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Abstract

Almost Einstein manifolds are conformally Einstein up to a scale singularity, in general. This notion comes from conformal tractor calculus. In the current paper we discuss almost Einstein structures on closed Riemannian product manifolds and on 4-manifolds of cohomogeneity one. Explicit solutions are found by solving ordinary differential equations. In particular, we construct three families of closed 4-manifolds with almost Einstein structure corresponding to the boundary data of certain unimodular Lie groups. Two of these families are Bach-flat, but neither (globally) conformally Einstein nor half conformally flat. On products with a 2-sphere we find an exotic family of almost Einstein structures with hypersurface singularity as well.

Keywords: Almost Einstein spaces; Poincaré-Einstein space.
MSC 2000: 53A30; 53C25

1 Introduction

The study of Einstein metrics g on manifolds is of much interest, both in geometry and physics. Einstein metrics are subject to the partial differential equation

$$Ric(g) \ = \ \lambda \cdot g$$

on the metric tensor g, where $Ric(g)$ denotes the Ricci curvature of g and $\lambda \in \mathbb{R}$ is some constant. In the literature, plenty of material on the investigation of Einstein metrics can be found, especially, in dimension 4. On homogeneous spaces and in cohomogeneity one explicit constructions of Einstein metrics are well known (cf. [17] and e.g.) Especially, $SU(2)$-invariant Einstein metrics of cohomogeneity one on 4-manifolds were explicitly constructed via the use of twistor methods and ordinary differential equations (cf. [28, 16, 31]). A special case is the setting of conformally compact Einstein manifolds, which were introduced in the work of Fefferman and Graham [12], and which play nowadays an important part in the AdS/CFT-correspondence of string theory and supergravity as proposed by Maldacena [23]. Thus the notion of conformally compact Einstein manifolds provides an important link to conformal differential geometry, and this is also a motivation for the investigations of the current paper.

Although the Einstein condition $Ric(g) = \lambda{\cdot}g$ should be seen as an equation of (semi)-Riemannian geometry, there is indeed a very natural interpretation from the point of view of conformal geometry. This interpretation is best explained by the invention of *tractor calculus*. Roughly speaking, there is a unique correspondence between Einstein metrics in a given conformal class on a manifold M and the so-called parallel *standard tractors* (cf. [30, 7, 13, 21]). Introducing an arbitrary metric g in the conformal class of a manifold, the parallel tractor equation is simply expressed by a single PDE for a real function $\sigma \in C^\infty(M)$, which is

$$Hess^g_o \sigma \ = \ \mathsf{P}^g_o{\cdot}\sigma \ ,$$

where $Hess^g_o\sigma$ and P^g_o denote the trace-free parts of the Hessian and the Schouten tensor, respectively (cf. Section 2). This is an overdetermined, conformally covariant PDE of second order for σ. If σ is a solution without zeros, then $\sigma^{-2}g$ is an Einstein metric in the conformal class. However, a solution σ might well admit zeros on a manifold M. Therefore, in general, we call a solution σ an *almost Einstein structure* of the conformal manifold M. Essentially, this notion is appropriate in order to include the instances of conformally compact and asymptotically locally flat Einstein manifolds into a unified discussion about (almost) Einstein metrics.

The main aim of this paper is to demonstrate the explicit construction of closed (= compact, without boundary) Riemannian conformal manifolds admitting an almost Einstein structure, i.e., we aim to solve $Hess^g_o\sigma = \mathsf{P}^g_o{\cdot}\sigma$ explicitly. Thereby, we focus on almost Einstein structures σ with hypersurface singularity (which is the case that gives rise to conformally compact Poincaré-Einstein metrics). In fact, we describe explicit solutions in two situations, namely, on Riemannian product manifolds, especially, on products with the 2-sphere, and on closed 4-manifolds with a cohomogeneity one group action. In both cases of our discussion the equation $Hess^g_o\sigma = \mathsf{P}^g_o{\cdot}\sigma$ reduces to a system of ordinary differential equations, whose explicit solutions are easily found. Thus we can decide, which of the solutions give rise to almost Einstein structures on closed manifolds.

Note that locally (and almost everywhere) the problem of solving $Hess^g_o\sigma = \mathsf{P}^g_o{\cdot}\sigma$ is, of course, equivalent to solving Einstein's equation $Ric(g) = \lambda{\cdot}g$. (And, in fact, the local solutions that we obtain are well known Einstein metrics as documented in the literature.) However, there is a subtle difference in our approach. Namely, since we are trying to solve the almost Einstein equation (which is conformally covariant), we do not have the *burden* to exactly find the Einstein metric in the conformal class. Instead, we have the *freedom* of solving for some suitable *Ansatz* in the conformal class, and then finally, conformally rescale the solution by the almost Einstein structure σ. On the first glance, this procedure seems to be circumstantial. However, there are two arguments in favour of it. First of all, we also derive solutions for σ with singularities, i.e., in particular, we can solve for conformally compact Poincaré-Einstein metrics. (And this was the original motivation for our considerations.) Moreover, it turns out that the derived ODE's and the explicit presentation of their solutions are very simple for the basic *Ansatz* that we use. This allows a uniform treatment for various families of initial conditions on the boundary/singularity set.

The course of the paper is as follows. In Section 2 we recall the basic notions of almost Einstein structures and parallel standard tractors. (Later in the text we mostly omit the application of tractors, and simply work with a function σ, which solves the almost Einstein equation with respect to some metric in the conformal class.) In Section 3 we discuss the almost Einstein equation on Riemannian product manifolds. This discussion will lead to a rough geometric description of the factors of such manifolds (cf. Proposition 3.4 and Theorem 3.5). In the following section we give a complete and explicit account on almost Einstein structures that occur on closed Riemannian products with a 2-sphere. If the product manifold has odd dimensions, we find an exotic family of almost Einstein structures with hypersurface singularity (cf. Theorem 4.3).

The remaining Sections 5 to 7 are concerned with almost Einstein structures on 4-manifolds of cohomogeneity one. In Section 5 we give a brief account on homogeneous almost Einstein spaces and motivate our basic *Ansatz* in cohomogeneity one. In Section 6 we explicitly solve this *Ansatz* on closed 4-manifolds when the conformal boundary is a unimodular 3-dimensional Lie group equipped with a certain family of left invariant metrics (cf. (24)). In Section 7 we discuss the *renormalised volume* of conformally compact Poincaré-Einstein metrics with boundary that arise from the basic *Ansatz* (24). We end with a summary and conclusions (cf. Section 8).

2 Almost Einstein structures

We recall in this section the notion of *almost Einstein structures*, which is an invariant concept of conformal differential geometry. As the naming suggests, almost Einstein structures are closely related to Einstein metrics in Riemannian geometry. The motivation for almost Einstein structures comes from *conformal tractor calculus*. We briefly explain this here.

Let M^n be a smooth manifold of dimension $n \geq 3$. Recall that a *Riemannian conformal structure* on M is a smooth $\mathbb{R}_+$-ray subbundle $\mathcal{Q} \subset S^2T^*M$, whose fibre over $p \in M$ consists of conformally related positive definite scalar products on T_pM. Then smooth sections of $\mathcal{Q}$ are metrics on M, and we denote the set of all such sections by c. Any two sections $g, \tilde{g} \in c$ are related by $\tilde{g} = e^{2f}g$ for some function $f \in C^\infty(M)$, i.e., g and $\tilde{g}$ are conformally equivalent metrics on M.

The principal $\mathbb{R}_+$-bundle $\pi : \mathcal{Q} \to M$ induces for any representation $t \in \mathbb{R}_+ \mapsto t^{-w/2} \in \mathrm{End}(\mathbb{R})$, $w \in \mathbb{R}$, a natural real line bundle $\mathcal{E}[w]$ over M, which is called the *conformal density bundle of weight* w. The conformal *standard tractor bundle* $\mathcal{T}$ of (M, c) is naturally defined as a quotient bundle of rank $n+2$ of the 2-jet prolongation $J^2(\mathcal{E}[1])$ of the weighted bundle $\mathcal{E}[1]$. The projection to $\mathcal{T}$ of the 2-jet of a section in $\mathcal{E}[1]$ gives naturally rise to a conformally covariant second order differential operator $\mathcal{D} : \Gamma(\mathcal{E}[1]) \to \Gamma(\mathcal{T})$. The standard tractor bundle $\mathcal{T}$ also admits a composition structure

$$\mathcal{T} \; = \; \mathcal{E}[1] \;\lhd\!\!\!+\; TM[-1] \;\lhd\!\!\!+\; \mathcal{E}[-1] \; ; \tag{1}$$

$\mathcal{E}[-1]$ may be naturally identified with a subbundle of $\mathcal{T}$ and $TM[-1] = TM \otimes \mathcal{E}[-1]$ is a subbundle of the quotient bundle $\mathcal{T}/\mathcal{E}[-1]$. We denote by Π the natural projection from $\mathcal{T}$ to $\mathcal{E}[1]$. The superposition $\Pi \circ \mathcal{D}$ acts as the identity on $\Gamma(\mathcal{E}[1])$. Moreover, the standard tractor bundle $\mathcal{T}$ is naturally equipped with an indefinite scalar product $\langle \cdot, \cdot \rangle_{\mathcal{T}}$ and a covariant derivative ∇, the so-called *tractor connection*, which is by definition conformally invariant and preserves the tractor metric $\langle \cdot, \cdot \rangle_{\mathcal{T}}$.

With respect to the choice of a metric g in the given conformal class c on M the weighted bundles $\mathcal{E}[w]$, $w \in \mathbb{R}$, are trivialised and the composition structure (1) splits into the direct sum

$$\mathcal{T} \; \cong_g \; \mathbb{R} \oplus TM \oplus \mathbb{R} \; .$$

Accordingly, the operator $\mathcal{D}$ splits with respect to g into the triple

$$\mathcal{D}^g \sigma \; = \; (\; \sigma \; , \; grad^g(\sigma) \; , \; \Box^g \sigma) \, , \tag{2}$$

where $grad^g(\sigma)$ denotes the gradient of the function $\sigma \in C^\infty(M)$ and $\Box^g := -\frac{1}{n}(\Delta^g - tr_g \mathsf{P}^g)$ with Laplacian $\Delta^g = tr_g(\nabla^g \circ d)$. (The Schouten tensor P^g will be defined in (6) below.)

It is a matter of fact that for densities $\omega \in \Gamma(\mathcal{E}[1])$ the equation

$$\nabla \mathcal{D} \omega \; = \; 0 \tag{3}$$

is equivalent to

$$\text{trace-free part of} \, (\nabla^g d\sigma \; - \; \mathsf{P}^g \cdot \sigma) \; = \; 0 \; , \tag{4}$$

where σ is the function which corresponds via $g \in c$ to the density ω. In turn, it is also true that if a tractor $T \in \Gamma(\mathcal{T})$ satisfies $\nabla T = 0$ then the projecting component $\sigma = \Pi_g(T) \in C^\infty(M)$ of T with respect to $g \in c$ satisfies (4) and $\mathcal{D}^g \sigma = T$.

Definition 2.1 (*a*) *Let (M^n, c) be a conformal Riemannian manifold of dimension $n \geq 3$. We call a density ω, which solves (3), an almost Einstein structure on (M, c).*

(*b*) *Alternatively, let (M^n, g) be Riemannian manifold of dimension $n \geq 3$. We call a function $\sigma \in C^\infty(M)$, which solves (4), an almost Einstein structure on (M, g).*

Remark to Definition 2.1. Note that a solution σ of (4) on (M, g) conformally rescales to a solution $e^f \cdot \sigma$ of (4) with respect to $\tilde{g} = e^{2f} g \in [g]$. If σ is a solution of (4) without zeros, then $h = \sigma^{-2} g$ is an Einstein metric in the conformal class $c = [g]$ on M (i.e., the Ricci curvature of h is a constant multiple of h). On the other hand, if $h = \sigma^{-2} g$ is an Einstein metric, then σ is a solution of (4) with respect to g on M. However, in general, the zero set $\Sigma(\sigma)$ of a non-trivial solution σ of (4) is non-empty. This suggests the naming *almost Einstein structure* for a general solution of (4). (In [21] we have called a solution of (4) a *nc-Killing function.*) In case $\Sigma(\sigma)$ is empty on (M, g), we also say that g is *conformally Einstein* and $(M, [g])$ is a *conformal Einstein space.* $\Diamond$

In what follows we usually consider almost Einstein structures as functions σ on a Riemannian manifold (M, g), i.e., we mainly adopt the view of part (*b*) of Definition 2.1.

Theorem 2.2 *[13] Let (M, g, σ) be an almost Einstein space of Riemannian signature with parallel tractor $I := \mathcal{D}^g \sigma$ and $S(\sigma) := -\langle I, I \rangle_{\mathcal{T}}$. If $S(\sigma) > 0$ then $\Sigma(\sigma)$ is empty and $(M, \sigma^{-2} g)$ is Einstein with positive scalar curvature; if $S(\sigma) = 0$ then $\Sigma(\sigma)$ is either empty or consists of isolated points and $(M \setminus \Sigma(\sigma), \sigma^{-2} g)$ is Ricci-flat; if $S(\sigma) < 0$ then the scale singularity set $\Sigma(\sigma)$ is either empty or else is a smooth hypersurface, and $(M \setminus \Sigma(\sigma), \sigma^{-2} g)$ is Einstein of negative scalar curvature.*

Note that the quantity $S(\sigma)$ is related to the scalar curvature of the corresponding Einstein metric $h = \sigma^{-2}g$ on $M^n \setminus \Sigma(\sigma)$ by

$$scal^h \;=\; n(n-1)S(\sigma)\ . \tag{5}$$

Moreover, note that, if $\Sigma(\sigma)$ is a hypersurface singularity on M and $S(\sigma) = -1$, then the Einstein metric $\sigma^{-2}g$ is *asymptotically hyperbolic* at $\Sigma(\sigma)$, and $\Sigma(\sigma)$ with induced conformal structure $[g|_{\Sigma(\sigma)}]$ can be interpreted as the *conformal infinity* space of $\sigma^{-2}g$. In fact, if M is a *closed* almost Einstein manifold, i.e., compact without boundary, then $\Sigma_{\geq}(\sigma) := \{\sigma \geq 0\}$ is a compact manifold with boundary $\Sigma(\sigma)$ and the interior $\Sigma_+(\sigma) = \{\sigma > 0\}$ with metric $\sigma^{-2}g$ is a compact *Poincaré-Einstein space* (or *AH Einstein space*) with conformal infinity $\Sigma(\sigma)$. In case $S(\sigma) = 0$ and $\Sigma(\sigma)$ consists of isolated points, the Ricci-flat metric $\sigma^{-2}g$ is *asymptotically locally flat* near the singular points.

Theorem 2.3 *[14] Let (M^n, g, σ) be a closed almost Einstein space of Riemannian signature with dimension $n \geq 3$. If ρ is another almost Einstein structure on (M, g) such that $I_\sigma := \mathcal{D}^g\sigma$ and $I_\rho := \mathcal{D}^g\rho$ are linearly independent parallel tractors, then either*

- (a) *(M^n, g) is conformally equivalent to the round unit sphere (S^n, g^o_{rd}) (cf. Remark to Theorem 3.3) or*
- (b) *$\Sigma(\sigma)$ and $\Sigma(\rho)$ are non-empty hypersurface singularities and both metrics $\sigma^{-2}g$ and $\rho^{-2}g$ are asymptotically hyperbolic at infinity.*

In particular, Theorem 2.3 says that any almost Einstein space (M^n, g, σ) of dimension $n \geq 3$ with $\Sigma(\sigma) \neq \emptyset$, which is simultaneously conformally Einstein, has to be conformally equivalent to the round unit n-sphere (S^n, g^o_{rd}). This is a useful statement, since it allows to argue whether a closed Riemannian manifold is conformally Einstein or not (in the presence of an almost Einstein structure with scale singularity). For example, any closed almost Einstein 4-space with scale singularity, which is not half conformally flat, is *Bach-flat* without being conformally Einstein (cf. Section 6.1 for the definition of the Bach tensor).

3 Almost Einstein structures on Riemannian products

We discuss here equation (4) about almost Einstein structures on closed Riemannian product spaces (cf. [10] for a discussion of the conformally Einstein case).

3.1 The equation on products.

Let (M^n, g) denote a Riemannian manifold of dimension n with Levi-Civita connection ∇^g and Riemannian curvature tensor R^g. (In the following, we assume that any closed manifold M is connected.) Any curvature quantity that can be derived from R^g will be denoted by a superscript g. For example, we have the Ricci curvature tensor Ric^g, the scalar curvature $scal^g$ and the Schouten tensor P^g, which is defined by

$$\mathsf{P}^g \;:=\; \frac{1}{n-2}\left(\frac{scal^g}{2(n-1)} - Ric^g\right)\ . \tag{6}$$

The Hessian of functions on M with respect to g is denoted by $Hess^g := \nabla^g \circ d = (\nabla^g)^2 : S^2(T^*M) \to \mathbb{R}$. The trace of the Hessian is the Laplacian $\Delta^g = tr(\nabla^g)^2$.

Now let us assume that (M^n, g) is a product of Riemannian manifolds (M^{n_1}, g_1) and (M^{n_2}, g_2) of dimensions $n_1 \geq 1$ and $n_2 \geq 1$, respectively, with $n = n_1 + n_2 \geq 3$. In the following, we shall suppress pull backs and inclusions of the product space $M = M_1 \times M_2$, i.e., we interpret any vector field or tensor on the factors M_i as vector field or tensor on the product manifold M via the natural projections $\pi_i : M \to M_i$ and inclusions $\iota_i : M_i \to M$, $i = 1, 2$, respectively. For example, the metric tensor g on M is the sum g_1+g_2 of the pull backs of the metric tensors g_i on the factors M_i, $i=1,2$. The Levi-Civita connection on M satisfies $\nabla^g_{X_i}Y_i=\nabla^{g_i}_{X_i}Y_i$ for $X_i,Y_i\in\mathfrak{X}(M_i)$, $i=1,2$, and $\nabla^g_{X_i}Y_j=0$ for $i\neq j$. A simple calculation shows that the Riemannian curvature tensor R^g of the product metric g on M is the product of the Riemannian curvature tensors R^{g_1} and R^{g_2}, i.e., $R^g=R^{g_1}+R^{g_2}$, which also implies

$$Ric^g = Ric^{g_1}+Ric^{g_2} \qquad \text{and} \qquad scal^g = scal^{g_1}+scal^{g_2}\ .$$

In general, we denote the trace-free part of a symmetric 2-tensor A with respect to a given metric g on a manifold M by a subscript with A_o. Then we have $A=A_o+\frac{1}{n}tr_gA\cdot g$, where tr_gA denotes the trace of A with respect to g. The following Lemma 3.1 is proved by simple calculations.

Lemma 3.1 (*a*) *The trace-free part Ric^g_o of Ric^g on M is given by*

$$\begin{aligned} Ric^g_o \;=\;& Ric^{g_1}_o + Ric^{g_2}_o \\ &+\frac{1}{n}\left(\frac{n_2}{n_1}scal^{g_1}-scal^{g_2}\right)g_1+\frac{1}{n}\left(\frac{n_1}{n_2}scal^{g_2}-scal^{g_1}\right)g_2\ , \end{aligned}$$

where $Ric^{g_i}_o$ denotes the trace-free part of Ric^{g_i} with respect to g_i, $i=1,2$. The trace-free part of the Schouten tensor of g satisfies $\mathsf{P}^g_o=\frac{-1}{n-2}Ric^g_o$.

(*b*) *Let $\rho_i: M_i\to\mathbb{R}$ be real smooth functions on M_i for $i=1,2$. We set $\rho:=\rho_1+\rho_2$. Then the relations*

$$\begin{aligned} grad^g\rho &= grad^{g_1}\rho_1+grad^{g_2}\rho_2 \\ Hess^g\rho &= Hess^{g_1}\rho_1+Hess^{g_2}\rho_2 \\ \Delta^g\rho &= \Delta^{g_1}\rho_1+\Delta^{g_2}\rho_2 \end{aligned}$$

and

$$\begin{aligned} Hess^g_o\rho \;=\;& Hess^{g_1}_o\rho_1 + Hess^{g_2}_o\rho_2 \\ &+\frac{1}{n}\left(\frac{n_2}{n_1}\Delta^{g_1}\rho_1-\Delta^{g_2}\rho_2\right)g_1+\frac{1}{n}\left(\frac{n_1}{n_2}\Delta^{g_2}\rho_2-\Delta^{g_1}\rho_1\right)g_2 \end{aligned}$$

hold.

In the previous section we have introduced the notion of almost Einstein structures on Riemannian (conformal) manifolds. An almost Einstein structure σ on (M,g) is a solution to the partial differential equation

$$Hess^g_o\sigma \;=\; \frac{-1}{n-2}Ric^g_o\cdot\sigma \tag{7}$$

(which is obviously equivalent to (4)). On a Riemannian product manifold (M,g) equation (7) splits into conditions for functions on the factors.

Lemma 3.2 *Let $(M^n,g)=(M_1^{n_1}\times M_2^{n_2},g_1+g_2)$ be a Riemannian product manifold of dimension $n=n_1+n_2\geq 3$. Then a smooth function σ on M solves (7) if and only if $\sigma=\sigma_1+\sigma_2$ is a sum of pull backs of functions σ_i on M_i, $i=1,2$, which are solutions to the equations*

$$Hess^{g_1}_o\sigma_1 \;=\; -\frac{1}{n-2}Ric^{g_1}_o\cdot(\sigma_1+\sigma_2) \tag{8}$$

$$Hess^{g_2}_o\sigma_2 \;=\; -\frac{1}{n-2}Ric^{g_2}_o\cdot(\sigma_1+\sigma_2) \tag{9}$$

$$n_2\Delta^{g_1}\sigma_1-n_1\Delta^{g_2}\sigma_2 \;=\; \frac{1}{n-2}\left(n_1scal^{g_2}-n_2scal^{g_1}\right)\cdot(\sigma_1+\sigma_2)\ . \tag{10}$$

Proof. First, we show that a solution σ of (7) must be a sum of the form $\sigma(x,y) = \sigma_1(x) + \sigma_2(y)$ for all $(x,y) \in M$, where σ_i is a smooth function on M_i, $i = 1,2$. In fact, equation (7) implies $Hess^g(\sigma)(X,Y) = X(Y(\sigma)) = 0$ for all $X \in \mathfrak{X}(M_1)$ and $Y \in \mathfrak{X}(M_2)$. Now let x_o be fixed in M and set $\varphi(x,y) := \sigma(x,y) - \sigma(x_o,y)$. Since $Y(\sigma)$ does not depend on x, we have $Y(\varphi)(x,y) = Y(\sigma)(x,y) - Y(\sigma)(x_o,y) = 0$. Hence we can set $\sigma_1(x) := \varphi(x,y)$ and $\sigma_2(y) := \sigma(x_o,y)$ for all $(x,y) \in M$.

Now, Lemma 3.1 immediately shows that equation (7) for $\sigma(x,y) = \sigma_1(x) + \sigma_2(y)$ is equivalent to the system of equations (8) - (10) for the functions σ_1 and σ_2 on M. □

3.2 A coarse geometric description.

Recall that, in general, a vector field V on a Riemannian manifold (N^m, h) of dimension $m \geq 2$ is called a *conformal Killing vector field* if the Lie derivative $\mathcal{L}$ of the metric h in direction of V satisfies $\mathcal{L}_V h = \lambda \cdot h$ for some function $\lambda \in C^\infty(N)$. Accordingly, a gradient field $V = grad^h f$, $f \in C^\infty(N)$, is called a *conformal gradient* if V is a conformal Killing vector field. A well known result states that any conformal gradient V with a zero is an *essential* conformal Killing vector field on (N,h), i.e., $\mathcal{L}_V \tilde{h} \neq 0$ for any metric $\tilde{h} = e^{2f} h$, $f \in C^\infty(N)$, in the conformal class $[h]$ of h (cf. e.g. [19]). (A vector field V with $\mathcal{L}_V \tilde{h} = 0$ is not essential and is called a *Killing vector field* for $\tilde{h}$.) Furthermore, a fundamental result of conformal differential geometry states that for any $m \geq 2$ the round unit sphere (S^m, g^o_{rd}) is (up to conformal equivalence) the only closed Riemannian manifold of dimension m, which admits an essential conformal Killing vector field (cf. e.g. [1]). (Basically, this result is also responsible for Theorem 2.3.)

Theorem 3.3 *Let (N^m, h) be a closed Riemannian manifold of dimension $m \geq 2$. If N admits a non-constant solution Φ to the partial differential equation*

$$Hess^h_o \Phi \;=\; 0 \qquad (\Phi \not\equiv const.), \tag{11}$$

then (N^m, h) is conformally equivalent to the round unit m-sphere (S^m, g^o_{rd}).

Proof. Obviously, equation (11) is equivalent to the condition that $grad^h \Phi$ is a non-trivial conformal gradient. Since N is a closed Riemannian manifold, any gradient of a function Φ on N admits a zero, i.e., $grad^h \Phi(p) = 0$ for some $p \in N$. This implies for any solution Φ of (11) that the conformal gradient $grad^h \Phi$ is essential. Hence, by the classical result for $m \geq 2$ about essential conformal vector fields, the closed Riemannian manifold (N^m, h) must be conformally diffeomorphic to the round unit m-sphere. □

Remark to Theorem 3.3. Note that the standard round sphere (S^m, g^o_{rd}) really admits non-trivial solutions of (11). In fact, let $\iota : S^m \hookrightarrow \mathbb{R}^{m+1}$ be the standard embedding of S^m as unit m-sphere into the Euclidean space of dimension $m+1$. Then the solutions of (11) on (S^m, g^o_{rd}) are exactly the linear functions Φ on $\mathbb{R}^{m+1}$ restricted to S^m plus a constant. Thus, up to isometries, any solution is of the form $\Phi = ax_0 + b$ with $a, b \in \mathbb{R}$. We have $\Delta^{g^o_{rd}} \Phi = -m \cdot \Phi$ for $\Phi = ax_0$.

Note that for $m \geq 3$ equation (11) is exactly the almost Einstein equation on the round unit m-sphere. In particular, the metric $x_0^{-2} \cdot g^o_{rd}$ on the two caps of $S^m \setminus \{x_0 = 0\}$ is hyperbolic. However, equation (11) is not conformally covariant. (It is just an equation of Riemannian geometry.) Nevertheless, there are certain other conformally flat metrics on the sphere apart from the round metrics, which admit non-constant solutions of (11). We will discuss these solutions in Section 4 for the case of a 2-sphere S^2. (The case of an arbitrary n-sphere behaves similar.) ◊

Here is a statement about closed Riemannian products admitting an almost Einstein structure. Notice that we regard a 2-dimensional manifold M^2 to be Einstein if the scalar curvature is constant.

Proposition 3.4 *Let $(M^n, g) = (M_1^{n_1} \times M_2^{n_2}, g_1 + g_2)$ be a closed Riemannian product manifold of dimension $n = n_1 + n_2 \geq 3$. If (M^n, g) admits an almost Einstein structure $\sigma = \sigma_1 + \sigma_2$, then exactly one of the following 3 cases holds true:*

(*i*) *Any solution σ of (7) is constant and g on M is a product of Einstein metrics g_1 and g_2 with $n_2 scal^{g_1} = n_1 scal^{g_2}$. In particular, (M^n, g) is an Einstein space.*

(*ii*) *For any solution $\sigma = \sigma_1 + \sigma_2$ of (7) exactly one summand is constant and the other summand, let us say σ_1, solves (11) $Hess_o^{g_1}\sigma_1 = 0$ on M_1. In particular, M_1 is some n_1-sphere with $n_1 \geq 1$ and (M_2, g_2) is either an Einstein space or some circle S^1.*

(*iii*) *For any solution $\sigma = \sigma_1 + \sigma_2$ of (7) exactly one summand is constant and the other summand, let us say σ_1 on M_1 with $n_1 \geq 3$, solves*

$$\begin{aligned} Hess_o^{g_1}\sigma_1 &= \tfrac{-1}{n-2} Ric_o^{g_1} \cdot \sigma_1 \neq 0 \, , \\ \Delta^{g_1}\sigma_1 &= \tfrac{1}{n-2}\left(\tfrac{n_1}{n_2} scal^{g_2} - scal^{g_1}\right)\sigma_1 \, . \end{aligned}$$

The factor (M_2, g_2) is Einstein or some circle S^1.

The case that a solution σ of (7) is non-constant on both factors M_1 and M_2 does not occur.

Proof. Our proof examines whether there exists a solution σ of (7) (resp. to the system (8) - (10)), which is constant along some factor M_i,$i = 1, 2$, or not.

First, let us assume that there exists a solution $\sigma = \sigma_1 + \sigma_2$ on a product with $n_1 \leq n_2$ and neither σ_1 nor σ_2 being constant. Then (8) and (9) of Lemma 3.2 imply $Ric_o^{g_1} = Ric_o^{g_2} = 0$ and $Hess_o^{g_1}\sigma_1 = Hess_o^{g_2}\sigma_2 = 0$, i.e., σ_1 and σ_2 are solutions of (11) on their respective factors, which both have to be conformally equivalent to a round sphere (or circle). In fact, if $n_2 \geq 3$ or $n_1 = 1$, then one of the factors is a round sphere (or circle) with constant scalar curvature. And, then (10) implies that both scalar curvatures are constant, i.e., both factors are round spheres. However, in this case we have $\Delta^{g_i}\sigma_i = -\lambda_i\sigma_i + const.$, $i = 1, 2$, for certain constants $\lambda_i > 0$. This shows that (10) cannot be satisfied for these dimensions. We conclude that the only possibility that σ_1 and σ_2 are non-constant is when M_1 and M_2 are both conformally equivalent to a round 2-sphere. We will see in Section 4 that this case is impossible as well (cf. Lemma 4.4).

Now let us assume that the solution $\sigma = \sigma_1$ is non-constant on the factor M_1 and $\sigma_2 = 0$. Then (9) and (10) show that (M_2, g_2) is Einstein or a circle S^1. If $Hess_o^{g_1}\sigma_1 = 0$ then M_1 is some n_1-sphere by Theorem 3.3 or a circle S^1. If $Hess_o^{g_1}\sigma_1 \neq 0$, then $n_1 \geq 3$ and equations (8) - (10) reduce to the two equations, which are given in case (*iii*) of Proposition 3.4.

If σ is a constant then the Riemannian product manifold (M, g) itself has to be Einstein. Equation (10) implies the relation for the corresponding scalar curvatures as stated in case (*i*). □

We are interested in this paper in solving case (*ii*) of Proposition 3.4. Theorem 3.5 below and the discussion of Section 4 will completely clarify the situation in this case. Note that the first partial differential equation of case (*iii*) is not the almost Einstein condition (7) for M_1, since the factor $1/(n-2)$ is different from $1/(n_1-2)$. In particular, this equation says that M_1 is not an Einstein space. We will discuss the system of equations for case (*iii*) elsewhere.

Theorem 3.5 *Let $(M^n, g) = (M_1^{n_1} \times M_2^{n_2}, g_1 + g_2)$ be a closed Riemannian product manifold of dimension $n = n_1 + n_2 \geq 3$ as described in case (ii) of Proposition 3.4 with $\sigma = \sigma_1 \not\equiv const.$ The following 3 cases occur:*

(*iia*) *(M^n, g) is (up to constant dilation) the product of the unit circle $(S^1, d\varphi^2)$ with an Einstein manifold (M^{n_2}, g_2) of negative scalar curvature such that $\lambda := \sqrt{\frac{-scal^{g_2}}{n_2(n_2-1)}} \in \mathbb{N}$. Any solution σ of (7) has zeros and can be expressed in the form $\sigma = \sigma_1 = a \cdot \cos(\lambda\varphi) + b \cdot \sin(\lambda\varphi) \not\equiv 0$ with $a, b \in \mathbb{R}$.*

(*iib*) *$(M^n, g) = (S^2 \times M_2^{n_2}, g_1 + g_2)$ is a product with some 2-sphere and (M_2, g_2) is Einstein. (See Section 4 for a discussion of the solutions $\sigma = \sigma_1$ and the corresponding metrics g_1 on S^2.)*

(*iic*) *(M^n, g) is (up to constant dilation) a product of the form $M^n = S^{n_1} \times M_2$ with a round unit n_1-sphere of dimension $n_1 \geq 3$, where (M_2, g_2) is either a circle S^1 or an Einstein space of negative scalar curvature $-n_2(n_2-1)$. The solutions $\sigma = \sigma_1$ on S^{n_1} are of the form $\Phi = ax_0$, $a \neq 0$ (up to isometries of (S^{n_1}, g^o_{rd}); cf. Remark to Theorem 3.3).*

Note that cases (*iia*) and (*iic*) of Theorem 3.5 are almost Einstein cases with hypersurface singularities. In fact, since any solution σ of (7) splits into $\sigma_1 + \sigma_2$, it is clear that almost Einstein structures with isolated singularities cannot occur on product manifolds, in general. Any product of the form $S^{n_1} \times H^{n_2}$ with a round unit sphere of dimension $n_1 \geq 3$ and a closed hyperbolic space H^{n_2} belongs to case (*iic*). These are conformally flat examples. Case (*iia*) could be seen as a special case of (*iic*) when the round unit n_1-sphere were allowed to be just a unit circle S^1. Of course, examples based on solutions of the form $\Phi = ax_0$ on a round unit 2-sphere S^2 exist and are contained in the statement of case (*iib*). However, the case (*iib*) of products with a 2-sphere is more rich. (The reason is that for any Riemannian 2-manifold (N^2, h) the Ricci curvature only has a trace part, i.e., we always have $Ric^h_o = 0$.) We will explicitly determine all solutions σ of (7) on products with S^2 by solving a certain ordinary differential equation in the next section. This will allow us to give a detailed description of that case.

4 On products with the 2-sphere

In this section we describe explicitly case (*iib*) of Theorem 3.5. The discussion will prove the existence of an *exotic* family of closed Riemannian products with the 2-sphere, which admit an almost Einstein structure with hypersurface singularity. We will also see that no further examples of conformally Einstein product metrics with the 2-sphere are possible (apart from the obvious products of Proposition 3.4 (*i*)).

4.1 Warped products with coordinate singularity.

An almost Einstein structure $\sigma = \sigma_1$ on a product $M^n = S^2 \times M_2^{n_2}$ with a 2-sphere, where (M_2, g_2) is some Einstein space or some circle S^1, is a solution of the equations

$$Hess^{g_1}_o \sigma_1 = 0 \qquad \text{and} \qquad \Delta^{g_1} \sigma_1 = \frac{1}{n_2}\left(\frac{2}{n_2} scal^{g_2} - scal^{g_1}\right)\sigma_1 \ . \tag{12}$$

In particular, $grad^{g_1}\sigma_1$ is a conformal gradient on (S^2, g_1). It is a well known fact that conformal gradient fields induce a *warped product structure* on the underlying manifold (outside of the critical points) (cf. e.g. [19]). We give below a slight variation of this description on S^2, which is more suitable for our purposes. (We will use this description also in the later sections.)

To begin with, let F be a smooth function on a closed interval $I = [a, b]$, $a < b \in \mathbb{R}$, i.e., F is smooth on (a, b) and all derivatives with respect to the coordinate x are continuous on I. We say F is an *even* function at $x_o \in I$ if the Taylor series admits only development terms with even exponents, i.e.,

$$F(x - x_o) \ \sim \ \sum_{l=0}^{\infty} p_l (x - x_o)^{2l} \ , \quad q_l \in \mathbb{R} \ .$$

Accordingly, F is *odd* at $x_o \in I$ if all development terms have odd exponents. These definitions are also valid at the boundary points a and b of the closed interval I.

Lemma 4.1 *Let g_1 be a smooth Riemannian metric on the 2-sphere S^2, which admits a non-constant solution σ_1 of (11). Then the conformal gradient $V := grad^{g_1}\sigma_1$ admits exactly two distinct zeros $z_1, z_2 \in S^2$, and*

(*a*) *there exists a smooth function $F \in C^\infty(I)$ on some closed interval $I = [a, b] \subset \mathbb{R}$ (with coordinate r), which is even at the boundary points a, b of I, such that g_1 on $S^2 \setminus \{z_1, z_2\}$ is isometric to the warped product*

$$dr^2 \ + \ (\dot{F})^2 \, d\varphi^2 \qquad on \quad (a, b) \times S^1 \ ,$$

where $\varphi \in [0, 2\pi)$ is the arc length on a unit circle S^1. The warping function $\dot{F} = dF/dr$ is positive on (a, b) and satisfies

$$\ddot{F}(a) \;=\; -\ddot{F}(b) \;=\; 1\ . \tag{13}$$

The solution σ_1 on $S^2 \setminus \{z_1, z_2\}$ equals F on $(a, b) \times S^1$ and the conformal gradient V is $\dot{F}{\cdot}\partial r$.

(b) *Alternatively, there exists a smooth function $L \in C^\infty(\tilde{I})$ on some closed interval $\tilde{I} = [\tilde{a}, \tilde{b}] \subset \mathbb{R}$ with $L > 0$ on $(\tilde{a}, \tilde{b})$ and $L(\tilde{a}) = L(\tilde{b}) = 0$ such that g_1 on $S^2 \setminus \{z_1, z_2\}$ is isometric to the (generalised) warped product*

$$L^{-1}\, dt^2 \;+\; L\, d\varphi^2 \qquad on \quad (\tilde{a}, \tilde{b}) \times S^1\ ,$$

where $t \in \tilde{I}$ and $(S^1, d\varphi^2)$ is the unit circle. The function L satisfies

$$\frac{dL}{dt}(\tilde{a}) \;=\; -\frac{dL}{dt}L(\tilde{b}) \;=\; 2\ . \tag{14}$$

The solution σ_1 is given by the coordinate t on $(\tilde{a}, \tilde{b}) \times S^1$ and the conformal gradient V is $L{\cdot}\partial t$.

Proof. First, notice that any conformal Killing vector field either has one or two zeros on S^2. Since $V \not\equiv 0$ is a gradient and S^2 is compact, it follows that V has exactly two distinct zeros z_1, z_2.

(a) The warped product structure for g_1 off the critical points z_1, z_2 on S^2 is well known, as many works in the literature show. In short, the reason for the induced warped product structure is that the conformal gradient V is a parallel vector field with respect to any metric in the conformal class $[g_1]$, which makes the norm of V constant on $S^2 \setminus \{z_1, z_2\}$. Obviously, using this warped product description, the solution σ_1 is an integral F of the positive warping function. In particular, F is a strictly increasing function of the coordinate r of the interval I. The interval I must be closed, since S^2 is a closed Riemannian manifold.

The evenness and condition (13) for F are directly implied by the smoothness of g_1 at the critical points z_1, z_2 of σ_1. In fact, we have the following general statement: Let (r, φ) be polar coordinates on a neighbourhood U of the origin in $\mathbb{R}^2$ and let $h(r)$ be a radial function on U. Then a metric of the form $dr^2 + h^2(r)d\varphi^2$ is smooth at the origin if and only if the function h vanishes at 0, $\dot{h}(0) = 1$ and h^2 is smooth in the coordinate r^2, i.e., h is an odd function at 0.

(b) Now, let us assume that g_1 is given by a warped product as described in part (a) of Lemma 4.1. We set $t := F(r)$, which is a strictly increasing function on I. Further, we set $\tilde{I} = [\tilde{a}, \tilde{b}] := [F(a), F(b)]$, which is a closed interval, since the solution σ_1 is bounded on S^2. Then we have $(\dot{F}(r))^{-1}dt = dr$. With $L(t) := (\dot{F}(r(t)))^2$ we obtain the warped product metric as stated in part (b) of Lemma 4.1. Condition (13) directly implies (14) $(dL/dt)(\tilde{a}) = -(dL/dt)(\tilde{b}) = 2$. Also note, since $\dot{F}$ is odd in r at the boundary points, the function $L(t)$ is by construction smooth in the coordinate t on the closed interval $\tilde{I}$. By definition, the solution σ_1 is given by t on $(\tilde{a}, \tilde{b}) \times S^1$. The conformal gradient is the g_1-dual vector field to dt, which is $L{\cdot}\partial t$. □

Remark to Lemma 4.1. Both the conditions of part (a) on the function $F \in C^\infty(I)$ and of part (b) on the function $L \in C^\infty(\tilde{I})$ are sufficient for the reconstruction of a smooth metric on the 2-sphere, i.e., given any such function $F(r)$ or $L(t)$, respectively, the corresponding warped product metrics, as defined in (a) and (b), extend smoothly to S^2.

For these reconstructions, the solutions of (11) are given by $F(r)$ and the coordinate t, respectively. In fact, $F(r)$ extends smoothly to S^2, since $F(r)$ is assumed to be even (i.e. smooth in r^2) at the boundary points a, b. And then we have $Hess_o^{g_1}F = 0$. On the other hand, one easily checks that the function t is even in the coordinate $r = a + \int_{\tilde{a}}^{t}\left(1/\sqrt{L(x)}\right)dx$ at $a = r(\tilde{a})$ and $b = r(\tilde{b})$. Hence, t extends to a smooth function on the 2-sphere, which solves (11) as well. In fact, note that, for the same reason, any smooth function $A(t)$ on $[\tilde{a}, \tilde{b}] \times S^1$ extends smoothly to the 2-sphere. ♢

4.2 Explicit solutions with the 2-sphere.

In order to find almost Einstein structures on a product $M^n = S^2 \times M_2^{n_2}$ with the 2-sphere, we still have to solve the second part of (12). With the description of Lemma 4.1 (b) this becomes an ordinary differential equation for $L(t)$. In fact, a straightforward calculation shows $scal^{g_1} = -L''$. (The prime denotes derivatives with respect to $t \in \tilde{I}$.) For the Laplacian of an arbitrary smooth function $A(t)$ on (S^2, g_1) we have $\Delta^{g_1} A = (A'L)' = A''L + A'L'$. This reads $\Delta^{g_1}\sigma_1 = L'$ for the solution $\sigma_1 = t$ of (11), and equation (12) is equivalent to the ODE

$$n_2 \cdot L'(t) \;=\; t \cdot \left(L''(t) \;+\; \frac{2 \cdot scal^{g_2}}{n_2} \right) . \qquad (15)$$

We set $S(g_2) := \frac{scal^{g_2}}{n_2(n_2-1)}$ if $n_2 > 1$, and $S(g_2) := 0$ for $n_2 = 1$. (This is the *normed* scalar curvature of (M_2, g_2) with dimension $n_2 = n - 2 \geq 1$; cf. (5).) Then the general solution of (15) is given by

$$L(t) \;=\; C_1 \cdot t^{n-1} + S(g_2) \cdot t^2 + C_2 \;\geq 0 \,, \qquad C_1, C_2 \in \mathbb{R} \,. \qquad (16)$$

In order to find smooth metrics g_1 on the closed 2-sphere S^2 admitting a solution of (12), we still have to ensure condition (14). This means we have to find adjacent roots $\tilde{a}, \tilde{b} \in \mathbb{R}$ with $\tilde{a} < \tilde{b}$ of the polynomial (16) such that L is positive on the interval $(\tilde{a}, \tilde{b})$. And we have to fix the constants C_1, C_2 such that for a given $S := S(g_2) \in \mathbb{R}$ the gradient of the polynomial L has absolute value 2 at both adjacent roots $\tilde{a}, \tilde{b} \in \mathbb{R}$ (cf. Figure 1). For example, the polynomial

$$L(t) \;=\; Ct^2 \;-\; C^{-1} \qquad \text{on} \quad \tilde{I} = [C^{-1}, -C^{-1}] \qquad (17)$$

admits these properties for any $C < 0$. However, there exists another (non-standard or exotic) family of such polynomials L.

Lemma 4.2 *Let $L(t) = C_1 t^{n-1} + St^2 + C_2$, $C_1 \neq 0$ and $C_2 \in \mathbb{R}$, be a polynomial of degree $n - 1 > 2$ with fixed $S \in \mathbb{R}$. Then $L(t)$ admits adjacent roots $\tilde{a}, \tilde{b}$ such that $L > 0$ on the interval $(\tilde{a}, \tilde{b})$ and $L'(\tilde{a}) = -L'(\tilde{b}) = 2$ if and only if n is odd and the coefficients of L are given by*

$$C_1 \;=\; \frac{-2}{n-1} Q^{2-n} \cdot (1 + QS) \qquad \text{and} \qquad C_2 \;=\; Q \cdot \left(\frac{2 - (n-3)QS}{n-1} \right) \,, \qquad (18)$$

where the parameter $Q \in \mathbb{R}_+$ is a positive number with $Q^{-1} > \frac{n-3}{2} S$. In this case the adjacent roots are $\tilde{a} = -Q$ and $\tilde{b} = Q$ (cf. Figure 1).

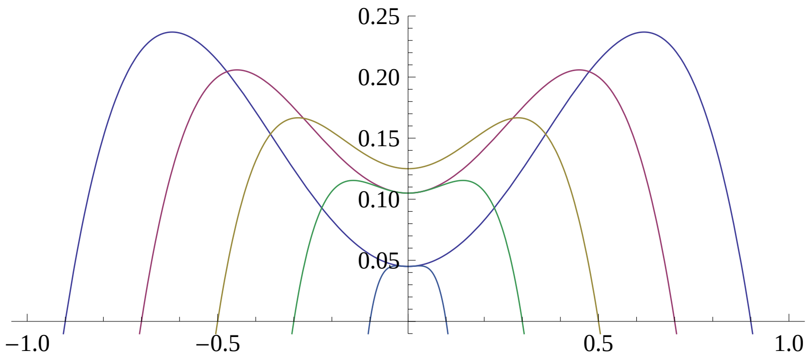

Figure 1: This graphic shows the graphs of the polynomial $L(t)$ for the values $Q = 0.9, 0.7, 0.5, 0.3$ and 0.1 when $n = 5$ and $S = 1$ are fixed. The absolute gradient of L is 2 at any root $\pm Q$. The graphic also demonstrates that, for certain different values of Q, the scale singularity sets at $t = 0$ of the corresponding closed almost Einstein spaces are conformally equivalent.

Proof. Let us assume that $L(t)$ is a polynomial with a pair of roots $\tilde{a} < \tilde{b} \in \mathbb{R}$ as required. If $n = 4$ then L has degree 3 and, since there are at least two roots, we can assume $L(t) =$

$(t-\tilde{a})(t-\tilde{b})(t-\tilde{c})$ for some $\tilde{c} \in \mathbb{R}$. Obviously, the condition $L'(\tilde{a}) = -L'(\tilde{b})$ implies $\tilde{a}-\tilde{c} = \tilde{b}-\tilde{c}$, which is a contradiction to $\tilde{a} < \tilde{b}$. Hence, $n = 4$ is not possible at all.

So let us assume $n \geq 5$. We notice that $t = 0$ cannot be a root of L with non-zero gradient. Moreover, we notice that for any $Q \neq 0$ the rescaled polynomial $Q^{-1} \cdot L(Qt)$ admits the pair $Q^{-1}\tilde{a}, Q^{-1}\tilde{b}$ of adjacent roots with the required properties. Thus we can assume $\tilde{a} < \tilde{b} = 1$ for the polynomial $L(t)$ without loss of generality.

Now, obviously, the conditions $L(1) = L(\tilde{a}) = 0$ and $-L'(1) = L'(\tilde{a}) = 2$ are satisfied if and only if there exists some polynomial P of degree $n-5$ such that

$$L(t) \;=\; (t-1)(t-\tilde{a})\left(\;(t-1)(t-\tilde{a})P(t) - \frac{2}{1-\tilde{a}}\;\right) .$$

Equivalently, this means $\tilde{b} = 1$ and $\tilde{a}$ are twofold roots of the polynomial $L(t) + \frac{2}{1-\tilde{a}}(t-1)(t-\tilde{a})$. This again is equivalent to the system

$$\begin{array}{rcrcrcl} C_1 & + & S & + & C_2 & = & 0 \\ (n-1)C_1 & + & 2S & + & 2 & = & 0 \\ \tilde{a}^{n-1}C_1 & + & \tilde{a}^2 S & + & C_2 & = & 0 \\ (n-1)\tilde{a}^{n-2}C_1 & + & 2\tilde{a}S & - & 2 & = & 0 \end{array} \tag{19}$$

of equations for $C_1, C_2, \tilde{a}$ with fixed $S \in \mathbb{R}$. The first two of these equations imply $C_1 = \frac{-2}{n-1}\cdot(1+S)$ and $C_2 = \frac{2-(n-3)S}{n-1}$. Furthermore, multiplying the fourth equation with $\frac{-\tilde{a}}{n-1}$ and adding this to the third equation above, gives the quadratic equation $(n-3)(\tilde{a}^2-1)S + 2(\tilde{a}+1) = 0$, whose only solutions are $\tilde{a}_1 = -1$ and $\tilde{a}_2 = 1 - \frac{2}{(n-3)S}$ (if $S \neq 0$).

Now let us assume $n \geq 5$ to be odd. In this case $\tilde{a}_1 = -1$ is in fact a solution of (19) (with C_1 and C_2 determined as above). On the other hand, since L is an even function at 0, if $\tilde{a}_2 = 1 - \frac{2}{(n-3)S}$ were a solution of (19), then $-\tilde{a}_2 = \frac{2}{(n-3)S} - 1$ had to be a solution as well. This implies $\tilde{a}_2 = -1, 1$ or 0. However, $\tilde{a}_2 = 0$ or 1 is impossible. Thus we can conclude that the only possible pair of roots is $\tilde{a} = -1$ and $\tilde{b} = 1$ for n odd.

We still need to ensure that the pair $\tilde{a} = -1$, $\tilde{b} = 1$ of roots is adjacent when $n \geq 5$ is odd. For this we notice that all roots of $L(t)$ are determined by the intersection of a straight line with a parabola of degree $\frac{n-1}{2}$. This shows that $L(t)$ can have at most 4 roots. Hence, $\tilde{a} = -1$ and $\tilde{b} = 1$ are adjacent roots if and only if $L(0) > 0$, i.e., $S < \frac{2}{n-3}$. If we rescale now the polynomial L by $Q \cdot L(t/Q)$ with $Q > 0$, we obtain exactly the case that is stated in Lemma 4.2.

It remains to discuss the case $n \geq 6$ even with C_1 and C_2 determined as above. Obviously, in this case $\tilde{a} = -1$ cannot solve the third equation of (19). Thus the only possibility for a root with gradient 2 is $\tilde{a}_2 = 1 - \frac{2}{(n-3)S}$ for $S \neq 0$. Inserting this value into the fourth equation of (19) implies the condition

$$(1+S)\cdot\left(S - \frac{2}{n-3}\right)^{n-2} \;-\; S^{n-1} \;+\; \left(1 + \frac{2}{n-3}\right)S^{n-2} \;=\; 0 \tag{20}$$

on $S \in \mathbb{R}$. With $\tilde{S} := S - \frac{1}{n-3}$ we obtain $(\tilde{S} - \frac{1}{n-3})^{n-1} - (\tilde{S} + \frac{1}{n-3})^{n-1} + \frac{n-1}{n-3}\left((\tilde{S} - \frac{1}{n-3})^{n-2} + (\tilde{S} + \frac{1}{n-3})^{n-2}\right) = 0$. Obviously, the polynomial in $\tilde{S}$ on the left hand of the latter equation is an even function at $\tilde{S} = 0$. Moreover, the Taylor coefficients of even degree at $\tilde{S} = 0$ are all positive. Hence, we see that (20) has no solutions for even n. This shows that the case $n \geq 6$ even does not provide adjacent roots for $L(t)$ as required. □

Note that for $S < 0$ the quadratic polynomial (17) with $C = S$ is the polynomial $L(t)$ with coefficients (18) and $Q = -S^{-1}$, i.e., the standard solution (17) is contained in the family of solutions described by Lemma 4.2 for odd $n \geq 5$. For even n solution (17) is a singular instance.

The following theorem describes the situation of case (*iib*) of Theorem 3.5 completely. The result follows directly from Lemma 4.1, its remark and Lemma 4.2.

Theorem 4.3 *Let $M^n = S^2 \times M_2^{n_2}$ be a product with metric $g = g_1 + g_2$, where g_2 is Einstein on M_2 or $M_2 = S^1$ with normed scalar curvature $S = S(g_2)$, such that $\sigma = \sigma_1$ is a non-constant almost Einstein structure on M.*

Then either $n \geq 3$ and g_1 is the round metric with Gaussian $-C$ given by $\frac{dt^2}{Ct^2 - C^{-1}} + (Ct^2 - C^{-1})d\varphi^2$ on $(\frac{1}{C}, -\frac{1}{C}) \times S^1$, where $C = S < 0$ for $n_2 > 1$ and $C < 0$ arbitrary for $n_2 = 1$ (i.e. $S = 0$). Or else $n = 2 + n_2 \geq 5$ is odd and the product metric g is given by

$$\left(\frac{dt^2}{C_1 t^{n-1} + St^2 + C_2} \;+\; (C_1 t^{n-1} + St^2 + C_2) \cdot d\varphi^2 \right) \quad + \quad g_2 \tag{21}$$

on $S^2 \setminus \{z_1, z_2\} \cong (-Q, Q) \times S^1$, where $(S^1, d\varphi^2)$ is the unit circle, Q is positive with $Q^{-1} > \frac{n-3}{2} S$ and the coefficients C_1, C_2 are given by (18).

In both cases the almost Einstein structure σ on M with metric g has the hypersurface singularity $\Sigma(\sigma) = \{t = 0\} \neq \emptyset$ with conformal infinity structure $[g|_{\Sigma(\sigma)}]$ given by $S^1 \times M_2$ with metric $C_2 \cdot d\varphi^2 + g_2$ (resp. $-C^{-1} \cdot d\varphi^2 + g_2$), i.e., $\Sigma(\sigma)$ is the product of (M_2, g_2) with a circle of radius $C_2 > 0$ (resp. $-C^{-1} > 0$).

Remark to Theorem 4.3. For $n \geq 5$ odd, the hypersurface singularity of the almost Einstein structure σ is given by $S^1 \times M_2$ with metric $C_2 \cdot d\varphi^2 + g_2$, where $C_2 = Q \cdot \left(\frac{2 - (n-3)QS}{n-1} \right)$. One easily checks that in case $S(g_2) > 0$ any constant C_2 with $0 < C_2 < \frac{1}{(n-1)(n-3)S}$ is realised by two different values for the parameter $Q > 0$. This shows that $S^1 \times M_2$ with metric $C_2 \cdot d\varphi^2 + g_2$ can be realised as the hypersurface singularity of two closed almost Einstein spaces, which are not conformally equivalent. ♢

Recall that so far there is still a gap in the proof of Proposition 3.4 in Section 3 about almost Einstein structures on $M = S^2 \times S^2$. This gap is closed by the following lemma.

Lemma 4.4 *Let $\sigma = \sigma_1 + \sigma_2$ be an almost Einstein structure on $M = S^2 \times S^2$ with product metric $g_1 + g_2$. Then at least one of the two functions σ_1, σ_2 is constant on its corresponding factor S^2.*

Proof. If $\sigma = \sigma_1 + \sigma_2$ is a solution of (8) - (10) on $S^2 \times S^2$ with $\sigma_1, \sigma_2 \not\equiv const.$ on their respective factors, then g_1 and g_2 can both be presented in the form of Lemma 4.1 (b) with certain functions $L_1(t_1), L_2(t_2)$ such that $\sigma_1 = t_1$ and $\sigma_2 = t_2$. Equation (10) is then equivalent to

$$2 \cdot (L_1' - L_2') \;=\; (L_1'' - L_2'') \cdot (t_1 + t_2) \;.$$

This equation is only solved when L_1 and L_2 are cubic polynomials in t_1 and t_2, respectively. However, there exist no cubic polynomials, which admit adjacent roots with absolute gradient 2 (cf. Proof of Lemma 4.2). Hence, such a solution $\sigma = \sigma_1 + \sigma_2$ does not exist on $S^2 \times S^2$. □

Proposition 3.4 and Theorem 4.3 also imply the following result.

Corollary 4.5 *(cf. [26, 8]) Let $(M, g) = (S^2 \times M_2^{n_2}, g_1 + g_2)$ be a closed Riemannian product manifold with a 2-sphere. Then any conformal Einstein scale σ is constant on M, g_1 is a round metric on S^2 and g_2 on M_2 is Einstein with $S(g_2) = S(g_1) > 0$.*

In particular, we observe that in dimension $n = 4$ any closed Riemannian product space admitting an almost Einstein structure σ is either a product $g_1 + g_2$ of Einstein metrics with $S(g_1) = S(g_2)$ or a product of a round n_1-sphere S^{n_1} with an Einstein space (M_2, g_2) such that $n_1(n_1 - 1) scal^{g_2} = -n_2(n_2 - 1) scal^{g_1}$. (Here we regard also a circle S^1 as an Einstein space with $S = scal = 0$). The latter examples are conformally flat.

Note that, if we allow compact spaces with boundary in dimension $n = 4$, then there are more examples with Einstein metrics. In fact, any cubic polynomial $L(t)$ with $L(0) > 0$ of the form (16) admitting a root $\tilde{b} > 0$, which is adjacent to $t = 0$, and $L'(\tilde{b}) = -2$, gives rise to a compact Poincaré-Einstein metric on (the completion of) $[0, \tilde{b}) \times M_2$ with boundary $S^1 \times M_2$. Different such spaces can have the same conformal boundary (cf. e.g. [3]). We will discuss explicit examples of compact Poincaré-Einstein spaces of dimension 4 with boundary in Section 7.

5 On homogeneous and cohomogeneity one spaces

In this section we discuss general aspects of almost Einstein structures σ on homogeneous and cohomogeneity one spaces. The homogeneous case of dimension $n \geq 3$ with $\Sigma(\sigma) \neq \emptyset$ basically reduces to the case of homogeneous Einstein spaces in dimension $n - 1$. The cohomogeneity one case is more interesting from the view point of almost Einstein structures. For instance, a hypersurface singularity $\Sigma(\sigma) \neq \emptyset$ has to be a (union of) *principle orbit* of the group action, i.e., the hypersurface is a closed Riemannian homogeneous manifold. Then the almost Einstein equation (7) reduces to a system of ordinary differential equations in the coordinate, which is transverse to the group action. If there is no *obstruction* on the conformal infinity structure, this ODE system has to have solutions. In fact, in the later Section 6 we will solve this ODE system explicitly on closed 4-manifolds for certain boundary data.

5.1 Homogeneous spaces with almost Einstein structure

We say a Riemannian manifold (M, g) is *G-homogeneous* if G is a closed subgroup (as topological space) of the isometry group $I(M, g)$ of (M, g), which acts transitively on M. Then the isotropy group K (at some point $x_o \in M$) is a compact subgroup of G and we may write $M = G/K$.

Analogously, in conformal geometry we say a space M with conformal structure $c = [g]$ is G-homogeneous if G is a closed subgroup of the conformal transformation group of (M, c), which acts transitively on M. It is a basic observation that, if G is a compact group, then there exists a G-invariant metric $\tilde{g} \in c$ in the conformal class, i.e., $(M, \tilde{g})$ is a Riemannian G-homogeneous space. (This G-invariant metric $\tilde{g} \in c$ can be defined via the *Haar integral* and is unique up to constant dilation.) On the other hand, if G does not leave any metric $g \in c$ invariant, then G is called *essential*. By the classical results about essential group actions, we know that this only happens if (M, g) is either the *Möbius sphere* (of dimension $n > 1$) or (M, g) is conformally equivalent to the Euclidean space $\mathbb{R}^n$. In both cases the conformal transformation group acts transitively and essentially. Nevertheless, for both spaces the standard metrics (i.e. Euclidean and round metrics) are homogeneous with respect to the action of some subgroup of the conformal transformation group. For these reasons, the notion of G-homogeneity in Riemannian and conformal geometry can be seen as equivalent (up to the phenomenon of essential group actions on S^n and $\mathbb{R}^n$). In the following we will usually consider G-homogeneity in the sense of Riemannian geometry.

Let us assume that $M = G/K$ is a closed, connected Riemannian homogeneous space with G-invariant metric g, which admits an almost Einstein structure $\sigma : M \to \mathbb{R}$. We denote by $L_q : M \to M$, $q \in G$, the isometric action of the element $g \in G$. Obviously, for any $q \in G$, the pull back $L_q^*\sigma : M \to \mathbb{R}$, $x \in M \mapsto \sigma(L_{q^{-1}}(x)) \in \mathbb{R}$, is again an almost Einstein structure of (M, g). This behaviour translates via $\mathcal{D}^g$ to the corresponding ∇-parallel standard tractors on M, i.e., the vector space $\mathcal{P}(\mathcal{T})$ of ∇-parallel standard tractors in $\Gamma(\mathcal{T})$ is in a natural way a G-representation space (cf. Section 2).

Lemma 5.1 *Let σ be an almost Einstein structure on a closed manifold M^n, $n \geq 3$, with G-homogeneous Riemannian metric g. Then either g is Einstein or $\dim\mathcal{P}(\mathcal{T}) \geq 2$.*

Proof. Let $I_\sigma = \mathcal{D}^g\sigma$ denote the ∇-parallel standard tractor that belongs to σ on M. The pull back $L_q^* I_\sigma$, $q \in G$, is the ∇-parallel standard tractor on M, which corresponds via $\mathcal{D}^g$ to the almost Einstein structure $L_q^*\sigma$. The connected component G_o of G is compact and acts transitively on M.

Now, if $\mathcal{P}(\mathcal{T})$ is 1-dimensional, then the representation of G_o on $\mathcal{P}(\mathcal{T})$ is trivial, i.e., $L_q^* I_\sigma = I_\sigma$ for all $q \in G_o$. Since G_o acts transitively, this is only possible if σ is constant on M, i.e., g is a G-homogeneous Einstein metric. □

Remark to Lemma 5.1. Note that, if $dim\mathcal{P}(\mathcal{T}) \geq 2$ in Lemma 5.1, then either g is a round metric on the sphere S^n or every almost Einstein structure on (M, g) has a hypersurface singularity. This follows from Theorem 2.3.

Here is the basic example of a Riemannian homogeneous metric on a closed space, which is not Einstein, but admits almost Einstein structures: Let (N^m, h) be a Riemannian homogeneous Einstein space of dimension $m \geq 2$, which is non-flat. Then the product metric $g = \eta d\varphi^2 + h$, $\eta > 0$ on $M = S^1 \times N$ is homogeneous as well, but not Einstein. However, if $scal^h$ is negative, then we have case (iia) of Theorem 3.5 at hand, where two almost Einstein structures with different hypersurface singularities occur. ♢

As we mentioned already in Section 4, any conformal gradient vector induces locally a warped product structure, and hence, a product with a parallel line element in the conformal class. The same result is true when $dim\mathcal{P}(\mathcal{T}) \geq 2$ on M.

Theorem 5.2 *Let $(M^n, [g])$, $n \geq 3$, be a conformally homogeneous, closed manifold admitting an almost Einstein structure σ. Then either there is a homogeneous Einstein metric in $[g]$, or else there exists a dense open submanifold in M, on which the conformal class $[g]$ is locally given by $dt^2 + h$ with h an Einstein metric of negative scalar curvature.*

In particular, for $n = 4$ the manifold $(M^4, [g])$ is either conformally Einstein or conformally flat.

Proof. By Lemma 5.1, we only have to consider the case when $dim\mathcal{P}(\mathcal{T}) \geq 2$ on M, i.e., the *conformal holonomy* (of the tractor connection ∇) is decomposable (with a 2-dimensional factor; cf. [20, 22]). This proves that $[g]$ is locally given by a *special Einstein product* $dt^2 + h$ (in the sense of [14] with a one-dimensional factor). If $(M^n, [g])$ is not the Möbius sphere, we have $\langle I, I \rangle > 0$ for any non-trivial $I \in \mathcal{P}(\mathcal{T})$. This implies that the factor h is Einstein with negative scalar curvature (cf. [20]). The same argument works on the Möbius sphere S^n as well, since there are (more than) two linearly independent $I_1, I_2 \in \mathcal{P}(\mathcal{T})$ with $\langle I_i, I_i \rangle > 0$, $i = 1, 2$. (Alternatively, let V be the conformal vector field, which corresponds to the ∇-parallel bitractor $I_1 \wedge I_2$ of $I_1, I_2 \in \mathcal{P}(\mathcal{T})$ on $(M, [g])$ with $\langle I_i, I_i \rangle > 0$, $i = 1, 2$. Then one can easily check that those metrics $\tilde{g}$ in the conformal class $[g]$, which make the norm of V constant (outside of its zero set), have the local form $dt^2 + h$.)

For $n = 4$, the metric h has to be hyperbolic, i.e., $dt^2 + h$ is conformally flat. Hence $(M^4, [g])$ is globally conformally flat. □

Note that the metric h in Theorem 5.2 is *locally homogeneous.* Basically, Theorem 5.2 reduces the case of closed conformally homogeneous almost Einstein spaces to the study of Riemannian homogeneous Einstein metrics. (In short, if σ has a singularity set, this σ cannot be G-invariant and, hence, a real line must split off.)

5.2 Almost Einstein in cohomogeneity one

We relax now the condition of homogeneity to that of cohomogeneity one. In this case a given metric g on M^n, $n \geq 1$, is invariant under the group action of some closed Lie subgroup G of $I(M, g)$, whose *principle orbits* are submanifolds of M with codimension 1. The quotient $B = M/G$ is a one-dimensional manifold (possibly with boundary points). The principle orbits (which correspond to the interior points of B) are Riemannian G-homogeneous spaces G/K, where K denotes the stabiliser of a point in the according orbit. (The stabiliser K is unique up to conjugation in G.) The orbits over the boundary points of B are called *special.* Any special orbit is G-homogeneous as well and can be written as G/H, where H is some compact subgroup of G, which contains the stabiliser K. In fact, for any special orbit, the quotient H/K is diffeomorphic to some m-sphere S^m with $0 \leq m < n$. This is a sufficient and necessary condition for the smoothness of the manifold M.

Now, let (M^n, g), $n \geq 3$, be a Riemannian manifold with a group action of cohomogeneity one by $G \subset I(M, g)$, and let $\mathcal{P}(\mathcal{T})$ be the space of ∇-parallel tractors on $(M, [g])$. We say an almost Einstein structure σ is *G-invariant* if the corresponding standard tractor I_σ is stable under the induced action of G on $\mathcal{P}(\mathcal{T})$. Obviously, an almost Einstein structure σ is G-invariant if and only if σ is a constant function on the G-orbits. And, if σ is an almost Einstein structure with $\Sigma(\sigma) \neq \emptyset$, then it is a necessary condition for σ to be G-invariant that the singularity set $\Sigma(\sigma)$ is G-invariant. Since we assume here cohomogeneity one, we can follow either $\Sigma(\sigma) \neq \emptyset$ is a (disjoint union of) principle G-orbit and $\sigma^{-2}g$ is asymptotically hyperbolic at this principle orbit, or else $\Sigma(\sigma) \neq \emptyset$ consists of special orbits of dimension 0. In the latter case the principle orbits are $(n-1)$-spheres which collapse at the isolated points of $\Sigma(\sigma)$ in M.

Let us consider the case when $\Sigma(\sigma) \neq \emptyset$ is a principle orbit of the G-action on (M, g). This assumption implies that $\Sigma(\sigma)$ (which we can understand as the conformal infinity of the asymptotically hyperbolic (= AH) metric $h = \sigma^{-2}g$ on $M \smallsetminus \Sigma(\sigma)$) is a G-homogeneous Riemannian space G/K. The following basic extension result for manifolds with boundary is known.

Theorem 5.3 *[5] Let $(\overline{F}^{n+1}, h)$ be a compact manifold with smooth boundary ∂F and AH Einstein metric h on $F = \overline{F} \smallsetminus \partial F$. Let γ be a smooth metric in the conformal infinity structure on ∂F, and let G be a closed connected subgroup of the isometry group $I(\partial F, \gamma)$. Then the action of G on the boundary ∂F extends to an isometric action on (F, h).*

Theorem 5.3 suggests the following *Ansatz* (22) for an explicit construction of almost Einstein spaces (without boundary) in cohomogeneity one: Let $\varepsilon > 0$ be some (small) number and let $\gamma(r)$, $r \in (-\varepsilon, \varepsilon)$, be a smooth family of G-homogeneous metrics on some closed manifold Σ^m of dimension $m \geq 2$. Furthermore, let $M^n := (-\varepsilon, \varepsilon) \times \Sigma$ be a tube neighbourhood of Σ with dimension $n = m + 1$ and let

$$g_\gamma \;:=\; dr^2 + \gamma(r) \tag{22}$$

be a smooth metric on M. Then, for an initial G-homogeneous metric $\gamma_0 = \gamma(0)$ on Σ, we aim to find the G-homogeneous family $\gamma(r)$ such that the metric g_γ admits an almost Einstein structure $\sigma = \sigma(r)$, whose singularity set is the hypersurface $\Sigma = \{r = 0\}$ in M. Our *Ansatz* (22) ensures the existence of a G-action on M as required by Theorem 5.3. In fact, one can even argue that (22) is the most general *Ansatz*, which ensures G-invariance. Hence, since γ_0 is analytic (as homogeneous metric) on the boundary, the existence of such a family $\gamma(r)$ with solution $\sigma(r)$ for small $\varepsilon > 0$ is guaranteed by the results of Fefferman and Graham for the case when m is odd, or in even dimensions m when the *obstruction tensor* of γ_0 on Σ^m vanishes as well (cf. [12]). In particular, without obstruction we can expect that the almost Einstein equation (7) $Hess^g_o\sigma = \frac{-1}{n-2} Ric^g_o \sigma$ reduces to a solvable system of ordinary differential equations for $\gamma(r)$. (Note that the solution σ depends by necessity only on the function r. The same *Ansatz* (22) is suitable to find conformally Einstein metrics if we do not assume that σ is singular at Σ.)

6 Explicit examples in dimension 4 of cohomogeneity one

Recall that Riemannian homogeneous Einstein spaces in dimension 4 are symmetric and in this way classified (cf. [17]), whereas Einstein 4-manifolds of cohomogeneity one seem not to be completely classified (cf. [4]). On the other hand, Theorem 5.2 says that any homogeneous almost Einstein space with scale singularity has to be conformally flat in dimension 4. The case of cohomogeneity one admitting an almost Einstein structure with hypersurface singularity (and nontrivial curvature) is already rather rich. Fortunately, as we demonstrate in this section, a special version of *Ansatz* (22) can be solved even explicitly on closed 4-manifolds of cohomogeneity one.

As we mentioned already above, the hypersurface singularity of an almost Einstein structure on a closed manifold of cohomogeneity one is a Riemannian homogeneous space. Simply connected

3-dimensional Riemannian homogeneous spaces are classified (cf. e.g. [27]). In particular, 3-dimensional Lie groups with left invariant metrics and the Riemannian products $\mathbb{R} \times S^2$ and $\mathbb{R} \times H^2$ (where H^2 denotes the hyperbolic plane) are homogeneous. We aim to examine here *Ansatz* (22) for $n = 4$ when $G = \Sigma$ is a 3-dimensional *unimodular* Lie group with Lie algebra $\mathfrak{g}$ and left invariant metric γ_0. Then the family of left invariant metrics $\gamma(t)$ on G is given by a family of positive definite inner products $B(t)$ on $\mathfrak{g}$, i.e., by a vector-valued function.

6.1 3-dimensional unimodular Lie groups

Recall that a Lie group G is called *unimodular* if its left invariant Haar measure is also right invariant. Only in this case the Lie group G admits a discrete subgroup Γ with compact quotient G/Γ (without boundary) (cf. e.g. [24]). This is a suitable assumption in order to find closed almost Einstein spaces with our *Ansatz* (22).

If a Lie group G is 3-dimensional, then (up to a sign) there exist a unique linear map $S_\times : \mathfrak{g} \to \mathfrak{g}$ such that $[x, y] = S_\times(x \times y)$ for any $x, y \in \mathfrak{g}$, where $\times$ denotes the cross product in $\mathfrak{g} \cong \mathbb{R}^3$ with respect to an inner product B_0 (and some orientation). It is a matter of fact that this map $S_\times$ is self adjoint with respect to B_0 if and only if G is unimodular. Thus, in the unimodular case there exists an orthonormal basis (X_1, X_2, X_3) of $(\mathfrak{g}, B_0)$ and some numbers $q_1, q_2, q_3 \in \mathbb{R}$ such that

$$[X_1, X_2] = q_3 X_3, \qquad [X_2, X_3] = q_1 X_1 \qquad \text{and} \qquad [X_3, X_1] = q_2 X_2 \ . \tag{23}$$

The 3-dimensional unimodular semisimple Lie algebras are $\mathfrak{sl}(2)$ and $\mathfrak{su}(2)$. Then we have the nilpotent Heisenberg algebra $\mathfrak{nil}_3$ and the Abelian algebra $\mathbb{R}^3$. Moreover, there is the Lie algebra $\mathfrak{euc}_2$ of the isometry group of the Euclidean plane, and the Lie algebra $\mathfrak{sol}_3$ of the isometry group of the Minkowski plane, which are both solvable Lie algebras.

We assume in the following that G is a 3-dimensional unimodular Lie group G with Lie algebra $\mathfrak{g}$, inner product B_0 and diagonalising basis (X_1, X_2, X_3) as in (23) to the eigenvalues

$$q_1 \ \in \mathbb{R} \qquad \text{and} \qquad \hat{q} := q_2 = q_3 \ \in \mathbb{R} \ .$$

Then $\gamma_0 := \theta_1^2 + \theta_2^2 + \theta_3^2$ is a left invariant metric on G, where $(\theta_1, \theta_2, \theta_3)$ denotes the dual basis to (X_1, X_2, X_3). Note that the conformal structure of such a left invariant metric on G with $q_1 \neq 0$ is uniquely determined by the quotient $\boldsymbol{q} := q_1/\hat{q}$. We consider the *Ansatz*

$$g \ := \quad L^{-1}\, dt^2 \ + \ L\,\theta_1^2 \ + \ \theta_2^2 \ + \ \theta_3^2 \tag{24}$$

on $M = (-\varepsilon, \varepsilon) \times G$ with positive function $L = L(t)$ on $(-\varepsilon, \varepsilon)$, which is a special case of (22) with $\sqrt{L}\cdot dr = dt$ (cf. Lemma 4.1 (b).)

The Levi-Civita connection $\nabla = \nabla^g$ of g on M is given with respect to the frame $(X_0 := \partial t, X_1, X_2, X_3)$ by

$$\begin{aligned}
\nabla_{X_0} X_0 &= L^{-2} \nabla_{X_1} X_1 = -\tfrac{L'}{2L} X_0, & \nabla_{X_1} X_0 &= \nabla_{X_0} X_1 = \tfrac{L'}{2L} X_1,\\
\nabla_{X_1} X_2 &= -\left(\tfrac{q_1}{2} L - \hat{q}\right) X_3, & \nabla_{X_2} X_1 &= -\tfrac{q_1}{2} L\cdot X_3,\\
\nabla_{X_1} X_3 &= \left(\tfrac{q_1}{2} L - \hat{q}\right) X_2, & \nabla_{X_3} X_1 &= \tfrac{q_1}{2} L\cdot X_2,\\
\nabla_{X_2} X_3 &= \tfrac{q_1}{2} X_1, & \nabla_{X_3} X_2 &= -\tfrac{q_1}{2} X_1 \quad .
\end{aligned}$$

A straightforward calculation shows that the non-vanishing components of the Riemannian curvature tensor with respect to the orthonormal basis $e = (\sqrt{L}\cdot\partial_t, (\sqrt{L})^{-1}\cdot X_1, X_2, X_3)$ are given by

$$\begin{aligned}
&R_{0110} := R(e_0, e_1, e_1, e_0) = -\tfrac{1}{2} L'',\\
&R_{1230} = R_{3120} = -\tfrac{1}{2} R_{2310} = -\tfrac{q_1}{4} L',\\
&R_{1221} = R_{1331} = \tfrac{q_1^2}{4} L, \qquad R_{2332} = -\tfrac{3q_1^2}{4} L + q_1 \hat{q} \ .
\end{aligned}$$

We obtain for the Ricci curvature of g on M:

$$Ric^g = \begin{pmatrix} -\frac{1}{2}L'' & 0 & 0 & 0 \\ 0 & -\frac{1}{2}L'' + \frac{1}{2}q_1^2 L & 0 & 0 \\ 0 & 0 & -\frac{1}{2}q_1^2 L + q_1\hat{q} & 0 \\ 0 & 0 & 0 & -\frac{1}{2}q_1^2 L + q_1\hat{q} \end{pmatrix}.$$

The scalar curvature is $scal^g = -L'' - \frac{1}{2}q_1^2 L + 2q_1\hat{q}$, and the components of the traceless part of the Schouten tensor $\mathsf{P}^g_o = -\frac{1}{2}Ric^g_o$ are

$$\begin{aligned} \mathsf{P}^g_o(e_0,e_0) &= \tfrac{1}{4}\left(\tfrac{1}{2}L'' - \tfrac{q_1^2 L}{4} + q_1\hat{q} \right) \\ \mathsf{P}^g_o(e_1,e_1) &= \tfrac{1}{4}\left(\tfrac{1}{2}L'' - \tfrac{5q_1^2 L}{4} + q_1\hat{q} \right) \\ \mathsf{P}^g_o(e_2,e_2) &= \mathsf{P}^g_o(e_3,e_3) = \tfrac{1}{4}\left(-\tfrac{1}{2}L'' + \tfrac{3q_1^2 L}{4} - q_1\hat{q} \right) . \end{aligned} \tag{25}$$

The Weyl tensor W^g (which is the traceless part of the Riemannian curvature tensor R^g) and the Cotton tensor C^g (which is the antisymmetrisation of $\nabla\mathsf{P}^g$) only have two independent components each:

$$\begin{aligned} W_{0110} &= -\tfrac{1}{3}\left(\tfrac{1}{2}L'' + q_1^2 L - q_1\hat{q} \right) \\ &= W_{2332} = -2W_{0220} = -2W_{0330} = -2W_{1221} = -2W_{1331} \\ W_{0123} &= -\tfrac{q_1}{2}L' = -2W_{0231} = -2W_{0312} , \end{aligned} \tag{26}$$

$$\begin{aligned} C_{110} &= -2C_{220} = -2C_{330} = -2\sqrt{L}\cdot\hat{\mathsf{P}}' \\ C_{132} &= -2C_{321} = -2C_{213} = -q_1\sqrt{L}\cdot(\mathsf{P}_{11} - \hat{\mathsf{P}}) , \end{aligned}$$

where we set $\hat{\mathsf{P}} := \mathsf{P}_{22} = \mathsf{P}_{33}$. Another straightforward computation shows that the Bach tensor B^g (which is given in abstract index notation by $B_{jk} = \nabla^l C_{jkl} - \mathsf{P}^{il}W_{ijkl}$) is diagonal with respect to (e_0, e_1, e_2, e_3). Since B has no trace and $B_{22} = B_{33}$, the Bach tensor only has two independent components as well, which are given by

$$\begin{aligned} B_{00} &= (\sqrt{L})'\cdot C_{101} + (\hat{\mathsf{P}} - \mathsf{P}_{11})W_{0110} \\ B_{11} &= \sqrt{L}\cdot(C'_{101} - \tfrac{3q_1}{2}C_{132}) + (\hat{\mathsf{P}} - \mathsf{P}_{00})W_{0110} . \end{aligned} \tag{27}$$

The gradient of an arbitrary function $\sigma = \sigma(t)$ is $grad^g\sigma = \sigma' L\partial_t$. For the Hessian, we have

$$Hess^g\sigma = (E/L)\cdot dt^2 + F\cdot L\theta_1^2 \qquad \text{with} \quad E = \sigma'' L + \frac{1}{2}\sigma' L', \quad F = \frac{1}{2}\sigma' L' .$$

The Laplacian is $\Delta^g\sigma = E + F = (\sigma' L)'$. The trace-free part of the Hessian is given by

$$Hess^g_o\sigma = \begin{pmatrix} \frac{3E-F}{4} & 0 & 0 & 0 \\ 0 & \frac{3F-E}{4} & 0 & 0 \\ 0 & 0 & -\frac{E+F}{4} & 0 \\ 0 & 0 & 0 & -\frac{E+F}{4} \end{pmatrix}. \tag{28}$$

Proposition 6.1 *Let g be the metric of Ansatz (24) on $M = (-\varepsilon, \varepsilon) \times G$ with $q_1, \hat{q} \in \mathbb{R}$.*

1. *A function* $\sigma(t)$ *is an almost Einstein structure for* g *on* M *if and only if* $\sigma'' = \frac{q_1^2}{4}\sigma$ *and* $L(t) > 0$ *solves the ODE*

$$\sigma' L' \;=\; \sigma \left(\frac{1}{2} L'' \;-\; q_1^2 L \;+\; q_1 \hat{q} \right) \;. \tag{29}$$

2. *For* $q_1 \neq 0$*, the general solution of (29) is given by*

$$\begin{aligned} L(t) \;&=\; \hat{q}/q_1 \;+\; C_1 e^{q_1 t} \;+\; C_2 e^{2q_1 t} \;+\; C_3 e^{-q_1 t} \;+\; C_4 e^{-2q_1 t} \;, \\ &C_1, C_2, C_3, C_4 \in \mathbb{R} \qquad \textit{with} \quad C_1C_3 - 4C_2C_4 = 0 \;, \end{aligned} \tag{30}$$

with corresponding almost Einstein structure

$$\sigma(t) \;=\; \begin{cases} A\cdot \left(2C_4 e^{-q_1 t/2} + C_3 e^{q_1 t/2}\right) & \textit{for } C_3 \textit{ or } C_4 \neq 0 \\ \tilde{A}\cdot \left(C_1 e^{-q_1 t/2} + 2C_2 e^{q_1 t/2}\right) & \textit{for } C_1 \textit{ or } C_2 \neq 0 \;, \end{cases}$$

$A, \tilde{A} \in \mathbb{R}$*. The scalar curvature of the Einstein metric* $h = \sigma^{-2} g$ *is given by*

$$scal^h \;=\; \begin{cases} -6A^2 q_1 \left(q_1 (C_3)^3 \;-\; 4\hat{q} C_3 C_4 \;+\; 4 q_1 C_1 (C_4)^2 \right) \\ -6A^2 q_1 \left(q_1 (C_1)^3 \;-\; 4\hat{q} C_1 C_2 \;+\; 4 q_1 C_3 (C_2)^2 \right) \;. \end{cases}$$

3. *For* $q_1 = 0$*, the general solution of (29) is the polynomial*

$$L(t) \;=\; \int (C_1 s + C_3)^2 \, ds \;=\; \frac{1}{3C_1} \left(C_1 s + C_3\right)^3 \;+\; C_4 \tag{31}$$

with almost Einstein structure $\sigma(t) = A\cdot(C_1 t + C_3)$ *and* $C_1, C_3, C_4,\ A \in \mathbb{R}$*. The scalar curvature* $scal^h$ *of the Einstein metric* $h = \sigma^{-2} g$ *is given by*

$$scal^h \;=\; 4A^2 C_1 \left((C_3)^3 \;-\; 3C_1 C_4 \right) \;.$$

4. *The metric* g *with (30) is half conformally flat if and only if either* $C_1 = C_2 = 0$ *or* $C_3 = C_4 = 0$*. The metric* g *given by (31) is conformally flat if and only if* $C_1 = 0$*.*

5. *The metric* g *is Bach-flat if and only if* g *admits an almost Einstein structure* σ*.*

Proof. The ordinary differential equation (29) for the existence of almost Einstein structures follows directly from (25) and (28). A simple computational exercise proves that the general solution is given by (30) and (31), respectively.

The expressions for the Weyl tensor of g in (26) show that half conformal flatness is equivalent to either

$$\begin{aligned} 2q_1^2 L \;+\; 3q_1 L' \;+\; L'' \;&=\; 2q_1\hat{q} \qquad \text{or} \\ 2q_1^2 L \;-\; 3q_1 L' \;+\; L'' \;&=\; 2q_1\hat{q} \;. \end{aligned}$$

The general solutions of these ODE's are

$$L_{\pm}(t) \;=\; \hat{q}/q_1 \;+\; C e^{\pm q_1 t} \;+\; D e^{\pm 2q_1 t} \qquad \text{resp.} \qquad L(t) \;=\; Ct + D \;.$$

The equations for Bach-flatness of g, which are derived from (27), are

$$\begin{aligned} &\left(5q_1^2 L' \;-\; 2L''' \right) L' \;+\; \left((L'')^2 \;-\; 4(q_1^2 L - q_1\hat{q})^2 \right) \;=\; 0 \qquad \text{and} \\ &2\left(5q_1^2 L'' \;-\; 2L'''' \right) L \;+\; \left(5q_1^2 L' \;-\; 2L''' \right) L' \\ &+\; \left((L'')^2 \;+\; 10q_1^2 L'' L \;-\; 4q_1^2 (\hat{q}^2 \;-\; 6q_1\hat{q} L \;+\; 5q_1^2 L^2) \right) \;=\; 0 \;. \end{aligned}$$

The first of these ODE's is a third order equation, which is solved by the 3-parameter families (30) and (31), respectively. We conclude that (30) and (31) are the most general solutions for Bach-flatness.

The formulae for the scalar curvature $scal^h$ of the resulting Einstein metrics $h = \sigma^{-2}g$ are derived by calculating the tractor metric of the ∇-parallel standard tractor $T := \mathcal{D}^g\sigma$. The relation is $scal^h = -12\langle T, T\rangle_{\mathcal{T}}$ (cf. (5)). □

Note that, for $C_1 = C_2 = C_3 = C_4 = 0$ in (30), any non-trivial solution $\sigma(t) = D_1 e^{-q_1 t/2} + D_2 e^{q_1 t/2}$ of $\sigma'' = \frac{q_1^2}{4}\sigma$ is an almost Einstein structure of g. The constant sectional curvature of $h = \sigma^{-2}g$ is then given by $q_1\hat{q}D_1D_2$.

In general, the solutions (30) and (31) have 3-parameters. However, via a parameter translation $t \mapsto \pm t + const.$, these solutions are actually reduced to 2-parameter families (for fixed $q_1, \hat{q}$). For $q_1 \neq 0$, the parameter translation can be arranged such that the almost Einstein structure is given by $\sigma(t) = \cosh(\frac{q_1}{2}t), \sinh(\frac{q_1}{2}t)$ or $\exp(\frac{q_1}{2}t)$ (up to a constant factor).

We discuss now the metrics (24) to the solutions (30) for the three cases of unimodular groups with $q_1 \neq 0$, in detail. In particular, we answer the question when these almost Einstein metrics extend smoothly to a closed 4-manifold.

6.2 Examples with $PSL(2,\mathbb{R})$

The projective linear group $PSL(2,\mathbb{R}) = SL(2,\mathbb{R})/\{\pm 1\}$ is a non-compact, 3-dimensional unimodular Lie group with Lie algebra $\mathfrak{sl}(2)$ spanned by $Z_1 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$, $Z_2 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ and $Z_3 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. The commutators are $[Z_1, Z_2] = -2Z_3$, $[Z_2, Z_3] = 2Z_1$ and $[Z_3, Z_1] = -2Z_2$. For a triple of real numbers $q_1 > 0$ and $q_2, q_3 < 0$, the corresponding left invariant metric on $PSL(2,\mathbb{R})$ is given by

$$\gamma_0 \;=\; \theta_1^2 + \theta_2^2 + \theta_3^2 \;=\; \frac{\omega_1^2}{q_2q_3} - \frac{\omega_2^2}{q_3q_1} - \frac{\omega_3^2}{q_1q_2}\,,$$

where $\omega_1, \omega_2, \omega_3$ denote the left invariant 1-forms on $PSL(2,\mathbb{R})$, which correspond to Z_1, Z_2, Z_3.

Note that $PSL(2,\mathbb{R})$ is the group of orientation-preserving isometries of the hyperbolic (upper half-)plane H^2 with corresponding isotropy group $SO(2) = \{e^{tZ_1} | t \in \mathbb{R}\}/\{\pm 1\}$. In fact, the group $PSL(2,\mathbb{R})$ is naturally identified with the unit tangent bundle $H^2 \times S^1$ of the hyperbolic plane. For the values $q_1 = 2$ and $\hat{q} = -1/2$, the induced $PSL(2,\mathbb{R})$-invariant metric on H^2 is the standard metric with constant sectional curvature $q_1\hat{q} = -1$, and the isotropy group $SO(2)$ has arc length 2π (like the unit tangent circles).

We apply now our *Ansatz* (24) to $G = PSL(2,\mathbb{R})$, i.e., we consider the metric $g = L^{-1}\,dt^2 + L\,\theta_1^2 + \theta_2^2 + \theta_3^2$ on $M = \tilde{I} \times PSL(2,\mathbb{R})$ with $L(t) > 0$ on some interval $\tilde{I} \subset \mathbb{R}$ and real numbers $q_1 > 0$, $\hat{q} := q_2 = q_3 < 0$. In fact, we can fix $q_1 = 2$ without loss of generality in the conformal setting. For this choice, we can assume that the almost Einstein structure is given (after some coordinate translation $t \mapsto \pm t + const.$) by either $\sigma_1(t) = \sinh(t)$, $\sigma_2 = \cosh(t)$ or $\sigma_3 = \exp(t)$. The corresponding solutions $L(t)$ of the ODE (29) on some interval $\tilde{I}$ (containing $0 \in \mathbb{R}$) are then given by the 2-parameter families

$$L_1(t) \;=\; \frac{\hat{q}}{2} + c_1\cdot(2\cosh(2t) - \cosh(4t)) + c_2\cdot(2\sinh(2t) - \sinh(4t)), \tag{32}$$

$$\begin{aligned} L_2(t) &= \tfrac{\hat{q}}{2} + c_1\cdot(2\cosh(2t) + \cosh(4t)) + c_2\cdot(2\sinh(2t) + \sinh(4t)), \\ L_3(t) &= \tfrac{\hat{q}}{2} + c_1\cdot\exp(-2t) + c_2\cdot\exp(4t) \end{aligned} \tag{33}$$

with $c_1, c_2 \in \mathbb{R}$. We concentrate here on the first family of solutions (32), which belongs to $\sigma_1(t) = \sinh(t)$. These solutions give rise to almost Einstein structures with hypersurface singularity at $t = 0$.

So let $L_1(t)$ be a function as in (32) with $\hat{q} < 0$, $c_1, c_2 \in \mathbb{R}$ such that $\tilde{a} < 0$ and $\tilde{b} > 0$ are adjacent zeros and $L_1(0) = \frac{\hat{q}}{2} + c_1 > 0$. We consider the metric g given by (24) with $q_1 = 2$ and

(32) on $(\tilde a,\tilde b)\times PSL(2,\mathbb{R})$. The boundary conformal structure for $\sigma_1=0$ at $t=0$ is given by the left invariant metric $\gamma_0=L_1(0)\theta_1^2+\theta_2^2+\theta_3^2$ on $PSL(2,\mathbb{R})$, which corresponds to the $S_\times$-eigenvalues $2\sqrt{L_1(0)}$ and $\hat q/\sqrt{L_1(0)}$. Their quotient is $\boldsymbol{q_\partial}=2L_1(0)/\hat q=1+2c_1/\hat q$. Recall that the integral curves $\mathcal{C}$ of Z_1 on $PSL(2,\mathbb{R})$ are closed circles. If $L_1(t)$ goes to zero, these circles collapse to a point. Thus, if we identify the circles $\mathcal{C}$ at the boundary of $[\tilde a,\tilde b]\times PSL(2,\mathbb{R})$ to single points, then we obtain a smooth 4-manifold Q, which is diffeomorphic to the 2-sphere bundle $H^2\times S^2$. The open subset $(\tilde a,\tilde b)\times PSL(2,\mathbb{R})$ in Q is the union of the principle orbits of the $PSL(2,\mathbb{R})$-action, and the two remaining special orbits are both copies of the hyperbolic plane $H^2=PSL(2,\mathbb{R})/SO(2)$.

We aim to extend the metric g on $(\tilde a,\tilde b)\times PSL(2,\mathbb{R})$ smoothly for $L_1=0$ to the special orbits in Q. For this purpose, similar as in the discussion of Section 4, we have to find adjacent zeros $\tilde a<0$ and $\tilde b>0$ for some solution L_1 with appropriate gradients. So we define the local vector fields

$$\begin{aligned} Y_1=\partial_\varphi &:= Z_1\\ Y_2=\partial_{y_2} &:= \sin(2\varphi)Z_2-\cos(2\varphi)Z_3\\ Y_3=\partial_{y_3} &:= \cos(2\varphi)Z_2+\sin(2\varphi)Z_3+2y_2Z_1 \end{aligned}$$

on some neighbourhood U (which contains the unit tangent circles $\mathcal{C}$) of an (arbitrary) point $p\in Q$ with $L_1=0$. By definition, we have $[Y_i,Y_j]=0$ for all $i,j=1,2,3$. Then, with respect to local coordinates (φ,y_2,y_3), the metric g is given around p by

$$\begin{aligned} g \;=\;& \hat q^{-1}\cdot\big(\,(L_1/\hat q)^{-1}dt^2\;+\;(L_1/\hat q)d\varphi^2\,\big)\\ &-(2\hat q)^{-1}\big(\,dy_2^2\;+\;(1+(2y_2/\hat q)^2L_1)dy_3^2\,\big)\;+\;(4y_2L_1/\hat q^2)d\varphi\circ dy_3\,. \end{aligned}\tag{34}$$

Note that $\varphi\in[0,2\pi]$ is a local coordinate, which parametrises $\mathcal{C}$.

Theorem 6.2 *Let $L_1(t)$ be a function as in (32) with suitable $c_1,c_2\in\mathbb{R}$ such that the gradient at two adjacent roots $\tilde a<0$ and $\tilde b>0$ satisfies $L_1'(\tilde a)=-L_1'(\tilde b)=-2\hat q$. Then*

$$g\;=\;L_1^{-1}\,dt^2\;+\;L_1\,\theta_1^2\;+\;\theta_2^2\;+\;\theta_3^2\qquad \textit{on}\;\;(\tilde a,\tilde b)\times PSL(2,\mathbb{R})$$

extends to a smooth metric on $Q\cong H^2\times S^2$, which admits an almost Einstein structure σ with hypersurface singularity. The conformal structure of the left invariant metric on $\Sigma(\sigma)\cong PSL(2,\mathbb{R})$ is determined by the quotient $\boldsymbol{q_\partial}=1+2c_1/\hat q$.

Proof. It is clear from (14) of Lemma 4.1 (and the subsequent remark) that $L_1'(\tilde a)=-L_1'(\tilde b)=-2\hat q$ is a necessary and sufficient condition for the smoothness of $-(L_1/\hat q)^{-1}dt^2-(L_1/\hat q)d\varphi^2$ on S^2 at the singular points with $L_1=0$. Also the function $L_1(t)$ smoothly extends to S^2. Hence, we can conclude that (34) extends to a smooth metric on (the special orbits of) Q. Note that the almost Einstein structure $\sigma=\sigma_1$ with singularity at $t=0$ smoothly extends to the special orbits of Q as well. □

Examples to Theorem 6.2. (1) Let $c_2=0$. Then $L_1(t)$ is an even function in $t=0$. Hence, in order to be able to smoothly extend g to Q, it is sufficient to solve

$$\begin{aligned} -\hat q/2 &= c_1(2\cosh(2\tilde a)-\cosh(4\tilde a)) \qquad\text{and}\\ -2\hat q &= 4c_1(\sinh(2\tilde a)-\sinh(4\tilde a)) \end{aligned}$$

for some $\tilde a<0$ and $c_1>-\hat q/2$. In fact, there exists exactly one solution $\tilde a=\log\sqrt{w_o}\approx-0.26$ to these equations, where w_o denotes the unique positive root of the polynomial w^3+3w-2. Then the parameter c_1 is uniquely determined by $-\hat q(4\cosh 2\tilde a-2\cosh 4\tilde a)^{-1}\approx-0.73\hat q>-\hat q/2$ (cf. Figure 2). Thus, for $q_1=2$ and $\hat q<0$, we obtain with this parameter c_1 a metric g on $(\tilde a,-\tilde a)\times PSL(2,\mathbb{R})$, which extends smoothly to $Q\cong H^2\times S^2$. Note that this metric g is not half conformally flat.

By construction, the function $\sigma_1(t) = \sinh(t)$ is an almost Einstein structure on (Q, g) with hypersurface singularity $PSL(2, \mathbb{R})$ at $t = 0$. The conformal class of the left invariant metric on that scale singularity set is determined by the quotient $\boldsymbol{q_\partial} = 1 - (2\cosh(2\tilde{a}) - \cosh(4\tilde{a}))^{-1} \approx -0.46$. This quotient does not depend on $\hat{q}$, which shows that up to a constant conformal factor we have constructed here exactly one almost Einstein space (Q, g).

(2) It is also possible to produce closed 4-manifolds admitting an almost Einstein structure with hypersurface singularity via the presented approach with $PSL(2, \mathbb{R})$. For this purpose, let $\Gamma \subset PSL(2, \mathbb{R})$ be some *cocompact Fuchsian group*, which acts without fixed points on H^2 (cf. e.g. [18]). Then the quotient H^2/Γ is a closed Riemannian surface. In addition, if Γ contains no *elliptic elements*, the quotient $G := PSL(2, \mathbb{R})/\Gamma$ is a closed homogeneous 3-space, which is naturally identified with the unit tangent bundle of the Riemannian surface H^2/Γ. For any $q_1 > 0$ and $q_2, q_3 < 0$, the quotient map $\pi : PSL(2, \mathbb{R}) \to G$ induces a $PSL(2, \mathbb{R})$-invariant Riemannian metric on G.

Now let $q_1 = 2$, $\hat{q} := q_2 = q_3 < 0$, and let $L_1(t)$ be some function on $(-\tilde{a}, \tilde{a})$ as constructed above in Example (1) (cf. Figure 2). Obviously, for the same reason as in Theorem 6.2, the corresponding $PSL(2, \mathbb{R})$-invariant metric g on $(\tilde{a}, -\tilde{a}) \times (PSL(2, \mathbb{R})/\Gamma)$ extends smoothly to the closed 4-manifold $\tilde{Q} := Q/\Gamma$, which is diffeomorphic to a 2-sphere bundle over the Riemannian surface H^2/Γ. As before, $\sigma_1(t) = \sinh(t)$ is an almost Einstein structure on $\tilde{Q}$. The corresponding hypersurface singularity is $G := PSL(2, \mathbb{R})/\Gamma$ with $PSL(2, \mathbb{R})$-invariant metric, whose conformal structure is determined by the quotient $\boldsymbol{q_\partial} \approx -0.46$ (for every $\hat{q} < 0$). Finally, note that $(\tilde{Q}, g)$ is a Bach-flat, closed Riemannian 4-manifold, which is neither half conformally flat nor conformally Einstein. ◊

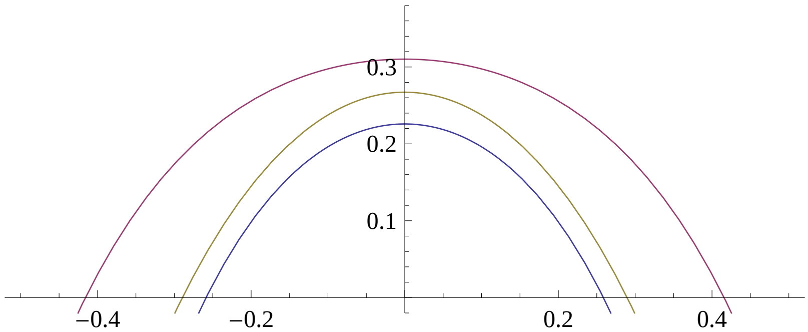

Figure 2: This graphic shows the unique solutions $L_{1,2}(t)$ with absolute gradient 2 at the zeros $\pm\tilde{a}$. The *smallest* graph corresponds to the almost Einstein structure, whose scale singularity is $PSL(2, \mathbb{R})/\Gamma$ with $\boldsymbol{q_\partial} \approx -0.46$. The *middle* graph describes the Einstein metric on $\mathbb{C}P^2 \sharp \overline{\mathbb{C}}P^2$. The *biggest* graph corresponds to the case of the Heisenberg group $Nil_3(\mathbb{R})$.

6.3 Examples with the Heisenberg group

Let $\mathbb{R}^3$ be given the metric

$$\gamma_0 = \theta_1^2 + \theta_2^2 + \theta_3^2 = q_1^2(dx - ydz)^2 + dy^2 + dz^2$$

with parameter $q_1 > 0$. This represents a family of left invariant metrics on the Heisenberg group $Nil_3(\mathbb{R}) \cong \mathbb{R}^3$. In fact, for the vector fields $X_1 := (1/q_1)\cdot\partial_x$, $X_2 := \partial_y$ and $X_3 := \partial_z + y\partial_x$ (which are dual to $\theta_1, \theta_2, \theta_3$), we have $[X_2, X_3] = q_1 X_1$ and $[X_1, X_j] = 0$ for $j = 2, 3$, i.e., $\hat{q} = q_2 = q_3 = 0$.

Now, let $\Gamma := Nil_3(\pi\mathbb{Z})$ denote the discrete subgroup of integral elements in $Nil_3(\mathbb{R})$ (up to multiplication with π). The quotient $G = Nil_3(\mathbb{R})/\Gamma$ is a closed homogeneous 3-space, which is diffeomorphic to some circle bundle over a 2-dimensional torus, where the S^1-fibres are the integral curves of X_1 (cf. e.g. [29]). And, the left invariant metric γ_0 on $Nil_3(\mathbb{R})$ passes down to a $Nil_3(\mathbb{R})$-invariant metric on this quotient space G, which we again denote by $\gamma_0 = \theta_1^2 + \theta_2^2 + \theta_3^2$. Then the circle fibres have length $q_1\pi$ with respect to γ_0 at every point of the base torus.

Our *Ansatz* (24) is given on $\tilde{I} \times G$ with $G = Nil_3(\mathbb{R})/\Gamma$. Again, we set $q_1 := 2$. This time, the solutions $L_1(t)$ corresponding to the almost Einstein structure $\sigma_1(t) = \sinh(t)$ are given by

$$L_1(t) \;=\; c_1{\cdot}(2\cosh(2t) - \cosh(4t)) + c_2{\cdot}(2\sinh(2t) - \sinh(4t)), \quad c_1, c_2 \in \mathbb{R}$$

(cf. (32)). The condition that $L_1(t)$ gives rise to a smooth metric on a closed 4-manifold Q of cohomogeneity one with respect to the $Nil_3(\mathbb{R})$-action is that $L_1(t)$ has two adjacent zeros with absolute gradient 2. Then the closed 4-manifold Q is by construction a 2-sphere bundle over some 2-torus.

Example. Let $c_2 = 0$. Then $L_1(t)$ is an even function at $t = 0$. And, if $c_1 > 0$, the solution $L_1(t)$ has exactly two zeros at $\tilde{a} = \log\sqrt{w_o} \approx -0.42$ and $\tilde{b} = -\tilde{a}$, where w_o is the smallest positive root of $w^4 - 2w^3 - 2w + 1$. Obviously, there exists a unique positive constant $c_1 \approx 0.31$ such that $L'(\tilde{a}) = 2$ at the negative zero $\tilde{a}$ (cf. Figure 2). For this choice of c_1, we obtain a smooth closed 4-manifold (Q, g), which is almost Einstein with hypersurface singularity $G = Nil_3(\mathbb{R})/\Gamma$ at $t = 0$. The space (Q, g) is neither conformally Einstein nor half conformally flat.

Note that, besides $\Gamma = Nil_3(\pi\mathbb{Z})$, other choices of cocompact subgroups Γ of $Nil_3(\mathbb{R})$ can be used for the construction. ◊

6.4 Examples with the Berger sphere

The special unitary group $SU(2)$ is a simply connected, compact and unimodular Lie group of dimension 3. As the group of units in $\mathbb{H}$, it is diffeomorphic to the 3-sphere S^3. For an arbitrary triple of positive real numbers $q_1, q_2, q_3 \in \mathbb{R}$, the corresponding left invariant metric on S^3 is given by

$$\gamma_0 \;=\; \theta_1^2 + \theta_2^2 + \theta_3^2 \;=\; 4\left(\frac{\omega_1^2}{q_2 q_3} + \frac{\omega_2^2}{q_3 q_1} + \frac{\omega_3^2}{q_1 q_2}\right)$$

with certain left invariant 1-forms $\omega_1, \omega_2, \omega_3$. The standard round metric g^o_{rd} on $S^3 = SU(2)$ with constant sectional curvature 1 corresponds to the values $q_1 = q_2 = q_3 = 2$.

The *Ansatz* (24) is given by $g = L^{-1}\,dt^2 + L\,\theta_1^2 + \theta_2^2 + \theta_3^2$ on $M = (-\varepsilon, \varepsilon) \times S^3$ with $L(t) > 0$. As before, we set $q_1 = 2$ without loss of generality in the conformal setting. The parameter $\hat{q} := q_2 = q_3$ is positive. For this choice, the ODE (29) is solved by $L_1(t)$, $L_2(t)$ and $L_3(t)$ as in (32) (and there below) with corresponding almost Einstein structures $\sigma_1(t) = \sinh(t)$, $\sigma_2 = \cosh(t)$ and $\sigma_3 = \exp(t)$, respectively.

When a solution $L(t)$ goes to zero for $t \to \tilde{a}$, the Hopf fibre along the vector field ∂_{ω_1}, which is dual to ω_1, collapses to a single point. In fact, S^3 times an interval $I = (\tilde{a}, \tilde{b})$ smoothly compactifies (with two special orbits) to the connected sum $Q = \mathbb{C}P^2 \sharp \overline{\mathbb{C}}P^2$. Around the special $SU(2)$-orbits in Q, the metric g of *Ansatz* (24) looks like

$$\begin{aligned} g \;=\; & (2/\hat{q}){\cdot}\big(\,(2L/\hat{q})^{-1}dt^2 \;+\; (2L/\hat{q})d\varphi^2\,\big) \\ & + (2/\hat{q})dy_2^2 + (1 + (4y_2/\hat{q})^2 L)dy_3^2 - (16 y_2 L/\hat{q}^2)d\varphi \circ dy_3 \end{aligned}$$

with respect to certain local coordinates φ, y_1, y_2, where the coordinate $\varphi \in [0, 2\pi)$ parametrises the collapsing Hopf fibre. This local expression shows that g extends smoothly to the special orbits (which are two copies of S^2) if and only if $L'(\tilde{a}) = -L'(\tilde{b}) = \hat{q}$. Thus, in order to construct closed almost Einstein 4-spaces via *Ansatz* (24), we aim to find solutions L_1, L_2 or L_3, which have two adjacent zeros with absolute gradient $\hat{q}$.

Another possibility for the construction of smooth closed almost Einstein 4-manifolds is that the principal orbits, which are diffeomorphic to S^3, collapse to a single point when t goes to infinity. In fact, $(-\infty, \infty) \times S^3$ smoothly compactifies to the 4-sphere S^4, whereas $(\tilde{a}, \infty) \times S^3$ and $(-\infty, \tilde{b}) \times S^3$ smoothly compactify to $\mathbb{C}P^2$ with a single additional point at infinity. A necessary condition for the conformal class of a metric of *Ansatz* (24) to smoothly extend to the point(s) at infinity is that $L(t)$ converges to $\hat{q}/q_1 = \hat{q}/2$, i.e., the left invariant metric on S^3 becomes a round metric when t goes to infinity. (Note that the round metrics are the only conformally flat ones among all left invariant metrics on S^3.)

We describe constructions for smooth closed almost Einstein 4-manifolds with $SU(2)$-action of cohomogeneity one via *Ansatz* (24) in the following examples. It turns out that all these examples are conformally Einstein with positive scalar curvature.

EXAMPLES. (1) Let $q_1 = 2$, $\hat{q} > 0$, $\tilde{a} < 0$ and $\sigma_2(t) = \cosh(t)$. We consider the metric g of *Ansatz* (24) on $(\tilde{a}, -\tilde{a}) \times S^3$ corresponding to a solution $L_2(t)$ with $c_2 = 0$. The condition that g smoothly extends to $Q = \mathbb{C}P^2 \sharp \overline{\mathbb{C}}P^2$ is

$$\begin{aligned} -\hat{q}/2 &= c_1(2\cosh(2\tilde{a}) + \cosh(4\tilde{a})) \qquad \text{and} \\ \hat{q} &= 4c_1(\sinh(2\tilde{a}) + \sinh(4\tilde{a})) \ . \end{aligned}$$

This is solved when $\tilde{a} = \log\sqrt{w_o} \approx -0.29$, where w_o is the unique positive root of $3w^4 + 4w^3 - 1$. The parameter c_1 is then uniquely determined by $\approx -\hat{q}/8.2$ (which lies between $-\hat{q}/2$ and 0; cf. Figure 2). By rescaling with $\sigma_2(t) = \cosh(t)$, we obtain a smooth Einstein metric on $\mathbb{C}P^2 \sharp \overline{\mathbb{C}}P^2$ of positive scalar curvature $scal^g = 6(\hat{q} - 2c_1)$, which is not half conformally flat. This construction works for any $\hat{q} > 0$. However, the resulting Einstein metrics on $\mathbb{C}P^2 \sharp \overline{\mathbb{C}}P^2$ are all isometric up to a constant conformal scaling factor. Thus we obtain a unique cohomogeneity one Einstein metric h on $\mathbb{C}P^2 \sharp \overline{\mathbb{C}}P^2$ with $scal^h = 12$. This metric was discovered in [25]. The isometry group of h is $U(2)$. (There is an additional S^1-action, since $q_2 = q_3$.) Up to finite covers, it is the only known cohomogeneity one Einstein metric on a closed 4-manifold (cf. [4]).

(2) For $q_1 = \hat{q} = 2$ and $c_1 = c_2 = 0$, we have the solution $L(t) \equiv 1$. Then the conformal class of the metric $dt^2 + \omega_1^2 + \omega_2^2 + \omega_3^2$ on $(-\infty, \infty) \times S^3$ smoothly extends to a conformally flat structure on the 4-sphere S^4 (by adding two single points at infinity). In fact, $(\cosh(t))^{-2}(dt^2 + \omega_1^2 + \omega_2^2 + \omega_3^2)$ is the standard round metric on S^4 with sectional curvature 1. The metric $(\sinh(t))^{-2}(dt^2 + \omega_1^2 + \omega_2^2 + \omega_3^2)$ on S^4 minus the equator at $t = 0$ is the hyperbolic metric on each of the two caps. And $\sigma_3 = \exp(t)$ is a Ricci-flat almost Einstein structure, which has an isolated zero at $t = -\infty$. The metric $e^{-2t} \cdot (dt^2 + \omega_1^2 + \omega_2^2 + \omega_3^2)$ extends smoothly to $t = \infty$, and thus we obtain the flat metric on $\mathbb{R}^4$. The derived metrics in this example are homogeneous.

(3) Let $q_1 = 2$, $\hat{q} > 0$, $\tilde{a} \in \mathbb{R}$ and $\sigma_3(t) = \exp(t)$. We consider the metric g of *Ansatz* (24) on $(\tilde{a}, \infty) \times S^3$ corresponding to a solution $L_3(t) = \hat{q}/2 + c_1 e^{-2t}$ with $c_2 = 0$. The only condition that the conformal class of g smoothly extends to $Q = \mathbb{C}P^2$ (by adding a 2-sphere at $t = \tilde{a}$ and a single point at $t = \infty$) is $-\hat{q}/2 = c_1 e^{-2\tilde{a}}$. Obviously, this has a unique solution $c_1 < 0$ for any $\hat{q} > 0$ and $\tilde{a} \in \mathbb{R}$. The resulting Einstein metric $e^{-2t} g$ on $\mathbb{C}P^2$ is by Proposition 6.1 half conformally flat and has positive scalar curvature $-24c_1$, i.e., this Einstein metric is up to a constant factor the standard metric on $\mathbb{C}P^2$ with constant holomorphic sectional curvature (which is again a homogeneous metric). For example, let $q_1 = \hat{q} = 2$, $\tilde{a} = 0$ and $r = e^t$. Then we have $c_1 = -1$, and $r^{-2} \cdot (dr^2/(r^2 - 1) + (1 - r^{-2})\omega_1^2 + \omega_2^2 + \omega_3^2)$ extends to a smooth Einstein metric on $\mathbb{C}P^2$.

(4) Let $q_1 = 2$, $\hat{q} > 0$ and $\tilde{b} \in \mathbb{R}$. We consider the solutions $L_3(t) = \hat{q}/2 + c_2 e^{4t}$ with $c_1 = 0$, $c_2 \neq 0$ and almost Einstein structure $\sigma_3(t) = \exp(t)$. Here the conformal class of the corresponding metric g on $(-\infty, \tilde{b}) \times S^3$ extends smoothly to a single point at minus infinity. However, there exists no $c_2 \in \mathbb{R}$ such that (the conformal class of) g extends smoothly to a closed 4-manifold. Instead, the almost Einstein structure $\sigma_3(t) = \exp(t)$ is Ricci-flat and admits an isolated zero at minus infinity. If we set $\hat{q} = 2$, $c_2 < 0$ and $r = e^{-t}$, we obtain the family $dr^2/(1 + c_2 r^{-4}) + r^2((1 + c_2 r^{-4})\omega_1^2 + \omega_2^2 + \omega_3^2)$ of the so-called *Eguchi-Hanson metrics*, which are half-conformally flat and, which have irreducible $SU(2)$-holonomy (cf. [11]). ♢

6.5 The remaining cases

So far we have discussed *Ansatz* (24) for the three unimodular Lie groups with $q_1 \neq 0$. In addition, there are the Abelian group $\mathbb{R}^3$ and the two solvable groups Euc_2 and Sol_3, which represent the isometry groups of the Euclidean and Minkowski plane, respectively. These three groups are unimodular with $q_1 = 0$. Their left invariant metrics are all flat. The assumption $q_2 = q_3$ is not suitable for Sol_3. For the other two cases this assumption is suitable and Proposition 6.1 can be applied for constructing almost Einstein structures in cohomogeneity one. However, Proposition 6.1 states that the corresponding solutions for $L(t)$ are cubic polynomials. These polynomials never admit adjacent roots with the same absolute gradient (cf. Proof of Lemma 4.2). Hence, these cases are not suitable for the construction of closed almost Einstein 4-manifolds. Instead, one can construct compact Poincaré-Einstein spaces with boundary (cf. Section 7). In particular, the *AdS* T^2 *black hole metrics* arise in this way.

Also note that $\mathbb{R} \times S^2$ and $\mathbb{R} \times H^2$ with product metric $d\varphi^2 + g_o$ (where g_o denotes the standard metric with constant sectional curvature on S^2 or H^2, respectively) are Riemannian homogeneous 3-spaces as well. These two models are also suitable for solving *Ansatz* (22), explicitly. In fact, if we take a compact quotient of the real line of these models, and consider a metric of the form $L^{-1}(t)dt^2 + L(t)d\varphi^2 + g_o$, then we obtain the ODE (15) with $n_2 = 2$. Up to an additional constant, this ODE is the same as (29) with $q_1 = 0$ and $\sigma = t$. We have seen in Section 4 that the general solution for $L(t)$ of (15) with $n_2 = 2$ is a cubic polynomial, and for certain parameters, $L(t)$ gives rise to compact Poincaré-Einstein spaces with boundary, in particular, the *AdS-Schwarzschild solution.*

7 Computations of the renormalised volume

In this final section we discuss the *renormalised volume* of compact Poincaré-Einstein spaces with boundary that arise via *Ansatz* (24).

Recall that a conformally compact Poincaré-Einstein metric h on a manifold X^{n+1} of dimension $n+1$ induces a conformal structure on its boundary ∂X. For a given representative g_0 on ∂X of that conformal structure, there exists a unique special defining function s such that the Poincaré-Einstein metric h is given in the form $s^{-2}(ds^2 + g_s)$ on a collar $[0, \delta) \times \partial X$ of ∂X in X. The smooth family g_s of metrics on ∂X that occurs in this description can be expanded into a Taylor series for $s = 0$. For $n = 3$ that expansion looks like

$$g_s \; = \; g_0 \; + \; g^{(2)} s^2 \; + \; g^{(3)} s^3 \; + \; O(s^4),$$

where, in fact, $g^{(2)}$ is the Schouten tensor P^{g_0} of the representative g_0 on ∂X. The tensor $g^{(3)}$ has no trace and is not an intrinsic quantity of the boundary. It determines the higher order terms of the expansion of h on the collar completely (cf. e.g. [15, 3]).

With respect to a special defining function s the Riemannian manifold (X, h) can be cut off at some small value $s = \epsilon$. Then the volume $Vol(\epsilon)$ of $X \setminus \{s < \epsilon\}$ with respect to h is finite. For $n = 3$, the asymptotic expansion of $Vol(\epsilon)$ for $\epsilon \to 0$ has the form

$$Vol(\epsilon) \; = \; c_o \epsilon^{-3} \; + \; c_2 \epsilon^{-1} \; + \; V_h \; + \; O(\epsilon) \; .$$

The constant V_h is the *renormalised volume* of the conformally compact Poincaré-Einstein 4-space (X, h). Via the generalised *Chern-Gauss-Bonnet* formula and with respect to an arbitrary smooth metric g in the conformal class of h on the 4-manifold X^4, the renormalised volume is given by

$$V_h \; = \; \frac{4}{3}\pi^2 \chi(X^4) \; - \; \frac{1}{24} \int_{X^4} \|W^g\|^2 dv_g \; , \tag{35}$$

where $\chi(X^4)$ denotes the *Euler characteristic* of X^4 (cf. [2]).

We demonstrate a computation of the renormalised volume for conformally compact Poincaré-Einstein metrics that arise from *Ansatz* (24). First, let us assume that G/Γ is a compact quotient of an unimodular Lie group G with $q_1 = 2$. A Poincaré-Einstein metric is given on $[0, \tilde{b}) \times G/\Gamma$ by the (induced) G-invariant metric

$$h \; = \; \sigma^{-2} g \; = \; \frac{dt^2}{\sigma^2 L_1} \; + \; \frac{L_1}{\sigma^2}\theta_1^2 \; + \; \frac{1}{\sigma^2}\theta_2^2 \; + \; \frac{1}{\sigma^2}\theta_3^2$$

subject to the parameters c_1, c_2 as in (32) and $\sigma(t) = \alpha^{-1}\cdot \sinh(t)$, $\alpha := \sqrt{L_1(0)}$. The conformal infinity structure on G/Γ is determined by the representative $g_0 := \alpha^2\theta_1^2 + \theta_2^2 + \theta_3^2$ corresponding to the values $q_1 = 2$ and $\hat{q} \in \mathbb{R}$. Now, there exists a unique special defining function s to that representative g_0, which only depends on the coordinate t. A straightforward calculation shows that the Taylor expansion of t with respect to s is

$$t(s) \;=\; \alpha\cdot s \;+\; \left(\frac{\alpha^3}{12} - \alpha c_1 \right) s^3 \;-\; \left(\frac{4}{3}\alpha^2 c_2 \right) s^4 \;+\; O(s^5).$$

The expansion of the family g_s is

$$\begin{aligned} g_s \;&=\; g_0 \;+\; \big(-(2c_1 + \alpha^2/2)\alpha^2\theta_1^2 \;+\; (2c_1 - \alpha^2/2)(\theta_2^2 + \theta_3^2) \big)\, s^2 \\ &\qquad +\; \frac{8c_2\alpha}{3} \big(-2\alpha^2\theta_1^2 \;+\; \theta_2^2 \;+\; \theta_3^2 \big)\, s^3 \quad +\; O(s^4)\ . \end{aligned}$$

If V_o denotes the volume of the compact quotient G/Γ with respect to the metric $\theta_1^2 + \theta_2^2 + \theta_3^2$ and the parameters $q_1 = 2$, $\hat{q} \in \mathbb{R}$, then we have for the volume expansion of h at $s = 0$:

$$Vol(\epsilon) \;=\; \frac{\alpha V_o}{3}\epsilon^{-3} \;+\; \left(\alpha c_1 - \frac{3}{4}\alpha^3 \right) V_o \epsilon^{-1} \;+\; V_h \;+\; O(\epsilon)\ .$$

Moreover, we deduce from (26) with $q_1 = 2$ that the norm of the Weyl tensor with respect to g is given by

$$\|W\|_g^2 \;=\; \frac{4}{3}\left(\frac{1}{2}L_1'' \;+\; 4L_1 - 2\hat{q} \right)^2 \;+\; 12(L_1')^2\ . \tag{36}$$

Similarly, for $q_1 = 0$ and $\hat{q} \in \mathbb{R}$ the Poincare-Einstein metrics on $[0, \tilde{b}) \times G/\Gamma$ for *Ansatz* (24) are given by $L_1(t) = -\frac{C_1^2}{3}t^3 + C_4$ with $C_4 > 0$ and $\sigma(t) = A\cdot C_1 t$ with $A = 1/\sqrt{C_1^2 C_4}$ (cf. Proposition 6.1). The Taylor expansion of t with respect to the special defining function s corresponding to the boundary metric $g_0 := C_4\theta_1^2 + \theta_2^2 + \theta_3^2$ is

$$t(s) \;=\; \sqrt{C_4}\cdot s \;+\; \frac{C_1^2 C_4}{18}s^4 \;+\; O(s^5)\ .$$

The expansion of the Poincare-Einstein metrics and their volume are

$$\begin{aligned} g_s \;&=\; g_0 \;-\; \tfrac{C_1^2\sqrt{C_4}}{9} \big(-2C_4\theta_1^2 \;+\; \theta_2^2 \;+\; \theta_3^2 \big)\, s^3 \quad +\; O(s^4)\ , \\ Vol(\epsilon) \;&=\; \tfrac{\sqrt{C_4}V_o}{3}\epsilon^{-3} \;+\; V_h \;+\; O(\epsilon)\ . \end{aligned}$$

Example. Let $G = SU(2)$ with $q_1 = 2$ and $\hat{q} > 0$. There exists no almost Einstein metric (with hypersurface singularity) of the form $L_1(t) = \hat{q}/2 + c_1\cdot x(t) + c_2\cdot y(t)$ with $x(t) := 2\cosh(2t) - \cosh(4t)$, $y(t) := 2\sinh(2t) - \sinh(4t)$ and $\sigma_1 = \sinh(t)$, which smoothly extends to $\mathbb{C}P^2 \sharp \overline{\mathbb{C}}P^2$. (Instead, we have seen in Section 6.4 the construction of a $SU(2)$-invariant conformally Einstein metric on $\mathbb{C}P^2 \sharp \overline{\mathbb{C}}P^2$.)

However, there exists a family of conformally compact Poincaré-Einstein metrics with boundary at $t = 0$ coming from solutions of the form $L_1(t)$. This is the well known family of *AdS-Kerr spaces.* In fact, let us assume that a solution L_1 is a positive function on an interval $[0, \tilde{b})$ with $\tilde{b} > 0$. The condition that $L_1(t)$ gives rise to a smooth Poincaré-Einstein metric on $Q = \mathbb{C}P^2 \setminus B^4$ (minus a ball B^4) with boundary S^3 is

$$-\hat{q}/2 \;=\; x(\tilde{b})c_1 \;+\; y(\tilde{b})c_2 \qquad \text{and} \qquad -\,\hat{q} \;=\; x'(\tilde{b})c_1 \;+\; y'(\tilde{b})c_2\ .$$

One easily checks that the correct parameters c_1, c_2, which satisfy this smoothness condition, are given in dependence on $\tilde{b} > 0$ by

$$\begin{pmatrix} c_1 \\ c_2 \end{pmatrix} \;=\; \frac{\hat{q}}{24(\sinh(\tilde{b}))^2} \begin{pmatrix} y'(\tilde{b}) & -y(\tilde{b}) \\ -x'(\tilde{b}) & x(\tilde{b}) \end{pmatrix} \begin{pmatrix} 1/2 \\ 1 \end{pmatrix}\ .$$

With these values for c_1, c_2, the conformal class of the Berger metric $g_0 = L_1(0)\theta_1^2 + \theta_2^2 + \theta_3^2$ on the boundary, is determined by the quotient $\boldsymbol{q_\partial} = 1 + \frac{y'(\tilde{b})-2y(\tilde{b})}{24(\sinh(\tilde{b}))^2}$, which is only positive for $\tilde{b} \in (0, \frac{\log(3)}{2})$. At the boundaries of this interval the quotient $\boldsymbol{q_\partial}$ goes to zero. The maximum value for $\boldsymbol{q_\partial}$ is $(2-\sqrt{3})/3$, which is attained for $\tilde{b}_{max} := \log(3)/4$. Any value of $\boldsymbol{q_\partial}$ between 0 and $(2-\sqrt{3})/3$ is attained by exactly two $\tilde{b}$, i.e., for all such $\boldsymbol{q_\partial}$ there exist two different Poincaré-Einstein metrics with the same conformal infinity structure in this family. Since $(2-\sqrt{3})/3 \approx 0.089 < 1$, we see that the round metrics cannot represent the conformal class on the boundary for any member in this family (cf. [9]). Also note that c_2 is always positive for $0 < \tilde{b} < \log(3)/2$, i.e., no Poincaré-Einstein metric of this family is even at the boundary.

With the Gauss-Bonnet formula (35) and expression (36) we obtain for the renormalised volume of the Poincaré-Einstein metric $h = (\sinh(t))^{-2}L(0)g$:

$$\begin{aligned} V_h &= \frac{4}{3}\pi^2\chi(\mathbb{C}P^2 \setminus B) - \frac{1}{24}\int_{\mathbb{C}P^2\setminus B} \|W\|^2 dv_g \\ &= \frac{4}{3}\pi^2\chi(\mathbb{C}P^2 \setminus B)\left(1 - \frac{1}{8\hat{q}^2}\int_0^{\tilde{b}} \|W\|^2 dt\right) \\ &= \frac{4}{3}\pi^2\chi(\mathbb{C}P^2 \setminus B)\cdot \hat{V}(\tilde{b}), \end{aligned}$$

where $\chi(\mathbb{C}P^2 \setminus B) = 2$ and

$$\hat{V}(\tilde{b}) = \frac{4\cosh(\tilde{b}) - 3\cosh(3\tilde{b}) - 5\cosh(5\tilde{b}) + 20\sinh(\tilde{b}) + 4\sinh(5\tilde{b})}{72\sinh(\tilde{b})}. \tag{37}$$

The maximum value of the function $\hat{V}(\tilde{b})$ is $(4-\sqrt{3})/18 \approx 0.126$, which is exactly attained for $\tilde{b}_{max}$ (when $\boldsymbol{q_\partial}$ is maximal). We conclude that the maximum value for the renormalised volume V_h is smaller then $\frac{1}{7}\cdot\frac{4\pi^2}{3}\chi(\mathbb{C}P^2 \setminus B)$ (cf. [9] and Figure 3).

8 Summary and conclusions

We discuss in this paper the almost Einstein equation $Hess_o^g\sigma = \mathsf{P}_o^g\cdot\sigma$ for a scaling function σ on a (closed) Riemannian manifold with metric tensor g. This equation can be seen as a conformally invariant replacement for the Einstein equation $Ric^g = \lambda\cdot g$, $\lambda \in \mathbb{R}$. The advantage of solving $Hess_o^g\sigma = \mathsf{P}_o^g\cdot\sigma$ can be seen in the fact that for a concrete *Ansatz*, it might be more convenient to solve for some metric in the conformal class, which is not exactly the Einstein metric. Moreover, the almost Einstein equation has a generalising feature, which relies on the phenomenon that a solution σ might have zeros. These zeros are automatically included in the process of solving the equation. (This is not the case when we try to solve Einstein's equation for an asymptotically hyperbolic metric.)

After two introductory sections, we discuss in the first part of this paper (Sections 3 and 4) the almost Einstein equation on Riemannian product spaces and derive a coarse geometric description for the corresponding factors. In case of a product $S^2 \times M_2$ with a 2-sphere, the almost Einstein equation is equivalent to the system

$$Hess_o^{g_1}\sigma_1 = 0 \qquad \text{and} \qquad \Delta^{g_1}\sigma_1 = \frac{1}{n_2}\left(\frac{2}{n_2}scal^{g_2} - scal^{g_1}\right)\sigma_1 \tag{38}$$

for a function σ_1 on the 2-sphere (S^2, g_1). The first of these equations implies the existence of a conformal gradient, which in turn induces a warped product structure on the factor S^2. With respect to this warped product structure the second equation becomes an ordinary differential equation. We are able to solve this ODE explicitly. This gives rise to a complete and explicit description of almost Einstein structures on products with S^2.

In fact, we find an exotic family of almost Einstein structures with hypersurface singularity on $S^2 \times M_2$ of odd dimension $n = 2+n_2 \geq 5$ (cf. Theorem 4.3). The Riemannian metrics in this family are obstruction flat (in the sense that the Fefferman-Graham obstruction tensor vanishes; cf. [12]) without being conformally Einstein. The corresponding Poincaré-Einstein metrics (off the scale singularity set) admit an even power series expansion with respect to a special defining function of the conformal boundary. For certain different (non-isometric) product metrics on $S^2 \times M_2$ the hypersuface singularities are equivalent as conformal manifolds.

On the other hand, our discussion also shows that there exists no conformally Einstein product metric on any closed product manifold with a 2-dimensional factor (apart from the obvious Einstein products). This reproves a result of Palais, Terng and Derdzinski (cf. [26, 8]). Our main argument for the proof of this result relies on the fact that conformal gradients on closed spaces are essential conformal vector fields, and such vector fields occur on the Möbius sphere only.

In the second part of the paper (Sections 5 to 7) we discuss almost Einstein structures on closed Riemannian 4-manifolds admitting an isometric group action of cohomogeneity one. In this setting the boundary data are given by a closed homogeneous 3-manifold and the almost Einstein equation is equivalently expressed by some ODE in a coordinate of the transverse direction of the principle orbits. Again, depending on the boundary data, we can easily solve this ODE, explicitly.

In fact, we exhibit three families of closed almost Einstein spaces in dimension 4 of cohomogeneity one. These three families correspond to compact quotients of the 3-dimensional unimodular Lie groups $SU(2)$, $PSL(2,\mathbb{R})$ and the Heisenberg group $Nil_3(\mathbb{R})$. None of the three families contains half conformally flat metrics. For the case of $SU(2)$ the boundary space is a Berger sphere, and we obtain an Einstein metric on $\mathbb{C}P^2\sharp\overline{\mathbb{C}P}^2$ of positive scalar curvature. This Einstein metric of cohomogeneity one was first discovered by Page in [25]. For the case of $PSL(2,\mathbb{R})$ and $Nil_3(\mathbb{R})$ the almost Einstein structures *really* admit a hypersurface singularity (which is the given boundary data). Thus, these two families give rise to Bach-flat, closed Riemannian 4-manifolds, which are neither half conformally flat nor conformally Einstein (but almost Einstein).

In the final section we also discuss compact Poincaré-Einstein spaces of dimension 4 with homogeneous boundary. There are 7 different families corresponding to compact quotients of 5 unimodular Lie groups of dimension 3 and the homogeneous models $\mathbb{R} \times S^2$ and $\mathbb{R} \times H^2$ with a real line. We calculate the first terms of these Poincaré-Einstein metrics in the asymptotic expansion with respect to a special defining function of the boundary. For the family with the Berger spheres, depending on the quotient $\boldsymbol{q_\partial}$, the renormalised volume is partly positive, but always less then $\frac{1}{7}\cdot\frac{4\pi^2}{3}\chi(\mathbb{C}P^2 \setminus B)$ (cf. Figure 3). This result confirms computations of [9].

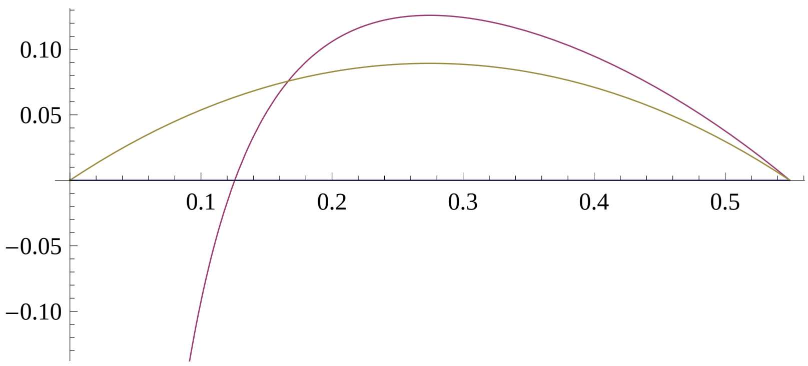

Figure 3: This graphic shows the quotient $\boldsymbol{q_\partial}$ and the corresponding value $\hat{V} < 1/7$ (cf. (37)) for the renormalised volume in dependence of the positive zero $\tilde{b}$ of $L_1(t)$ for the example of Section 7. The maximum values are both attained at $\tilde{b} = \log(3)/4$. For $\tilde{b} \to 0$ the renormalised volume goes to $-\infty$, and for $\tilde{b} \to \log(3)/2$, it goes to 0.

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