+$ (cf. Section 4 of [8]). This implies that the charts of the form $\varphi_r$ for the special defining functions $r$ of $g_+$ define an even smooth structure $\mathcal{D}_{g_+}$ on $\overline{M}$. The even smooth structure $\mathcal{D}_{g_+}$ on $\overline{M}$ is uniquely determined in this way by $g_+$.

**Definition 4.1.** *Let $g_+$ be an even asymptotically hyperbolic metric on the interior $M$ of a smooth manifold $(\overline{M}, \mathcal{D})$ with boundary $N$. Then we call $\mathcal{D}_{g_+}$ the natural even smooth structure on $\overline{M}$ induced by $g_+$.*

4.2. **The conformal structure on $D_\ell\overline{M}$.** Now let us consider the collapsing $\ell$-sphere product $D_\ell\overline{M}$, $\ell \geq 0$, for some manifold $(\overline{M}, \mathcal{D})$ with boundary $N$ and even asymptotically hyperbolic metric $g_+$ on the interior $M$. In this situation we always assume in the following that $\overline{M}$ is equipped with $\mathcal{D}_{g_+}$ and $D_\ell\overline{M}$ carries the smooth structure $\mathcal{D}^\ell_{g_+}$ induced by $\mathcal{D}_{g_+}$. Also remember that any continuous function $f$ on $\overline{M}$ induces via $\Lambda$ a continuous function $f_\ell$ on $D_\ell\overline{M}$. In particular, a smooth defining function $r$ for $N$ in $\overline{M}$ induces a continuous function $r_\ell$ on $D_\ell\overline{M}$. (Often it will be convenient to denote $r_\ell$ simply by $r$ again.) Note that $r_\ell$ is not smooth with respect to $\mathcal{D}^\ell_{g_+}$ at the pole $N_p$.

**Lemma 4.2.** *Let $r$ be a generalised special defining function for the boundary $N$ in $\overline{M}$ and let $r_\ell$ be the induced function on $D_\ell\overline{M}$. Then the function $(r_\ell)^2$ is smooth on $(D_\ell\overline{M}, \mathcal{D}^\ell_{g_+})$.*

Proof. We have to prove the smoothness of $r_\ell^2$ at the pole $N_p$. For this, let $\varphi : U \to U'$ be some smooth chart on $N$. In combination with $r$ and some small $\varepsilon$, the chart $\varphi$ gives rise to a chart $\varphi_r : U_\varepsilon \to [0, \varepsilon) \times U'$ in $\mathcal{D}_{g_+}$. Then the chart $\varphi_r$ induces a chart $(\varphi_r)_\ell : (U_\varepsilon)_\ell \to B^{\ell+1}_\varepsilon \times U'$ of $\mathcal{D}^\ell_{g_+}$ as given by (1) of Section 3. Now, by construction, the pullback of $r_\ell$ on $D_\ell\overline{M}$ to $B^{\ell+1}_\varepsilon \times U'$ via $(\varphi_r)^{-1}_\ell$ is given by $|x|$, where $x = (x^1, \ldots, x^{\ell+1})$ are the standard coordinates in $B^{\ell+1}_\varepsilon$. Since $|x|^2$ is smooth on $B^{\ell+1}_\varepsilon \times U'$, we can conclude that $(r_\ell)^2$ is smooth on $(U_\varepsilon)_\ell$ in $D_\ell\overline{M}$ with respect to $\mathcal{D}^\ell_{g_+}$. This argument works for any smooth chart $\varphi$ on the boundary $N$, which proves that $(r_\ell)^2$ is smooth at the pole $N_p$. □

The bulk of $D_\ell\overline{M}$ is the open smooth submanifold $S^\ell \times M$. The submanifold $S^\ell \times M$ is equipped with the Riemannian product metric $g^\ell_+ := g_{rd} \times g_+$, where $g_{rd}$ denotes the round metric of constant sectional curvature 1 on $S^\ell$. We want to demonstrate now that the conformal class of this metric $g_+^\ell$ on the bulk $S^\ell \times M$ extends smoothly to the collapsing $\ell$-sphere product $D_\ell\overline{M}$ for any $\ell \geq 0$. For this purpose, let $r$ be a generalised special defining function for the boundary $N$ in $\overline{M}$. Then we have the smooth metric $r_\ell^2 \cdot g_+^\ell$ on the bulk $S^\ell \times M$ of $D_\ell\overline{M}$, which belongs to the conformal class of $g_+^\ell$.

**Lemma 4.3.** *Let $r$ be a generalised special defining function for the boundary $N$ in $\overline{M}$. Then the metric $r_\ell^2 \cdot g_+^\ell$ on the bulk $S^\ell \times M$ extends to a smooth metric $\overline{g}$ on $(D_\ell\overline{M}, \mathcal{D}_{g_+}^\ell)$ for any $\ell \geq 0$.*

Proof. Let $N_\varepsilon$ be a collar of $N$ in $\overline{M}$ with respect to $r$ and some $\varepsilon > 0$ such that $|dr|_{r^2 g_+} \equiv 1$ on $N_\varepsilon$. We set $N_{\varepsilon,\ell} := \Lambda(S^\ell \times N_\varepsilon)$, which is a tubular collar of the pole $N_p$ in $D_\ell\overline{M}$. For convenience, we write $r = r_\ell$. The defining function $r$ induces a natural diffeomorphism $N_{\varepsilon,\ell} \cong B_\varepsilon^{\ell+1} \times N$ and the metric $r^2 \cdot g_+^\ell$ is given on $N_{\varepsilon,\ell} \smallsetminus N_p$ by

$$dr^2 \; + \; r^2 g_{rd} \; + \; g_r \; . \tag{3}$$

Note that $dr^2 + r^2 g_{rd}$ is the flat Euclidean metric on $B_\varepsilon^{\ell+1} \smallsetminus \{0\}$, which extends smoothly to the origin. In our situation, this shows that the symmetric 2-tensor $dr^2 + r^2 g_{rd}$ extends smoothly to the solid cylinder $B_\varepsilon^{\ell+1} \times N$. It is also clear that $g_r$ is a continuous tensor on $N_{\varepsilon,\ell}$. Since $dr^2 + r^2 g_{rd}$ on $B_\varepsilon^{\ell+1}$ and $g_{r=0}$ on $N_p$ are non-degenerate, we can conclude that $r^2 \cdot g_+^\ell$ is a continuous non-degenerate and symmetric 2-tensor on $D_\ell\overline{M}$. It remains to show that the symmetric 2-tensor $g_r$ is smooth at the pole $N_p$.

To see the latter point, we note that the definition of evenness for $g_+$ implies directly that $r^2 g_+$ on $\overline{M}$ is the restriction of some smooth metric $h$ on the doubling $(D\overline{M}, \mathcal{D}_{g_+}^d)$, which is invariant under the smooth involution $I_{D\overline{M}}$. This means that with respect to any chart $\varphi \in \mathcal{D}_{g_+}$ of the form $\varphi : U \to [0, \varepsilon) \times U'$ all the coefficients of $r^2 g_+$ are restrictions of certain smooth functions on $(-\varepsilon, \varepsilon) \times U'$, which are even in the coordinate of the interval $(-\varepsilon, \varepsilon)$. Application of the Lemmata 3.2 and 3.3 to these coefficient functions shows that the symmetric tensor $g_r$ depends smoothly on $r^2$. Since $r^2 = r_\ell^2$ is a smooth function on $D_\ell\overline{M}$ (cf. Lemma 4.2), it follows that $g_r$ is smooth at the pole in $(D_\ell\overline{M}, \mathcal{D}_{g_+}^\ell)$. □

The conformal class of $\overline{g}$ on $D_\ell\overline{M}$ is a smooth extension of the conformal class of $g_+^\ell$ on the bulk $S^\ell \times M$, defined via a generalised special defining function $r$. Since the bulk is dense in $D_\ell\overline{M}$, this observation implies immediately that the conformal class of $\overline{g}$ on $D_\ell\overline{M}$ does not depend on the choice of the generalised special defining function $r$. We conclude that the asymptotically hyperbolic metric $g_+$ on the interior $M$ of $\overline{M}$ induces naturally for any $\ell \geq 0$ a smooth conformal structure on the collapsing $\ell$-sphere product $(D_\ell\overline{M}, \mathcal{D}_{g_+}^\ell)$ such that the natural embedding of (the bulk) $S^\ell \times M$ with metric $g_+^\ell = g_{rd} \times g_+$ into $D_\ell\overline{M}$ is a smooth conformal map. We denote the conformal class of $\overline{g}$ on $D_\ell\overline{M}$ by $c_\ell[g_+]$.

**Definition 4.4.** *Let $g_+$ be an even asymptotically hyperbolic metric on the interior $M$ of a manifold $\overline{M}$ with boundary $N$. Then we call $(D_\ell\overline{M}, c_\ell[g_+])$ the collapsing $\ell$-sphere product of $g_+$. And we call the pole $N_p$ with restricted conformal structure $c_\ell[g_+]|_{N_p}$ a conformal infinity of $(D_\ell\overline{M}, c_\ell[g_+])$ of codimension $\ell + 1$.*

In shorter notation, we simply say $(D_\ell\overline{M}, c_\ell[g_+])$ is the $S^\ell$-*doubling* of $g_+$. Note that the pole $N_p$ with restricted conformal structure $c_\ell[g_+]|_{N_p}$ is conformally equivalent to the conformal infinity of $g_+$ on the boundary $N$ of $\overline{M}$.

The proof of Lemma 4.3 also provides the local normal form (3) for a class of representatives in $c_\ell[g_+]$ around the pole $N_p$. This normal form is characteristic for the collapsing sphere product construction.

**Theorem 4.5.** *Let $g_t$, $t \in [0,\varepsilon)$, be any family of metrics on a manifold $N$ such that $g^\ell := (dr^2 + r^2 g_{rd}) + g_r$ is a smooth metric on the product space $B_\varepsilon^{\ell+1} \times N$, where $r$ denotes the radial coordinate of the ball $B_\varepsilon^{\ell+1}$. Then $g_+ := t^{-2}(dt^2 + g_t)$ is a smooth even asymptotically hyperbolic metric on $(0,\varepsilon) \times N$, and the metric $g^\ell$ represents the conformal class $c_\ell[g_+]$ of the $S^\ell$-doubling of $g_+$ on the interior of $\overline{M} = [0,\varepsilon) \times N$.*

Proof. It follows directly from the assumptions that the restriction of the metric $g^\ell = (dr^2 + r^2 g_{rd}) + g_r$ to $B_\varepsilon^1 \times N$, where $B_\varepsilon^1$ is some straight line in $B_\varepsilon^{\ell+1}$ through the origin, is a smooth metric. This shows that $dt^2 + g_t$ is a smooth metric on $(-\varepsilon,\varepsilon) \times N$, which implies that $g_+ = t^{-2}(dt^2 + g_t)$ on $(0,\varepsilon) \times N$ is a smooth asymptotically hyperbolic metric. Also note that the map $t \mapsto -t$ on $(-\varepsilon,\varepsilon) \times N$ is by construction an isometry for $dt^2 + g_t$. We conclude that $g_+$ is even.

Now, in turn, we can produce the $S^\ell$-doubling of the even asymptotically hyperbolic metric $g_+ = t^{-2}(dt^2 + g_t)$ on the interior of $\overline{M} = [0,\varepsilon) \times N$. By definition of $c_\ell[g_+]$ on $B_\varepsilon^{\ell+1} \times N$, it is clear that the metric $g^\ell = (dr^2 + r^2 g_{rd}) + g_r$ is a representative of $c_\ell[g_+]$. □

In conformal geometry, a standard problem is the existence and construction of compact Poincaré-Einstein spaces $\overline{M}$ for a given conformal infinity structure (cf. [14, 18, 4, 1]). In this respect, we can think of the $S^\ell$-doubling as a solution of an analogous problem for a given pole $N$ with conformal infinity structure in higher codimension $\ell + 1$. This view will be justified in Section 7 when we show that the $S^\ell$-doubling of an AH Einstein metric $g_+$ is an (multiple) *almost Einstein space*. This problem in higher codimension seems to be intimately related to the theory of totally umbilic submanifolds in almost Einstein spaces (cf. Section 9).

## 5. The model example – $D_\ell\overline{\mathbb{H}}^{n+1}$

The Poincaré hyperbolic disk is a standard model for asymptotically hyperbolic geometry. To describe this model briefly, let us denote by $\overline{B}_1^{n+1}$ the closed unit ball in $\mathbb{R}^{n+1}$. Its boundary is the unit $n$-sphere $S^n$ and the corresponding interior is the open unit ball $B_1^{n+1}$. The hyperbolic metric $g_+$ is given on $B_1^{n+1}$ by

$$g_+ \;=\; \frac{4}{(1-|x|^2)^2}\, \Sigma_{i=1}^{n+1}(dx^i)^2 \;,$$

where $|x|^2$ denotes the square of the Euclidean norm on $\mathbb{R}^{n+1}$. A special defining function $r$ for the boundary $S^n$ in $\overline{B}_1^{n+1}$ with respect to $g_+$ on the collar $\overline{B}_1^{n+1} \setminus \{0\}$ is given by $r = 2\cdot\frac{1-|x|}{1+|x|}$. Writing $\Sigma_{i=1}^{n+1}(dx^i)^2$ in polar coordinates and expressing everything in terms of $r$ then gives

$$g_+ \;=\; r^{-2}\left(dr^2 + (1-(r/2)^2)^2 g_{rd}\right), \tag{4}$$

where $g_{rd}$ is the round metric on $S^n$ of scalar curvature $n(n-1)$ (cf. [13]). In particular, we see that the boundary conformal structure of $g_+$ induced on $S^n$ is equivalent to the Möbius sphere. In the following, we will denote the hyperbolic model with boundary simply by $\overline{\mathbb{H}}^{n+1}$.

By the results and constructions of the previous sections, we know how to construct the collapsing $\ell$-sphere products $D_\ell\overline{\mathbb{H}}^{n+1}$ of the hyperbolic space $\overline{\mathbb{H}}^{n+1}$ for any $\ell, n \geq 0$. We want to argue here that $D_\ell\overline{\mathbb{H}}^{n+1}$ (with conformal structure $c_\ell[g_+]$) is conformally equivalent to the sphere $S^{n_\ell}$ of dimension $n_\ell := n+\ell+1$ with standard metric $g_{rd}$, i.e., $D_\ell\overline{\mathbb{H}}^{n+1}$ is a realisation of the $n_\ell$-dimensional Möbius sphere.

To start with, let $\gamma$ be an arbitrary loop (= closed path) in $D_\ell\overline{\mathbb{H}}^{n+1}$. A tubular collar of the pole $N_p = S^n$ in $D_\ell\overline{\mathbb{H}}^{n+1}$ is homeomorphic to $B_1^{\ell+1} \times S^n$. In case $\gamma$ intersects the pole $N_p$, the radial coordinate $r$ of $\gamma$ on the ball $B_1^{\ell+1}$ becomes zero (at some time). Obviously, if $\ell \geq 1$ we can deform the loop $\gamma$ homotopically, so that the radial coordinate $r$ of the deformation on $B_1^{\ell+1} \times S^n$ does not vanish to any time. This shows that any loop in $D_\ell\overline{\mathbb{H}}^{n+1}$ is homotopic to a loop $\tilde{\gamma}$, which does not intersect the pole. Now, if $\ell \geq 2$ the bulk of $D_\ell\overline{\mathbb{H}}^{n+1}$, which is homeomorphic to $S^\ell \times B_1^{n+1}$, is simply connected. We can conclude that $\tilde{\gamma}$, and hence $\gamma$, are homotopic to the trivial loop.

In case $\ell = 1$ we can still argue that $\tilde{\gamma}$ on the bulk $S^1 \times B_1^{n+1}$ is null-homotopic. In fact, this follows, since $\tilde{\gamma}$ on $S^1 \times B_1^{n+1}$ is homotopic to a loop $\hat{\gamma}$, which is constant on the factor $B_1^{n+1}$, and which lies in a tube $B_1^2 \times S^n$ of the pole. Such a loop is null-homotopic, since $B_1^2$ is simply connected. It follows that $D_\ell\overline{\mathbb{H}}^{n+1}$ is a simply connected space for any $\ell \geq 1$. Note that, if $\ell = 0$, then $D_\ell\overline{\mathbb{H}}^{n+1}$ is homeomorphic to the doubling of a closed unit ball in $\mathbb{R}^{n+1}$, i.e., $D_0\overline{\mathbb{H}}^{n+1}$ is homeomorphic to the $(n+1)$-sphere, which is simply connected except for $n = 0$. Altogether, yet we know that $D_\ell\overline{\mathbb{H}}^{n+1}$ is a simply connected and closed space for any $n_\ell \geq 2$.

Next we want to show that the conformal structure $c_\ell[g_+]$ on the $S^\ell$-doubling $D_\ell\overline{\mathbb{H}}^{n+1}$ is flat. This is clear for $n + \ell \leq 1$. So let us assume $n_\ell \geq 3$. To prove the statement in this case, it is sufficient to observe that the Riemannian product metric $g_{rd} \times g_+$, which lies by construction in the conformal class $c_\ell[g_+]$ on the bulk $S^\ell \times \mathbb{H}^{n+1}$ of $D_\ell\overline{\mathbb{H}}^{n+1}$, is conformally flat. There are various ways to see this. For us, it is most natural to argue here that $g_{rd} \times g_+$ is a special Einstein product (cf. Section 7), which implies that the conformal holonomy group of $g_{rd} \times g_+$ on $S^\ell \times \mathbb{H}^{n+1}$ is the product of the Riemannian holonomy groups of the space- and timelike metric cones of the factors $g_+$ and $g_{rd}$ (cf. [20]). Since the factors $g_+$ and $g_{rd}$ both have constant sectional curvature $\pm 1$, the metric cones are flat and their holonomy groups are trivial. Hence the conformal holonomy group of $g_{rd} \times g_+$ is trivial, which is only possible if the conformal curvature (i.e. the Weyl and Cotton tensor) vanishes. This shows the conformal flatness of $c_\ell[g_+]$ on the bulk $S^\ell \times \mathbb{H}^{n+1}$, hence on the $S^\ell$-doubling $D_\ell\overline{\mathbb{H}}^{n+1}$.

It is well known from the theory of Riemannian surfaces that any simply connected closed surface with smooth conformal structure is equivalent to the Riemannian sphere. The analogue statement in any higher dimension $m \geq 3$ is true as well. One way to see this uses the so-called *developing map* as follows (cf. e.g. [25]). It is a matter of fact that any simply connected, conformally flat smooth manifold $X^m$ of dimension $m \geq 3$ admits a conformal immersion (or development map) $\iota$ into the $m$-dimensional Möbius sphere $L\mathbb{P}^m \cong S^m$. This immersion $\iota$ is uniquely determined up to a conformal transformation (with an element of the Möbius group) of $S^m$. Now, in case $X$ is a closed conformally flat manifold, the developing map $\iota$ is a smooth covering of the sphere $S^m$ (with standard smooth structure). Thus, since $S^m$ is simply connected, the developing map $\iota$ is a conformal diffeomorphism. This argument applies to $D_\ell\overline{\mathbb{H}}^{n+1}$ for $n_\ell \geq 2$, which shows that $D_\ell\overline{\mathbb{H}}^{n+1}$ is conformally equivalent to the Möbius sphere $L\mathbb{P}^{n_\ell}$ and, in particular, diffeomorphic to the standard sphere $S^{n_\ell}$. (For $n_\ell = 1$ it is anyway clear that $D_0\overline{\mathbb{H}}^1$ is a circle $S^1$.)

**Theorem 5.1.** *The $S^\ell$-doubling space $D_\ell\overline{\mathbb{H}}^{n+1}$ of the hyperbolic metric $g_+$ on $\mathbb{H}^{n+1}$ is conformally equivalent to the round $n_\ell$-sphere $(S^{n_\ell}, g_{rd})$ for any $n, \ell \geq 0$.*

We aim to present an explicit conformal diffeomorphism between $D_\ell\overline{\mathbb{H}}^{n+1}$ and the Möbius sphere $L\mathbb{P}^{n_\ell}$ (for any $n, \ell \geq 0$). For this purpose, we remind of the ambient model for the Möbius sphere $L\mathbb{P}^n$ (with $n \geq 0$), which is also the basic construction for the treatments of Section 8. So let us consider the Minkowski space $\mathbb{R}^{1,n+1}$ of dimension $n+2$ with standard metric

$$\langle x, \tilde{x}\rangle_{1,n+1} = -x^0\tilde{x}^0 \ + \ \Sigma_{i=1}^{n+1} x^i\tilde{x}^i,$$

$x, \tilde{x} \in \mathbb{R}^{n+2}$, of signature $(1, n+1)$. Here the standard coordinates of a point $x$ in $\mathbb{R}^{1,n+1}$ are denoted by $(x^0, x^1, \cdots, x^{n+1})$, where $x^0$ is the timelike coordinate. The projective null cone $L\mathbb{P}^n$ of $\mathbb{R}^{1,n+1}$ is diffeomorphic to the sphere $S^n$ and the Minkowski metric $\langle\cdot,\cdot\rangle_{1,n+1}$ induces the standard conformally flat structure on $L\mathbb{P}^n$. The upper half $L_+$ of the null cone in $\mathbb{R}^{1,n+1}$ is in a natural way a $\mathbb{R}_+$-ray bundle over $L\mathbb{P}^n$ with projection $\pi$. Any smooth section of this $\mathbb{R}_+$-ray bundle induces a conformal embedding of $L\mathbb{P}^n \cong S^n$ into the Minkowski space $\mathbb{R}^{1,n+1}$, and thus represents a metric in the conformal class of $L\mathbb{P}^n$. For example, the intersection of $L_+$ with the hyperplane $\{x^0 = 1\}$ represents a round metric of constant sectional curvature 1 on $L\mathbb{P}^n$.

The hyperbolic space $\mathbb{H}^{n+1}$ can be realised in $\mathbb{R}^{1,n+1}$ as well. In fact, the hyperquadric $\{x \in \mathbb{R}^{n+2} \ : \ \langle x, x\rangle_{1,n+1} = -1\}$ with induced metric is isometric to the Poincaré disk and the projective null cone $L\mathbb{P}^n$ can naturally be seen as its boundary. A defining function for the boundary $L\mathbb{P}^n$ with respect to $g_+$ on $\mathbb{H}^{n+1}$ in $\mathbb{R}^{1,n+1}$ is then given by

$$r = x^0 - \sqrt{(x^0)^2 - 1}\ .$$

(Note that $\sqrt{(x^0)^2 - 1} = \sqrt{\Sigma_{i=1}^{n+1}(x^i)^2}$ is the spacelike radial coordinate of $\mathbb{R}^{1,n+1}$ restricted to $\mathbb{H}^{n+1}$.) One can easily check that $r$ is the special defining function on $\mathbb{H}^{n+1}$ realised in $\mathbb{R}^{1,n+1}$, which belongs to the round standard metric $g_{rd}$ on the projective null cone $L\mathbb{P}^n$.

As we mentioned already above, the cone $\mathbb{R}_+ \times \mathbb{H}^{n+1}$ with metric $-dt^2 + t^2 g_+$ is flat. In fact, this metric cone is realised in $\mathbb{R}^{1,n+1}$ by the upper half timelike cone $\{x \in \mathbb{R}^{n+2} \mid \langle x, x\rangle_{1,n+1} < 0, x^0 > 0\}$. Also note that the cone $\mathbb{R}_+ \times S^\ell$, $\ell \geq 0$, with metric $ds^2 + s^2 g_{rd}$ is isometric to the Euclidean space $\mathbb{R}^{\ell+1} \setminus \{0\}$ (with removed origin), and the round sphere $(S^\ell, g_{rd})$ is isometrically embedded as the unit sphere into the cone $\mathbb{R}^{\ell+1} \setminus \{0\}$.

From our discussion so far, we obtain now an obvious isometric embedding of the Riemannian product space $(S^\ell, g_{rd}) \times (\mathbb{H}^{n+1}, g_+)$ into $\mathbb{R}^{1,n_\ell+1}$ for any $n, \ell \geq 0$. In fact, we can simply take the product of the embeddings of $\mathbb{H}^{n+1}$ into $\mathbb{R}^{1,n+1}$ and $(S^\ell, g_{rd})$ into $\mathbb{R}^{\ell+1}$:

$$\begin{array}{cccc} \iota : & S^\ell \times \mathbb{H}^{n+1} & \to & \mathbb{R}^{1,n_\ell+1}, \\ & (y^1, \ldots, y^{\ell+1}) \times (x^0, \ldots, x^{n+1}) & \mapsto & (x^0, \ldots, x^{n+1}, y^1, \ldots, y^{\ell+1})\ , \end{array}$$

where $\langle x, x\rangle_{1,n+1} = -1$ and $\langle y, y\rangle_{\ell+1} = 1$. The image $\iota(S^\ell \times \mathbb{H}^{n+1})$ is then a submanifold of the upper half null cone $L_+$ in $\mathbb{R}^{1,n_\ell+1}$ and a smooth section of $\pi : L_+ \to L\mathbb{P}^{n_\ell}$ restricted to the base space $\pi \circ \iota(S^\ell \times \mathbb{H}^{n+1})$. The base $\pi \circ \iota(S^\ell \times \mathbb{H}^{n+1})$ is $L\mathbb{P}^{n_\ell}$ minus the set $L\mathbb{P}^n = \{[x :$

$y] \in L\mathbb{P}^{n_\ell} \mid y = 0\}$. The latter space is the projective null cone of $\mathbb{R}^{1,n+1}$ and the boundary of $\mathbb{H}^{n+1}$ (i.e. the pole of the collapsing $\ell$-sphere product of $\overline{\mathbb{H}}^{n+1}$ as we see next).

Now, by rescaling the section $\iota(S^\ell \times \mathbb{H}^{n+1})$ of $L_+$ over the base $\pi \circ \iota(S^\ell \times \mathbb{H}^{n+1})$ with the factor $\frac{1}{x^0}$ such that the $x^0$-coordinate is normed to 1, we obtain a conformal embedding $\tilde{\iota}$ of $S^\ell \times \mathbb{H}^{n+1}$ into the round sphere $(S^{n_\ell}, g_{rd})$ (minus $L\mathbb{P}^n$). Note that the special defining function $r = x^0 - \sqrt{(x^0)^2 - 1}$ gives rise to an identification of $S^\ell \times (\mathbb{H}^{n+1} \smallsetminus \{x^0 = 1\})$ with $(B_1^{\ell+1} \smallsetminus \{0\}) \times S^n$. Via this identification the embedding $\tilde{\iota}$ is explicitly given by

$$\begin{array}{lccc} \tilde{\iota}: & (B_1^{\ell+1} \smallsetminus \{0\}) \times S^n & \to & \mathbb{R}^{1,n_\ell+1}, \\ & (r, y^1, \cdots, y^{\ell+1}) \times (x^1, \cdots, x^{n+1}) & \mapsto & (1, \frac{1-r^2}{1+r^2} x^i, \frac{2r}{1+r^2} y^j) \end{array}$$

with $|x|^2 = |y|^2 = 1$. It is obvious that the map $\tilde{\iota}$ extends smoothly to the singular set $\{0\} \times S^n$ in $B_1^{\ell+1} \times S^n$, and its inverse is also smooth at the image of $\{0\} \times S^n$. This shows that the conformal embedding $\tilde{\iota}$ of the bulk $S^\ell \times \mathbb{H}^{n+1}$ into the Möbius sphere $L\mathbb{P}^{n_\ell} \cong (S^{n_\ell}, [g_{rd}])$ admits a smooth extension to the collapsing $\ell$-sphere product $D_\ell \overline{\mathbb{H}}^{n+1}$ of $\overline{\mathbb{H}}^{n+1}$ such that the image of the pole $N_p$ is $L\mathbb{P}^n = \{[x : y] \in L\mathbb{P}^{n_\ell} \mid y = 0\}$ in $L\mathbb{P}^{n_\ell}$. This extension with image $L\mathbb{P}^{n_\ell}$ is smoothly invertible. Thus we have another (constructive) proof of the conformal equivalence of $D_\ell \overline{\mathbb{H}}^{n+1}$ and the Möbius sphere $L\mathbb{P}^{n_\ell}$ for any $n, \ell \geq 0$.

## 6. Tractor calculus and almost Einstein structures

We introduce here some standard constructions from conformal tractor calculus (cf. [26, 3]). Tractor calculus provides us with the right setting for a discussion of almost Einstein structures, which are closely related to Poincaré-Einstein metrics (in the case of negative scalar curvature). We will see later indeed that the collapsing sphere product of a Poincaré-Einstein space admits by construction multiple almost Einstein structures.

6.1. **Tractor calculus.** Let $M^n$ be a smooth manifold of dimension $n \geq 3$. Recall that a Riemannian conformal structure on $M$ is a smooth $\mathbb{R}_+$-ray subbundle $\mathcal{Q} \subset S^2T^*M$, whose fibre over $p \in M$ consists of conformally related positive definite metrics at $p$. Smooth sections of $\mathcal{Q}$ are metrics on $M$, which define an equivalence class $c = [g]$ of conformally related metrics $g$ on $M$. The principal $\mathbb{R}_+$-bundle $\pi : \mathcal{Q} \to M$ induces for any representation $t \in \mathbb{R}_+ \mapsto t^{-w/2} \in \mathrm{End}(\mathbb{R})$, $w \in \mathbb{R}$, a natural line bundle $\mathcal{E}[w]$ over $M$, which we call the conformal density bundle of weight $w$. Then, if $\mathcal{V}$ is any vector bundle of tensors on $M$, we denote by $\mathcal{V}[w] := \mathcal{V} \otimes \mathcal{E}[w]$ the corresponding tensor bundle of conformal weight $w \in \mathbb{R}$. We write $\boldsymbol{g}$ for the conformal metric on $(M, c)$, that is the tautological section of $S^2T^*M[2] := S^2T^*M \otimes \mathcal{E}[2]$ determined by $\mathcal{Q}$. Note that $\mathcal{E}[w]$, $w \in \mathbb{R}$, is trivialised by any metric $g \in c$.

Now let $(M^n, c)$ be a Riemannian conformal manifold of dimension $n \geq 3$. For a given choice of metric $g \in c$, the tractor bundle $\mathcal{T}$ of the conformal manifold $(M, c)$, may be identified with the direct sum

$$\mathcal{T} \ \cong_g \ \mathbb{R} \oplus TM \oplus \mathbb{R} \ ,$$

i.e., a section $T$ in $\mathcal{T}$ consists of a triple $(a, \psi, b)$, where $a, b$ are real functions and $\psi$ is a vector field on $M$. Under a conformal rescaling of $g$ to $\tilde{g} = e^{2\varphi} \cdot g$ with respect to a smooth function $\varphi$ the triple $(a, \psi, b)$ transforms by

$$(\tilde{a}, \tilde{\psi}, \tilde{b}) \ = \ (\ e^{\varphi} a \ , \ e^{-\varphi} \cdot (\psi + a \cdot grad^g(\varphi)) \ , \ e^{-\varphi} \cdot (b - d\varphi(\psi) - \frac{a}{2} \| grad^g \varphi \|_g^2) \ ) \ , \tag{5}$$

i.e., the metric $\tilde{g}$ gives rise to a different isomorphism for the tractor bundle $\mathcal{T}$ with $\mathbb{R}\oplus TM\oplus\mathbb{R}$. The transformation rule (5) shows that $\mathcal{T}$ invariantly admits a composition structure

$$\mathcal{T} \;=\; \mathcal{E}[1] \;\subset\!\!\!+\; TM[-1] \;\subset\!\!\!+\; \mathcal{E}[-1]\ ;$$

$\mathcal{E}[-1]$ may be naturally identified with a subbundle of $\mathcal{T}$ and $TM[-1]$ is a subbundle of the quotient bundle $\mathcal{T}/\mathcal{E}[-1]$. We denote the natural projection from $\mathcal{T}$ to $\mathcal{E}[1]$ by $\Pi$.

The tractor bundle $\mathcal{T}$ carries an invariant metric $\langle\cdot,\cdot\rangle_{\mathcal{T}}$ of signature $(1,n{+}1)$ and a canonical invariant connection $\nabla$, which preserves this tractor metric $\langle\cdot,\cdot\rangle_{\mathcal{T}}$. The metric $\langle\cdot,\cdot\rangle_{\mathcal{T}}$ is given for $T,\hat{T}\in\mathcal{T}$ with respect to a metric $g\in c$ by

$$\langle T,\hat{T}\rangle_{\mathcal{T}} \;=\; a\hat{b}+\hat{a}b \;+\; g(\psi,\hat{\psi})\ .$$

The tractor connection $\nabla$ is given by

$$\nabla_X \begin{pmatrix} a \\ \psi \\ b \end{pmatrix} = \begin{pmatrix} X(a)-g(X,\psi) \\ \nabla^g_X\psi + b{\cdot}X - a{\cdot}\mathsf{P}^g(X) \\ X(b)+\mathsf{P}^g(X,\psi) \end{pmatrix}$$

for any $X\in TM$, where $\nabla^g$ denotes the Levi-Civita connection of $g$ on $TM$, and

$$\mathsf{P}^g = \frac{1}{n-2}\left(\frac{scal^g}{2(n-1)} - Ric^g\right)$$

is the Schouten tensor in terms of the Ricci tensor $Ric^g$ and the scalar curvature $scal^g$ of $g$. With $\mathsf{P}^g(X)$ we denote the vector in $TM$, which is dual to $\mathsf{P}^g(X,\cdot)$ via $g$.

6.2. **Almost Einstein structures.** As discussed in [3], there exists a conformally covariant second order differential operator $\mathcal{D}:\Gamma(\mathcal{E}[1])\to\Gamma(\mathcal{T})$ for densities of conformal weight 1. This differential operator $\mathcal{D}$ acts with respect to a metric $g\in c$ on real functions $s\in C^\infty(M)$ by

$$\mathcal{D}^g s \;=\; \begin{pmatrix} s \\ grad^g(s) \\ \Box^g s \end{pmatrix},$$

where $\Box^g := -\frac{1}{n}(\Delta^g - tr_g\mathsf{P}^g)$ with Laplacian $\Delta^g s = tr_g\nabla^g ds$. For an invariant construction of $\mathcal{D}$ see [9]. It is a matter of fact that for densities $\sigma\in\Gamma(\mathcal{E}[1])$ the equation $\nabla\mathcal{D}\sigma=0$ is equivalent to

$$\text{trace-free part of}\,(\nabla^g ds \;-\; s{\cdot}\mathsf{P}^g) \;=\; 0\ , \tag{6}$$

where $s$ corresponds to $\sigma$ via $g\in c$. In turn, it is true that if a tractor $T=(s,\psi,b)$ satisfies $\nabla T=0$ then the component $s\in C^\infty(M)$ of $T$ with respect to $g\in c$ satisfies (6) and $\mathcal{D}^g s = T$.

It is also well known that a solution $s$ of (6) without zeros on $M$ has the property that the rescaled metric $\tilde{g}=s^{-2}{\cdot}g$ is an Einstein metric, i.e., $\tilde{g}$ satisfies $Ric^{\tilde{g}} = \frac{scal^{\tilde{g}}}{n}{\cdot}\tilde{g}$. Equivalently, this means that the metric $\sigma^{-2}\boldsymbol{g}$ is Einstein. On the other hand, if a metric $s^{-2}{\cdot}g$ in the conformal class $c$ on $M$ is Einstein then $s$ satisfies (6) with respect to $g$. However, in general one has to expect that a solution of (6) admits zeros on $M$.

**Definition 6.1.** *Let $(M^n,c)$, $n\geq 3$, be a conformal manifold with standard tractor bundle $\mathcal{T}$.*

(1) *We call $(M^n, c)$ an almost Einstein space if a $\nabla$-parallel standard tractor $I \neq 0$ exists.*
(2) *If $I \neq 0$ in $\Gamma(\mathcal{T})$ is $\nabla$-parallel, then we call $\sigma = \Pi(I)$ an almost Einstein structure of $(M, c)$. Accordingly, if $g \in c$ is a choice of metric, then we call a non-trivial solution $s \in C^\infty(M)$ of (6) an almost Einstein structure of $(M, c)$.*
(3) *If $I \neq 0$ in $\Gamma(\mathcal{T})$ is $\nabla$-parallel, then we denote the zero set of the almost Einstein structure $\sigma = \Pi(I)$ by $\Sigma(\sigma)$ (resp. $\Sigma(I)$). We call $\Sigma(\sigma)$ the scale singularity set of $\sigma$ (resp. $I$).*
(4) *We denote by $\mathcal{P}(\mathcal{T})$ the subspace of $\Gamma(\mathcal{T})$, which consists of all $\nabla$-parallel standard tractors $I$ on $(M^n, c)$. Note that $\mathcal{P}(\mathcal{T})$ is a vector space of finite dimension $\mathcal{N} \leq n+2$, on which the tractor metric $\langle \cdot, \cdot \rangle_{\mathcal{T}}$ naturally induces a symmetric bilinear form.*
(5) *Let $\mathcal{S}$ be an Euclidean subspace of $\mathcal{P}(\mathcal{T})$. Then we set $\Sigma(\mathcal{S}) = \bigcap_{I \in \mathcal{S}} \Sigma(I)$.*

On an almost Einstein manifold $(M, c, I)$ (or $(M, c, \sigma)$ with $\sigma = \Pi(I)$) we shall write $S(I)$ (or $S(\sigma)$) as a shorthand for $-\langle I, I \rangle_{\mathcal{T}}$. This may be viewed as scalar curvature quantity for the structure, since off the singularity set $\Sigma(I)$ we have $S(I) = \frac{scal^g}{n(n-1)}$ for the metric $g = \sigma^{-2}\boldsymbol{g}$.

**Theorem 6.2.** [10] *Let $(M, c, I)$ be an almost Einstein space of Riemannian signature with $\sigma = \Pi(I)$ and $S(\sigma) = -\langle I, I \rangle_{\mathcal{T}}$. If $S(\sigma) > 0$ then $\Sigma(\sigma)$ is empty and $(M, \sigma^{-2}\boldsymbol{g})$ is Einstein with positive scalar curvature; if $S(\sigma) = 0$ then $\Sigma(\sigma)$ is either empty or consists of isolated points and $(M \smallsetminus \Sigma(\sigma), \sigma^{-2}\boldsymbol{g})$ is Ricci-flat; if $S(\sigma) < 0$ then the scale singularity set $\Sigma(\sigma)$ is either empty or else is a smooth hypersurface, and $(M \smallsetminus \Sigma(\sigma), \sigma^{-2}\boldsymbol{g})$ is Einstein of negative scalar curvature.*

6.3. **The relation to Poincaré-Einstein geometry.** The case of an almost Einstein structure $\sigma = \Pi(I)$ on $(M^n, c)$ with $S(\sigma) = -1$ and $\Sigma(\sigma) \neq \emptyset$ is closely related to the geometry of Poincaré-Einstein spaces as follows. If $g \in c$ is a metric on $M$, then the corresponding solution $s$ of (6) to $\sigma$ vanishes exactly on $\Sigma(\sigma)$ and we have $ds \neq 0$ for any $p \in \Sigma(\sigma)$. Hence the real function $s$ has positive and negative values on $M$. We set $\overline{M_+}(s) := \{x \in M | s(x) \geq 0\}$. By construction, the space $\overline{M_+}(s)$ is a smooth manifold with boundary $\Sigma(\sigma)$, for which $s$ serves as a defining function. The interior $M_+(s)$ of $\overline{M_+}(s)$ is the open subset of $M$, where $s$ is positive. Note that, since $\langle \mathcal{D}\sigma, \mathcal{D}\sigma \rangle_{\mathcal{T}} = 1$, we have $|grad^g s|_g = 1$ on the hypersurface $\Sigma(\sigma)$. This shows that the metric $g_+ = s^{-2}g$ is asymptotically hyperbolic and, moreover, satisfies $Ric^{g_+} = -(n-1)g_+$ on the interior $M_+(s)$. Thus

$$(\overline{M_+}(s), g_+) \tag{7}$$

is a Poincaré-Einstein space as introduced in Section 4. In case $M$ is a closed space the Poincaré-Einstein metric $g_+$ on $M_+(s)$ is conformally compact. Note that the construction can also be applied to the almost Einstein structure $-s$ on $(M, g)$. In general, the manifolds $M_+(s)$ and $M_+(-s)$ with metric $g_+ = s^{-2}g$ are not isometric, even not locally near the boundary $\Sigma(\sigma)$. In the following, if the Poincaré-Einstein metric $g_+$ is even (at the hypersurface singularity $\Sigma(\sigma)$), then we call the corresponding almost Einstein structure $\sigma$ and the parallel standard tractor $I = \mathcal{D}\sigma$ *even* as well.

6.4. **Conformal holonomy.** We remark here that almost Einstein structures can be detected by conformal holonomy. In fact, let $(M, c)$ be a connected conformal manifold, and note that $\nabla$ on $\mathcal{T}$ over $(M, c)$ admits a uniquely defined holonomy group, which we denote by $Hol(\mathcal{T})$ (cf. [20, 21, 2]). Since the tractor connection $\nabla$ is canonical, we call $Hol(\mathcal{T})$ the *conformal holonomy group* of $(M, c)$. In general, $Hol(\mathcal{T})$ is a subgroup of the structure group $\mathrm{O}(1, n+1)$ of $\mathcal{T}$, and $\mathbb{R}^{1,n+1}$ is the standard representation space of $Hol(\mathcal{T})$. (Note that a concrete realisation of a representation of $Hol(\mathcal{T})$ on $\mathbb{R}^{1,n+1}$ needs the choice of a tractor frame at a point of $M$.) The conformal holonomy representation on $\mathbb{R}^{1,n+1}$ has the following useful property, which is a direct consequence of the very definition of holonomy.

**Lemma 6.3.** *Let $(M^n, c)$ be a conformal manifold of dimension $n \geq 3$ with conformal holonomy group $Hol(\mathcal{T})$. Let $\mathcal{P}(\mathcal{T}) \subset \Gamma(\mathcal{T})$ be the space of $\nabla$-parallel standard tractors on $(M, c)$, and let $Fix(\mathbb{R}^{1,n+1})$ be the space of fixed vectors under that action of $Hol(\mathcal{T})$ on $\mathbb{R}^{1,n+1}$ (with respect to the choice of some tractor frame). Then there exists a uniquely determined vector space isomorphism*

$$\begin{array}{ccc} \mathcal{P}(\mathcal{T}) & \cong & Fix(\mathbb{R}^{1,n+1}), \\ I & \leftrightarrow & v(I) \ . \end{array}$$

In particular, Lemma 6.3 give rise to a 1-to-1 correspondence for almost Einstein structures on $(M, c)$ and the space $Fix(\mathbb{R}^{1,n+1}) \setminus \{0\}$ of non-trivial $Hol(\mathcal{T})$-fixed vectors in $\mathbb{R}^{1,n+1}$.

6.5. **An example.** The Möbius sphere $L\mathbb{P}^n \cong (S^n, [g_{rd}])$, $n \geq 3$, is simply connected and conformally flat. This implies that the conformal holonomy of $L\mathbb{P}^n$ is trivial, and the space $\mathcal{P}(\mathcal{T})$ of $\nabla$-parallel standard tractors on $L\mathbb{P}^n$ has dimension $\mathcal{N} = n + 2$. There is a clear geometric picture for the almost Einstein structures on the Möbius sphere $L\mathbb{P}^n$ via the ambient model (cf. e.g. [10]). We simply describe here a certain set of almost Einstein structure $\sigma_i$, $i = 1, \ldots, n+1$, with negative scalar curvature $S(\sigma_i) = -1$ on the round sphere $(S^n, g_{rd})$. For this purpose, let $S^n$ be embedded as the unit sphere in the Euclidean space $\mathbb{R}^{n+1}$. We denote the standard coordinate functions on $\mathbb{R}^{n+1}$ by $(x^1, \ldots, x^{n+1})$, and by $s_i$, $i = 1, \ldots, n + 1$, we denote the restriction of $x^i$ to the sphere $S^n$. One easily checks that every $s_i$, $i = 1, \ldots, n+1$, is an almost Einstein structure on the round sphere $(S^n, g_{rd})$, and the corresponding parallel tractors $I_i = \mathcal{D}^{g_{rd}}(s_i)$ have norm $-1$ with respect the tractor metric $\langle \cdot, \cdot \rangle_{\mathcal{T}}$. Obviously, the singularity set $\Sigma(I_i)$, $i = 1, \ldots, n + 1$, is the equator, which is the intersection of $S^n$ with the hyperplane $\{x^i = 0\}$. Then each metric $s_i^{-2} \cdot g_{rd}$, $i = 1, \ldots, n + 1$, on $S^n \setminus \Sigma(I_i)$ represents two copies of the hyperbolic metric $g_+$ on $\mathbb{H}^n$. Also note that, if $\mathcal{J} \subset \{1, \ldots, n + 1\}$ is any collection of $\ell+1$ indices $\{i_1, \ldots, i_{\ell+1}\}$, then the metric $\left(\Sigma_{j=1}^{\ell+1} s_{i_j}^2\right)^{-1} \cdot g_{rd}$ on $S^n \setminus \{\Sigma_{j=1}^{\ell+1} s_{i_j}^2 = 0\}$ is isometric to the product metric $g_{rd} \times g_+$ on $S^\ell \times \mathbb{H}^{n-\ell}$. This again demonstrates that the sphere $(S^n, [g_{rd}])$ is conformally equivalent to the collapsing $\ell$-sphere product of the hyperbolic space $\overline{\mathbb{H}}^{n-\ell}$.

## 7. Almost Einstein structures on $D_\ell \overline{M}$

In this section we describe characteristic properties of the $S^\ell$-doubling $D_\ell \overline{M}$ of an asymptotically hyperbolic space. For example, the $S^\ell$-doubling $D_\ell \overline{M}$ admits a natural action of the orthogonal group $\mathrm{O}(\ell + 1)$ by conformal transformations. If $\overline{M}$ is a Poincaré-Einstein space, the $S^\ell$-doubling admits almost Einstein structures, whose scale singularities intersect at the pole $N_p$. The latter feature is explained by the conformal holonomy.

7.1. **Special Einstein products.** First, recall that in dimensions 1 and 2 a canonical tractor connection (equivalently Cartan connection) is not determined locally by the conformal structure in the usual way. However, in [12] we describe a definition for $\mathcal{T} \to M$, $dim(M) = 1, 2$, and a connection $\nabla$ via a given Einstein metric on $M$. Note that in dimension 2 a metric is Einstein if it has constant scalar curvature. In dimension 1 any metric is Einstein, by convention. In the construction of [12], the given Einstein metric on $M$ is represented by a $\nabla$-parallel tractor $I$ in $\Gamma(\mathcal{T})$. In both cases $dim(M)=1,2$ the connection $\nabla$ is flat.

Next recall that $(M_1^{n_1}\times M_2^{n_2}, g_1\times g_2)$, with $n_1,n_2\geq 1$ and $n_1+n_2\geq 3$, is a so-called *special Einstein product* if $(M_i,g_i)$, $i=1,2$, is Einstein with corresponding Einstein tractor $I_i$ and $|I_1|^2=-|I_2|^2\neq 0$. The latter condition simply means that the relation $n_2(n_2-1)scal^{g_1}=-n_1(n_1-1)scal^{g_2}\neq 0$ holds for the scalar curvatures of the factors. By pull-back, we view each of the tractor bundles $\mathcal{T}_i$, for $i=1,2$, as bundles on the product $M_1\times M_2$. By $\mathcal{T}_i^{\perp}$, $i=1,2$, we denote the orthogonal complement to $I_i$ in $\mathcal{T}_i$ with respect to the tractor metric $\langle\cdot,\cdot\rangle_{\mathcal{T}_i}$. The tractor connection $\nabla$ on $\mathcal{T}_i$ restricts to a connection $\nabla^i$, for $i=1,2$, on the subbundle $\mathcal{T}_i^{\perp}$. We have the following general result for special Einstein products (cf. [20, 2, 12]).

**Proposition 7.1.** *Let $(M_1\times M_2, g_1\times g_2)$ be a special Einstein product. We define $\mathcal{T}:=\mathcal{T}_1^{\perp}\oplus\mathcal{T}_2^{\perp}$, $\nabla:=\nabla^1\oplus\nabla^2$ and $\langle\cdot,\cdot\rangle_{\mathcal{T}}:=\langle\cdot,\cdot\rangle_{\mathcal{T}_1^{\perp}}+\langle\cdot,\cdot\rangle_{\mathcal{T}_2^{\perp}}$. Then $(\mathcal{T},\langle\cdot,\cdot\rangle_{\mathcal{T}},\nabla)$ is the normal standard tractor bundle with canonical metric and connection.*

Note that this construction of $\mathcal{T}$ is also applicable for special Einstein products with $n_1\leq 2$ or $n_2\leq 2$ if we use the tractor bundles in low dimensions as mentioned above.

7.2. **Multiple almost Einstein structures on $D_\ell\overline{M}$.** Now let $(\overline{M}^{n+1},g_+)$ be an even Poincaré-Einstein space of dimension $n+1\geq 1$, and let $D_\ell\overline{M}$ be an $S^\ell$-doubling with conformal structure $c_\ell[g_+]$ of dimension $n_\ell=n+\ell+1\geq 3$. We denote by $Hol(\mathcal{T})$, $\mathcal{P}(\mathcal{T})$ and $Hol_\ell(\mathcal{T})$, $\mathcal{P}_\ell(\mathcal{T})$ the conformal holonomy group and the space of $\nabla$-parallel tractors on $(M,[g_+])$ and $(D_\ell\overline{M},c_\ell[g_+])$, respectively. The dimension of $\mathcal{P}(\mathcal{T})$ is $\mathcal{N}\geq 1$.

**Theorem 7.2.** *Let $(\overline{M}^{n+1},g_+)$ be a connected even Poincaré-Einstein space with $S^\ell$-doubling $D_\ell\overline{M}$ of dimension $n_\ell\geq 3$ and pole $N_p$ of codimension $\ell+1$. Then*

1. *$Hol_\ell(\mathcal{T})=Hol(\mathcal{T})$.*
2. *Every almost Einstein structure $\sigma=\Pi(I)$, $I\neq 0$, on the bulk $S^\ell\times M^{n+1}$ extends smoothly to $D_\ell\overline{M}$. If $\ell=0$ then every almost Einstein structure on $M$ extends smoothly to the doubling $D_0\overline{M}$.*
3. *The space $\mathcal{P}_\ell(\mathcal{T})$ has dimension $\mathcal{N}+\ell$ and splits naturally into an orthogonal direct sum $\mathcal{S}_\ell\oplus\mathcal{P}_\ell^{\perp}$, where $\mathcal{S}_\ell\subset\mathcal{P}_\ell(\mathcal{T})$ is Euclidean with $dim(\mathcal{S}_\ell)=\ell+1$, $\Sigma(\mathcal{S}_\ell)=N_p$ and $\sigma=\Pi(I)$ is even for every $0\neq I\in\mathcal{S}_\ell$.*

The main ingredient for the proof of Theorem 7.2 is the following general result for the holonomy group of a connection.

**Lemma 7.3.** *Let $\mathcal{V}\to M$ be a smooth vector bundle over a connected manifold $M$ with smooth connection $D$. Let $N$ be a smooth submanifold of codimension $k>1$ in $M$. Then the holonomy groups of $(\mathcal{V},D)$ on $M$ and of the restriction $(\mathcal{V},D)|_{M\smallsetminus N}$ on $M\smallsetminus N$ (naturally) coincide.*

Proof. First, note that, since the codimension of $N$ in $M$ is greater then 1, the set $M\smallsetminus N$ is a connected open dense submanifold of $M$. Moreover, any continuous path $\gamma_0:[0,1]\to M$ is homotopic to a continuous path $\gamma_1:[0,1]\to M\smallsetminus N$ via a homotopy equivalence $\gamma:[0,1]^2\to M$, for which $\gamma_s$ is a path in $M\smallsetminus N$ for any $s\in(0,1]$. Also note that the holonomy group of $(\mathcal{V},D)$ on $M$ contains naturally the holonomy group of the restriction $(\mathcal{V},D)|_{M\smallsetminus N}$. We want to show that they coincide.

For this purpose, let us consider the holonomy algebra $\mathfrak{hol}_p((\mathcal{V},D)_{M\smallsetminus N})$ of the restriction at a base point $p\in M\smallsetminus N$. We denote by $\mathcal{L}_{q,p}$ the set of continuous paths from $q$ to $p$ in

$M$. By $\tilde{\mathcal{L}}_{q,p}$ we denote the subset of those paths in $\mathcal{L}_{q,p}$, which do not meet $N$. Due to the classical holonomy theorem of Ambrose and Singer the holonomy algebra $\mathfrak{hol}_p((\mathcal{V},D)|_{M\smallsetminus N})$ is spanned by the parallel translations of the curvature endomorphisms $R^D_q$ of $D$ at any $q \in M \smallsetminus N$ along some path of $\tilde{\mathcal{L}}_{q,p}$. Note that the set $\mathcal{R}_{M\smallsetminus N}$ of curvature endomorphisms, which spans $\mathfrak{hol}_p((\mathcal{V},D)|_{M\smallsetminus N})$, is a closed subspace of $\mathfrak{gl}(\mathcal{V})$.

Now, since for any path $\gamma_0$ a homotopy exists such that $\gamma_s$ is a path in $M \smallsetminus N$ for all $s \in (0,1]$, and since the curvature $R^D$ is smooth and $\mathcal{R}_{M\smallsetminus N}$ is closed in $\mathfrak{gl}(\mathcal{V})$, it follows that the parallel translations of any curvature endomorphism $R^D_q$, $q \in M$, along some path of $\mathcal{L}_{q,p} \smallsetminus \tilde{\mathcal{L}}_{q,p}$ do not contribute any further element to $\mathcal{R}_{M\smallsetminus N}$. This proves $\mathfrak{hol}_p(\mathcal{V},D) = \mathfrak{hol}_p((\mathcal{V},D)|_{M\smallsetminus N})$. In particular, the connected holonomy groups $Hol^o_p(\mathcal{V},D)$ and $Hol^o_p((\mathcal{V},D)|_{M\smallsetminus N})$ coincide. In fact, since any loop at $p$ in $M$ is homotopy equivalent to a loop at $p$ in $M \smallsetminus N$, it follows that the holonomy groups of $(\mathcal{V},D)$ and $(\mathcal{V},D)|_{M\smallsetminus N}$ coincide as well. □

Proof of Theorem 7.2. (1) First, we assume $\ell > 0$. Note that the holonomy group of $(\mathcal{T},\nabla)$ on the round sphere $S^\ell$ is trivial, and from Proposition 7.1, it follows directly that the conformal holonomy group of the bulk $S^\ell \times M$ of $D_\ell\overline{M}$ is the product of the holonomy groups of the factors $(\mathcal{T}_i^\perp,\nabla^i)$, $i = 1,2$. This shows that the conformal holonomy group of the bulk $S^\ell \times M$ is equal to the holonomy group $Hol(\mathcal{T})$ of $(\mathcal{T},\nabla)$ on $(M,g_+)$. However, the pole $N_p$ in $D_\ell\overline{M}$ has codimension $\ell+1 > 1$. Hence, application of Lemma 7.3 proves that $Hol(\mathcal{T}) = Hol_\ell(\mathcal{T})$ for $\ell > 0$. This identification is natural.

Note that for $\ell = 0$ the bulk of the doubling $D_0\overline{M}$ is disconnected. Hence an application of Lemma 7.3 is not possible. Instead, we have the following argument. In Section 8 we will see that $D_0\overline{M}$ is naturally embedded in $D_1\overline{M}$ as the hypersurface singularity of an almost Einstein structure. Moreover, note that the normal bundle of $D_0\overline{M}$ in $D_1\overline{M}$ is trivial, by construction. Then Theorem 4.5 of [10] states that in this situation the intrinsic tractor parallel transport along curves in $D_0\overline{M}$ coincides with the extrinsic tractor parallel transport in $D_1\overline{M}$. This fact implies that $Hol_0(\mathcal{T})$ is naturally contained in $Hol_1(\mathcal{T})$. Since $M$ is an open submanifold of $D_0\overline{M}$, it is also clear that $Hol(\mathcal{T})$ is naturally contained in $Hol_0(\mathcal{T})$. Hence, with $Hol(\mathcal{T}) = Hol_1(\mathcal{T})$, we obtain $Hol(\mathcal{T}) = Hol_0(\mathcal{T})$ as well.

(2) We prove this statement on the level of $\nabla$-parallel standard tractors, i.e., we show that any $\nabla$-parallel tractor on the bulk extends smoothly to $D_\ell\overline{M}$. In fact, we have already seen in the prove of statement (1) that the conformal holonomy group of the bulk $S^\ell \times M$ naturally coincides with $Hol_\ell(\mathcal{T})$. Thus we obtain via the natural vector space isomorphism of Lemma 6.3 a natural isomorphism $\Psi : \mathcal{P}_\times(\mathcal{T}) \cong \mathcal{P}_\ell(\mathcal{T})$, where $\mathcal{P}_\times(\mathcal{T})$ denotes the space of $\nabla$-parallel tractors on the bulk. By construction, this isomorphism has the property that the restriction of $\Psi(I) \in \mathcal{P}_\ell(\mathcal{T})$ to the bulk $S^\ell \times M$ is $I \in \mathcal{P}_\times(\mathcal{T})$ . In particular, the almost Einstein structure $\Pi(\Psi(I))$ is a smooth extension of $\Pi(I)$ to $D_\ell\overline{M}$ for any $0 \neq I \in \mathcal{P}_\times(\mathcal{T})$. For $\ell = 0$ the same argument works if we consider one connected component $M$ of $S^0 \times M$ in $D_0\overline{M}$.

(3) Let $\ell > 0$. Note that, by construction, the conformal class $c_\ell[g_+]$ restricted to the bulk $S^\ell \times M^{n+1}$ of $D_\ell\overline{M}$ is represented by the special Einstein product $g_{rd} \times g_+$, i.e., by Proposition 7.1, the tractor bundle $\mathcal{T}$ on $S^\ell \times M^{n+1}$ splits into a direct sum $\mathcal{T} = \mathcal{T}_1^\perp \oplus \mathcal{T}_2^\perp$ with connection $\nabla = \nabla^1 \oplus \nabla^2$. Then it is also clear that $\mathcal{P}_\times(\mathcal{T})$ splits into a direct orthogonal sum $\mathcal{P}^1_\times \oplus \mathcal{P}^2_\times$, where $\mathcal{P}^1_\times \cong \mathbb{R}^{\ell+1}$ denotes (the pull-back of) the space of $\nabla$-parallel tractors

in $\Gamma(\mathcal{T}_1^{\perp})$ on $(S^\ell, g_{rd})$ and $\mathcal{P}^2_\times$ denotes (the pull-back of) the $\nabla$-parallel tractors in $\Gamma(\mathcal{T}_2^{\perp})$ on $(M, g_+)$. Via the isomorphism $\Psi$ we obtain the natural splitting $\mathcal{P}_\ell^{\perp} \oplus \mathcal{S}_\ell$ of $\mathcal{P}_\ell(\mathcal{T})$, where we denote $\mathcal{P}_\ell^{\perp} := \Psi(\mathcal{P}^2_\times)$ and $\mathcal{S}_\ell := \Psi(\mathcal{P}^1_\times)$. The latter space has dimension $\ell + 1$. In particular, we see that $dim(\mathcal{P}_\ell(\mathcal{T})) = \mathcal{N} + \ell$.

Let us to take a closer look at the parallel tractors in $\mathcal{S}_\ell$. First, note that any non-trivial $I \in \mathcal{S}_\ell$ has negative scalar curvature $S(I) < 0$, i.e., $\mathcal{S}_\ell$ is an Euclidean subspace of $\mathcal{P}_\ell(\mathcal{T})$ with respect to the tractor metric. Then it follows from Theorem 5.2 in [22] that $\Sigma(\mathcal{S}_\ell)$ is either empty or a submanifold of codimension $\ell+1$ in $D_\ell\overline{M}$. Moreover, the proof of Theorem 5.2 in [22] shows that $\Sigma(\mathcal{S}_\ell)$ cannot lie in the bulk, i.e., $\Sigma(\mathcal{S}_\ell) \subset N_p$. Next we describe the almost Einstein structures, which correspond to the tractors in $\mathcal{S}_\ell$, in explicit form near the pole $N_p$. This will show that $\Sigma(\mathcal{S}_\ell) = N_p$.

So let us consider $S^\ell$ as the unit sphere in $\mathbb{R}^{\ell+1}$. We have shown at the end of Section 5 that the restriction $s_i$ for $i = 1, \ldots, \ell+1$ of the $i$th coordinate function on $\mathbb{R}^{\ell+1}$ is an almost Einstein structure on the round unit sphere $S^\ell$. In fact, the corresponding $\nabla$-parallel standard tractors $\{I_1, \ldots, I_{\ell+1}\}$ span $\mathcal{P}(\mathcal{T})$ of $S^\ell$. Via pull-back we can understand the $I_i$'s and $s_i$'s as parallel tractors and almost Einstein structures on the bulk $S^\ell \times M$ with metric $g_{rd} \times g_+$, respectively. Now let $r$ be any special defining function on an $\varepsilon$-collar of the boundary $N$ in $\overline{M}$. The special defining function $r$ defines a chart with image $B^{\ell+1}_\varepsilon \times N$ around the pole $N_p$ of $D_\ell\overline{M}$. We denote the standard coordinates of the $\varepsilon$-ball $B^{\ell+1}_\varepsilon$ by $(y^1, \ldots, y^{\ell+1})$. The conformal class $c_\ell[g_+]$ on $B^{\ell+1}_\varepsilon \times N$ is represented by (3) $r^2{\cdot}(g_{rd} \times g_+) = dr^2 + r^2 g_{rd} + g_r$, i.e., here the rescaled functions $r{\cdot}s_i$, $i = 1, \ldots, \ell+1$, are almost Einstein structures. Note that $y^i = r{\cdot}s_i$ for $i = 1, \ldots, \ell+1$ on $(B^{\ell+1}_\varepsilon \setminus \{0\}) \times N$. This shows that the coordinate function $y^i$ is the almost Einstein structure on $B^{\ell+1}_\varepsilon \times N$ with metric $dr^2 + r^2 g_{rd} + g_r$, which corresponds to $\Psi(I_i) \in \mathcal{S}_\ell$ for $i = 1, \ldots, \ell+1$. With Theorem 5.2 of [22] we obtain $\Sigma(\mathcal{S}_\ell) = \bigcap_{i=1}^{\ell+1} \Sigma(I_i) = N_p$. Also note that the map $(y^i, n) \in B^{\ell+1}_\varepsilon \times N \mapsto (-y^i, n) \in B^{\ell+1}_\varepsilon \times N$ is an isometry of $r^2{\cdot}(g_{rd} \times g_+)$. This proves that any almost Einstein structure $\sigma_i = \Pi(I_i)$, $i = 1, \ldots, \ell+1$, is even. Hence any almost Einstein structure $\sigma = \Pi(I)$, $I \in \mathcal{S}_\ell$, is even.

Finally, let $\ell = 0$. In this case we know by (1) that the almost Einstein structures on $M$ correspond uniquely to the almost Einstein structures on $D_0\overline{M}$, i.e., $dim(\mathcal{P}_0(\mathcal{T})) = \mathcal{N}$. The natural splitting of $\mathcal{P}_0(\mathcal{T})$ is simply given by $\mathcal{S}_0 := \mathbb{R}I$ and $\mathcal{P}_0^{\perp} = I^{\perp}$, where $I$ is the $\nabla$-parallel tractor, which induces the Poincaré-Einstein metric $g_+$ on $M$. By assumption, $\Pi(I)$ is even. □

7.3. **Two further properties.** Theorem 7.2 establishes the existence of multiple almost Einstein structures with intersecting scale singularities on the collapsing $\ell$-sphere product of a Poincaré-Einstein space for $\ell > 0$. In turn, we show in [22] that any closed Riemannian conformal manifold admitting multiple almost Einstein structures has locally the form of a collapsing $\ell$-sphere product near the intersecting scale singularities. Also note that by Theorem 7.2 $D_\ell\overline{M}$ has decomposable conformal holonomy if $M$ is a Poincaré-Einstein space (cf. [20, 2]). However, if the conformal holonomy $Hol(\mathcal{T})$ of $(M, g_+)$ acts irreducibly on the orthogonal complement of the $Hol(\mathcal{T})$-fixed vector of the standard representation, which corresponds to the Einstein metric $g_+$, then $D_\ell\overline{M}$ is not (even locally) conformal to a special Einstein product at the pole $N_p$!

Finally, we observe that there is a natural action of the orthogonal group $\mathrm{O}(\ell+1)$ on the $S^\ell$-doubling $D_\ell\overline{M}$ of any asymptotically hyperbolic space $(\overline{M}, g_+)$. In fact, the standard action $\gamma$ of $\mathrm{O}(\ell+1)$ on the round sphere $S^\ell$ extends trivially to an action $\tilde{\gamma}$ on the product space $S^\ell \times M$, which is the bulk of $D_\ell\overline{M}$. If we define $\bar\gamma|_{N_p}$ to be the trivial action of $\mathrm{O}(\ell+1)$ on the pole $N_p$ of $D_\ell\overline{M}$, then $\bar\gamma|_{N_p}$ and $\tilde\gamma$ give rise to a smooth group action $\bar\gamma$ of $\mathrm{O}(\ell+1)$ on the $S^\ell$-doubling $D_\ell\overline{M}$. The action $\bar\gamma$ on $(D_\ell\overline{M}, c_\ell[g_+])$ consists only of conformal transformations. The quotient $D_\ell\overline{M}/\bar\gamma$ by this action is the smooth manifold $\overline{M}$ with boundary $N$. The conformal action $\bar\gamma$ plays a crucial role in the considerations of [22]. In case $(\overline{M}, g_+)$ is a Poincaré-Einstein space the action $\bar\gamma$ (restricted to the connected component $\mathrm{SO}(\ell+1)$) is generated by the flow of those *normal conformal Killing vector fields*, which correspond to the wedge products $I \wedge J$ of two Einstein tractors $I, J \in \mathcal{S}_\ell$ (cf. [19]).

## 8. An ambient metric of $D_\ell\overline{M}$

Let $(D_\ell\overline{M}, c_\ell[g_+])$ be the $S^\ell$-doubling of some even AH Einstein metric $g_+$. We discuss in this section explicit constructions of a Poincaré-Einstein metric with conformal infinity $(D_\ell\overline{M}, c_\ell[g_+])$ and a Fefferman-Graham ambient metric for the conformal manifold $(D_\ell\overline{M}, c_\ell[g_+])$.

8.1. **A Poincaré-Einstein metric for $D_\ell\overline{M}$.** Let $\ell \geq 0$ be arbitrary. The Euclidean space $\mathbb{R}^{\ell+1}$ is naturally identified via the inclusion $i$ with the hypersurface $\{x^{\ell+2} = 0\}$ in $\mathbb{R}^{\ell+2}$. The restriction of $i$ to the unit $\ell$-sphere in $\mathbb{R}^{\ell+1}$ gives naturally rise to an embedding $\iota : S^\ell \hookrightarrow S^{\ell+1}$. We observed already at the end of Section 3 that the inclusion $\iota$ induces a smooth embedding $\iota_{\ell,\ell+1}$ of the $S^\ell$-doubling $(D_\ell\overline{M}, c_\ell[g_+])$ into the $S^{\ell+1}$-doubling $(D_{\ell+1}\overline{M}, c_{\ell+1}[g_+])$ of an even AH Einstein metric $g_+$. In fact, on the bulk $S^\ell \times M$ of $(D_\ell\overline{M}, c_\ell[g_+])$ the embedding $\iota_{\ell,\ell+1}$ is just given by $\iota \times id|_M$. On an $\varepsilon$-tube around the pole $N_p$ of $D_\ell\overline{M}$ with respect to any special defining function $r$ of $g_+$ the embedding $\iota_{\ell,\ell+1}$ is given by

$$i \times id|_N : B^{\ell+1}_\varepsilon \times N \to B^{\ell+2}_\varepsilon \times N \ .$$

Since $c_\ell[g_+]$ on the bulk $S^\ell \times M$ is the conformal class of $g_{rd} \times g_+$ and $\iota$ is an isometric embedding, it is obvious that $\iota_{\ell,\ell+1}$ is a conformal embedding of $D_\ell\overline{M}$ into $D_{\ell+1}\overline{M}$.

At the end of Section 5 we have discussed the almost Einstein structures on the sphere. Obviously, there exists (up to a sign) a unique almost Einstein structure $\sigma_{\ell+2}$ on $S^{\ell+1}$ with (scalar curvature) $S(\sigma_{\ell+2}) = -1$ and scale singularity $\Sigma(\sigma_{\ell+2}) = \iota(S^\ell)$. The almost Einstein structure $\sigma_{\ell+2}$ gives naturally rise to an almost Einstein structure on the $S^{\ell+1}$-doubling $D_{\ell+1}\overline{M}$, which we also denote by $\sigma_{\ell+2}$ (cf. Theorem 7.2). By construction, the scale singularity $\Sigma(\sigma_{\ell+2})$ on $D_{\ell+1}\overline{M}$ is the image of the $S^\ell$-doubling $D_\ell\overline{M}$ under $\iota_{\ell,\ell+1}$. In particular, if $\bar g \in c_{\ell+1}[g_+]$ on $D_{\ell+1}\overline{M}$ and $s_{\ell+2}$ corresponds to $\sigma_{\ell+2}$ via $\bar g$, then the space $\overline{D_{\ell+1}\overline{M}_+}(s_{\ell+2})$ admits $D_\ell\overline{M}$ as its boundary (cf. (7)). The metric $g_{\ell+1} := s^{-2}_{\ell+2}\cdot\bar g$ on the interior $(D_{\ell+1}\overline{M})_+(s_{\ell+2})$ is AH Einstein. This proves that

$$(8) \qquad \left(\ \overline{D_{\ell+1}\overline{M}_+}(s_{\ell+2})\ ,\ g_{\ell+1}\ \right)$$

is a Poincaré-Einstein space with conformal infinity $(D_\ell\overline{M}, c_\ell[g_+])$. Note that the Poincaré-Einstein metric $g_{\ell+1}$ is even.

With respect to a special defining function $r$ of $g_+ = r^{-2}(dr^2 + g_r)$ the conformal infinity $c_\ell[g_+]$ of $\overline{D_{\ell+1}\overline{M}_+}(s_{\ell+2})$ restricted to a tubular collar $B^{\ell+1}_\varepsilon \times N$ of the pole $N_p \subset D_\ell\overline{M}$ is represented by a metric $\tilde g_0 = dr_\ell^2 + r_\ell^2 g_{rd} + g_{r_\ell}$ of the form (3), where $r_\ell$ is the radial coordinate on $B^{\ell+1}_\varepsilon$ induced by $r$. The corresponding normal form of $g_{\ell+1}$ is then given by

$$(9) \qquad x^{-2}\cdot\left(dx^2\ +\ (dr_\ell^2 + r_\ell^2 g_{rd} + g_{r_{\ell+1}(x)})\right)\ ,$$

where $x$ denotes the additional standard coordinate of the collar $B_\varepsilon^{\ell+2} \times N$ of $B_\varepsilon^{\ell+1} \times N$ and $r_{\ell+1}(x) = \sqrt{r_\ell^2 + x^2}$ is the radial coordinate of $B_\varepsilon^{\ell+2} \times N$. Off the pole $N_p$ the conformal infinity $c_\ell[g_+]$ is represented by $\tilde{g}_0 = g_{rd} \times g_+$. The corresponding normal form of $g_{\ell+1}$ is given on a collar of the bulk of $D_\ell \overline{M}$ by

$$x^{-2} \cdot \left(dx^2 + \left(1 - x^2/4\right)^2 \cdot g_{rd} + \left(1 + x^2/4\right)^2 \cdot g_+\right) \tag{10}$$

(cf. [11]).

8.2. **An ambient metric for $D_\ell\overline{M}$.** Next we recall the notion of ambient metric spaces due to Fefferman and Graham (cf. [6]). Let $(M^n, [g])$ be a Riemannian conformal manifold with corresponding $\mathbb{R}_+$-ray subbundle $\mathcal{Q} \xrightarrow{\pi} M$ in $S^2T^*M$ of conformally related metrics. The bundle $\mathcal{Q}$ admits the $\mathbb{R}_+$-action $\gamma(s)(x, g_x) = (x, s^2 g_x)$, $x \in M$. A manifold $\tilde{M}$ of dimension $n + 2$ with a free $\mathbb{R}_+$-action, which is also denote by $\gamma$, and an $\mathbb{R}_+$-equivariant embedding $i : \mathcal{Q} \to \tilde{M}$ is called an ambient space of $(M, [g])$. We write $\boldsymbol{X} \in \mathfrak{X}(\tilde{M})$ for the fundamental vector field generating the $\mathbb{R}_+$-action, i.e., for $f \in C^\infty(\tilde{M})$ and $u \in \tilde{M}$ we have $\boldsymbol{X} f(u) = \frac{d}{dt} f(\gamma(e^t)u)|_{u=0}$. An ambient metric $\boldsymbol{h}$ is a pseudo-Riemannian metric on an ambient space $(\tilde{M}, i)$ of signature $(1, n + 1)$ such that

(i) the metric $\boldsymbol{h}$ is homogeneous of degree 2 with respect to the $\mathbb{R}_+$-action $\gamma$, i.e., $\mathcal{L}_{\boldsymbol{X}}\boldsymbol{h} = 2\boldsymbol{h}$ and $\boldsymbol{X}$ is a homothetic vector field.
(ii) For $u = (x, g_x) \in \mathcal{Q}$ and $\xi, \eta \in T_u\mathcal{Q}$, we have $\boldsymbol{h}(i_*\xi, i_*\eta) = g_x(\pi_*\xi, \pi_*\eta)$.

The second condition implies that the tangent direction along the fibres of $\mathcal{Q} \xrightarrow{\pi} M$ is null. To simplify the notation we will usually identify $\mathcal{Q}$ with its image in $\tilde{M}$ and suppress the embedding map $i$. In order to link the geometry of the ambient manifold to the underlying conformal structure on $M$ one requires further conditions on $\boldsymbol{h}$. A suitable condition is the Ricci-flatness of $\boldsymbol{h}$ on $\tilde{M}$ (at least up to a certain order of an expansion at $\mathcal{Q}$; cf. [6]).

In [8] Fefferman and Graham pointed out that for any conformal manifold $(N, c)$ there exists a natural 1-to-1 correspondence between Ricci-flat ambient metrics $\boldsymbol{h}$ and even Poincaré-Einstein metrics $g_+$ (which have conformal infinity $(N, c)$). We briefly describe this correspondence. Let $g_+ = r^{-2}(dr^2 + g_r)$ be an even Poincaré-Einstein metric on some collar $[0, \varepsilon) \times N$, which is smooth up to the boundary $N$ with conformal infinity $c = [g_0]$, where $g_r$ denotes some 1-parameter family of metrics with respect to a special defining function $r$. Then the smooth metric

$$\boldsymbol{h} \ := \ 2\rho dt^2 + 2t dt d\rho + t^2 g_{r(\rho)} \tag{11}$$

on $\tilde{N} := N \times (0, \infty) \times (-\frac{\varepsilon^2}{2}, \frac{\varepsilon^2}{2})$ with $r(\rho) = \sqrt{2|\rho|}$ is Ricci-flat and ambient for $(N, c)$. Here the coordinate $t \in (0, \infty)$ parametrises the fibre of the ray bundle $\mathcal{Q} \to N$ and $\rho$ is the coordinate of a thickening interval. In turn, for $\rho \leq 0$ the Poincaré-Einstein metric $g_+$ is given by the restriction of the metric (11) to the hypersurface $\{-2\rho t^2 = 1\}$ in $\tilde{N}$. Note that with $s = rt$ and $-2\rho = r^2$ the metric (11) takes on the open subset $\{\rho < 0\} \subset \tilde{N}$ the form

$$-ds^2 + s^2 g_+ \ , \tag{12}$$

i.e., the ambient metric $\boldsymbol{h}$ is (off the ray $\mathcal{Q}$) a metric cone over the Poincaré-Einstein metric $g_+$.

We can apply now the normal form (11) for a Ricci-flat ambient metric directly to the even Poincaré-Einstein metrics (9) and (10) of $(D_\ell\overline{M}, c_\ell[g_+])$. We obtain

$$\boldsymbol{h} \;=\; 2\rho dt^2 + 2tdtd\rho + t^2(dr_\ell^2 + r_\ell^2 g_{rd} + g_{r(\rho)}) \tag{13}$$

with $r(\rho) = \sqrt{r_\ell^2 + 2|\rho|}$ around the pole $N_p$, and moreover, we have

$$\boldsymbol{h} \;=\; 2\rho dt^2 + 2tdtd\rho + t^2\left((1+\rho/2)^2 g_{rd} + (1-\rho/2)^2 g_+\right) \tag{14}$$

for the bulk of $(D_\ell\overline{M}, c_\ell[g_+])$ (cf. [11]). This is a sufficient answer for an ambient metric of $(D_\ell\overline{M}, c_\ell[g_+])$. Note that (13) and (14) represent the same ambient metric off the pole $N_p$, i.e., (13) is a smooth extension of (14) over the pole. (Also note that the ambient metrics of the form (13) are up to a $\mathcal{Q}$-fixing diffeomorphism isometric for any two special defining functions $r_1$ and $r_2$ of $(\overline{M}, g_+)$.) However, we also want to explain (13) and (14) from a more geometric point of view. In fact, recall that we used in Section 5 the ambient space $\mathbb{R}^{1,n_\ell+1}$ of the Möbius sphere in order to explain the smooth collapsing of $(D_\ell\overline{\mathbb{H}}^{n+1}, c_\ell[g_+])$. The following description generalises this picture.

First, we recall from [11] that the ambient metric (14) of the bulk, which is a special Einstein product, is isometric to the product of cones of the factors $(S^\ell, g_{rd})$ and $(M, g_+)$, i.e., we have

$$\boldsymbol{h} \;=\; (dv^2 + v^2 g_{rd}) \times (-ds^2 + s^2 g_+) \tag{15}$$

via the coordinate change $t = \frac{v+s}{2}, \rho = \frac{2(v-s)}{v+s}$. We aim to extend the ambient metric (15) smoothly to the pole $N_p$ of $D_\ell\overline{M}$. For this, let $r$ be an arbitrary special defining function of $g_+$, which defines an $\varepsilon$-collar around $N$. An ambient metric for the conformal boundary $N$ of $(\overline{M}, g_+)$ is then given by (11) on $\tilde{N}$. Recall that (11) is a smooth extension of the metric cone (12). Hence we set

$$\boldsymbol{h} \;:=\; (dv^2 + v^2 g_{rd}) \times \left(2\rho dt^2 + 2tdtd\rho + t^2 g_{r(\rho)}\right) \tag{16}$$

on $\mathbb{R}^{\ell+1} \times \tilde{N}$, which is by construction a smooth and Ricci-flat extension of the ambient metric (15) (resp. (14)).

We claim that (16) is an ambient metric on $\mathbb{R}^{\ell+1} \times \tilde{N}$ for a tubular collar of the pole $N_p$ in $D_\ell\overline{M}$. This is clear off the pole, where the ray bundle $\mathcal{Q}$ is given by the hypersurface $\{rt = v\}$, and the Euler vector $\boldsymbol{X}$ is $v\partial v + t\partial t$. Obviously, the hypersurface $\{rt = v\}$ and $\boldsymbol{X} = v\partial v + t\partial t$ are also smoothly defined over the pole $N_p$. In fact, $\mathcal{Q}$ restricted to $N_p$ is the set $\{r, v = 0\}$ and the Euler vector $\boldsymbol{X}$ over $N_p$ is just $t\partial t$. Factoring $\mathcal{Q} = \{rt = v\}$ through the integral curves of $\boldsymbol{X}$ then produces a smooth $n_\ell$-dimensional manifold, which is diffeomorphic to the collapsing $\ell$-sphere product of the collar $[0, \varepsilon) \times N$ in $\overline{M}$, and the section in $\mathcal{Q}$ with $t = 1$ represents a smooth metric in the $S^\ell$-doubling of $g_+$. We conclude that (16) is a Ricci-flat ambient metric for $(D_\ell\overline{M}, c_\ell[g_+])$ around the pole $N_p$ (which is by construction isometric to (13)). Again, the ambient metrics of the form (16) are (up to a $\mathcal{Q}$-fixing diffeomorphism) isometric for any two special defining functions $r_1$ and $r_2$ of $(\overline{M}, g_+)$. In words, an ambient metric (around the pole $N_p$) for the $S^\ell$-doubling of an even AH Einstein metric is given by the product of (the completion to the origin of) the cone over $S^\ell$ with an ambient metric of the even boundary $N$.

**Theorem 8.1.** *Let $(\overline{M}^{n+1}, g_+)$ be an even AH Einstein space and $(D_\ell\overline{M}, c_\ell[g_+])$ an $S^\ell$-doubling of $g_+$ for some $\ell \geq 0$. The $S^\ell$-doubling $D_\ell\overline{M}$ admits a Ricci-flat ambient metric $\boldsymbol{h}$,* *which is given for the bulk of $D_\ell\overline{M}$ by (14) (resp. (15)). Moreover, with respect to a special defining function $r$ (such that $g_+ = r^{-2}(dr^2 + g_r)$ for $r < \varepsilon$)*

1. *the ambient metric $\boldsymbol{h}$ is given on a tubular collar of the pole $N_p$ by (16) with $r(\rho) = \sqrt{2|\rho|}$.*
2. *Alternatively, $\boldsymbol{h}$ is given around the pole $N_p$ by (13) with $r(\rho) = \sqrt{r_\ell^2 + 2|\rho|}$.*

## 9. The pole $N_p$ as minimal submanifold

Let $(D_\ell\overline{M}, c_\ell[g_+])$, $\ell \geq 0$, be the $S^\ell$-doubling of an even asymptotically hyperbolic space $(\overline{M}, g_+)$. We show in this section that the pole $N_p$ is a totally umbilic submanifold of $D_\ell\overline{M}$. In fact, with respect to any normal form metric (3) in the conformal class $c_\ell[g_+]$ the pole $N_p$ is minimal.

### 9.1. **Total umbilicity and minimality.**

Let $(M^n, g)$ be an arbitrary Riemannian manifold and let $N^m$, $dim(N) = m > 0$, be a submanifold of $M$ with codimension $s := n - m > 0$. The restriction of the tangent bundle $TM$ to the submanifold $N$ admits a natural $g$-orthogonal decomposition into the tangential part $TN$ with projection $pr$ and the bundle $T^\perp N$ of normal vectors on $N$ in $M$ with projection $pr_\perp$. The restriction $g_N := g|_{TN}$ is a Riemannian metric on $N$, and the Levi-Civita connection $\nabla^{g_N}$ of $g_N$ is the tangential part of $\nabla^g$ (restricted to tangent vector fields on $N$).

**Definition 9.1.** *The second fundamental form of a submanifold $N^m$ in $(M, g)$ is given by the normal part*

$$II^g(X, Y) := pr_\perp \circ \nabla^g_X Y, \quad X, Y \in \mathfrak{X}(N),$$

*of the Levi-Civita connection of $g$ on $M$. The mean curvature of $N$ in $(M, g)$ is the trace $H^g := \frac{1}{m} tr_g II^g$.*

1. *If $m > 1$ and $II^g$ has only a trace part, i.e., $II = H^g \otimes g_N$ on $N$, then we call $N$ a totally umbilic submanifold of $(M, g)$.*
2. *If $H^g = 0$ then $N$ is called a minimal submanifold of $(M, g)$.*

Note that the property of total umbilicity for a submanifold $N$ does not depend on the metric in the conformal class of the ambient manifold $(M, c)$, i.e., total umbilicity is a well defined, invariant notion for submanifolds of a conformal manifold.

**Theorem 9.2.** *Let $(D_\ell\overline{M}, c_\ell[g_+])$, $\ell \geq 0$, be the $S^\ell$-doubling of an even asymptotically hyperbolic metric $g_+$ on the interior of a manifold $\overline{M}^{n+1}$. Then*

1. *the pole $N_p$ is a totally umbilic submanifold of codimension $\ell + 1$ in $D_\ell\overline{M}$ (if $n \geq 2$).*
2. *The pole is a minimal submanifold in $D_\ell\overline{M}$ with respect to a metric of the form (3) in $c_\ell[g_+]$ (if $n \geq 1$).*

Proof. With respect to any special defining function $r$ on an $\varepsilon$-collar of the boundary $N$ in $(\overline{M}, g_+)$ we obtain a normal form metric (3) $g = dr^2 + r^2 g_{rd} + g_r$ in $c_\ell[g_+]$ on a tubular collar $B^{\ell+1}_\varepsilon \times N$. Let $n_i := \partial x^i$, $i = 1, \ldots, \ell + 1$, denote the standard coordinate vectors of the factor $B^{\ell+1}_\varepsilon$, and let $e_i := \partial y^i$, $i = 1, \ldots, n$, be some local coordinate vectors on $N$. Let $\Psi : (x, y) \mapsto (-x, y)$ be the isometric involution on $B^{\ell+1}_\varepsilon \times N$ (due to the evenness of $g_+$). We have

$$g(\nabla^g_{e_i} e_j, n_k) \;=\; g(\Psi_*(\nabla^g_{e_i} e_j), \Psi_*(n_k)) \;=\; -g(\nabla^g_{e_i} e_j, n_k) \;=\; 0$$

for any $i, j \in \{1, \dots, n\}$, $k \in \{1, \dots, \ell+1\}$. This proves that $N_p$ is totally geodesic in $D_\ell\overline{M}$ with respect to the metric $g = dr^2 + r^2 g_{rd} + g_r$. In particular, $N_p$ is totally umbilic in $(D_\ell\overline{M}, c_\ell[g_+])$ and minimal with respect to $g$. □

In the following, we denote by $W^g$ and $C^g$ the Weyl and Cotton tensor, respectively, of a metric $g \in c_\ell[g_+]$ on $D_\ell\overline{M}$. The restrictions of $W^g$, $C^g$ to the pole $N_p$ are denoted by $W|_{N_p}$, resp., $C|_{N_p}$. On the other hand, we have the (intrinsic) Weyl and Cotton tensor $W^{N_p}$ and $C^{N_p}$ of the pole $N_p$, respectively. The pole $N_p$ admits further extrinsic curvature conditions in $(D_\ell\overline{M}, c_\ell[g_+])$ in case $(\overline{M}, g_+)$ is a Poincaré-Einstein space.

**Theorem 9.3.** *Let $N_p$ be the pole of an $S^\ell$-doubling $(D_\ell\overline{M}, c_\ell[g_+])$ for a Poincaré-Einstein space $(\overline{M}^{n+1}, g_+)$ with $n \geq 3$. Let $g \in c_\ell[g_+]$ be a metric, and let $T^\perp N_p$ be the normal bundle of $N_p$ in $D_\ell\overline{M}$. Then*

1. *$n \lrcorner W^g = 0$ for any normal vector $n \in T^\perp N_p$.*
2. *If $dim(M) \geq 5$, then $n \lrcorner C^g = 0$ for any $n \in T^\perp N_p$.*
3. *$W^{N_p} = W^g|_{N_p}$ and $C^{N_p} = C^g|_{N_p}$.*

Theorem 9.3 follows directly from Theorem 8.5 of [22]. The proof relies essentially on the fact that, if $(\overline{M}^{n+1}, g_+)$ is Poincaré-Einstein, any unit normal vector on $N_p$ in $D_\ell\overline{M}$ with metric $g \in c_\ell[g_+]$ is the gradient at $N_p$ of an almost Einstein structure on $D_\ell\overline{M}$!

9.2. **Strong umbilicity.** In [22] we have introduced the notion of a *normal s-form tractor* $I_N$ for any submanifold $N^m$, $m \geq 1$, of codimension $s > 0$ in a conformal manifold $(M, c)$. The normal $s$-form tractor is by definition a section of the $s$-form tractor bundle $\Lambda^s\mathcal{T}$ of $M$ restricted to $N$. In case $s = 1$ the definition of $I_N$ in [22] coincides with the classical definition in [3] for the normal tractor of a hypersurface. With respect to a metric $g \in c$ on the ambient manifold $M$ the normal $s$-form tractor $I_N$ of a submanifold $N$ is given by

$$I_N := \begin{pmatrix} & 0 & \\ vol(g|_{T^\perp N}) & & 0 \\ & H^g \lrcorner vol(g|_{T^\perp N}) & \end{pmatrix}, \tag{17}$$

where $vol(g|_{T^\perp N})$ denotes the volume form of the (oriented) normal bundle $(T^\perp N, g)$ (cf. (34) in [22]). (Note that the normal bundle $T^\perp N$ admits a local orientation in any case.)

If $I_N$ is $\nabla$-parallel along $N$ with respect to the tractor connection $\nabla$ on $(M, c)$, i.e.,

$$\nabla_X I_N = 0 \qquad \forall X \in TN ,$$

then we say $N$ is a *strongly umbilic* submanifold of the conformal manifold $(M, c)$. Let us set $\kappa(X) := \Sigma_{i=1}^s (X(\nu_i) + \mathsf{P}^g(X, n_i))n_i$ for $X \in TN$ with respect to $g \in c$, where $\nu_i := g(H^g, n_i)$ and $\{n_1, \dots, n_s\}$ is some orthonormal basis of the normal bundle $T^\perp N$. We have shown in [22] that $N$ is strongly umbilic in $(M, c)$ if and only if $N$ is totally umbilic and

$$\kappa(X) \lrcorner vol(g|_{T^\perp N}) = 0$$

for any $X \in TN$ with respect to $g \in c$.

**Theorem 9.4.** *Let $N_p$ be the pole of the $S^\ell$-doubling $(D_\ell\overline{M}, c_\ell[g_+])$, $\ell \geq 0$, of an even asymptotically hyperbolic space $(\overline{M}^{n+1}, g_+)$, $n \geq 1$. Then $N_p$ is strongly umbilic in $D_\ell\overline{M}$.*

PROOF. Let $g = dr^2 + r^2 g_{rd} + g_r$ be a normal form metric (3) in $c_\ell[g_+]$ on a tube $B_\varepsilon^{\ell+1} \times N$ of the pole $N_p$. We know already that $N_p$ is minimal with respect to $g$. Hence, in order to show strong umbilicity for the pole, it is sufficient to show $Ric^g(X, n) = 0$ for any $X \in TN_p$, $n \in T^\perp N_p$ with respect to the normal form metric $g$.

So let $n_i := \partial x^i$, $i = 1, \ldots, \ell + 1$, denote the standard coordinate vectors of the factor $B_\varepsilon^{\ell+1}$, and let $e_i := \partial y^i$, $i = 1, \ldots, n$, be some local coordinate vectors on $N$ (as in the proof of Theorem 9.2). Note that $g(\nabla^g_{e_i} n_j, n_k) = 0$ and $\nabla^g_{n_j} n_k = 0$ on $B_\varepsilon^{\ell+1} \times N$ for all $i \in \{1, \ldots, n\}$ and $j, k \in \{1, \ldots, \ell + 1\}$. Furthermore, we have $\nabla^g_{e_i} e_j = \Sigma_k f^k_{ij} e_k + \Sigma_s b^s_{ij} n_s$ for certain functions $f^k_{ij}$ and $b^s_{ij}$ on $B_\varepsilon^{\ell+1} \times N$. The functions $b^s_{ij}$ are zero for $r = 0$. We obtain $g(\nabla^g_{e_i} \nabla^g_{e_j} e_k, n_l) = 0$ at the pole for all $i, j, k \in \{1, \ldots, n\}$, $l \in \{1, \ldots, \ell + 1\}$. This shows $Ric^g(X, n) = \Sigma_i R^g(X, e_i, e_i, n) = 0$ for any $X \in TN_p$, $n \in T^\perp N_p$. □

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