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Finite Groups**

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# On Torsion Subgroups in Integral Group Rings of Finite Groups\*

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*In honour of K. W. Gruenberg*

## 1 Introduction

Throughout  $G$  is a finite group. The integral group ring of  $G$  is denoted by  $\mathbb{Z}G$  and  $V(\mathbb{Z}G)$  is the subgroup of the unit group  $U(\mathbb{Z}G)$  consisting of the units with augmentation 1.

The question whether a finite subgroup  $H$  of  $V(\mathbb{Z}G)$  is isomorphic to a subgroup of  $G$  may be seen as a question on  $G$  however as well as on  $H$  in the sense of [15, Problem 19]. J. A. Cohn and D. Livingstone showed that if  $H$  is a cyclic group of prime power order then the question has always an affirmative answer [5, Theorem 4.1]. Till 2006 no other general result was known without specifying  $G$ . In the mean time it has been proved that  $V(\mathbb{Z}G)$  has a subgroup isomorphic to  $C_p \times C_p$  if, and only if,  $G$  has such a subgroup [17], [12]. The main reason why there are no further general results is that very little is known on the torsion subgroups of  $V(\mathbb{Z}G)$  when  $G$  is a simple or almost simple group.

Therefore it is necessary to strengthen efforts on integral group rings of finite simple groups. Quite satisfactory results have been obtained for the series  $\mathrm{PSL}(2, q)$  [14]. In this article we continue this study with respect to the series of the Suzuki groups  $\mathrm{Sz}(q) = {}^2B_2(q)$ ,  $q = 2^{2m+1}$ , in order to get results for all minimal simple groups  $G$ .

Note that in integral group rings of finite groups subgroups  $H$  of finite order are not always contained in a group basis.<sup>1</sup> This is another reason for the failure of general results. H. Zassenhaus conjectured that each finite subgroup  $H$  of  $V(\mathbb{Z}G)$  is conjugate within  $\mathbb{Q}G$  to a subgroup of  $G$ . This conjecture known as third Zassenhaus conjecture, usually denoted by ZC3, does not hold. Even group bases need not be isomorphic [10]. ZC3 has been verified by A. Weiss when  $G$  is nilpotent [25]. It is open in the case when  $H$  is a  $p$ -group or abelian. The subcase when  $H$  is cyclic - known as first Zassenhaus conjecture ZC1 - has been extensively studied in the last years. A positive solution to ZC1 is known mainly for special classes of soluble groups. The weaker question whether cyclic subgroups of  $V(\mathbb{Z}G)$  are isomorphic to a subgroup of  $G$  [21, Problem 8] has an affirmative answer when  $G$  is soluble [11]. Again with respect to simple or almost simple groups almost nothing was known till 2006.

It is obvious to study first the weaker question whether the prime graph  $\Pi(V(\mathbb{Z}G))$  - nowadays called the Gruenberg-Kegel graph - coincides with that one of  $G$  [15, Problem 21]. Note that the vertices of the Gruenberg-Kegel graph are the primes dividing the order of a torsion element of the group. Two different vertices  $p$  and  $q$  are connected by an edge if, and only if, there is an element of order  $p \cdot q$ .

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\*Typing errors corrected, June 2010

<sup>1</sup>A subgroup  $H$  of  $V(\mathbb{Z}G)$  with  $|H| = |G|$  is a  $\mathbb{Z}$ -basis of  $\mathbb{Z}G$  and is called a group basis. It consists of linearly independent elements.

An algorithm described by I. S. Luthar and I. B. S. Passi [18] for proving ZC1 for the alternating group  $A_5$  and an extension of this algorithm with Brauer characters developed by M. Hertweck [13] led to positive answers for several simple or almost simple groups of the question on  $\Pi(V(\mathbb{Z}G))$  within the last years, cf. [16], [4].

The article is organized as follows. In Section 2 we study the Gruenberg-Kegel graph of specific subgroups of  $V(\mathbb{Z}G)$  for an arbitrary finite group  $G$ . Among other this leads to the following

**Theorem A.** Let  $H$  be a group basis of  $\mathbb{Z}G$  and let  $U$  be an isolated subgroup of  $H$ . Then the normalized torsion elements of the centralizer ring  $C_{\mathbb{Z}G}(U)$  are the central elements of  $U$ . Moreover  $\Pi(N_{V(\mathbb{Z}G)}(U)) = \Pi(N_H(U))$ .

In Section 3 we consider torsion subgroups of  $V(\mathbb{Z}G)$  for groups  $G$  with abelian Sylow subgroups. We show that finite 2-subgroups of  $V(\mathbb{Z}G)$  are rationally conjugate to a subgroup of  $G$  provided  $G$  has elementary abelian Sylow subgroups of order at most 8. In the next section we study with the Luthar-Passi-Hertweck method torsion units of  $\mathbb{Z}\text{PSL}(3, 3)$  and obtain that  $\Pi(\text{PSL}(3, 3)) = \Pi(V(\mathbb{Z}\text{PSL}(3, 3)))$ . The results on the torsion units of the minimal simple group  $\text{PSL}(3, 3)$  are used in the final section. There we show with the generic character table of  $\text{Sz}(q)$  that elementary abelian subgroups of  $V(\mathbb{Z}\text{Sz}(q))$  are isomorphic to subgroups of  $\text{Sz}(q)$  and consider the same question with respect to  $\text{PSL}(3, 3)$ . Together with the results on  $\mathbb{Z}\text{PSL}(2, q)$  obtained in [14] we get the following conclusions:

**Theorem B.** Let  $G$  be a simple group which admits a non-trivial partition. Let  $H$  be a finite subgroup of  $V(\mathbb{Z}G)$ . Then for each prime  $p$  the  $p$ -rank of  $H$  is less or equal than that one of  $G$ .

**Proposition C.** Let  $G$  be a minimal simple group and let  $H$  be an elementary abelian subgroup of  $V(\mathbb{Z}G)$ . Then  $H$  is isomorphic to a subgroup of  $G$  except possibly the case that  $H \cong C_3^3$  and  $G = \text{PSL}(3, 3)$ .

## 2 Gruenberg-Kegel graph

By [5] we know that the primes dividing the order of a torsion unit of  $V(\mathbb{Z}G)$  are primes dividing  $|G|$ . Thus  $\Pi(V(\mathbb{Z}G))$  and  $\Pi(G)$  have the same vertices. The Gruenberg-Kegel graphs of  $G$  and  $V(\mathbb{Z}G)$  can therefore differ only when the latter has more edges.

Our first observation shows that such an additional edge must arise between torsion units which are both not contained in a group basis of  $\mathbb{Z}G$ .

**Lemma 2.1.** *Let  $W$  be a subgroup of a group basis  $H$  of  $V(\mathbb{Z}G)$ . Let  $p$  and  $q$  be different primes. Assume that  $w \in W$  with  $o(w) = p$  commutes with a unit  $u \in V(\mathbb{Z}G)$  of order  $q$ . Then there is an element  $h \in C_H(W)$  of order  $q$ .*

If  $u = \sum_{g \in C} z_g g \in \mathbb{Z}G$  is a unit and  $C$  is a conjugacy class of  $G$  the partial augmentation of  $u$  with respect to  $C$  is

$$\varepsilon_C(u) := \sum_{g \in C} z_g.$$

The augmentation map  $\varepsilon : \mathbb{Z}G \rightarrow \mathbb{Z}$ ,  $\sum_{g \in G} z_g g \mapsto \sum_{g \in G} z_g$  shows that the sum over all partial augmentations of a unit  $u$  of  $V(\mathbb{Z}G)$  is 1. Moreover by S. D. Berman and G. Higman the 1-coefficient of  $u$  is zero provided  $u \neq 1$ . For the proof of Lemma 2.1 we use the following result on partial augmentations due to M. Hertweck. For the convenience of the reader we state it because it is frequently used in this article.

**Theorem 2.2.** [13, Theorem 2.3] *Let  $G$  be a finite group and  $u \in V(\mathbb{Z}G)$  a torsion unit of order  $k$ . Let  $C$  be a conjugacy class of  $G$  and let  $g \in C$ . Then  $\varepsilon_C(u) = 0$  provided  $o(g) \nmid k$ .*

**Proof of Lemma 2.1.** Let  $u = \sum_{g \in H} z_g g$ . Because  $u$  has order  $q$  it follows from Theorem 2.2

that  $\varepsilon_C(u) = 0$  for each  $H$ -conjugacy class  $C$  which does not consist of elements of order  $q$ . Thus

$$s := \sum_{i=1}^k \varepsilon_{C_i}(u) = 1,$$

where  $C_1, \dots, C_k$  denote the conjugacy classes of  $H$  of the elements of order  $q$ . By assumption  $w^{-1}uw = u$ . Thus  $w$  acts on the coefficients  $z_g$  of  $u$ . If there is no element of order  $q$  of  $H$  which commutes with  $w$  each class  $C_i$  consists of  $\langle w \rangle$ -orbits of length  $p$ . Hence for each  $i$  the partial augmentation  $\varepsilon_{C_i}(u)$  is divisible by  $p$  and therefore  $s$  as well. This contradiction completes the proof.  $\square$

Lemma 2.1 suggests to define a subgraph  $\Pi_r(\mathbb{V}(\mathbb{Z}G))$  of  $\Pi(\mathbb{V}(\mathbb{Z}G))$  as follows. It has the same vertices as  $\Pi(\mathbb{V}(\mathbb{Z}G))$  but two different vertices  $p$  and  $q$  are linked by an edge if, and only if, the centralizer  $C_{\mathbb{V}(\mathbb{Z}G)}(g)$  of an element  $g \in G$  of order  $p$  contains an element of order  $q$ . Then we get immediately from Lemma 2.1

**Corollary 2.3.** *Let  $G$  be a finite group and let  $H$  be a group basis of  $\mathbb{V}(\mathbb{Z}G)$ . Then*

$$\Pi(H) = \Pi_r(\mathbb{V}(\mathbb{Z}G)).$$

**Proof.** By the class sum correspondence  $\Pi(H) = \Pi(G)$ . Then the corollary follows from Lemma 2.1.  $\square$

A subgroup  $U$  of a finite group  $G$  is called isolated, if for each non-trivial element  $u$  of  $U$  the centralizer  $C_G(u)$  is contained in  $U$  and if for each  $g \in G$  the intersection of  $U$  with  $U^g$  is trivial or coincides with  $U$ . A finite group has an isolated subgroup if, and only if, its Gruenberg-Kegel graph is disconnected and this is the case if, and only if, its augmentation ideal decomposes [9, Thm. 1 and Prop. 4] and [26, Theorem 6]. Note that the proof of these equivalences requires the classification of the finite simple groups. The question on the decomposition of augmentation ideals was certainly the motivation for K. W. Gruenberg and O. Kegel to determine precisely the structure of finite groups  $G$  with disconnected  $\Pi(G)$  [26, Theorem A]. Isolated subgroups are Hall subgroups. Thus the first part of Theorem A of the introduction is a special case of the following

**Theorem 2.4.** *Let  $H$  be a group basis of  $\mathbb{Z}G$  and let  $W$  be a Hall subgroup of  $H$ . Assume that for each prime  $p \in \pi(W)$  there is a  $w \in W$  of order  $p$  such that  $C_H(w) \leq W$ . Then the normalized torsion elements of the centralizer ring  $C_{\mathbb{Z}G}(W)$  are the central elements of  $W$ .*

**Proof.** Let  $u \in C_{\mathbb{V}(\mathbb{Z}G)}(W)$  be of prime order  $q$ . Assume there is a  $p \in \pi(W)$  different from  $q$ . By assumption there is  $w \in W$  with  $o(w) = p$  whose  $H$ -centralizer is contained in  $W$ . By Lemma 2.1 we see that  $q$  divides the order of  $C_H(w)$ . It follows that  $q \in \pi(W)$ .

Let  $Q$  be a Sylow  $q$ -subgroup of  $W$  and let  $v \in C_{\mathbb{V}(\mathbb{Z}G)}(W)$  of prime power order  $q^m$ . Assume that  $v$  is not contained in  $W$ , then  $\langle v, Q \rangle$  is a finite  $q$ -group of order strictly bigger than  $Q$ . But  $W$  is supposed to be a Hall subgroup of  $H$  and it is well known that the order of a torsion subgroup of  $\mathbb{V}(\mathbb{Z}G)$  divides the order of  $G$ . This shows that  $|Q| < |\langle v, Q \rangle|$  is impossible and we get that  $v \in W$ . Now it follows that each torsion element of  $C_{\mathbb{V}(\mathbb{Z}G)}(W)$  is contained in  $W$ .  $\square$

The next result is mentioned in [16].

**Proposition 2.5.** *Let  $G$  be a Frobenius group and  $C$  be a Frobenius complement of  $G$ . Then*

$$\Pi(\mathbb{V}(\mathbb{Z}G)) = \Pi(G) \quad \text{and} \quad \Pi(\mathbb{V}(\mathbb{Z}C)) = \Pi(C).$$

**Proof.** By [16, Proposition 4.3]  $\Pi(\mathbb{V}(\mathbb{Z}G)) = \Pi(G)$  provided  $G$  has a normal soluble subgroup  $N$  such that  $\Pi(\mathbb{V}(\mathbb{Z}H/N)) = \Pi(H/N)$ . Thus, if  $G$  is soluble, the statement follows immediately. An insoluble Frobenius group is a soluble extension of  $A_5$  or  $S_5$ . The same holds for  $C$ . But for  $A_5$  and  $S_5$  the first Zassenhaus conjecture ZC1 is valid [18], [19]. This completes the proof.  $\square$

**Proof of Theorem A.** It remains to prove the second part. So let  $U$  be an isolated subgroup of the group basis  $H$ . Put  $V = V(\mathbb{Z}G)$ . If  $N_H(U) = U$  then  $H$  is a Frobenius group and  $U$  a Frobenius complement. By Proposition 2.5 we get that  $\Pi(V) = \Pi(H)$ . Because  $\Pi(U)$  is a full subgraph of  $\Pi(H)$  the Gruenberg-Kegel graph of  $N_V(U)$  must coincide with that one of  $U$ .

Assume now that  $N_H(U) > U$ . Then  $N_H(U)$  is a Frobenius group with kernel  $U$  and by a well known result of Thompson the Frobenius kernel  $U$  is nilpotent. Let  $x \in N_V(U)$  be of order  $r \cdot q$  where  $r$  and  $q$  are different primes. If  $r$  and  $q$  divide  $|U|$  then they are linked by an edge even in  $\Pi(U)$  because of the nilpotency of  $U$ .

Suppose  $q \in \pi(U)$  and  $r \notin \pi(U)$ . Clearly we may write  $x = x_r \cdot x_q$  with  $o(x_r) = r, o(x_q) = q$  and  $x_r \cdot x_q = x_q \cdot x_r$ . Let  $Q$  be a Sylow  $q$ -subgroup of  $U$ . Because  $U$  is isolated it follows that  $N_H(Q) = N_H(U)$ . Now Coleman's Lemma [6] says that

$$N_V(Q) = N_H(U) \cdot C_V(Q).$$

Note that  $N_V(U) \leq N_V(Q)$ . Thus we can write  $x_r = g_r \cdot c_r$  and  $x_q = g_q \cdot c_q$  with  $g_r, g_q \in N_H(U)$  and  $c_r, c_q \in C_V(Q) \cap N_V(U)$ . Consider the surjective group homomorphism

$$\varphi : N_V(U) \longrightarrow N_H(U)/C_H(Q).$$

Clearly  $C_V(Q) \cap N_V(U)$  coincides with the kernel of  $\varphi$ .

Now  $\varphi(x) = \varphi(g_q) \cdot \varphi(g_r)$ . Because  $q$  and  $r$  are not connected in  $\Pi(N_H(U))$  we see that  $\varphi(g_q) = 1$  or  $\varphi(g_r) = 1$ . If the latter holds then  $x_r$  is a torsion unit in  $C_V(Q)$  of order  $r$ . Consequently by Lemma 2.1 we get that there are elements of order  $r$  in the centralizer  $C_H(w)$  of a non-trivial element  $w$  of  $Q$ . But  $C_H(w) \leq U$  because  $U$  is isolated contradicting that  $r \notin \pi(U)$ . This shows that we must have  $\varphi(g_q) = 1$ . So  $x_q \in C_V(Q)$  and by Theorem 2.4 we get that  $x_q \in Z(Q)$ . By Lemma 2.1 we see that  $C_H(x_q)$  has an element of order  $r$ . But  $C_H(x_q) \leq U, r \notin \pi(U)$  and we have reached at a contradiction.

Suppose  $q \notin \pi(U)$  and  $r \notin \pi(U)$ . As in the preceding case it follows that  $\varphi(g_r) \neq 1$  and  $\varphi(g_q) \neq 1$ . Then  $\varphi(x) = \varphi(g_r) \cdot \varphi(g_q)$  is an element of order  $r \cdot q$ . But then  $N_H(U)$  has an element of order  $r \cdot q$  and there is an edge between the vertices  $r$  and  $q$  in  $\Pi(N_H(U))$ .  $\square$

### 3 Groups with abelian Sylow $p$ -subgroups

**Proposition 3.1.** *Assume that  $G$  has an elementary abelian Sylow 2-subgroup. Then 2-subgroups of  $V(\mathbb{Z}G)$  are isomorphic to a subgroup of  $G$ .*

**Proof.** By [5] we know that 2-elements of  $V(\mathbb{Z}G)$  are involutions. Thus any finite 2-subgroup of  $V(\mathbb{Z}G)$  is elementary abelian. Because the order of a finite subgroup of  $V(\mathbb{Z}G)$  divides the order of  $G$  the result follows.  $\square$

The next proposition in this section is an obvious generalization of a result of [7, Proposition 2.11] on soluble groups.

**Proposition 3.2.** *Assume that  $G$  is a finite  $p$ -constrained group. Suppose that  $G$  has abelian Sylow  $p$ -subgroups. Then a  $p$ -subgroup of  $V(\mathbb{Z}G)$  is rationally conjugate to a subgroup of  $G$ .*

**Proof.** By [7, Lemma 2.1] we may assume that  $O_{p'}(G) = 1$ . By assumption  $G$  is  $p$ -constrained. Thus the generalized Fitting subgroup coincides with the Fitting subgroup  $F(G)$ . But  $F(G) = O_p(G)$  because  $O_{p'}(G) = 1$ . Thus we get  $C_G(O_p(G)) \subset O_p(G)$ . By assumption  $G$  has abelian Sylow  $p$ -subgroups. Thus it follows that  $G$  has a normal Sylow  $p$ -subgroup  $P$ .

Now by a theorem of A. Weiss [21, 38.12] we get that all finite  $p$ -subgroups of  $V(\mathbb{Z}G)$  are conjugate within  $\mathbb{Q}G$  to a subgroup of  $P$ .  $\square$

**Corollary 3.3.** *Let  $G$  be a group with abelian Sylow 2-subgroup whose invariants are pairwise different. Then finite 2-subgroups of  $V(\mathbb{Z}G)$  are rationally conjugate to a subgroup of  $G$ .*



**Proof.** It follows from Burnside's transfer theorem that  $G$  has a normal 2-complement. Thus we may apply Proposition 3.2 and the result follows.  $\square$

Note, if one likes to use the theorem of Feit and Thompson, the corollary follows also from [7, Proposition 2.11].

The next result indicates that conjugacy may also hold in the case when  $G$  is not  $p$ -constraint.

**Proposition 3.4.** *Let  $G$  be a group whose Sylow 2-subgroups are Kleinian four groups or elementary abelian of order 8. Then each 2-subgroup of  $V(\mathbb{Z}G)$  is rationally conjugate to a subgroup of  $G$ .*

**Proof.** If  $G$  is soluble the result follows from Proposition 3.2. We may assume that  $G$  has no normal subgroup of odd order and that  $G$  is insoluble.

If Sylow 2-subgroups of  $G$  are isomorphic to  $C_2 \times C_2$  then  $G$  has a simple non-abelian normal subgroup  $S$  and  $G/S$  is of odd order. By Walter's classification of the simple groups with abelian Sylow 2-subgroups [24, Theorem I] it follows that  $S$  is isomorphic to  $\text{PSL}(2, q)$ , where  $q$  is congruent to 3 or 5 modulo 8.

All involutions in a simple group  $S$  with abelian Sylow 2-subgroups are conjugate. By Theorem 2.2 we get that all involutions in  $V(\mathbb{Z}G)$  are rationally conjugate. By Proposition 3.1 an elementary abelian 2-subgroup  $H$  of  $V(\mathbb{Z}G)$  is isomorphic to a subgroup  $U$  of  $G$ . Let  $\sigma$  be such an isomorphism then  $\chi(\sigma(h)) = \chi(h)$  for each irreducible character of  $G$ . By [21, Lemma 37.6]  $H$  and  $U$  are rationally conjugate.

Assume now that the Sylow 2-subgroups of  $G$  are isomorphic to  $C_2^3$ . Then - again assuming that  $G$  is insoluble and has no normal subgroup of odd order -  $G$  has either a simple normal subgroup with elementary abelian Sylow 2-subgroups of order 8 or a normal subgroup isomorphic to  $C_2 \times S$  and  $S$  is isomorphic to  $\text{PSL}(2, q)$ , where  $q$  is congruent to 3 or 5 modulo 8. In the first case we see as in the foursubgroup case that all involutions in  $V(\mathbb{Z}G)$  are conjugate and the result follows analogously.

In the latter case  $G$  has a central subgroup  $C = \langle t \rangle$  of order 2. Because  $G$  has abelian Sylow 2-subgroups it follows that  $G \cong C \times A$ , where  $A$  is a group of automorphisms of  $S$  which contains  $S = \text{Inn}S$ .  $A$  has only one conjugacy class of involutions. Thus all involutions in  $V(\mathbb{Z}A)$  are conjugate within  $\mathbb{Q}A$ . By Theorem 2.2 it follows that the first Zassenhaus conjecture holds for elements of order two in  $G$ .

$G$  has three conjugacy classes of involutions represented by  $t, t \cdot u$  and  $u$ , where  $u$  is an involution of  $S$ . Let  $H$  be an elementary abelian subgroup of  $V(\mathbb{Z}G)$  of order 4. Assume that  $H$  has two involutions  $h_1, h_2$  with the same partial augmentations as  $u$ . Then define an isomorphism between  $H$  and a Sylow 2-subgroup  $P$  of  $S$  by sending  $h_i$  to two different involutions of  $P$ . It follows as above that  $H$  and  $P$  are conjugate within  $\mathbb{Q}G$ .

Assume now that  $H$  has two involutions  $h_1, h_2$  with the same partial augmentations as  $t \cdot u$ . Let  $P = \langle u_1, u_2 \rangle$  be a Sylow 2-subgroup of  $S$ . Let  $K = \langle t \cdot u_1, t \cdot u_2 \rangle$ . Then define an isomorphism between  $H$  and  $K$  by sending  $h_i$  to  $t \cdot u_i$ . Note that  $H$  must have at least one involution  $h$  with the same partial augmentations as  $u$ . Thus as before we see that  $H$  and  $K$  are conjugate within  $\mathbb{Q}G$ .

Finally suppose that  $H$  is elementary abelian of order 8. The surjective ring homomorphism  $\kappa : \mathbb{Z}G \rightarrow \mathbb{Z}G/C$  induces for each torsion subgroup  $U \leq V(\mathbb{Z}G)$  a group homomorphism  $\tau_U$  from  $U$  into  $V(\mathbb{Z}G/C)$ . If  $u$  is a non-trivial element of the kernel of  $\tau_U$  then  $u$  must have a non-trivial partial augmentation on the class  $\{t\}$ . Because  $t$  is central we get by a well known result of S. D. Berman [2] that  $u$  coincides with  $t$ . On the other hand because the order of a torsion subgroup of  $V(\mathbb{Z}G/C)$  divides  $|G/C|$  the order of the image of  $H$  under  $\tau_H$  is at most 4. Therefore we may write  $H = \langle t \rangle \times H_0$ . By the previous we know that  $H_0$  is rationally conjugate to a

subgroup of  $G$ . Because  $t$  is central the same conjugation takes  $H$  into a subgroup of  $G$ .  $\square$

## 4 Torsion units of $V(\mathbb{Z}\text{PSL}(3, 3))$

First we explain briefly the method of Luthar and Passi extended by Hertweck.

Let  $u \in V(\mathbb{Z}G)$  be a torsion unit of order  $k$ . Let  $\zeta$  be a primitive  $k$ -th root of unity and denote by  $\text{Irr } G = \{\chi_1, \dots, \chi_h\}$  the set of the ordinary irreducible characters of  $G$ . Denote by  $\varepsilon_{C_j}(u)$ ,  $j \in \{1, \dots, h\}$  the partial augmentation of  $u$  with respect to the conjugacy class  $C_j$ . In order to show that  $u$  is rationally conjugate to a group element  $g \in G$ , it suffices to show that  $\varepsilon_C(u^d) = \varepsilon_C(g^d)$  for all conjugacy classes  $C$  of  $G$  and all divisors  $d$  of  $k$  [20, Theorem 2.5]. In particular this means  $\varepsilon_{C_j}(u) = 1$  for one index  $j$  and that all other partial augmentations vanish. If  $k = p$  is a prime then the last condition suffices to show rational conjugacy by Theorem 2.2.

Denote by  $\mu_\ell(u, \chi_i)$  the multiplicity of  $\zeta^\ell$  as eigenvalue of  $D_i(u)$ , where  $D_i$  is a complex representation affording  $\chi_i$ .

For each  $i \in \{1, \dots, h\}$  and each  $\ell \in \{0, \dots, k-1\}$  the following equation admits the calculation of this multiplicity via values of the irreducible characters on powers of  $u$ .

$$\begin{aligned} \mu_\ell(u, \chi_i) &= \frac{1}{k} \sum_{d|k} \text{Tr}_{\mathbb{Q}(\zeta^d)/\mathbb{Q}}(\chi_i(u^d)\zeta^{-d\ell}) \\ &= \frac{1}{k} \sum_{\substack{d|k \\ d \neq 1}} \text{Tr}_{\mathbb{Q}(\zeta^d)/\mathbb{Q}}(\chi_i(u^d)\zeta^{-d\ell}) + \frac{1}{k} \text{Tr}_{\mathbb{Q}(\zeta)/\mathbb{Q}}(\chi_i(u)\zeta^{-\ell}). \end{aligned} \quad (1)$$

Inductively the first term in (1), denoted by  $a_{i,\ell}$ , may be assumed to be known.

$$a_{i,\ell} = \frac{1}{k} \sum_{\substack{d|k \\ d \neq 1}} \text{Tr}_{\mathbb{Q}(\zeta^d)/\mathbb{Q}}(\chi_i(u^d)\zeta^{-d\ell}) = \frac{1}{k} \sum_{\substack{d|k \\ d \neq 1}} \text{Tr}_{\mathbb{Q}(\zeta^d)/\mathbb{Q}}(\chi_i(g^d)\zeta^{-d\ell}).$$

Let  $g_j$  be a representative of  $C_j$ . Because of  $\chi_i(u) = \sum_{j=1}^h \varepsilon_{C_j}(u)\chi_i(g_j)$  the second term in (1) is

$$\frac{1}{k} \text{Tr}_{\mathbb{Q}(\zeta)/\mathbb{Q}}(\chi_i(u)\zeta^{-\ell}) = \frac{1}{k} \sum_{j=1}^h \varepsilon_{C_j}(u) \text{Tr}_{\mathbb{Q}(\zeta)/\mathbb{Q}}(\chi_i(g_j)\zeta^{-\ell}).$$

This leads to a system of linear equations in the unknowns  $\varepsilon_{C_j}(u)$  of the form

$$T\varepsilon + a = \mu,$$

where

$$\begin{aligned} T &= \frac{1}{k} \begin{pmatrix} \text{Tr}_{\mathbb{Q}(\zeta)/\mathbb{Q}}(\chi_1(g_1)\zeta^{-0}) & \dots & \text{Tr}_{\mathbb{Q}(\zeta)/\mathbb{Q}}(\chi_1(g_h)\zeta^{-0}) \\ \text{Tr}_{\mathbb{Q}(\zeta)/\mathbb{Q}}(\chi_1(g_1)\zeta^{-1}) & \dots & \text{Tr}_{\mathbb{Q}(\zeta)/\mathbb{Q}}(\chi_1(g_h)\zeta^{-1}) \\ \vdots & & \vdots \\ \vdots & & \vdots \\ \text{Tr}_{\mathbb{Q}(\zeta)/\mathbb{Q}}(\chi_h(g_1)\zeta^{-k+1}) & \dots & \text{Tr}_{\mathbb{Q}(\zeta)/\mathbb{Q}}(\chi_h(g_h)\zeta^{-k+1}) \end{pmatrix} \in \mathbb{Q}^{hk \times h}, \\ \varepsilon &= \begin{pmatrix} \varepsilon_{C_1}(u) \\ \vdots \\ \varepsilon_{C_h}(u) \end{pmatrix} \in \mathbb{Q}^h, \quad a = \begin{pmatrix} a_{1,0} \\ a_{1,1} \\ \vdots \\ \vdots \\ a_{h,k-1} \end{pmatrix}, \quad \mu = \begin{pmatrix} \mu_0(u, \chi_1) \\ \mu_1(u, \chi_1) \\ \vdots \\ \vdots \\ \mu_{k-1}(u, \chi_h) \end{pmatrix} \in \mathbb{Q}^{hk}. \end{aligned}$$

Because the multiplicities  $\mu_\ell(u, \chi_i)$  are non-negative integers, bounded above by the degree  $\chi_i(1)$ , this gives restrictions on the partial augmentations which have sometimes the desired partial augmentations of  $u$  as the only solutions.

Hertweck extended the method of Luthar and Passi to  $p$ -regular elements and Brauer characters. Let  $p$  be a prime and  $\varphi$  a Brauer character.  $\varphi$  can be extended to  $p$ -regular torsion elements of  $\mathbb{Z}G$ .

Let  $C_1, \dots, C_r$  the conjugacy classes of  $p$ -regular elements. Then

$$\varphi(u) = \sum_{j=1}^r \varepsilon_{C_j}(u) \varphi(g_j),$$

where  $g_j$  denotes a representative of the conjugacy class  $C_j$ . [13, Theorem 3.2].

Let  $(K, R, \overline{R})$  be a  $p$ -modular system sufficiently large for  $G$  with  $\text{char } K = 0$  and let  $u$  be a  $p$ -regular torsion unit of order  $k$  of  $V(\mathbb{Z}G)$ . Denote by  $\zeta$  a primitive  $k$ -th root of unity of  $\overline{R}$  (identify the  $k$ -th roots of unity of  $\overline{R}$  and that one of  $K$ ). Let  $d$  be a representation affording  $\varphi$ , then  $\mu_\ell(u, \varphi, p)$  denotes the multiplicity of  $\zeta^\ell$  as eigenvalue of  $d(u)$ .

Then analogously to the case of ordinary characters

$$\mu_\ell(u, \varphi, p) = \frac{1}{k} \sum_{d|k} \text{Tr}_{\mathbb{Q}(\zeta^d)/\mathbb{Q}}(\varphi(u^d) \zeta^{-d\ell})$$

for  $0 \leq \ell \leq k - 1$ .

The Luthar-Passi-Hertweck method is of course supported by general results on partial augmentations of a torsion unit  $u$  of  $V(\mathbb{Z}G)$ . Some of them are stated at the beginning of Section 2. For the following note that  $\exp \text{PSL}(3, 3) = 8 \cdot 3 \cdot 13$  and  $\text{PSL}(3, 3)$  has elements of order 6.

**Proposition 4.1.** *Let  $G = \text{PSL}(3, 3)$  and  $u \in V(\mathbb{Z}G)$ . Then*

- a)  *$u$  is rationally conjugate to an element of  $G$  provided  $o(u) \in \{2, 4, 8, 13\}$ .*
- b) *There are no elements of order 26 and 39 in  $V(\mathbb{Z}G)$ .*

**Proof.** We use parts of the ordinary character table shown in table 2 and parts of the 3-modular Brauer table as shown in table 1. These character tables have been taken from [8].<sup>2</sup> Throughout  $u$  denotes a torsion unit of the considered order and we apply the Luthar-Passi-Hertweck method. Note that  $\sum_{j=1}^h \varepsilon_{C_j}(u) = 1$  because  $u$  is normalized.

- *Torsion units of order 2 of  $V(\mathbb{Z}G)$  are conjugate within  $\mathbb{Q}G$  to group elements.*  
By Berman and Higman and Theorem 2.2 the only possibly non-zero partial augmentation is  $\varepsilon_{2a}(u)$ .<sup>3</sup>

- *Torsion units of order 4 of  $V(\mathbb{Z}G)$  are conjugate within  $\mathbb{Q}G$  to group elements.*

We get

$$\begin{aligned} 2\mu_0(u, \varphi_2, 3) &= -\varepsilon_{2a}(u) + \varepsilon_{4a}(u) + 1 \geq 0, \\ 2\mu_2(u, \varphi_2, 3) &= \varepsilon_{2a}(u) - \varepsilon_{4a}(u) + 1 \geq 0. \end{aligned}$$

Adding the first inequation with  $\varepsilon_{2a}(u) + \varepsilon_{4a}(u) - 1 \geq 0$  and the second with  $-\varepsilon_{2a}(u) - \varepsilon_{4a}(u) + 1 \geq 0$  it follows that  $0 \leq \varepsilon_{4a}(u) \leq 1$ .

As  $\varepsilon_{2a}(u) + \varepsilon_{4a}(u) = 1$ , we have  $(\varepsilon_{2a}(u), \varepsilon_{4a}(u)) \in \{(0, 1), (1, 0)\}$ . By a result of A. A. Bovdi [3] the sum over all partial augmentations on classes of order  $p^m$  of a unit of order  $p^n$  is divisible by  $p$  provided  $m < n$ . Thus the latter pair is impossible.

<sup>2</sup>obtained in GAP by the commands `CharacterTable("PSL(3,3)"), CharacterTable("PSL(3,3) mod 3` resp.

<sup>3</sup>Throughout we use the GAP - notation for the conjugacy classes

	1a	2a	4a	8a	8b	13a	13b	13c	13d
$\varphi_2$	3	-1	1	A	$\overline{A}$	C	$\overline{C}$	G	$\overline{G}$
$\varphi_4$	6	2	.	B	$-\overline{B}$	D	$\overline{D}$	H	$\overline{H}$

$$\begin{aligned}
A &= -1 + \zeta_8 + \zeta_8^3 = -1 + \sqrt{2}i & B &= -\zeta_8 - \zeta_8^3 = -\sqrt{2}i \\
C &= \zeta_{13} + \zeta_{13}^3 + \zeta_{13}^9 & D &= \zeta_{13}^2 + \zeta_{13}^4 + \zeta_{13}^5 + \zeta_{13}^6 + \zeta_{13}^{10} + \zeta_{13}^{12} \\
G &= \zeta_{13}^2 + \zeta_{13}^5 + \zeta_{13}^6 & H &= \zeta_{13}^4 + \zeta_{13}^7 + \zeta_{13}^8 + \zeta_{13}^{10} + \zeta_{13}^{11} + \zeta_{13}^{12}
\end{aligned}$$

Table 1: Parts of the 3-modular Brauer table of  $\mathrm{PSL}(3, 3)$

- *Torsion units of order 8 of  $V(\mathbb{Z}G)$  are conjugate within  $\mathbb{Q}G$  to group elements.*  
Using  $\varphi_4$  we obtain

$$\begin{aligned}
\mu_0(u, \varphi_4, 3) &= \varepsilon_{2a}(u) + 1 \geq 0 \\
\mu_4(u, \varphi_4, 3) &= -\varepsilon_{2a}(u) + 1 \geq 0
\end{aligned}$$

and hence  $-1 \leq \varepsilon_{2a}(u) \leq 1$ . This forces  $\varepsilon_{2a}(u) = 0$ , again by [3]. With this restriction we get

$$\begin{aligned}
2\mu_0(u, \varphi_2, 3) &= \varepsilon_{4a}(u) - \varepsilon_{8a}(u) - \varepsilon_{8b}(u) + 1 \geq 0, \\
2\mu_4(u, \varphi_2, 3) &= -\varepsilon_{4a}(u) + \varepsilon_{8a}(u) + \varepsilon_{8b}(u) + 1 \geq 0.
\end{aligned}$$

Adding with  $\varepsilon_{4a}(u) + \varepsilon_{8a}(u) + \varepsilon_{8b}(u) - 1 \geq 0$  and  $-\varepsilon_{4a}(u) - \varepsilon_{8a}(u) - \varepsilon_{8b}(u) + 1 \geq 0$ , respectively, we get

$$0 \leq \varepsilon_{4a}(u) \leq 1$$

and again by [3] this yields  $\varepsilon_{4a}(u) = 0$ . The inequalities

$$\begin{aligned}
2\mu_1(u, \varphi_2, 3) &= \varepsilon_{8a}(u) - \varepsilon_{8b}(u) + 1 \geq 0 \\
2\mu_5(u, \varphi_2, 3) &= -\varepsilon_{8a}(u) + \varepsilon_{8b}(u) + 1 \geq 0
\end{aligned}$$

together with  $\varepsilon_{8a}(u) + \varepsilon_{8b}(u) - 1 \geq 0$  and  $-\varepsilon_{8a}(u) - \varepsilon_{8b}(u) + 1 \geq 0$  show  $\varepsilon_{8a}(u), \varepsilon_{8b}(u) \in \{0, 1\}$ . Thus  $u$  is conjugate to an element of  $G$  within  $\mathbb{Q}G$ .

- *Torsion units of order 13 of  $V(\mathbb{Z}G)$  are conjugate within  $\mathbb{Q}G$  to group elements.*  
We get the inequalities

$$13\mu_1(u, \varphi_2, 3) = 10\varepsilon_{13a}(u) - 3\varepsilon_{13b}(u) - 3\varepsilon_{13c}(u) - 3\varepsilon_{13d}(u) + 3 \geq 0, \quad (2)$$

$$13\mu_2(u, \varphi_2, 3) = -3\varepsilon_{13a}(u) - 3\varepsilon_{13b}(u) + 10\varepsilon_{13c}(u) - 3\varepsilon_{13d}(u) + 3 \geq 0, \quad (3)$$

$$13\mu_4(u, \varphi_2, 3) = -3\varepsilon_{13a}(u) + 10\varepsilon_{13b}(u) - 3\varepsilon_{13c}(u) - 3\varepsilon_{13d}(u) + 3 \geq 0, \quad (4)$$

$$13\mu_7(u, \varphi_2, 3) = -3\varepsilon_{13a}(u) - 3\varepsilon_{13b}(u) - 3\varepsilon_{13c}(u) + 10\varepsilon_{13d}(u) + 3 \geq 0. \quad (5)$$

Adding (5) and three times  $\varepsilon_{13a}(u) + \varepsilon_{13b}(u) + \varepsilon_{13c}(u) + \varepsilon_{13d}(u) - 1 \geq 0$  gives  $\varepsilon_{13d}(u) \geq 0$ . By adding (2), (3), (4) and four times  $-\varepsilon_{13a}(u) - \varepsilon_{13b}(u) - \varepsilon_{13c}(u) - \varepsilon_{13d}(u) + 1 \geq 0$  we get  $\varepsilon_{13d}(u) \leq 1$ . By symmetry in the inequations (2), (3), (4), (5) and the augmentation we get the same results for  $\varepsilon_{13a}(u), \varepsilon_{13b}(u)$  and  $\varepsilon_{13c}(u)$ . Hence there is only one non-vanishing partial augmentation and we are done.

- *There is no torsion unit of order 26 in  $V(\mathbb{Z}G)$ .*  
 $\chi_8$  vanishes on all elements of order 13. Thus we get

$$\begin{aligned}
13\mu_0(u, \chi_8) &= 12\varepsilon_{2a}(u) + 14 \geq 0 \\
13\mu_{13}(u, \chi_8) &= -12\varepsilon_{2a}(u) + 12 \geq 0
\end{aligned}$$

	1a	2a	3a	3b	4a	6a	8a	8b	13a	13b	13c	13d
$\chi_2$	12	4	3	.	.	1	.	.	-1	-1	-1	-1
$\chi_3$	13	-3	4	1	1	.	-1	-1	.	.	.	.
$\chi_8$	26	2	-1	-1	2	-1	.	.	.	.	.	.

Table 2: Parts of the ordinary character table of  $\text{PSL}(3, 3)$

and hence  $-1 \leq \varepsilon_{2a}(u) \leq 1$ . This contradicts the fact that

$$\mu_0(u, \chi_3) = -\frac{18}{13}\varepsilon_{2a}(u) + \frac{5}{13}$$

is a non-negative integer.

- *There is no torsion unit of order 39 in  $V(\mathbb{Z}G)$ .*

As  $\chi_8$  vanishes on the classes of elements of order 13 and  $\chi_8(3a) = \chi_8(3b)$  the  $a_{s,\ell}$  are independent of the partial augmentations of  $u^3$  and  $u^{13}$  and the inequalities

$$\begin{aligned} 13\mu_0(u, \chi_8) &= -8\varepsilon_{3a}(u) - 8\varepsilon_{3b}(u) + 8 \geq 0 \\ 13\mu_{13}(u, \chi_8) &= 4\varepsilon_{3a}(u) + 4\varepsilon_{3b}(u) + 9 \geq 0 \end{aligned}$$

hold for every set of partial augmentations of  $u^3$  and  $u^{13}$ .

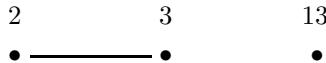
Hence we see  $-2 \leq \varepsilon_{3a}(u) + \varepsilon_{3b}(u) \leq 1$ . But this violates

$$\mu_1(u, \chi_8) = -\frac{1}{39}\varepsilon_{3a}(u) - \frac{1}{39}\varepsilon_{3b}(u) + \frac{9}{13} \in \mathbb{Z}$$

in all of the possible cases.

□

**Corollary 4.2.** *The Gruenberg-Kegel graphs of  $\text{PSL}(3, 3)$  and  $V(\mathbb{Z}\text{PSL}(3, 3))$  coincide:*



## 5 Elementary abelian subgroups

**Proposition 5.1.** *Let  $G = \text{PSL}(3, 3)$ ,  $p \in \{2, 3, 13\}$  and  $H \leq V(\mathbb{Z}G)$  an elementary abelian  $p$ -group,*

$$H \cong \underbrace{C_p \times \dots \times C_p}_{k\text{-times}}$$

a) *If  $p = 2$ , then  $k \leq 2$  and  $H$  is isomorphic to a subgroup of  $G$ .*

b) *If  $p = 3$ , then  $k \leq 3$ .*

c) *If  $p = 13$ , then  $k \leq 1$  and  $H$  is isomorphic to a subgroup of  $G$ .*

**Proof.** We use again parts of the ordinary character table of  $G$ , cf. table 2. We extend the representation affording irreducible character  $\chi$  of  $G$  to a representation  $\rho$  of  $\mathbb{C}G$ . Then  $\rho$  restricted to  $H$  has character  $\chi_H$ .

- $p = 2$ . Consider  $(\chi_3)_H$ . Because every involution of  $V(\mathbb{Z}G)$  is rationally conjugate to an

element of the class **2a** we get

$$\begin{aligned}\mathbb{N}_0 \ni (1_H, (\chi_3)_H)_H &= \frac{1}{|H|} \sum_{h \in H} 1_H(h) \chi_3(h^{-1}) = \frac{1}{|H|} (\chi_3(1) + (|H| - 1)(-3)) \\ &= \frac{1}{2^k} (13 + (2^k - 1)(-3)).\end{aligned}$$

Thus  $13 + (2^k - 1)(-3) \geq 0$  and it follows that  $k \leq 2$ .

- $p = 3$ . As the sum of augmentations of the elements of order 3 on the classes of elements of order 3 is 1 and  $\chi_8$  takes the value  $-1$  on all group elements of order 3 we get

$$\begin{aligned}\mathbb{N}_0 \ni (1_H, (\chi_8)_H)_H &= \frac{1}{|H|} \sum_{h \in H} 1_H(h) \chi_8(h^{-1}) = \frac{1}{|H|} (\chi_8(1) + (|H| - 1)(-1)) \\ &= \frac{1}{3^k} (26 + (3^k - 1)(-1)).\end{aligned}$$

This shows  $26 + (3^k - 1)(-1) \geq 0$  and therefore  $k \leq 3$ .

- $p = 13$ . A similar argument as in the above case gives

$$\begin{aligned}\mathbb{N}_0 \ni (1_H, (\chi_2)_H)_H &= \frac{1}{|H|} \sum_{h \in H} 1_H(h) \chi_2(h^{-1}) = \frac{1}{|H|} (\chi_2(1) + (|H| - 1)(-1)) \\ &= \frac{1}{13^k} (12 + (13^k - 1)(-1)).\end{aligned}$$

Hence  $12 + (13^k - 1)(-1) \geq 0$  which implies  $k \leq 1$ .

□

**Remark 5.2.** The case  $p = 13$  in Proposition 5.1 follows also from [12, Corollary 1] because Sylow 13-subgroups of  $\text{PSL}(3, 3)$  are cyclic.

Looking solely at ordinary characters it is not possible to show that  $C_3^3$  does not occur as a subgroup of  $V(\mathbb{Z}\text{PSL}(3, 3))$ .

**Proposition 5.3.** *Each elementary abelian subgroup of  $V(\mathbb{Z}\text{Sz}(q))$  is isomorphic to a subgroup of  $\text{Sz}(q)$ .*

**Proof.** Let  $H \leq V(\mathbb{Z}\text{Sz}(q))$  be an elementary abelian  $p$ -group of order  $p^k$ . If  $p$  is odd the Sylow  $p$ -subgroups of  $\text{Sz}(q)$  are cyclic. By [12, Corollary 1] all finite  $p$ -subgroups of  $V(\mathbb{Z}\text{Sz}(q))$  are cyclic and isomorphic to a subgroup of  $\text{Sz}(q)$ .

Assume now that  $p = 2$ . Consider the generic character table of  $\text{Sz}(q)$  as given in [8], cf. table 3. As in the proof of Proposition 5.1 we extend the representation of  $\text{Sz}(q)$  which affords  $\delta_1$  to a representation  $\rho$  of  $\mathbb{C}\text{Sz}(q)$ . Then  $\rho$  restricted to  $H$  has character  $(\delta_1)_H$ . Because  $\text{Sz}(q)$  has only one conjugacy class of involutions all non-trivial elements of  $H$  are rationally conjugate to an involution  $t$  of  $\text{Sz}(q)$ . Thus for the multiplicity of the trivial character  $1_H$  of  $H$  in  $(\delta_1)_H$  with  $r = 2^m$  we get

$$\begin{aligned}(1_H, (\delta_1)_H)_H &= \frac{1}{|H|} \sum_{h \in H} 1_H(h) \delta_1(h^{-1}) = \frac{1}{|H|} (\delta_1(1) + (|H| - 1)\delta_1(t)) \\ &= \frac{1}{2^k} (r(q - 1) + (2^k - 1)(-r)) = \frac{1}{2^k} (rq - 2^k r).\end{aligned}$$

Since  $(1_H, (\delta_1)_H)_H \in \mathbb{N}_0$  we see that  $rq - 2^k r \geq 0$ . Thus it follows that  $k \leq 2m + 1$ .

On the other hand the centre of a Sylow 2-subgroup of  $\text{Sz}(q)$  is isomorphic to  $C_2^{2m+1}$ . Consequently  $H$  is isomorphic to a subgroup of  $\text{Sz}(q)$ . □

	1	$x^a$	$y^b$	$z^c$	$t$	$f$	$f^{-1}$
1	1	1	1	1	1	1	1
$\alpha$	$q^2$	1	-1	-1	.	.	.
$\beta_j$	$q^2 + 1$	$\omega^{ja} + \omega^{-ja}$	.	.	1	1	1
$\gamma_k^{(1)}$	$(q-1)(q-2r+1)$	.	$g(b, k)$	.	$2r-1$	-1	-1
$\gamma_\ell^{(2)}$	$(q-1)(q+2r+1)$	.	.	$h(c, \ell)$	$-2r-1$	-1	-1
$\delta_1$	$r(q-1)$	.	1	-1	$-r$	$ri$	$-ri$
$\delta_2$	$r(q-1)$	.	1	-1	$-r$	$-ri$	$ri$

with  $q = 2^{2m+1}$ ,  $r = 2^m$ ,  $x, y, z, t, f \in G$  and  $\omega, \zeta, \eta, i \in \mathbb{C}$ , where

$x$  has order  $q-1$ ,  $\omega$  is a primitive  $(q-1)$ -th root of unity  
 $y$  has order  $q+2r+1$ ,  $\zeta$  is a primitive  $(q+2r+1)$ -th root of unity  
 $z$  has order  $q-2r+1$ ,  $\eta$  is a primitive  $(q-2r+1)$ -th root of unity  
 $t$  is an involution,  
 $f$  has order 4,  $i$  is a primitive 4-th root of unity.

Further

$$\begin{aligned}
g(b, k) &= -\zeta^{kb} - \zeta^{kbq} - \zeta^{-kb} - \zeta^{-kbq}, \\
h(c, \ell) &= -\eta^{\ell c} - \eta^{\ell cq} - \eta^{-\ell c} - \eta^{-\ell cq},
\end{aligned}$$

and  $1 \leq a, j \leq \frac{1}{2}(q-2)$ ,  $b, k \in I_1$  and  $c, \ell \in I_2$ .  $I_1$  and  $I_2$  are appropriate index sets with  $|I_1| = \frac{q+2r}{4}$ ,  $|I_2| = \frac{q-2r}{4}$ .

Table 3: Generic character table of  $\text{Sz}(q)$

**Corollary 5.4.** *Let  $H$  be an elementary abelian 2- or 5-subgroup of  $V(\mathbb{Z}\text{Sz}(q))$ . Then  $H$  is rationally conjugate to a subgroup  $U$  of  $\text{Sz}(q)$ .*

**Proof.** By 5.3 there is an isomorphism  $\varphi : H \rightarrow U \leq \text{Sz}(q)$ . Then for  $h \in H$  and all  $\chi \in \text{Irr}(\text{Sz}(q))$  we see that  $\chi(h) = \chi(\varphi(h))$  because  $\text{Sz}(q)$  has only one conjugacy class of elements of order 2, 5 resp. Thus we may apply [21, Lemma 37.6] and it follows that  $H$  and  $U$  are conjugate within  $\mathbb{Q}G$ .  $\square$

**Remark 5.5.** In [1, Korollar 4.2] it is shown that  $\Pi(V(\mathbb{Z}\text{Sz}(8))) = \Pi(\text{Sz}(8))$ . Even each cyclic subgroup of  $V(\mathbb{Z}\text{Sz}(8))$  is isomorphic to a subgroup of  $\text{Sz}(8)$ .

**Proof of Theorem B.** It suffices to show the result when  $H$  is elementary abelian. M. Suzuki classified the simple groups with a non-trivial partition [22]. These are the groups  $\text{PSL}(2, q)$  and  $\text{Sz}(q)$ . By [14, Theorem 2.1] we get that the 2-rank of an elementary abelian subgroup  $H$  is less or equal to the 2-rank of  $\text{PSL}(2, q)$ . For odd primes  $p$  not dividing  $q$  the Sylow subgroups of  $\text{PSL}(2, q)$  are cyclic. Again by [12, Corollary 1] all finite  $p$ -subgroups of  $V(\mathbb{Z}\text{PSL}(2, q))$  are cyclic and therefore isomorphic to a subgroup of  $\text{PSL}(2, q)$ . If the characteristic  $q = p^f$  is odd the Sylow  $p$ -subgroups are elementary abelian. Because  $|H|$  divides  $|G|$  the result follows also in this case. Finally Proposition 5.3 concludes the proof.  $\square$

**Proof of Proposition C.** By a famous result of J. Thompson a minimal simple group is isomorphic to a group of the series  $\text{Sz}(q)$ ,  $\text{PSL}(2, q)$  or to  $\text{PSL}(3, 3)$  [23]. The result follows now from Proposition 5.1, Proposition 5.3 and from [12], [14] as in the proof of Theorem B.  $\square$

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