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On semiclassical and universal inequalities for
eigenvalues of quantum graphs

Preprint 2009/006

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Fachbereich Mathematik

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WWW: <http://www.mathematik.uni-stuttgart.de/preprints>

ISSN 1613-8309

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L^AT_EX-Style: Winfried Geis, Thomas Merkle

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Abstract

We study the spectra of quantum graphs with the method of trace identities (sum rules), which are used to derive inequalities of Lieb-Thirring, Payne-Pólya-Weinberger, and Yang types, among others. We show that the sharp constants of these inequalities and even their forms depend on the topology of the graph. Conditions are identified under which the sharp constants are the same as for the classical inequalities; in particular, this is true in the case of trees. We also provide some counterexamples where the classical form of the inequalities is false.

On semiclassical and universal inequalities for eigenvalues of quantum graphs

Semra Demirel and Evans M. Harrell II

January 15, 2010

1 Introduction

This article is focused on inequalities for the means, moments, and ratios of eigenvalues of quantum graphs. A quantum graph is a metric graph with one-dimensional Schrödinger operators acting on the edges and appropriate boundary conditions imposed at the vertices and at the finite external ends, if any. Here we shall define the Hamiltonian H on a quantum graph as the minimal (Friedrichs) self-adjoint extension of the quadratic form

$$\phi \in C_c^\infty \mapsto E(\phi) := \int_\Gamma |\phi'|^2 ds, \quad (1.1)$$

which leads to vanishing Dirichlet boundary conditions at the ends of exterior edges and to the conditions at each vertex v_k that ϕ is continuous and moreover

$$\sum_j \frac{\partial \phi}{\partial x_{kj}}(0^+) = 0, \quad (1.2)$$

where the sum runs over all edges emanating from v_k , and x_{kj} designates the distance from v_k along the j -th edge. (Edges connecting v_k to itself are accounted twice.) In the literature these vertex conditions are usually known as Kirchhoff or Neumann conditions. Other vertex conditions are possible, and are amenable to our methods with some complications, but they will not be considered in this article. For details about the definition of H we refer to [15].

Quantum mechanics on graphs has a long history in physics and physical chemistry [21, 24], but recent progress in experimental solid state physics has renewed attention on them as idealized models for thin domains. While the problem of quantum systems in high dimensions has to be solved numerically, since quantum graphs are locally one dimensional their spectra can often be determined explicitly. A large literature on the subject has arisen, for which we refer to the bibliography given in [3, 7].

The subject of inequalities for means, moments, and ratios of eigenvalues is rather well developed for Laplacians on domains and for Schrödinger operators, and it is our aim to determine the extent to which analogous theorems apply to quantum graphs. For example, when there is a potential energy $V(x)$ in appropriate function spaces, Lieb-Thirring inequalities provide an upper bound for the moments of the negative eigenvalues $E_j(\alpha)$ of the Schrödinger operator $H(\alpha) = -\alpha \nabla^2 + V(x)$ in $L^2(\mathbb{R}^d)$, $\alpha > 0$, of the form

$$\alpha^{d/2} \sum_{E_j(\alpha) < 0} (-E_j(\alpha))^\gamma \leq L_{\gamma,d} \int_{\mathbb{R}^d} (V_-(x))^{\gamma+d/2} dx \quad (1.3)$$

for some constant $L_{\gamma,d} \geq L_{\gamma,d}^{cl}$, where $L_{\gamma,d}^{cl}$, known as the *classical constant*, is given by

$$L_{\gamma,d}^{cl} = \frac{1}{(4\pi)^{d/2}} \frac{\Gamma(\gamma+1)}{\Gamma(\gamma+d/2+1)}.$$

It is known that (1.3) holds true for various ranges of $\gamma \geq 0$ depending on the dimension d ; see [5, 13, 19, 20, 23, 27]. In particular, in [18] Laptev and Weidl proved that $L_{\gamma,d} = L_{\gamma,d}^{cl}$ for all $\gamma \geq 3/2$ and $d \geq 1$, and Stubbe [25] has recently given a new proof of sharp Lieb-Thirring inequalities for $\gamma \geq 2$ and $d \geq 1$ by showing monotonicity with respect to coupling constants. His proof is based on general trace identities for operators [10, 11] known as sum rules, which will again be used as the foundation of the present article.

When there is no potential energy but instead the Laplacian is given Dirichlet conditions on the boundary of a bounded domain, then the means of the first n eigenvalues are bounded from below by the Berezin-Li-Yau inequality in terms of the volume of the domain, and in addition there is a large family of universal bounds on the spectrum, dating from the work of Payne, Pólya, and Weinberger [22], which constrain the spectrum without any reference to properties of the domain. (For a review of the subject, see [2].) It turns out that there are far-reaching analogies between these “universal” inequalities for Dirichlet Laplacians and Lieb-Thirring inequalities, which have led to common proofs based on sum rules [8–12, 25]. More precisely, some sharp Lieb-Thirring inequalities and some universal inequalities of the PPW family can be viewed as corollaries of a “Yang-type” inequality like (2.5) below, which in turn follows from a sum rule identity.

In one dimension a domain is merely an interval and the spectrum of the Dirichlet Laplacian is a familiar elementary calculation, for which the question of universal bounds is trivial and uninteresting. A quantum graph, however, has a spectrum that responds in complex ways to its connectedness; if the total length is finite and appropriate boundary conditions are imposed at exterior vertices, then the spectrum is discrete, and questions about counting functions, moments, etc. and their relation to the topology of the graph become interesting, even in the absence of a potential energy. Below we shall prove several inequalities for the spectra of finite quantum graphs, with the aid of the same trace identities we use to derive Lieb-Thirring inequalities.

For Lieb-Thirring inequalities on quantum graphs the essential question is whether a form of (1.3) holds with the sharp constant for $d = 1$, or whether the connectedness of the graph can change the state of affairs. In [6] T. Ekholm, R. Frank and H. Kovařík proved Lieb-Thirring inequalities for Schrödinger operators on regular metric trees for any $\gamma \geq 1/2$, but without sharp constants. We shall show below that trees enjoy a Lieb-Thirring inequality with the sharp constant when $\gamma \geq 2$, but that this circumstance depends on the topology of the graph.

We begin with some simple explicit examples showing that neither the expected Lieb-Thirring inequality nor the analogous universal inequalities for finite quantum graphs without potential hold in complete generality. As it will be convenient to have a uniform way of describing examples, we shall let x_{ij} denote the distance from vertex v_i along the j -th edge Γ_j emanating from v_i . We note that every edge corresponds to two distinct coordinates $x_{ij} = L - x_{i'j'}$ where L is the length of the edge, and that a homoclinic loop from a vertex v_i to itself is accounted as two edges.

For the operator $-\frac{d^2}{dx^2}$ on an interval, with vanishing Dirichlet boundary conditions, the universal inequality of Payne-Pólya-Weinberger reduces to $E_2/E_1 \leq 5$, and the Ashbaugh-Benguria theorem becomes $E_2/E_1 \leq 4$, both of which are trivial in one dimension. But for which quantum graphs do these classic inequalities continue to be valid? We shall show below that the classic PPW and related inequalities can be proved for the case of trees, with Dirichlet boundary conditions imposed at all external ends of edges, using the method of sum rules. The sum-rule proof does not work for every graph, however, so the question naturally arises whether the topology makes a real difference, or whether a better method of proof is required. The following examples show that the failure of the sum-rule proof in the case of multiply connected graphs is not an artifact of the method but due to a true topological effect.

We refer to graphs consisting of a circle attached to a single external edge as “simple balloon graphs.” The external edge may either be infinite or of finite length with a vanishing boundary condition at its exterior end. Consider first the graph $\Gamma := \Gamma_1 \cup \Gamma_2$, which consists of a loop Γ_1 to which a finite external interval Γ_2 is attached at a vertex v_1 . Without loss of generality we may fix the length of the loop as 2π , while the “string” will be of length L .

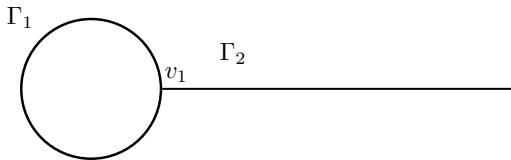


Figure 1: “balloon graph”

Example 1.1. (Violation of the analogue of PPW.) Let us begin with the case of a balloon graph with $L < \infty$, and assume that there is no potential. We set $\alpha = 1$. Thus H locally has the form $-\frac{d^2}{dx^2}$ with Dirichlet conditions at the end of the string Γ_2 and vertex conditions (1.2) at v_1 connecting it to the loop.

For convenience we slightly simplify the coordinate system, letting $x_s := x_{12}$ be the distance on $\Gamma_s = \Gamma_2$ from the node, and $x_\ell := x_{11} - \pi$ on Γ_1 . Thus x_ℓ increases from $-\pi$ at v_1 to $x_2 = +\pi$ when it joins it again. It is possible to analyze the eigenvalues of the balloon graph quite explicitly: With a Dirichlet condition at $x_s = L$, any eigenfunction must be of the form $a \sin(k(L - x_s))$ on Γ_2 . On Γ_1 symmetry dictates that the eigenfunction must be proportional to either $\sin kx_\ell$ or $\cos kx_\ell$. There are thus two categories of eigenfunctions and eigenvalues. Eigenfunctions of the form $\sin kx_\ell$ contribute nothing to the vertex condition (1.2) (because the outward derivatives at the node are equal in magnitude with opposite signs), and therefore the derivative of $a \sin(k(L - x_s))$ must vanish at $x_s = 0$. If k is a positive integer, then k^2 is an eigenvalue corresponding to an eigenfunction that vanishes on Γ_2 . Otherwise, the conditions on Γ_2 cannot be achieved without violating the condition of continuity with the eigenfunction on Γ_1 . To summarize: the eigenvalues of the first category are the squares of positive integers.

The second category of eigenfunctions match $\cos kx_\ell$ on the loop to $a \sin(k(L - x_s))$ on the interval. The boundary conditions and continuity lead after a standard calculation to the transcendental equation

$$\cot kL = 2 \tan k\pi. \tag{1.4}$$

There are three interesting situations to consider. In the limit $L \rightarrow 0$, an asymptotic analysis of (1.4) shows that the eigenvalues tend to $\{(\frac{n}{2})^2\}$. In the limit $L \rightarrow \infty$, the lower eigenvalues tend to $\{(n + \frac{1}{2})^2 \frac{\pi^2}{L^2}\}$, which are the eigenvalues of an interval of length L with Dirichlet conditions at L and Neumann conditions at 0. The ratio of the first two eigenvalues in this limit is approximately 9, which is already greater than the classically anticipated value of 5 or 4. The highest value of the ratio is, somewhat surprisingly, attained for an intermediate value of L , viz., $L = \pi$, for which (1.4) can be easily solved, yielding $k = \pm \frac{1}{\pi} \arctan \frac{1}{\sqrt{2}} + j$ for a positive integer j . The corresponding fundamental ratio of the lowest two eigenvalues becomes

$$\frac{E_2}{E_1} = \left(\frac{\pi - \arctan \frac{1}{\sqrt{2}}}{\arctan \frac{1}{\sqrt{2}}} \right)^2 \doteq 16.8453.$$

(We spare the reader the direct calculation showing that the critical value of the ratio occurs precisely at $L = \pi$, establishing this value as the maximum among all simple balloons.)

Example 1.2. (Showing that E_2/E_1 can be arbitrarily large.) A modification of Example 1.1 with more complex topology shows that no upper bound on the ratio of the first two eigenvalues is possible for the graph analogue of the Dirichlet problem. We again set $\alpha = 1$ and assume $V = 0$, and consider a “fancy balloon” graph consisting of an external edge, Γ_s , the “string,” of length π joined at v_1 to N edges $\Gamma_m, m = 1 \dots N$ of length π , all of which meet at a second vertex v_2 . We observe that the eigenfunctions may be chosen either even or odd under pairwise permutation of the edges Γ_m . This is because if Pf represents the linear transformation of a function f defined on the graph by permuting two of the variables $\{x_{21}, \dots, x_{2N}\}$, and ϕ_j is an eigenfunction of the quantum graph with eigenvalue E_j , then so are $\phi_j \pm P\phi_j$. (In particular, continuity and (1.2) are

preserved by these superpositions.) Moreover, the fundamental eigenfunction is even under any permutation, because it is unique and does not change sign.

By continuity and the conditions (1.2) at the vertices, as in Example 1.1, a straightforward exercise shows that $E_1 = \left(\frac{1}{\pi} \arctan\left(\frac{1}{\sqrt{N}}\right)\right)^2$, and that there are other even-parity eigenvalues

$$\left(j \pm \frac{1}{\pi} \arctan\left(\frac{1}{\sqrt{N}}\right)\right)^2$$

for all positive integers j . Odd parity, when combined with continuity, forces the eigenfunctions to vanish at the nodes, and thus leads to eigenvalues of the form j^2 , for positive integers j . The fundamental ratio E_2/E_1 for this example can be seen to be

$$\left(\frac{\pi - \arctan\left(\frac{1}{\sqrt{N}}\right)}{\arctan\left(\frac{1}{\sqrt{N}}\right)}\right)^2,$$

which is roughly $\pi^2 N$ for large N .

Remarks

1. With no external edges, the lowest eigenvalue of a quantum graph is $E_1 = 0$, so one might intuitively argue that for a graph with a large and complex interior part the effect of an exterior edge with a boundary condition is small. The theorems and examples given below, however, point towards a more nuanced intuition.

2. Another instructive example is the “bunch-of-balloons” graph, with many nonintersecting loops attached to the string at v_1 . We leave the details to the interested reader.

Example 1.3. (Violation of classical Lieb-Thirring.) Next consider a balloon graph with $L = \infty$ and the Schrödinger operator $H(l) := -\frac{d^2}{dx^2} + V(x)$ on $L_2(\Gamma)$ with vertex conditions (1.2). Let the potential V be given by

$$V(x) := \begin{cases} V_1(x) := \frac{-2a^2}{\cosh^2(ax)}, & x \in \Gamma_1 = [-\pi, \pi] \\ V_2(x) := 0, & x \in \Gamma_2 = [0, \infty) \end{cases}.$$

Then the eigenfunction corresponding to the eigenvalue $-a^2$ is given by $C \cosh^{-1}(ax_\ell)$ on Γ_1 and by e^{-ax_s} on Γ_2 . The continuity condition gives $C = \cosh(a\pi)$ and the condition (1.2) at v_1 leads to the equation

$$\tanh(a\pi) = \frac{1}{2}. \tag{1.5}$$

Denoting the ratio

$$\mathcal{Q}(\gamma, V) := \frac{|E_1|^\gamma}{\int_\Gamma |V(x)|^{\gamma+1/2} dx},$$

we compute

$$\mathcal{Q}(3/2, V) = \frac{a^3}{2 \int_0^\pi \frac{4a^4}{\cosh^4(ax_\ell)} dx_\ell} = \left(8 \int_0^{a\pi} \frac{1}{\cosh^4(y)} dy\right)^{-1} = \left(\frac{8}{3} \tanh(a\pi)(2 + \operatorname{sech}^2(a\pi))\right)^{-1}.$$

Because of (1.5), $\operatorname{sech}^2(a\pi) = 1 - \tanh^2(a\pi) = \frac{3}{4}$, and therefore

$$\mathcal{Q}(3/2, V) = \frac{3}{11} > \frac{3}{16} = L_{3/2,1}^{cl}. \tag{1.6}$$

Note that the ratio $\mathcal{Q}(3/2, V)$ is independent of the length of the loop, as expected because any length L can be achieved by a change of scale.

The ratio $\mathcal{Q}(\gamma, V)$ can also be calculated explicitly for the case $\gamma = 2$. In this case

$$\begin{aligned}\mathcal{Q}(2, V) &= \left[2^{7/2} \left(\frac{3}{4} \arctan(\tanh(a\pi/2)) + \frac{3}{16} \operatorname{sech}(a\pi) + \frac{1}{8} \operatorname{sech}^3(a\pi) \right) \right]^{-1} \\ &\doteq 0.2009 > L_{2,1}^{cl} = \frac{8}{15\pi} \doteq 0.1697.\end{aligned}$$

2 Lieb-Thirring inequalities for quantum graphs

2.1 Classical Lieb-Thirring inequality for metric trees

Our point of departure is the family of sum-rule identities from [10, 11]. Let H and G be abstract self-adjoint operators satisfying certain mapping conditions. We suppose that H has nonempty discrete spectrum lying below the continuum, $\{E_j : H\phi_j = E_j\phi_j\}$. In the situations of interest in this article the spectrum will either be entirely discrete, in which case we focus on spectral subsets of the form $J := \{E_j, j = 1 \dots k\}$, or else, when there is a continuum, it will lie on the positive real axis and we shall take J as the negative part of the spectrum. Let P_A denote the spectral projector associated with H and a Borel set A .

Then, given a pair of self-adjoint operators H and G with domains $D(H)$ and $D(G)$, such that $G(\mathcal{J}) \subset D(H) \subset D(G)$, where \mathcal{J} is the subspace spanned by the eigenfunctions ϕ_j corresponding to the eigenvalues E_j , it is shown in [10, 11] that:

$$\begin{aligned}\sum_{E_j \in J} (z - E_j)^2 \langle [G, [H, G]]\phi_j, \phi_j \rangle - 2(z - E_j) \langle [H, G]\phi_j, [H, G]\phi_j \rangle \\ = 2 \sum_{E_j \in J} \int_{\kappa \in J^c} (z - E_j)(z - \kappa)(\kappa - E_j) dG_{j\kappa}^2,\end{aligned}\tag{2.1}$$

where $dG_{j\kappa}^2 := |\langle G\phi_j, dP_\kappa G\phi_j \rangle|$ corresponds to the matrix elements of the operator G with respect to the spectral projections onto J and J^c . Because of our choice of J ,

$$\sum_{E_j \in J} (z - E_j)^2 \langle [G, [H, G]]\phi_j, \phi_j \rangle - 2(z - E_j) \langle [H, G]\phi_j, [H, G]\phi_j \rangle \leq 0.\tag{2.2}$$

In this section H is the Schrödinger operator on the graph Γ , namely

$$H_\Gamma(\alpha) = -\alpha \frac{d^2}{dx^2} + V(x) \text{ in } L^2(\Gamma), \quad \alpha > 0,$$

with the usual conditions (1.2) at each vertex v_i . In particular, if any leaves (i.e. edges with one free end) are of finite length, vanishing Dirichlet boundary conditions are imposed at their ends. Without loss of generality we may assume that $V \in C_0^\infty$ for the operator $H_\Gamma(\alpha)$. Under this assumption, for any $\alpha > 0$, $H_\Gamma(\alpha)$ has at most a finite number of negative eigenvalues. We denote negative eigenvalues of $H_\Gamma(\alpha)$ by $E_j(\alpha)$ corresponding to the normalized eigenfunctions ϕ_j .

We shall be able to derive inequalities of the standard one-dimensional type when it is possible to choose G to be multiplication by the arclength along some distinguished subsets of the graph. This depends on the following:

Lemma 2.1. *Suppose that there exists a continuous, piecewise-linear function G on the graph Γ , such that at each vertex v_k*

$$\sum_j \frac{\partial G}{\partial x_{kj}}(0^+) = 0.\tag{2.3}$$

Suppose that $\Gamma = \cup_m \Gamma_m$ with $(G')^2 = a_m$ on Γ_m . If the spectrum has nonempty essential spectrum, assume that $z \leq \inf \sigma_{\text{ess}}(H)$. Then

$$\sum_{j,m} (z - E_j)_+^2 a_m \|\chi_{\Gamma_m} \phi_j\|^2 - 4\alpha (z - E_j)_+ a_m \|\chi_{\Gamma_m} \phi'_j\|^2 \leq 0. \quad (2.4)$$

We observe that $\chi_{\Gamma_m} = 1 \Leftrightarrow a_m \neq 0$.

Proof. The formula (2.4) is a direct application of (2.2), when we note that locally, $[H, G] = -2G' \frac{d}{dx_j} - G''$ and $[G, [H, G]] = 2(G')^2$. (A factor of 2α has been divided out.) The reason for the condition (2.3) is that $G\phi_j$ must be in the domain of definition of H , which requires that at each vertex,

$$0 = \sum_j \frac{\partial G \phi_j}{\partial x_{kj}}(0^+) = G \sum_j \frac{\partial \phi_j}{\partial x_{kj}}(0^+) + \phi_j \sum_j \frac{\partial G}{\partial x_{kj}}(0^+) = \phi_j \sum_j \frac{\partial G}{\partial x_{kj}}(0^+).$$

□

If we are so fortunate that $(G')^2$ is the same constant on every edge, then (2.4) reduces to the quadratic inequality

$$\sum_j (z - E_j)_+^2 - 4\alpha (z - E_j)_+ \|\phi'_j\|^2 \leq 0, \quad (2.5)$$

familiar from [8–11, 25], where it was shown that it implies universal spectral bounds for Laplacians and Lieb-Thirring inequalities for Schrödinger operators in routine ways. Equation (2.5) can be considered as a Yang-type inequality, after [?].

Stubbe's monotonicity argument. In [25] Stubbe showed that some of the classical sharp Lieb-Thirring inequalities follow from the quadratic inequality (2.5). Here we apply the same argument to quantum graphs: For any $\alpha > 0$, the functions $E_j(\alpha)$ are non-positive, continuous and increasing. $E_j(\alpha)$ is continuously differentiable except at countably many values where $E_j(\alpha)$ fails to be isolated or enters the continuum. By the Feynman-Hellman theorem,

$$\frac{d}{d\alpha} E_j(\alpha) = \langle \phi_j, -\phi''_j \rangle = \|\phi'_j\|^2.$$

Setting $z = 0$, (2.5) reads

$$\alpha \sum_{E_j(\alpha) < 0} (-E_j(\alpha))^2 + 2\alpha^2 \frac{d}{d\alpha} \sum_{E_j(\alpha) < 0} (-E_j(\alpha))^2 \leq 0.$$

For any $\alpha \in]\alpha_{N+1}, \alpha_N[$ the number of eigenvalues is constant, and therefore

$$\frac{d}{d\alpha} \left(\alpha^{1/2} \sum_{E_j(\alpha) < 0} (-E_j(\alpha))^2 \right) \leq 0.$$

This means that $\alpha^{1/2} \sum_{E_j(\alpha) < 0} (-E_j(\alpha))^2$ is monotone decreasing in α . Hence, by Weyl's asymptotics (see [4, 28]),

$$\alpha^{1/2} \sum_{E_j(\alpha) < 0} (-E_j(\alpha))^2 \leq \lim_{\alpha \rightarrow 0^+} \alpha^{1/2} \sum_{E_j(\alpha) < 0} (-E_j(\alpha))^2 = L_{2,1}^{cl} \int_{\Gamma} (V_-(x))^{2+1/2} dx.$$

Remark 2.2. Strictly speaking the Feynman-Hellman theorem only holds for nondegenerate eigenvalues. In the case of degenerate eigenvalues one has to take the right basis in the corresponding degeneracy space and to change the numbering if necessary, see e.g. [26].

The balloon counterexamples given above might lead one to think that the existence of cycles poses a barrier for a quantum graph to have an inequality of the form (2.5). Consider, however the following example.

Example 2.3. (Hash graphs.) Let Γ be a planar graph consisting of (or metrically isomorphic to) the union of a closed family of vertical lines and line segments \mathfrak{F}_v and a closed family of horizontal lines and line segments \mathfrak{F}_h . We assume that for some $\delta > 0$ the distance between any two lines or line segments in \mathfrak{F}_v is at least δ , and that the same is true of \mathfrak{F}_h . (The assumption on the spacing of the lines allows an unproblematic definition of the vertex conditions (1.2).) We impose Dirichlet boundary conditions at any ends of finite line segments. We also suppose a “crossing condition,” that there are no vertices touching exactly three edges. (I.e., no line segment from \mathfrak{F}_v has an end point in \mathfrak{F}_h and *vice versa*.)

Regarding the graph as a subset of the xy -plane, we let $G(x, y) = x + y$. It is immediate from the crossing condition that G satisfies (2.3). Furthermore, the derivative of G along every edge is 1, and therefore the quadratic inequality (2.5) holds.

A quadratic inequality (2.5) can arise in a different way, if there is a family of piecewise affine functions G_ℓ each with a range of values $a_{\ell m}$, but such that $\sum_\ell a_{\ell m} = 1$ (or any other fixed positive constant). This occurs in our next example. Even when this is not possible, if we can arrange that $0 < a_{\min} \leq \sum_\ell a_{\ell m} \leq a_{\max}$, then the resulting weaker quadratic inequality

$$\sum_j (z - E_j)_+^2 - 4\alpha \frac{a_{\max}}{a_{\min}} (z - E_j)_+ \|\phi'_j\|^2 \leq 0, \quad (2.6)$$

will still lead to universal spectral bounds that may be useful. We speculate about this circumstance below.

Example 2.4. (*Y*-graph) As the next example we consider a simple graph, namely the *Y*-graph, which is a star-shaped graph with three positive halfaxes Γ_i , $i = 1, 2, 3$, joined at a single vertex v_1 . If we set

$$G_1(x) := \begin{cases} g_1 := 0, & x_{11} \in \Gamma_1 \\ g_2 := -x_{12}, & x_{12} \in \Gamma_2, \\ g_3 := x_{13}, & x_{13} \in \Gamma_3 \end{cases}$$

then obviously $G(\mathcal{J}) \subset D(H_\Gamma(\alpha))$ holds, and with Lemma 2.1 we get

$$\sum_j (z - E_j)_+^2 (\|\chi_{\Gamma_2} \phi_j\|^2 + \|\chi_{\Gamma_3} \phi_j\|^2) - 4\alpha (z - E_j)_+ (\|\chi_{\Gamma_2} \phi'_j\|^2 + \|\chi_{\Gamma_3} \phi'_j\|^2) \leq 0. \quad (2.7)$$

As Γ_1 doesn't contribute to this inequality, we cyclically permute the zero part of G , i.e. we next choose $G_2(x)$, such that $g_2 = 0$, $g_1 = x_{11}$ and $g_3 = -x_{13}$, and finally $G_3(x)$, such that $g_3 = 0$, $g_1 = x_{11}$ and $g_2 = -x_{12}$. These give us two further inequalities analogous to (2.7). Summing all three inequalities, and noting that on every edge, $\sum_{\ell=1}^3 a_{\ell m} = 2$, we finally obtain

$$\sum_j 2(z - E_j)_+^2 - 8\alpha (z - E_j)_+ \|\phi'_j\|^2 \leq 0, \quad (2.8)$$

which when divided by 2 yields the quadratic inequality (2.5).

We next extend the averaging argument to prove (2.5) for arbitrary metric trees. A metric tree Γ consists of a set of vertices, a set of leaves and a set of edges, i.e., segments of the real axis, which connect the vertices, such that there is exactly one path connecting any two vertices. It is common in graph theory to distinguish between edges and leaves; a leaf is joined to a vertex at only one of its endpoints, i.e. there is a free end, at which we shall set Dirichlet boundary conditions. (When the distinction is not material we shall refer to both edges and leaves as edges. It is also common to regard one free end as the distinguished “root” r of the tree, but for our purposes all free ends of the graph have the same status.) We denote the vertices by v_i , $i = 1, \dots, n$. The edges including leaves will be denoted by e . We shall explicitly write l_j for leaves when the distinction matters.

Theorem 2.5. For any tree graph with a finite number of vertices and edges, the mapping

$$\alpha \mapsto \alpha^{1/2} \sum_{E_j(\alpha) < 0} (-E_j(\alpha))^2$$

is nonincreasing for all $\alpha > 0$. Consequently

$$\alpha^{1/2} \sum_{E_j(\alpha) < 0} (-E_j(\alpha))^2 \leq L_{2,1}^{cl} \int_{\Gamma} (V_-(x))^{2+1/2} dx$$

for all $\alpha > 0$.

Remark 2.6. By the monotonicity principle of Aizenman and Lieb (see [1]), Theorem 2.5 is also true with the sharp constant for higher moments of eigenvalues. Alternatively, the extension to higher values of γ can be obtained directly from the trace inequality of [12] for power functions with $\gamma > 2$. Furthermore, Theorem 2.5 can be extended by a density argument to potentials $V \in L^{\gamma+1/2}(\Gamma)$.

To prepare the proof of Theorem 2.5, we first formulate some auxiliary results.

Lemma 2.7. For all $n \in \mathbb{N}$,

$$\sum_{k=0}^{\lfloor \frac{n-1}{2} \rfloor} \binom{n-1}{2k} = \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor - 1} \binom{n-1}{2k+1}. \quad (2.9)$$

Proof. This is a simple computation. \square

Definition 2.8. Let \mathcal{E} be the set of all edges $e \subset \Gamma$. We call the mapping $\mathcal{C} : \mathcal{E} \rightarrow \{0, 1\}$ a *coloring* and say that \mathcal{C} is an *admissible coloring* if at each vertex $v \in \Gamma$ the number

$$\#\{e : e \text{ emanates from } v : \mathcal{C}(e) = 1\}$$

is even. We let $\mathcal{A}(\Gamma)$ denote the set of all admissible colorings on Γ .

Theorem 2.9. Let Γ_n be a metric tree with n vertices. For an edge $e \subset \Gamma_n$, we denote by

$$a(e, n) := \#\{\mathcal{C}(\Gamma_n) \in \mathcal{A} : \mathcal{C}(e) = 1\}$$

the number of all admissible mappings $\mathcal{C} \in \mathcal{A}(\Gamma_n)$, such that $\mathcal{C}(e) = 1$ for $e \subset \Gamma_n$. Then

$$a(e, n) \text{ is independent of } e \subset \Gamma_n. \quad (2.10)$$

Proof. We shall prove (2.10) by induction over the number of vertices of Γ . The case with one vertex v_1 is trivial because of the symmetry of the graph. Given a metric tree Γ_n with n vertices, we can decompose it as follows. Γ_n consists of a metric tree Γ_{n-1} with $n-1$ vertices to which $m-1$ leaves l_j , $j = 2, \dots, m$, are attached to the free end of a leaf $l_1 \subset \Gamma_{n-1}$. We call the vertex at which the leaves l_j , $j = 1, \dots, m$, are joined v_n . Hence,

$$\Gamma_n := \Gamma_{n-1} \cup v_n \cup \bigcup_{j=2}^m l_j.$$

By the induction hypothesis,

$$a(e, n-1) := \#\{\mathcal{C} \in \mathcal{A}(\Gamma_{n-1}) : \mathcal{C}(e) = 1\} \text{ is independent of } e \subset \Gamma_{n-1}. \quad (2.11)$$

Obviously for every edge or leaf $e \neq l_1$ in Γ_{n-1} , we have

$$a(e, n-1) = \#\{\mathcal{C} \in \mathcal{A}(\Gamma_{n-1}) : \mathcal{C}(e) = 1 \wedge \mathcal{C}(l_1) = 1\} + \#\{\mathcal{C} \in \mathcal{A}(\Gamma_{n-1}) : \mathcal{C}(e) = 1 \wedge \mathcal{C}(l_1) = 0\}. \quad (2.12)$$

Now, we have to show that $a(e, n)$ is independent of $e \in \Gamma_n$. Note first that for each fixed leaf l_j of the subgraph $\Gamma^* = v_n \cup \bigcup_{j=1}^m l_j$, we have

$$\mu_1 := \#\{\mathcal{C} \in \mathcal{A}(\Gamma^*) : \mathcal{C}(l_j) = 1, l_j \in \Gamma^*\} = \sum_{k=0}^{\lfloor \frac{m}{2} \rfloor - 1} \binom{m-1}{2k+1} \quad (2.13)$$

and

$$\mu_0 := \#\{\mathcal{C} \in \mathcal{A}(\Gamma^*) : \mathcal{C}(l_j) = 0, l_j \in \Gamma^*\} = \sum_{k=0}^{\lfloor \frac{m-1}{2} \rfloor} \binom{m-1}{2k}. \quad (2.14)$$

Hence, for arbitrary neighboring edges $e', e'' \in \Gamma_{n-1}$ the following equality holds,

$$\begin{aligned} a(e', n) &= \mu_1 \#\{\mathcal{C} \in \mathcal{A}(\Gamma_{n-1}) : \mathcal{C}(e') = 1 \wedge \mathcal{C}(l_1) = 1\} \\ &\quad + \mu_0 \#\{\mathcal{C} \in \mathcal{A}(\Gamma_{n-1}) : \mathcal{C}(e') = 1 \wedge \mathcal{C}(l_1) = 0\}, \end{aligned} \quad (2.15)$$

and respectively

$$\begin{aligned} a(e'', n) &= \mu_1 \#\{\mathcal{C} \in \mathcal{A}(\Gamma_{n-1}) : \mathcal{C}(e'') = 1 \wedge \mathcal{C}(l_1) = 1\} \\ &\quad + \mu_0 \#\{\mathcal{C} \in \mathcal{A}(\Gamma_{n-1}) : \mathcal{C}(e'') = 1 \wedge \mathcal{C}(l_1) = 0\}. \end{aligned} \quad (2.16)$$

By Lemma 2.7, $\mu := \mu_0 = \mu_1$. Therefore, with (2.12) the equalities (2.15) and (2.16) read

$$\begin{aligned} a(e', n) &= \mu a(e', n-1), \\ a(e'', n) &= \mu a(e'', n-1). \end{aligned}$$

Furthermore, by the induction hypothesis,

$$a(e', n-1) = a(e'', n-1),$$

from which it immediately follows that

$$a(e', n) = \mu a(e', n-1) = \mu a(e'', n-1) = a(e'', n).$$

This proves Theorem 2.9. \square

Proof of Theorem 2.5. In order to apply Stubbe's monotonicity argument [25], we need to establish inequality (2.5) for metric trees. To do this, we proceed as for the example of the Y -graph. Let \mathcal{J} denote the subspace spanned by the eigenfunctions ϕ_j on $L^2(\Gamma)$ corresponding to the eigenvalues E_j . Note first that there exist self-adjoint operators G , which are given by piecewise affine functions g_i on the edges (or leaves) of Γ , such that $G(\mathcal{J}) \subset D(H(\alpha)) \subset D(G)$. Edges (or leaves) on which constant functions g_i are given, do not contribute to the sum rule. Therefore we average over a family of operators G , such that every edge e (or leaf) of the tree appears equally often in association with an affine function having $G' = \pm 1$ on e . We let \mathcal{G} denote the set of continuous operators $G(x) = \{g_i(x) \text{ affine, } x \in e_i \text{ (or } l_i)\}$, which satisfy (1.2) at the vertices v of Γ . Indeed it is not necessary to average over all the operators $G \in \mathcal{G}$, because it makes no difference in Lemma 2.1, for instance, whether $g'_i = 1$ or $g'_i = -1$. Therefore we define an equivalence relation $\sim_{\mathcal{G}}$ on \mathcal{G} as follows: Let $\tilde{G} = \{\tilde{g}_i(x) \text{ affine, } x \in e_i, \text{ (or } l_i)\}$ be another operator in \mathcal{G} . We say that $G \sim \tilde{G} \Leftrightarrow \forall i \in \{1, \dots, n\} : |g'_i(x)| = |\tilde{g}'_i(x)|$. We define $\mathcal{G}^* := \mathcal{G} / \sim$. Then we can consider the isomorphism

$$\mathcal{I} : \mathcal{A}(\Gamma) \rightarrow \mathcal{G}^*, \quad (2.17)$$

where for each $\mathcal{C} \in \mathcal{A}(\Gamma)$ we choose an affine function $G_{\mathcal{C}} \in \mathcal{G}^*$ on Γ , such that $|G'_{\mathcal{C}}(e)| = \mathcal{C}(e)$ for every $e \in \Gamma$. By Theorem 2.9, we know that $\#\{\mathcal{C} \in \mathcal{A}(\Gamma) : \mathcal{C}(e) = 1\}$ is independent of $e \in \Gamma$.

This means that summing up all inequalities corresponding to (2.4), which we get from each $G_C \in \mathcal{G}^*$, leads to

$$\sum_j (z - E_j)_+^2 p - 4\alpha(z - E_j)_+ p \|\phi_j'\|^2 \leq 0, \quad (2.18)$$

where $p := \sum_\ell a_{\ell m} = \#\{\mathcal{C} \in \mathcal{A}(\Gamma) : \mathcal{C}(e) = 1\}$ and we have used the normalization $\|\phi_j\| = 1$. Having the analogue of inequality (2.5) for metric trees, we can reformulate the monotonicity argument for our case. This proves Theorem 2.5. \square

Remark 2.10. The proof applies equally to metric trees with leaves of infinite lengths.

2.2 Modified Lieb-Thirring inequalities for one-loop graphs

In this section we consider the graph Γ consisting of a circle to which two leaves are attached. It is not hard to see that the construction leading to Lieb-Thirring inequalities with the sharp classical constant fails for one-loop graphs, because no family of auxiliary functions G_ℓ exists with the side condition that $\sum_\ell a_{\ell m} = 1$ throughout Γ . Unlike the case of the balloon graph, it is possible to replace the classical inequality with a weakened version (2.6) as mentioned above. There is, however another option, based on commutators with exponential functions, following an idea of [12]: As usual, we define the one-parameter family of Schrödinger operators

$$H(\alpha) = -\alpha \frac{d^2}{dx^2} + V(x), \quad \alpha > 0,$$

in $L^2(\Gamma)$ with the usual conditions (1.2) at each vertex v_i of Γ . The leaves are denoted by $\Gamma_1 := [0, \infty)$ and $\Gamma_2 := [0, \infty)$, while we write Γ_3 and Γ_4 for the semicircles with lengths L . Let ϕ_j be the eigenfunctions of $H(\alpha)$ corresponding to the eigenvalues $E_j(\alpha)$.

Theorem 2.11. *Let $q := 2\pi/L$. For all $\alpha > 0$ the mapping*

$$\alpha \mapsto \alpha^{\frac{1}{2}} \sum_{E_j(\alpha) < 0} \left(z - \frac{3}{16} \alpha q^2 - E_j \right)_+^2 \quad (2.19)$$

is nonincreasing. Furthermore, for all $z \in \mathbb{R}$ and all $\alpha > 0$ the following sharp Lieb-Thirring inequality holds:

$$R_2(z, \alpha) \leq \alpha^{-1/2} L_{2,1}^{cl} \int_\Gamma \left(V(x) - \left(z + \frac{3}{16} q^2 \alpha \right) \right)_-^{2+1/2} dx, \quad (2.20)$$

where

$$R_2(z, \alpha) := \sum_{E_j(\alpha) < z} (z - E_j(\alpha))_+^2.$$

Remark 2.12. Once again, Theorem 2.11 can be extended to potentials $V \in L^{\gamma+1/2}(\Gamma)$ and is true for all $\gamma \geq 2$, either by the monotonicity principle of Aizenman and Lieb [1] or by the trace formula of [12] for $\gamma \geq 2$.

For the proof of Theorem 2.11, we make use of a theorem of Harrell and Stubbe:

Theorem 2.13 ([12, Theorem 2.1]). *Let H be a self-adjoint operator on \mathcal{H} , with a nonempty set J of finitely degenerate eigenvalues lying below the rest of the spectrum J^c and $\{\phi_j\}$ an orthonormal set of eigenfunctions of H . Let G be a linear operator with domain \mathcal{D}_G and adjoint G^* defined on \mathcal{D}_{G^*} such that $G(\mathcal{D}_H) \subseteq \mathcal{D}_H \subseteq \mathcal{D}_G$ and $G^*(\mathcal{D}_H) \subseteq \mathcal{D}_H \subseteq \mathcal{D}_{G^*}$, respectively. Then*

$$\begin{aligned} & \frac{1}{2} \sum_{E_j \in J} (z - E_j)^2 (\langle [G^*, [H, G]] \phi_j, \phi_j \rangle + \langle [G, [H, G^*]] \phi_j, \phi_j \rangle) \\ & \leq \sum_{E_j \in J} (z - E_j) (\|[H, G] \phi_j\|^2 + \|[H, G^*] \phi_j\|^2). \end{aligned} \quad (2.21)$$

Remark 2.14. Strictly speaking, in [12] it was assumed that the spectrum was purely discrete. However, the extension to the case where continuous spectrum is allowed in J^c follows exactly as in Theorem 2.1 of [11].

Proof of Theorem 2.11. In this case it is not possible to get a quadratic inequality from Lemma 2.1 without worsening the constants. This follows from the fact that the conditions $\phi_3(0) = \phi_4(0)$ and $\phi_3(L) = \phi_4(L)$ imply that the piecewise linear function G has to be defined equally on Γ_3 and Γ_4 . Consequently, the condition (1.2) can be satisfied only with different values of a_m as in (2.6), namely $a_1 = a_2 = 4a_3 = 4a_4$. Our proof of Theorem 2.11 consists of three steps. First we apply Lemma 2.1, after which we apply Theorem 2.13. Finally we combine both results and apply the line of argument given in [12].

First step: Using Lemma 2.1 with the choice,

$$G(x) := \begin{cases} g_1 := -2x_{11} , & x_{11} \in \Gamma_1 \\ g_2 := 2x_{22} + L , & x_{22} \in \Gamma_2 \\ g_3 := x_{13} , & x_{13} \in \Gamma_3 \\ g_4 := x_{14} , & x_{14} \in \Gamma_4 \end{cases} ,$$

we obtain

$$4 \left(\sum_{E_j(\alpha) < 0} (z - E_j(\alpha))_+^2 p_{12}(j) - 4\alpha \sum_{E_j(\alpha) < 0} (z - E_j(\alpha))_+ p'_{12}(j) \right) + \sum_{E_j(\alpha) < 0} (z - E_j(\alpha))_+^2 p_{34}(j) - 4\alpha \sum_{E_j(\alpha) < 0} (z - E_j(\alpha))_+ p'_{34}(j) \leq 0, \quad (2.22)$$

where $p_{ik}(j) := \|\chi_{\Gamma_i} \phi_j\|^2 + \|\chi_{\Gamma_k} \phi_j\|^2$ and $p'_{ik}(j) := \|\chi_{\Gamma_i} \phi'_j\|^2 + \|\chi_{\Gamma_k} \phi'_j\|^2$.

Second step: Next, in Theorem 2.13 we set

$$G(x) := \begin{cases} g_1 := 1 , & x_{11} \in \Gamma_1 \\ g_2 := 1 , & x_{22} \in \Gamma_2 \\ g_3 := e^{-i2\pi x_{13}/L} , & x_{13} \in \Gamma_3 \\ g_4 := e^{i2\pi x_{14}/L} , & x_{14} \in \Gamma_4 \end{cases} .$$

It is easy to see that $G\phi_j \in D(H_\alpha)$. With $q := 2\pi/L$, the first commutators work out to be

$$[H_j, g_j] = 0, \quad j = 1, 2,$$

$$[H_3, g_3] = e^{-iqx_{13}} \alpha (q^2 + 2iqd/dx), \quad [H_4, g_4] = e^{iqx_{14}} \alpha (q^2 - 2iqd/dx);$$

whereas for the second commutators,

$$[g_j^*, [H_j, g_j]] = [g_j, [H_j, g_j^*]] = 0, \quad j = 1, 2, \quad (2.23)$$

$$[g_j^*, [H_j, g_j]] = [g_j, [H_j, g_j^*]] = 2\alpha q^2, \quad j = 3, 4.$$

From inequality (2.21), we get

$$\sum_{E_j(\alpha) \in J} (z - E_j(\alpha))^2 p_{34}(j) \leq \alpha \sum_{E_j(\alpha) \in J} (z - E_j(\alpha)) (q^2 p_{34}(j) + 4p'_{34}(j)). \quad (2.24)$$

Third step: Adding (2.22) and (2.24) we finally obtain

$$2 \left(R_2(z, \alpha) + 2\alpha \frac{d}{d\alpha} R_2(z, \alpha) \right) \leq \alpha q^2 \frac{3}{2} \sum_{E_j \in J} (z - E_j) p_{34}(j), \quad (2.25)$$

or

$$2R_2(z, \alpha) + 4\alpha \frac{d}{d\alpha} R_2(z, \alpha) - \alpha q^2 \frac{3}{2} R_1 \leq 0, \quad (2.26)$$

which is equivalent to

$$\frac{\partial}{\partial \alpha} \left(\alpha^{1/2} R_2(z, \alpha) \right) \leq \frac{3q^2}{8} \alpha^{1/2} R_1(z, \alpha). \quad (2.27)$$

Letting $U(z, \alpha) := \alpha^{1/2} R_2(z, \alpha)$, the inequality has the form

$$\frac{\partial U}{\partial \alpha} \leq \frac{3}{16} q^2 \frac{\partial U}{\partial z}. \quad (2.28)$$

Since the expression in (2.20) can be written as $U(z - \frac{3}{16} q^2 \alpha, \alpha)$, an application of the chain rule shows that the monotonicity claimed in (2.20) follows from (2.28). (We note that (2.28) can be solved by changing to characteristic variables $\xi := \alpha - \frac{16z}{3q^2}$, $\eta := \alpha + \frac{16z}{3q^2}$, in terms of which

$$\frac{\partial U}{\partial \xi} \leq 0. \quad (2.29)$$

(i.e., U decreases as ξ increases while η is fixed.) By shifting the variable in (2.29), we also obtain

$$U(z, \alpha) \leq U\left(z + \frac{3}{16} q^2 (\alpha - \alpha_s), \alpha_s\right) \quad (2.30)$$

for $\alpha \geq \alpha_s$. By Weyl's asymptotics, for all $\gamma \geq 0$,

$$\lim_{\alpha \rightarrow 0^+} \alpha^{\frac{d}{2}} \sum_{E_j(\alpha) < z} (z - E_j(\alpha))^\gamma = L_{\gamma, d}^{cl} \int_{\Gamma} (V(x) - z)_-^{\gamma+d/2} dx, \quad (2.31)$$

see [4, 28]. Hence, as $\alpha_s \rightarrow 0$, the right side of (2.30) tends to

$$L_{2,1}^{cl} \int \left(V(x) - \left(z + \frac{3}{16} q^2 \alpha \right) \right)_-^{2+1/2} dx,$$

so the conclusion of Theorem 2.11 follows. \square

Remark 2.15. Theorem 2.11 can be generalized to one-loop graphs to which $2n$, $n \in \mathbb{N}$ equidistant halfaxes are attached.

To summarize, in this section we have seen that for some classes of quantum graphs a quadratic inequality (2.5) can be proved with the classical constants, and that for some other classes of graphs similar statements can be proved at the price of worse constants as in (2.6), or of a shift in the zero-point energy as in (2.20).

It is reasonable to ask whether one can look at the connectness of a graph and say whether a weak Yang-type inequality (2.6) can be proved. As we have seen, this is the case if there exists a family of continuous functions G_ℓ on the graph such that

- On each edge, all the derivatives $\{G'_\ell\}$ are constant.
- At each vertex v_k , each function G_ℓ satisfies

$$\sum_j \frac{dG_\ell}{dx_{kj}}(0^+) = 0.$$

- For each edge e there exists at least one function G_ℓ with $G'_\ell \neq 0$.

Interestingly, the question of the existence of such a family of functions can be rephrased in terms of the theory of electrical resistive circuits, a subject dating from the mid nineteenth century [14]. We first note that for a suitable family of functions to exist, there must be at least two leaves, which can be regarded as external leads of an electric circuit, bearing some resistance. (In the finite case let the resistance be equivalent to the length of the leaf, and in the infinite case let it be some fixed finite value, at least as large as the length of any finite leaf.) Each internal edge is regarded as a wire bearing a resistance equal to the length of the edge. If we regard the value of G'_ℓ as a current, then Kirchhoff's condition at the vertex of an electric circuit is exactly the condition (1.2) that $\sum_j \frac{dG_\ell}{dx_{k_j}}(0^+) = 0$, and the condition that the electric potential G_ℓ must be uniquely defined at all vertices is equivalent to global continuity of G_ℓ . It has been known since Weyl [29] that the currents and potentials in an electric circuit are uniquely determined by the voltages applied at the leads. There are, however, circuits such that no matter what voltages are applied to the external leads, there will be an internal wire where no current flows; the most well-known of these is the Wheatstone bridge. (See, for instance, the Wikipedia article on the Wheatstone bridge.)

Let us call a metric graph a *generalized Wheatstone bridge* when the corresponding circuit has exactly two external leads and a configuration for which no current will flow in at least one of its wires. Then we conjecture that there are only two impediments to the existence of a suitable family of functions G_ℓ , and therefore to a weakened quadratic inequality (2.6), namely: Unless a quantum graph contains either

- a) a subgraph that can be disconnected from all leaves by the removal of one point (such as a balloon graph or a graph shaped like the letter α); or
- b) a subgraph that when disconnected from the graph by cutting two edges is a generalized Wheatstone bridge,

then an inequality of the form (2.6) holds. Otherwise the best that can be obtained may be a modified quadratic inequality with a variable shift, as in Theorem 2.11.

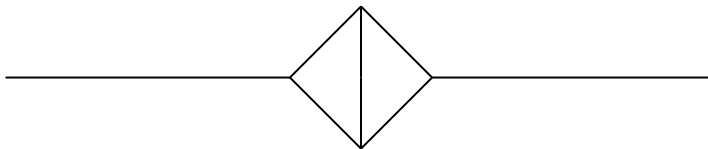


Figure 2: “Wheatstone bridge”

3 Universal bounds for finite quantum graphs

In this section we derive differential inequalities for Riesz means of eigenvalues of the Dirichlet Laplacian on bounded metric trees Γ with at least one leaf (free edge). From these inequalities we derive Weyl-type bounds on the averages of the eigenvalues of the Dirichlet Laplacian

$$H_D := \left(-\frac{d^2}{dx^2} \right)_D \text{ in } L^2(\Gamma),$$

with the conditions (1.2) at each vertex v_i . At the ends of the leaves, vanishing Dirichlet boundary conditions are imposed. We recall that with the methods of [9, 10] these are consequences of the same quadratic inequality (2.5) as was used above to prove Lieb-Thirring inequalities. When the

total length of the graph is finite, the operator H_D on $D(H_D)$ has a positive discrete spectrum $\{E_j\}_{j=1}^\infty$, allowing us to define the *Riesz mean of order ρ* ,

$$R_\rho(z) := \sum_j (z - E_j)_+^\rho \quad (3.1)$$

for $\rho > 0$ and real z .

Theorem 3.1. *Let Γ be a metric tree of finite length and with finitely many edges and vertices, and let H_D be the Dirichlet Laplacian in $L^2(\Gamma)$ with domain $D(H_D)$. Then for $z > 0$,*

$$R_1(z) \geq \frac{5}{4z} R_2(z); \quad (3.2)$$

$$R_2'(z) \geq \frac{5}{2z} R_2(z); \quad (3.3)$$

and consequently

$$\frac{R_2(z)}{z^{5/2}}$$

is a nondecreasing function of z .

Proof. The claims are vacuous for $z \leq E_1$, so we henceforth assume $z > E_1$. The line of reasoning of the proof of Theorem 2.5 applies just as well to the operator H_D on $D(H_D)$, yielding

$$\sum_j (z - E_j)_+^2 - 4(z - E_j)_+ \|\phi_j'\|^2 \leq 0. \quad (3.4)$$

Since $V \equiv 0$, $\|\phi_j'\|^2 = E_j$. Observing that

$$\sum_j (z - E_j)_+ E_j = z R_1(z) - R_2(z),$$

we get from (3.4)

$$5R_2(z) - 4zR_1(z) \leq 0.$$

This proves (3.2). Inequality (3.3) follows from (3.2), as $R_2'(z) = 2R_1(z)$. \square

Since by the Theorem 3.1, $R_2(z)z^{-5/2}$ is a nondecreasing function, we obtain a lower bound of the form $R_2(z) \geq Cz^{5/2}$ for all $z \geq z_0$ in terms of $R_2(z_0)$. Upper bounds can be obtained from the limiting behavior of $R_2(z)$ as $z \rightarrow \infty$, as given by the Weyl law. In the following, we to follow [9] to derive Weyl-type bounds on the averages of the eigenvalues of H_D in $L^2(\Gamma)$.

Corollary 3.2. *For $z \geq 5E_1$,*

$$16E_1^{-1/2} \left(\frac{z}{5}\right)^{5/2} \leq R_2(z) \leq L_{2,1}^{cl} |\Gamma| z^{5/2},$$

where $L_{2,1}^{cl} := \frac{\Gamma(3)}{(4\pi)^{1/2}\Gamma(7/2)}$, and $|\Gamma|$ is the total length of the tree.

Proof. By Theorem 3.1, for all $z \geq z_0$,

$$\frac{R_2(z)}{z^{5/2}} \geq \frac{R_2(z_0)}{z_0^{5/2}}. \quad (3.5)$$

As $R_2(z_0) \geq (z_0 - E_1)_+^2$ for any $z_0 > E_1$, it follows from (3.5) that

$$R_2(z) \geq (z_0 - E_1)_+^2 \left(\frac{z}{z_0}\right)^{5/2}.$$

The coefficient $\frac{(z_0 - E_1)_+^2}{z_0^{5/2}}$ is maximized when $z_0 = 5E_1$. Thus we get

$$16E_1^{-1/2} \left(\frac{z}{5}\right)^{5/2} \leq R_2(z).$$

For metric trees with total length $|\Gamma|$, the Weyl law states that

$$\lim_{n \rightarrow \infty} \frac{\sqrt{E_n}}{n} = \frac{\pi}{|\Gamma|}, \quad (3.6)$$

(see [16]). It follows that

$$\frac{R_2(z)}{z^{5/2}} \rightarrow L_{2,1}^{cl} |\Gamma|,$$

as $z \rightarrow \infty$. Since $\frac{R_2(z)}{z^{5/2}}$ is nondecreasing, we get

$$\frac{R_2(z)}{z^{5/2}} \leq L_{2,1}^{cl} |\Gamma|, \quad \forall z < \infty.$$

□

In summary, we get from Theorem 3.1 and Corollary 3.2 the following two-sided estimate:

$$4E_1^{-1/2} \left(\frac{z}{5}\right)^{3/2} \leq \frac{5}{4z} R_2(z) \leq R_1(z). \quad (3.7)$$

In order to obtain similar estimates, related to higher eigenvalues, we introduce the notation

$$\overline{E}_j := \frac{1}{j} \sum_{\ell \leq j} E_\ell$$

for the means of eigenvalues E_ℓ ; similarly, the means of the squared eigenvalues are denoted

$$\overline{E}_j^2 := \frac{1}{j} \sum_{\ell \leq j} E_\ell^2.$$

For a given z , we let $ind(z)$ be the greatest integer i such that $E_i \leq z$. Then obviously,

$$R_2(z) = ind(z)(z^2 - 2z\overline{E}_{ind(z)} + \overline{E}_{ind(z)}^2).$$

As for any integer j and all $z \geq E_j$, $ind(z) \geq j$, we get

$$R_2(z) \geq \mathcal{D}(z, j) := j(z^2 - 2z\overline{E}_j + \overline{E}_j^2).$$

Using Theorem 3.1 for $z \geq z_j \geq E_j$, it follows that

$$R_2(z) \geq \mathcal{D}(z_j, j) \left(\frac{z}{z_j}\right)^{5/2}. \quad (3.8)$$

Furthermore, $\overline{E}_j^2 \leq \overline{E}_j^2$ by the Cauchy-Schwarz inequality, and hence

$$\mathcal{D}(z, j) = j \left((z - \overline{E}_j)^2 + \overline{E}_j^2 - \overline{E}_j^2 \right) \geq j(z - \overline{E}_j)^2. \quad (3.9)$$

This establishes the following

Corollary 3.3. *Suppose that $z \geq 5\overline{E_j}$. Then*

$$R_2(z) \geq \frac{16jz^{5/2}}{25(5\overline{E_j})^{1/2}} \quad (3.10)$$

and, therefore,

$$R_1(z) \geq \frac{4jz^{3/2}}{5(5\overline{E_j})^{1/2}}. \quad (3.11)$$

Proof. Combining equations (3.8) and (3.9), we get

$$R_2(z) \geq j(z_j - \overline{E_j})^2 \left(\frac{z}{z_j} \right)^{5/2}.$$

Inserting $z_j = 5\overline{E_j}$ the first statement follows. (This choice of z_j maximizes the constant appearing in (3.10).) The second statement results from substituting the first statement into (3.7). \square

The Legendre transform is an effective tool for converting bounds on $R_\rho(z)$ into bounds on the spectrum, as has been realized previously, e.g., in [17]. Recall that if $f(z)$ is a convex function on \mathbb{R}^+ that is superlinear in z as $z \rightarrow +\infty$, its Legendre transform

$$\mathcal{L}[f](w) := \sup_z \{wz - f(z)\}$$

is likewise a superlinear convex function. Moreover, for each w , the supremum in this formula is attained at some finite value of z . We also note that if $f(z) \geq g(z)$ for all z , then $\mathcal{L}[g](w) \leq \mathcal{L}[f](w)$ for all w . The Legendre transform of the two sides of inequality (3.11) is a straightforward calculation (e.g., see [9]). The result is

$$(w - [w])E_{[w]+1} + [w]\overline{E_{[w]}} \leq \frac{w^3}{j^2} \frac{125}{108} \overline{E_j}, \quad (3.12)$$

for certain values of w and j . In Corollary 3.3 it is supposed that $z \geq 5\overline{E_j}$. Let z_{\max} be the value for which $\mathcal{L}[f](w) = wz_{\max} - f(z_{\max})$, where f is the right side of (3.11). Then by an elementary calculation,

$$w = \frac{6j}{5} \left(\frac{z_{\max}}{5\overline{E_j}} \right)^{1/2}.$$

It follows that inequality (3.12) is valid for $w \geq 6j/5$. Meanwhile, for any w we can always find an integer k such that on the left side of (3.12), $k-1 \leq w < k$. If $k > 6j/5$ and if we let approach k from below, we obtain from (3.12)

$$E_k + (k-1)\overline{E_{k-1}} \leq \frac{k^3}{j^2} \frac{125}{108} \overline{E_j}.$$

The left side of this equation is the sum of the eigenvalues E_1 through E_k , so we get the following:

Corollary 3.4. *For $k \geq \frac{6}{5}j$, the means of the eigenvalues of the Dirichlet Laplacian on an arbitrary metric tree with finitely many edges and vertices satisfy a universal Weyl-type bound,*

$$\frac{\overline{E_k}}{\overline{E_j}} \leq \frac{125}{108} \left(\frac{k}{j} \right)^2. \quad (3.13)$$

In [12] it was shown that a similar inequality with a different constant can be proved for all $k \geq j$ in the context of the Dirichlet Laplacian on Euclidian domains. The very same argument

applies to quantum graphs with $V = 0$. With this assumption $\|\phi'_j\|^2 = E_j$, so with $\alpha = 1$ (2.5) can be rewritten as a quadratic inequality,

$$P_j(z) := \sum_{\ell=1}^j (z - E_\ell)(z - 5E_\ell) \leq 0 \quad (3.14)$$

for $z \in [E_j, E_{j+1}]$ (cf. [12], eq. (4.6)). From (3.2) and (3.5) for $z \geq z_0 \geq E_j$,

$$R_1(z) \geq \frac{5}{4z} R_2(z) \geq \frac{5}{4} z^{3/2} z_0^{-5/2} \sum_{\ell=1}^j (z_0 - E_\ell)^2. \quad (3.15)$$

The derivative of the right side of (3.15) with respect to z_0 , by a calculation, is a negative quantity times $P_j(z_0)$, and therefore an optimal choice for the value of (3.15) is the root

$$z_0 = 3\overline{E}_j + \sqrt{D_j} \leq 5\overline{E}_j, \quad (3.16)$$

where D_j is the discriminant of P_j . The inequality in (3.16) results from the Cauchy-Schwarz inequality as in [10, 12]. Because $P_j(z_0) = 0$,

$$0 = \sum_{\ell=1}^j (z_0 - E_\ell)(z_0 - 5E_\ell) = 5 \sum_{\ell=1}^j (z_0 - E_\ell)^2 - 4z_0 \sum_{\ell=1}^j (z_0 - E_\ell),$$

so (3.15) reads

$$R_1(z) \geq \left(\frac{z}{z_0}\right)^{3/2} \sum_{\ell=1}^j (z_0 - E_\ell) = \left(\frac{z}{z_0}\right)^{3/2} j(z_0 - \overline{E}_j).$$

From the left side of (3.16), $z_0 - \overline{E}_j \geq \frac{2}{3}z_0$, so

$$R_1(z) \geq \left(\frac{2}{3}jz_0^{-1/2}\right) z^{3/2}. \quad (3.17)$$

The Legendre transform of (3.17) is

$$k\overline{E}_k \leq \frac{z_0}{3j^2} k^3, \quad (3.18)$$

and a calculation of the maximizing z in the Legendre transform of the right side of (3.17) shows that (3.18) is valid for all $k > j$. In particular, with the inequality on the right side of (3.16), we have established the following:

Corollary 3.5. *For $k \geq j$, the means of the eigenvalues of H_D in $L_2(\Gamma)$ satisfy*

$$\frac{\overline{E}_k}{\overline{E}_j} \leq \frac{5}{3} \left(\frac{k}{j}\right)^2. \quad (3.19)$$

Remark 3.6. Relaxing the assumption to $k \geq j$ comes at the price of making the constant on the right side larger. It would be possible to interpolate between (3.19) and (3.13) for $k \in [j, 6j/5]$ with a slightly better inequality.

Acknowledgments. The authors are grateful to several people for useful comments, including Rupert L. Frank, Lotfi Hermi, Thomas Morley, Joachim Stubbe, and Timo Weidl, and to Michael Music for calculations and insights generated by them. We also wish to express our appreciation to the Mathematisches Forschungsinstitut Oberwolfach for hosting a workshop in February, 2009, where this collaboration began, and to the Erwin Schrödinger Institut for hospitality.

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