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Abstract

The problem of exercising an American option in discrete time in an optimal way is considered. In contrast to existing literature the algorithm proposed in this paper is completely nonparametric in the sense that it does not rely on any specific model for the generation of the asset values. It is shown that the algorithm is universally consistent in the sense that the achieved expected payoff converges to the optimal value whenever the returns of the underlying asset are stationary and ergodic. The algorithm is illustrated by applying it to simulated data.

AMS classification: Primary 91G70; secondary 60G40, 62G08.

Key words and phrases: American options, ergodicity, optimal exercising, optimal stopping, stationarity, universal consistency.

1 Introduction

In this paper the problem of exercising an American option in discrete time (also called Bermudan option) in an optimal way is considered. This problem can be formulated in a mathematical way as following: Let $(X_j)_{j\in\mathbb{Z}}$ be positive random variables defined on the same probability space describing the values of the underlying asset of the option at time points $j \in \mathbb{Z}$. For simplicity we consider only the case that X_j be real-valued, i.e., we consider only options on a single asset. Let $f : \mathbb{R} \to \mathbb{R}_+$ be the payoff function of the option, which we assume to be nonnegative, bounded and measurable, e.g., $f(x) = \max\{K - x, 0\}$ in case of an American put option with strike K. Let r be the riskless interest rate. If we get the payoff at time $t > 0$ we discount it towards zero by the factor $e^{-r\cdot t}$, so for asset value x at time t the discounted payoff of the option is $e^{-r\cdot t} \cdot f(x)$.

Let $L > 0$ be the expiration date of our option. In the sequel we renormalize the payoff function such that we can assume $X_0 = 100$, and we consider an American option on X_j with exercise opportunities restricted to $\{0, 1, \ldots, L\}$ (sometimes also called Bermudan option). Any rule for exercising such an option within $\{0, 1, \ldots, L\}$ can rely only on the values of the asset at times $j \le k$ in case that it decides to exercise the option at time k. Therefore it can be described by a stopping time τ, i.e., by a measurable function of $\ldots, X_{-1}, X_0, \ldots, X_L$ where the event $[\tau = k]$ is contained in the σ-algebra $\mathcal{F}(\ldots, X_{-1}, X_0, \ldots, X_k)$ generated by $\ldots, X_{-1}, X_0, \ldots, X_k$. Let $\mathcal{T}(0, \ldots, L)$ be the set of all such stopping times. Any stopping time τ describing the exercising of an American option yields in the mean the payoff

$$\mathbf{E}\left(e^{-r\cdot\tau} \cdot f(X_\tau)\right),$$

and it is this quantity which we want to maximize, i.e., we want to construct a stopping time $\tau^* \in \mathcal{T}(0, \ldots, L)$ such that

$$V_0 := \sup_{\tau \in \mathcal{T}(0,\ldots,L)} \mathbf{E}\left\{e^{-r\cdot\tau} \cdot f(X_\tau)\right\} = \mathbf{E}\left\{e^{-r\cdot\tau^*} \cdot f(X_{\tau^*})\right\}.$$

It is well-known that in complete and arbitrage-free markets V_0 is exactly the price of the above option (see, e.g., Karatzas and Shreve (1995)). However, in this article we are not so much interested in determing the price of the option in a complete and arbitrage-free market, instead we want to find an optimal strategy for exercising a given American option in discrete time even if the market is not complete or not arbitrage-free. So in this sense we are mainly dealing with the situation of the holder of the option.

The standard approach to exercise an American option in an optimal way is to assume some kind of more or less restrictive mathematical model for the financial market (e.g., a Black-Scholes model) and to determine the optimal exercising strategy in this model (e.g. by solving free-boundary problems, see, e.g., Elliott and Kopp (1999)). In this article we are interested in a completely nonparametric approach to this problem, i.e., we want to avoid any model assumption for the values of the underlying asset. Instead we assume only that the corresponding returns form a stationary and ergodic sequence.

More precisely at time n we assume that $X_{-n}, \ldots, X_0$ are given and we want to construct a stopping time

$$\hat{\tau}_n = \hat{\tau}_n(X_{-n}, \ldots, X_0, \ldots, X_L) \in \mathcal{T}(0, 1, \ldots, L)$$

such that

$$\hat{V}_{0,n} := \mathbf{E}\left\{e^{-r\cdot\hat{\tau}_n} \cdot f(X_{\hat{\tau}_n})\right\}$$

converges to V_0.

In the definition of our estimates we firstly use results from the general theory of optimal stopping showing that an optimal stopping time can be constructed by computing so-called continuation values, which describe the value of the option given an observed value of the underlying stock under the contraint of holding the option rather than exercising it (cf., e.g., Chow, Robbins and Siegmund (1971) or Shiryayev (1978)). Secondly we use that these continuation values can be represented as conditional expectations (cf., e.g., Tsitsiklis and van Roy (1999), Longstaff and Schwarz (2001) or Egloff (2005)), and our algorithm uses techniques from nonparametric regression to estimate these conditional expectations from stationary and ergodic data. This is in general a rather challenging task, where usually extremely complex and data consuming algorithms are necessary (cf., e.g., Morvai, Yakowitz and Györfi (1996)). But in case that it is enough to construct algorithms which converge in the so-called Cesàro sense, a relatively simple and nice algorithm exists (cf., e.g., Section 27.5 in Györfi et al. (2002)), which uses techniques from the theory of prediction of individual sequences (cf., e.g., Cesa-Bianchi and Lugosi (2006)). These techniques have already been used successfully in the contex of portfolio optimization (cf., e.g., Györfi, L., Lugosi, G. and Udina, F. (2006), Györfi, L., Udina, F. and Walk, H. (2008) and the references therein). In this paper we introduce as main trick an averaging of such estimates and show that by using this trick we can derive a consistency result of our estimated stopping rule from Cesàro consistency of the underlying regression estimates. So in the definition of our estimate we thirdly apply estimates defined by use of ideas from the theory of prediction of individual sequences.

In order to simplify our notation we ignore throughout this article measurability problems occuring in connection with suprema of random variables over uncountable sets of stopping times. These problems are well understood and could be treated rigorously by using essential suprema (cf., e.g., Section 1.6 in Chow, Robbins and Siegmund (1971)). Furthermore we do not indicate in our notation that relations in connection with conditional expectations hold only almost sureley or almost everywhere.

The algorithm computing estimates of the optimal stopping time is described in Section 2, the main result is formulated in Section 3, and the algorithm is illustrated by applying it to simulated data in Section 4. Section 5 contains the proof of the main result, proofs of auxiliary results are given in the appendix.

2 Construction of an approximation of the optimal stopping time

Our first idea is to use results from the general theory of optimal stopping in order to determine the optimal stopping time τ^*. Define the so-called continuation value describing the value of the option given $\dots, X_{-1} = x_{-1}, X_0 = x_0, \dots, X_t = x_t$ subject to the constraint of holding it at time $t \in \{0, \dots, L-1\}$ rather than exercising it by

$$\begin{aligned} & q_t(\dots, x_{-1}, x_0, \dots, x_t) \\ & := \sup_{\tau \in \mathcal{T}(t+1,\dots,L)} \mathbf{E}\left(e^{-r\cdot\tau} f(X_\tau) \middle| \dots, X_{-1} = x_{-1}, X_0 = x_0, \dots, X_t = x_t\right), \end{aligned}$$

where $\mathcal{T}(t+1, \dots, L)$ is the set of all stopping times with values in $\{t+1, t+2, \dots, L\}$, and by

$$q_L(\dots, x_0, \dots, x_L) := 0,$$

and define the so-called value function which describes the value we get in the mean if we sell the option in an optimal way after time $t-1$ given $\dots, X_{-1} = x_{-1}, X_0 = x_0, \dots, X_t = x_t$ by

$$\begin{aligned} & V_t(\dots, x_{-1}, x_0, \dots, x_t) \\ & := \sup_{\tau \in \mathcal{T}(t,t+1,\dots,L)} \mathbf{E}\left\{e^{-r\cdot\tau} \cdot f(X_\tau) \middle| \dots, X_{-1} = x_{-1}, X_0 = x_0, \dots, X_t = x_t\right\}. \end{aligned} \tag{1}$$

For $t \in \{-1, 0, \dots, L-1\}$ set

$$\tau_t^* := \inf\{s \geq t+1 \,:\, q_s(\dots, X_{-1}, X_0, \dots, X_s) \leq e^{-r\cdot s} \cdot f(X_s)\}. \tag{2}$$

We can conclude from the general theory of optimal stopping (see, e.g., Chow, Robbins and Siegmund (1971) or Shiryayev (1978)):

Lemma 1 *It holds*

$$V_t(\dots, x_{-1}, x_0, \dots, x_t) = \mathbf{E}\left\{e^{-r\cdot\tau_{t-1}^*} f(X_{\tau_{t-1}^*}) \middle| \dots, X_{-1} = x_{-1}, X_0 = x_0, \dots, X_t = x_t\right\} \tag{3}$$

for $t \in \{0, \dots, L\}$. *Furthermore*

$$V_0 := \sup_{\tau \in \mathcal{T}(0,\dots,L)} \mathbf{E}\left\{e^{-r\cdot\tau} \cdot f(X_\tau)\right\} = \mathbf{E}\left\{e^{-r\cdot\tau^*} \cdot f(X_{\tau^*})\right\} \tag{4}$$

is fulfilled for

$$\tau^* := \tau^*_{-1} = \inf\left\{j \in \{0,1,\dots,L\} \,:\, e^{-r\cdot j}\cdot f(X_j) \geq q_j(\dots,X_{-1},X_0,\dots,X_j)\right\}.$$

For the sake of completeness a proof of Lemma 1 is given in the appendix.

From Lemma 1 we get that it suffices to compute the continuation values $q_0, \dots, q_{L-1}$ in order to construct the optimal stopping rule τ^*. In Tsitsiklis and van Roy (1999), Longstaff and Schwarz (2001) and Egloff (2005) it is shown that in case of Markovian processes the continuation values can be computed recursively by evaluation of conditional expectations. In our next lemma we show that this is also true in the setting considered in this paper.

Lemma 2 *The continuation values satisfy*

$$\begin{aligned} &q_j(\dots,x_{-1},x_0,\dots,x_j)\\ &= \mathbf{E}\Big\{ \max\left\{e^{-r\cdot(j+1)}\cdot f(X_{j+1}), q_{j+1}(\dots,X_{-1},X_0,\dots,X_{j+1})\right\} \\ &\qquad\qquad \Big| \dots, X_{-1}=x_{-1}, X_0=x_0,\dots,X_j=x_j\Big\}, \end{aligned} \tag{5}$$

and

$$q_j(\dots,x_{-1},x_0,\dots,x_j) = \mathbf{E}\left\{e^{-r\cdot\tau^*_j} f(X_{\tau^*_j}) | \dots, X_{-1}=x_{-1}, X_0=x_0,\dots,X_j=x_j\right\} \tag{6}$$

for any $j \in \{0,1,\dots,L-1\}$.

Lemma 2 can be proven as in the case of Markovian processes, for the sake of completeness the proof is given in the appendix.

Usually in applications the (joint) distribution of the asset prices is unknown and therefore it is impossible to use (5) (or (6)) in order to compute the continuation values. In the sequel we will try to estimate them by using (recursively defined) regression estimates in order to approximate the conditional expectations in (5). To do this we use for any $n \in \mathbb{N}$ the asset values until time $-n$ in order to construct an estimate of the optimal stopping rule for selling the American option on the data $X_0, \dots, X_L$.

Next we describe how we construct estimates $\hat{q}_j^{(n)}(X_{-n},\dots,X_j)$ of $q_j(\dots,X_0,\dots,X_j)$. Here the estimates will depend only on the returns of the arguments $X_{-n},\dots,X_j$.

The estimates are defined recursively with respect to $j \in \{0, \ldots, L\}$. For $j = L$ we have $q_L = 0$ and in this case we set

$$\hat{q}_L^{(n)} := 0.$$

Given $\hat{q}_{j+1}^{(n)}$ (defined on $(0,\infty)^{j+n+2}$) for some $j \in \{0, 1, \ldots, L-1\}$ we define $\hat{q}_j^{(n)}$ as follows.

We start with defining $\hat{q}_{j,(k,h)}^{(n)}$. The definition will depend on parameters $k \in \mathbb{N}$ and $h > 0$ and a kernel function $K : \mathbb{R}^{j+k+1} \to \mathbb{R}_+$ which we define by

$$K(v) := H\left(\|v\|_2^{j+k+1} \right),$$

where $\|v\|_2$ denotes the Euclidean norm of v and $H : \mathbb{R}_+ \to \mathbb{R}_+$ is a given nonincreasing and continuous function satisfying

$$H(0) > 0 \quad \text{and} \quad t \cdot H(t) \to 0 \quad (t \to \infty),$$

(e.g., $H(v) = e^{-v^2}$). Using these parameters we use local averaging to define

$$\begin{aligned}
&\hat{q}_{j,(k,h)}^{(n)}(u_{-n}, \ldots, u_0, \ldots, u_j) \\
&:= \sum_{i=-n+k+1}^{-(j+1)} \max\left\{ e^{-r\cdot(j+1)} \cdot f\left(100 \cdot \frac{u_{i+j+1}}{u_i}\right), \hat{q}_{j+1}^{(n+i)}(u_{-n}, \ldots, u_i, \ldots, u_{i+j+1}) \right\} \\
&\qquad \cdot \frac{K\left(\frac{\left(\frac{u_{i-k}}{u_{i-k-1}}, \ldots, \frac{u_{i+j}}{u_{i+j-1}}\right) - \left(\frac{u_{-k}}{u_{-k-1}}, \ldots, \frac{u_j}{u_{j-1}}\right)}{h} \right)}{\sum_{l=-n+k+1}^{-(j+1)} K\left(\frac{\left(\frac{u_{l-k}}{u_{l-k-1}}, \ldots, \frac{u_{l+j}}{u_{l+j-1}}\right) - \left(\frac{u_{-k}}{u_{-k-1}}, \ldots, \frac{u_j}{u_{j-1}}\right)}{h} \right)}
\end{aligned}$$

for $u_{-n} \in (0,\infty)$, $\ldots$, $u_j \in (0,\infty)$. Here we set

$$\hat{q}_{j,(k,h)}^{(n)} := 0$$

for $k \geq n - j - 1$, and $\frac{0}{0} := 0$.

Let $h_r > 0$ be such that $h_r \to 0$ for $r \to \infty$ and set

$$\mathcal{P} := \{(k, h_r) \quad : \quad k, r \in \mathbb{N}\}.$$

For $(k,h) \in \mathcal{P}$ define the cumulative loss of the corresponding estimate by

$$\begin{aligned}
&\hat{L}_{n,j}(k,h) := \hat{L}_{n,j}((x_{-n}, \ldots, x_j), k, h) \\
&:= \frac{1}{n} \sum_{i=1}^{n-1} \Bigg(\hat{q}_{j,(k,h)}^{(i)}(x_{-n}, \ldots, x_{-n+i}, \ldots, x_{-n+i+j}) \\
&\qquad - \max\left\{ e^{-r\cdot(j+1)} \cdot f\left(100 \cdot \frac{x_{-n+i+j+1}}{x_{-n+i}}\right), \hat{q}_{j+1}^{(i)}(x_{-n}, \ldots, x_{-n+i}, \ldots, x_{-n+i+j+1}) \right\} \Bigg)^2.
\end{aligned}$$

Put $c = 8B^2$ (where we assume that the payoff function is bounded by B), let $(p_{k,r})_{k,r}$ be a probability distribution such that $p_{k,r} > 0$ for all $k, r \in \mathbb{N}$, and define weights, which depend on these cumulative losses, by

$$w_{n,k,r}^{(j)} := w_{n,k,r}^{(j)}(x_{-n}, \dots, x_j) := p_{k,r} \cdot e^{-n \cdot \hat{L}_{n,j}(k,h_r)/c}$$

and their normalized values

$$v_{n,k,r}^{(j)} := v_{n,k,r}^{(j)}(x_{-n}, \dots, x_j) := \frac{w_{n,k,r}^{(j)}}{\sum_{s,t=1}^{\infty} w_{n,s,t}^{(j)}}.$$

The estimate $\hat{q}_j^{(n)}$ is defined on $(0,\infty)^{j+n+1}$ as the convex combination of the estimates $\hat{q}_{j,(k,h_r)}^{(n)}$ using the weights $v_{n,k,r}^{(j)}$, i.e., $\hat{q}_j^{(n)}$ is defined by

$$\hat{q}_j^{(n)}(x_{-n}, \dots, x_j) := \sum_{k,r=1}^{\infty} v_{n,k,r}^{(j)} \cdot \hat{q}_{j,(k,h_r)}^{(n)}(x_{-n}, \dots, x_j).$$

For the computation of our estimated stopping rule we use the arithmetic mean of the first n estimates, i.e., we use

$$\hat{q}_{j,n}(x_{-n}, \dots, x_j) := \frac{1}{n} \sum_{l=1}^{n} \hat{q}_j^{(l)}(x_{-l}, \dots, x_j) \tag{7}$$

for $j \in \{0, 1, \dots, L-1\}$ and $\hat{q}_{L,n} := q_L = 0$.

With this estimate of q_j we estimate the optimal stopping rule

$$\tau^* := \inf\left\{ j \in \{0, 1, \dots, L\} \, : \, e^{-r \cdot j} \cdot f(X_j) \geq q_j(\dots, X_{-1}, X_0, \dots, X_j) \right\}$$

by

$$\hat{\tau}_n := \inf\left\{ j \in \{0, 1, \dots, L\} \, : \, e^{-r \cdot j} \cdot f(X_j) \geq \hat{q}_{j,n}(X_{-n}, \dots, X_{-1}, X_0, \dots, X_j) \right\}.$$

3 Main theoretical result

Given $(X_j)_{j \in \mathbb{Z}}$ with $X_j > 0$ $(j \in \mathbb{Z})$ we define returns

$$Z_j := \frac{X_j}{X_{j-1}} \quad (j \in \mathbb{Z}).$$

Z_j describes the money we get at time j if we invest one Euro in the asset at time $j-1$. In Theorem 1 below we assume that the returns $(Z_j)_{j \in \mathbb{Z}}$ are stationary and ergodic.

Let the estimate $\hat{\tau}_n$ of the optimal stopping rule τ^* be defined as in the previous section. Then the following result is valid:

Theorem 1 *Let $(X_j)_{j\in\mathbb{Z}}$ be an arbitrary sequence of positive random variables such that the corresponding returns are stationary and ergodic. Assume that the payoff function is measurable, nonnegative and bounded by $B > 0$. Let the estimate be defined as in Section 2, where the kernel K is given by*

$$K(v) = H\left(\|v\|_2^{j+k+1}\right),$$

for some $H : \mathbb{R}_+ \to \mathbb{R}_+$ which is a nonincreasing and continuous function satisfying

$$H(0) > 0 \quad \text{and} \quad t \cdot H(t) \to 0 \quad (t \to \infty).$$

Then

$$\hat{V}_{0,n} := \mathbf{E}\left\{e^{-r\cdot\hat{\tau}_n} \cdot f(X_{\hat{\tau}_n})\right\} \to V_0^* = \mathbf{E}\left\{e^{-r\cdot\tau^*} \cdot f_{\tau^*}(X_{\tau^*})\right\}$$

for $n \to \infty$.

4 Application to simulated data

In this section we evaluate the behaviour of our newly proposed estimate for finite sample size by applying it to simulated data. In order to simplify the computation of the algorithm, we modify it such that we do not use the final averaging step (7) believing that the averaging should not destroy a convergence property of the estimate, so if we show that the algorithm works even without averaging, it should work with averaging, too. Furthermore we do not use returns relative to the previous day as x-values for our regression estimates, instead we use returns relative to the beginning of the time interval of an option. With the later modification it can be shown that the theoretical result above is still valid, because a consecutive sequence of these modified returns generates the same σ-algebra as the corresponding original returns. Furthermore, we ignore the first $n_{0,t} = (L-1-t)\cdot 200$ data points during the computation of $\hat{q}_j^{(t)}$ since we think that the first $n_{0,t}$ values of $\hat{q}_j^{(t+1)}$ are not reliable because they are based on too few data points.

We consider options on a single stock starting at $x_0 = 100$ which can be exercised on five equidistant time points $t_0 = 0$, $t_1 = 0.25$, $t_2 = 0.5$, $t_3 = 0.75$ and $t_4 = 1$. For the payoff function we use a strangle spread payoff function depending on four exercise points K_1, K_2, K_3 and K_4 where $K_2 - K_1 = K_4 - K_3$ (cf. Figure 1). We consider three

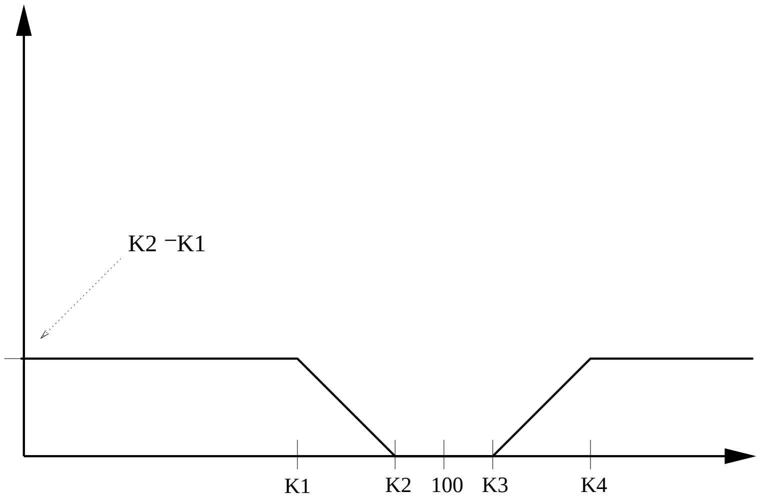

Figure 1: Strangle spread payoff with strike prices K_1, K_2, K_3 and K_4.

different models for generating the stock values, namely a geometric Brownian motion with a fixed volatility according to Black and Scholes (1973), a jump diffusion model according to Merton (1975), and a GARCH(1,1) in the form of Duan (1995). For each of these models we choose strike prices of the strangle spread payoff function, and determine the price of the corresponding option in three ways: Firstly we use a regression-based Monte Carlo estimate of the price based on the true price process, where we extend the state space in case of the GARCH(1,1)-model in order to get a 3-dimensional Markovian process (i.e., we use $(X_i, \sigma_i, \epsilon_i)$, see below). As regression-based Monte Carlo procedure we use the smoothing spline algorithm described in Kohler (2008), which gives results which are usually at least comparable but often better than the algorithms of Tsitsiklis and Van Roy (1999) and Longstaff and Schwarz (2001) based on parametric regression (cf. Kohler (2010)). Here the estimate is based on 1000 paths of length 5 of the price process. Secondly, we use the same procedure based on a Black-Scholes model adapted to a path of length 2000 generated from the original model via estimation of the volatility from this historical data in the usual way. And thirdly, we apply our newly proposed estimate to a path of length $n = 1500$, where we consider as bandwidths $h \in \{0.001, 0.01, 0.1\}$ and use the $k \in \{0, 1, 2\}$ last values of the returns for prediction of the value at the next time step. Each of these $3 \cdot 3 = 9$ models gets the same probability $p_{k,r} = \frac{1}{9}$, and for the constant used for computing the weights of the estimate from the cumulative empirical losses we use

$c = 8B^2$ where B is the maximal value of the payoff function. From our newly proposed estimate we construct an estimate of the option price which - in contrast to the estimates above - does not rely on future values of the pricing process via

$$\hat{V}_0 = \max\{f_0(X_0), \hat{q}_0^{(n)}(X_{-n}, \ldots, X_0)\}.$$

Here the estimate is motivated by the formula

$$V_0 = \mathbf{E}\left\{\max\{f_0(X_0), q_0(\ldots, X_{-1}, X_0)\}\right\},$$

which can be proven analogously to Lemma 2. For each of these three estimates we replicate the estimation of the price 100 times and present boxplots of the corresponding values.

In case of the geometric Brownian motion we simulate the price process according to

$$X_i = x_0 \cdot \exp\left((r - \frac{1}{2}\sigma^2) \cdot \frac{i}{4} + \frac{\sigma}{2} \cdot W_i\right) \quad (i \in \mathbb{N}_0),$$

where $x_0 = 100$, $r = 0.05$, $\sigma^2 = 0.25$ and where $(W_i)_{i \in \mathbb{N}_0}$ is a Wiener process starting with $W_0 = 0$. As payoff function we use in this case a strangle spread payoff function with strikes $K_1 = 75$, $K_2 = 90$, $K_3 = 110$ and $K_4 = 125$. Figure 2 describes the results of the three estimates.

In case of the jump diffusion model we simulate the price process according to

$$X_i = x_0 \cdot \exp\left((r - \frac{1}{2}\sigma^2) \cdot i + \sigma \cdot W_i\right) \cdot \exp\left((-\mu - \lambda \cdot (e^{\sigma_2^2/2} - 1)) \cdot \frac{i}{4} + \frac{\sigma_2}{2} \cdot \sum_{i=1}^{N_i} Y_i\right),$$

where $x_0 = 100$, $r = \mu = 0.05$, $\sigma^2 = 0.4$, $\sigma_2^2 = 0.1$, $\lambda = 4$, and where $(W_i)_{i \in \mathbb{N}_0}$ is a Wiener process starting with $W_0 = 0$, $(N_i)_{i \in \mathbb{N}_0}$ is a Poisson process with parameter λ starting with $N_0 = 0$, and Y_1, Y_2, ... are independent normally distributed random variables with expectation 0 and variance σ_2^2. As payoff function we use in this case a strangle spread payoff function with strikes $K_1 = 50$, $K_2 = 90$, $K_3 = 120$ and $K_4 = 160$. Figure 3 describes the results of the three estimates.

In case of the GARCH(1,1) model we simulate the price process according to

$$\begin{aligned} X_{i+1} &= X_i \cdot \exp\left(\frac{r}{4} - \frac{1}{2} \cdot \sigma_{i+1}^2 + \sigma_{i+1} \cdot \epsilon_{i+1}\right), \\ \sigma_{i+1}^2 &= \delta_0 + \delta_1 \cdot (\sigma_i \cdot \epsilon_i - \lambda \cdot \sigma_i)^2 + \xi_1 \cdot \sigma_i^2, \end{aligned}$$

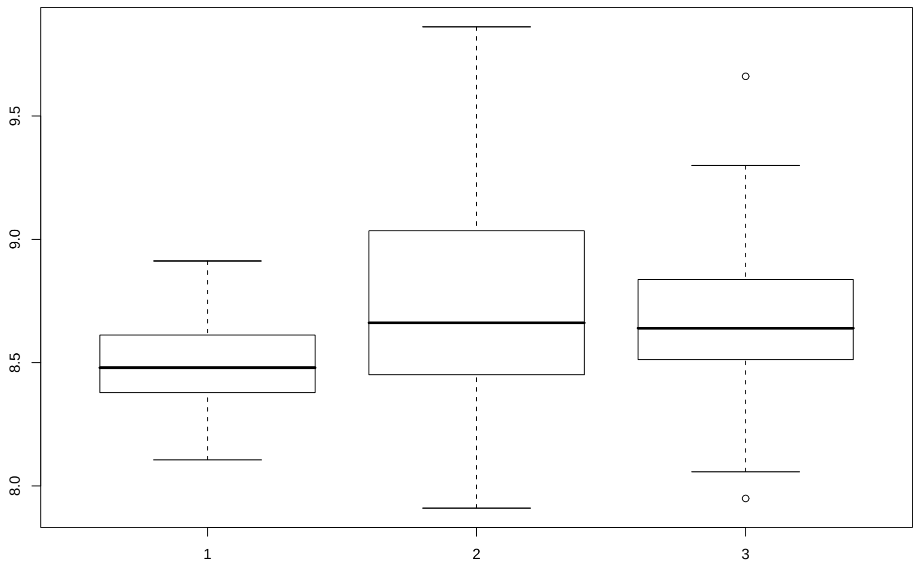

Figure 2: Results of the three different estimates in case of a standard Black-Scholes model. Estimate 1 is the regression-based Monte Carlo estimate from Kohler (2008) based on the true Black-Scholes model, for estimate 2 a Black-Scholes model is adapted to this true model, and estimate 3 is the new estimate proposed in this paper.

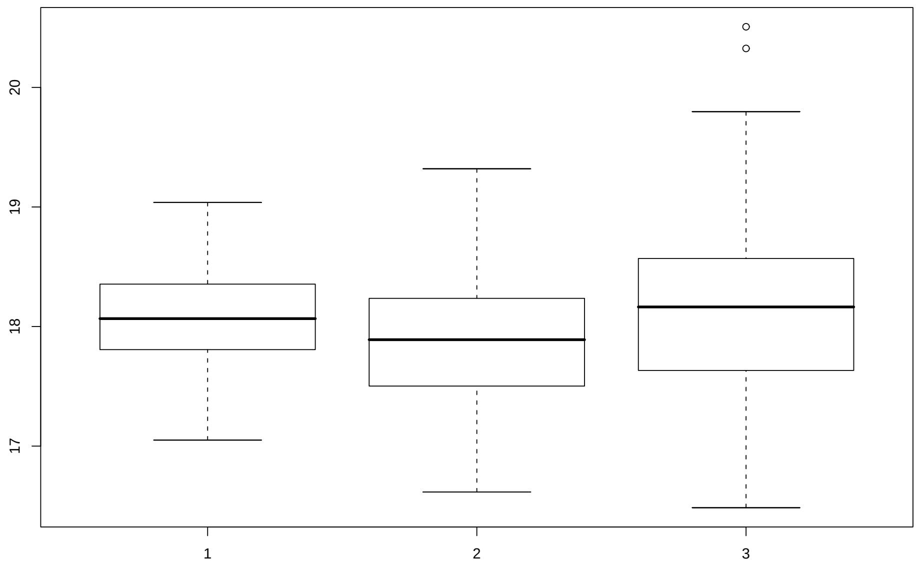

Figure 3: Results of the three different estimates in case of a jump diffusion model. Estimate 1 is the regression-based Monte Carlo estimate from Kohler (2008) based on the true jump diffusion model, for estimate 2 a Black-Scholes model is adapted to this true model, and estimate 3 is the new estimate proposed in this paper.

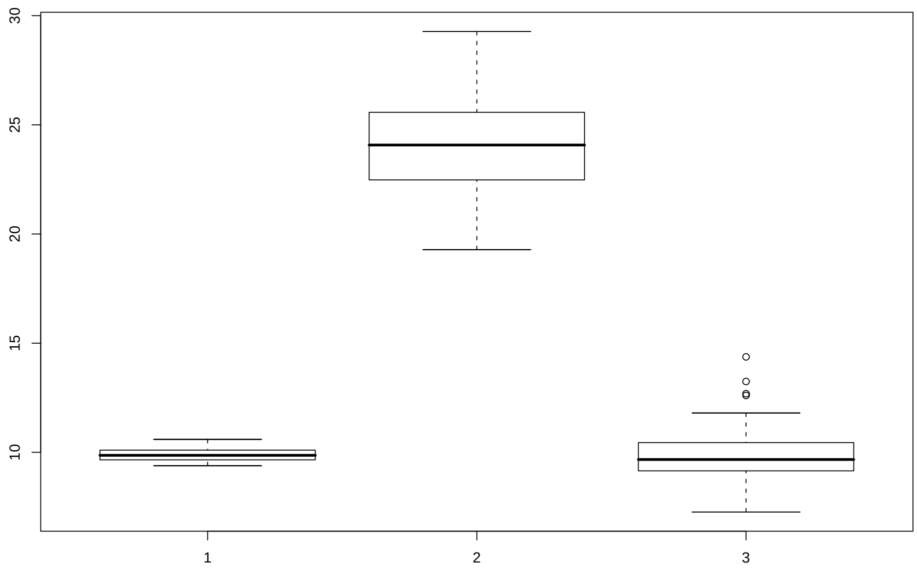

Figure 4: Results of the three different estimates in case of a GARCH(1,1) model. Estimate 1 is the regression-based Monte Carlo estimate from Kohler (2008) based on the true GARCH(1,1) model, for estimate 2 a Black-Scholes model is adapted to this true model, and estimate 3 is the new estimate proposed in this paper.

where $r = 0.05$, $\lambda = 0.7136$, $\delta_0 = 0.0000664$, $\delta_1 = 0.144$, $\xi_1 = 0.776$ and where $(\epsilon_t)_{t\in\mathbb{Z}}$ are independent normally distributed random variables with expectation zero and variance one. Here we start our simulation with $X_0 = x_0 = 100$. For σ_0 we use the random value we get if we start the second recursion with $\sigma^2_{-100} = 0$. As payoff function we use in this case a strangle spread payoff function with strikes $K_1 = 70$, $K_2 = 100$, $K_3 = 100$ and $K_4 = 130$ (which is in fact a capped straddle payoff function). Figure 4 describes the results of the three estimates.

Comparing the results in the three figures above we see that our estimate always produces values which are somehow close to the values of the regression-based Monte Carlo procedure based on the true model. In contrast, by just assuming a Black-Scholes model, fitting this to the data and using a corresponding regression-based Monte Carlo method, this might lead (as is the case for the GARCH(1,1) model) to estimates which are far away from the value to be estimated.

5 Proofs

5.1 Preliminaries to the proof of Theorem 1

Once we have constructed approximations

$$\hat{q}_j(\ldots, x_{-1}, x_0, \ldots, x_j)$$

of the continuation values $q_j(\ldots, x_{-1}, x_0, \ldots, x_j)$ we can use them to construct an approximation

$$\hat{\tau} = \inf\left\{j \in \{0, 1, \ldots, L\} \, : \, e^{-r\cdot j} \cdot f(X_j) \geq \hat{q}_j(\ldots, X_{-1}, X_0, X_1, \ldots, X_j)\right\}$$

of the optimal stopping time τ^*.

As our next lemma shows the errors of the estimates $\hat{q}_j$ determine the quality of the constructed stopping time.

Lemma 3 *Assume $\hat{q}_L = 0$. Then*

$$\begin{aligned}
&\mathbf{E}\left\{e^{-r\cdot\tau^*} \cdot f(X_{\tau^*}) | \ldots, X_{-2}, X_{-1}\right\} - \mathbf{E}\left\{e^{-r\cdot\hat{\tau}} \cdot f(X_{\hat{\tau}}) | \ldots, X_{-2}, X_{-1}\right\} \\
&\leq \sum_{j=0}^{L-1} \mathbf{E}\left\{ |\hat{q}_j(\ldots, X_{-1}, X_0, \ldots, X_j) - q_j(\ldots, X_{-1}, X_0, \ldots, X_j)| \, \middle| \, \ldots, X_{-2}, X_{-1}\right\}.
\end{aligned}$$

The assertion follows from a modification of the proof of Proposition 21 in Belomestny (2010). For the sake of completeness a complete proof is given in the appendix.

5.2 Proof of Theorem 1

Before we start the proof of Theorem 1 we introduce some notation. Observe that from the returns we can reconstruct the values of the asset via

$$X_j = X_0 \cdot Z_1 \cdot Z_2 \cdot \ldots \cdot Z_j = 100 \cdot Z_1 \cdot Z_2 \cdot \ldots \cdot Z_j$$

for $j \geq 0$. We describe the price of the asset normalized such that at time i it is equal to 100 by

$$X_j^{(i)} := 100 \cdot Z_{i+1} \cdot Z_{i+2} \cdot \ldots \cdot Z_{i+j} \quad \text{for } j \geq 0$$

and

$$X_j^{(i)} := 100 \cdot Z_i^{-1} \cdot Z_{i-1}^{-1} \cdot \ldots \cdot Z_{i+j+1}^{-1} \quad \text{for } j < 0.$$

Clearly,

$$X_i^{(0)} = X_i \quad \text{for } i \in \mathbb{Z}. \tag{8}$$

Furthermore, since we assume that the returns are stationary, we have also that

$$(\ldots, X_0^{(j)}, \ldots, X_k^{(j)}) \quad \text{and} \quad (\ldots, X_0^{(j+l)}, \ldots, X_k^{(j+l)}) \quad \text{have the same distribution.} \tag{9}$$

In the sequel we want to bound

$$\begin{aligned} V_0 - \hat{V}_{0,n} &= \mathbf{E}\left\{e^{-r\cdot\tau^*} \cdot f(X_{\tau^*}) - e^{-r\cdot\hat{\tau}_n} \cdot f(X_{\hat{\tau}_n})\right\} \\ &= \mathbf{E}\left\{\mathbf{E}\left\{e^{-r\cdot\tau^*} \cdot f(X_{\tau^*}) - e^{-r\cdot\hat{\tau}_n} \cdot f(X_{\hat{\tau}_n}) | \ldots, X_{-2}, X_{-1}\right\}\right\}. \end{aligned}$$

By Lemma 3 we have

$$V_0 - \hat{V}_{0,n} \leq \sum_{j=0}^{L-1} \mathbf{E}\left\{|\hat{q}_{j,n}(X_{-n}, \ldots, X_{-1}, X_0, \ldots, X_j) - q_j(\ldots, X_{-1}, X_0, \ldots, X_j)|\right\},$$

so it suffices to show

$$\mathbf{E}\left\{|\hat{q}_{j,n}(X_{-n}, \ldots, X_{-1}, X_0, \ldots, X_j) - q_j(\ldots, X_{-1}, X_0, \ldots, X_j)|\right\} \to 0 \quad (n \to \infty) \tag{10}$$

for $j \in \{0, 1, \ldots, L-1\}$.

Using the definition of $\hat{q}_{j,n}$ as arithmetic mean and the triangle inequality we get

$$\begin{aligned} &\mathbf{E}\left\{|\hat{q}_{j,n}(X_{-n}, \ldots, X_{-1}, X_0, \ldots, X_j) - q_j(\ldots, X_{-1}, X_0, \ldots, X_j)|\right\} \\ &= \mathbf{E}\left\{\left|\frac{1}{n}\sum_{l=1}^{n} \hat{q}_j^{(l)}(X_{-l}, \ldots, X_{-1}, X_0, \ldots, X_j) - q_j(\ldots, X_{-1}, X_0, \ldots, X_j)\right|\right\} \\ &\leq \frac{1}{n}\sum_{l=1}^{n} \mathbf{E}\left\{\left|\hat{q}_j^{(l)}(X_{-l}, \ldots, X_{-1}, X_0, \ldots, X_j) - q_j(\ldots, X_{-1}, X_0, \ldots, X_j)\right|\right\} \\ &\overset{(8)}{=} \frac{1}{n}\sum_{l=1}^{n} \mathbf{E}\left\{\left|\hat{q}_j^{(l)}(X_{-l}^{(0)}, \ldots, X_{-1}^{(0)}, X_0^{(0)}, \ldots, X_j^{(0)}) - q_j(\ldots, X_{-1}^{(0)}, X_0^{(0)}, \ldots, X_j^{(0)})\right|\right\} \\ &\overset{(9)}{=} \frac{1}{n}\sum_{l=1}^{n} \mathbf{E}\left\{\left|\hat{q}_j^{(l)}(X_{-l}^{(l)}, \ldots, X_{-1}^{(l)}, X_0^{(l)}, \ldots, X_j^{(l)}) - q_j(\ldots, X_{-1}^{(l)}, X_0^{(l)}, \ldots, X_j^{(l)})\right|\right\} \\ &= \mathbf{E}\left\{\frac{1}{n}\sum_{l=1}^{n} \left|\hat{q}_j^{(l)}(X_{-l}^{(l)}, \ldots, X_{-1}^{(l)}, X_0^{(l)}, \ldots, X_j^{(l)}) - q_j(\ldots, X_{-1}^{(l)}, X_0^{(l)}, \ldots, X_j^{(l)})\right|\right\}. \end{aligned}$$

Because of the Cauchy-Schwarz inequality it suffices to show

$$\mathbf{E}\left\{\frac{1}{n}\sum_{l=1}^{n} \left|\hat{q}_j^{(l)}(X_{-l}^{(l)}, \ldots, X_{-1}^{(l)}, X_0^{(l)}, \ldots, X_j^{(l)}) - q_j(\ldots, X_{-1}^{(l)}, X_0^{(l)}, \ldots, X_j^{(l)})\right|^2\right\} \to 0 \tag{11}$$

$(n \to \infty)$ for all $j \in \{0, \dots, L-1\}$. And because of boundedness of the estimates and of q_j this in turn follows from

$$\frac{1}{n} \sum_{l=1}^{n} \left| \hat{q}_j^{(l)}(X_{-l}^{(l)}, \dots, X_{-1}^{(l)}, X_0^{(l)}, \dots, X_j^{(l)}) - q_j(\dots, X_{-1}^{(l)}, X_0^{(l)}, \dots, X_j^{(l)}) \right|^2 \to 0 \qquad (12)$$

in probability for all $j \in \{0, \dots, L-1\}$.

The idea is now to use techniques from Section 27.5 (in particular Corollary 27.1) in Györfi et al. (2002). To simplify the notation, we reformulate the whole problem using returns. Set

$$m_L = 0.$$

Given m_{j+1} for $j \in \{0, 1, \dots, L-1\}$ we define m_j by

$$\begin{aligned} &m_j(\dots, z_{-1}, z_0, \dots, z_j) \qquad (13)\\ &:= \mathbf{E}\Big\{ \max\Big\{ e^{-r\cdot(j+1)} \cdot f(100 \cdot z_1 \cdot \ldots \cdot z_j \cdot Z_{j+1}), m_{j+1}(\dots, z_{-1}, z_0, \dots, Z_{j+1}) \Big\} \\ &\qquad \Big| \dots, Z_{-1} = z_{-1}, Z_0 = z_0, \dots, Z_j = z_j \Big\}. \end{aligned}$$

m_j can be considered as continuation value defined using returns (cf. Lemma 2, which is valid also if the payoff function depends on all previous state variables). Since $Z_j, Z_{j-1}, \dots$ generate the same σ-algebra as $X_j, X_{j-1}, \dots$ (which follows from $Z_j = X_j/X_{j-1}$ and $X_0 = 100$) this implies

$$m_j(\dots, Z_{-1}, Z_0, \dots, Z_j) = q_j(\dots, X_{-1}, X_0, \dots, X_j) \quad a.s. \qquad (14)$$

Next we define estimates of m_j using realizations $z_1, \dots, z_n$ of the returns $Z_1, \dots, Z_n$, with arguments $u_1, \dots, u_{n+j}$. We start with

$$\hat{m}_L^{(n)} := 0.$$

Given $\hat{m}_{j+1}^{(n)}$ for $j \in \{0, 1, \dots, L-1\}$ we define $\hat{m}_j^{(n)}$ as follows:

We start with defining $\hat{m}_{j,n,(k,h)}$ for parameters $k \in \mathbb{N}$ and $h > 0$ using local averaging by

$$\begin{aligned} &\hat{m}_{j,n,(k,h)}(z_1, \dots, z_n; u_1, \dots, u_{n+j}) \\ &:= \sum_{i=k+1}^{n-j-1} \max\left\{ e^{-r\cdot(j+1)} \cdot f(100 \cdot \prod_{r=i+1}^{i+j+1} z_r), \hat{m}_{j+1}^{(i)}(z_1, \dots, z_i; u_1, \dots, u_{i+j+1}) \right\} \end{aligned}$$

$$\cdot \frac{K\left(\frac{(z_{i-k},\ldots,z_{i+j})-(u_{n-k},\ldots,u_{n+j})}{h}\right)}{\sum_{i=k+1}^{n-j-1} K\left(\frac{(z_{i-k},\ldots,z_{i+j})-(u_{n-k},\ldots,u_{n+j})}{h}\right)}.$$

Here we set

$$\hat{m}_{j,n,(k,h)} := 0$$

for $k \geq n - j - 1$.

For $(k, h) \in \mathcal{P}$ (where $\mathcal{P}$ is the parameter set in the definition of the estimate) define the cumulative loss of the estimate with parameter (k, h) by

$$\hat{L}_{n,j}(k,h) := \hat{L}_{n,j}(k,h; z_1,\ldots,z_{n-1}; u_1,\ldots,u_{n+j}) =$$
$$\frac{1}{n}\sum_{i=1}^{n-1}\Bigg(\hat{m}_{j,i,(k,h)}(z_1,\ldots,z_i; u_1,\ldots,u_{i+j})$$
$$-\max\left\{e^{-r\cdot(j+1)}\cdot f(100\cdot\prod_{r=i+1}^{i+j+1}u_r), \hat{m}_{j+1}^{(i)}(z_1,\ldots,z_i; u_1,\ldots,u_{i+j+1})\right\}\Bigg)^2.$$

Put $c := 8B^2$ (where B is the bound on the payoff function), let $(p_{k,r})_{k,r}$ be the probability distribution used in the definition of the estimate (which satisfies $p_{k,r} > 0$ for all $k, r \in \mathbb{N}$) and define weights, which depend on these cumulative losses, by

$$w_{n,k,r}^{(j)} := w_{n,k,r}^{(j)}(z_1,\ldots,z_{n-1}; u_1,\ldots,u_{n+j}) = p_{k,r}\cdot e^{-n\hat{L}_{n,j}(k,h_r)/c}$$

and their normalized values by

$$v_{n,k,r}^{(j)} := v_{n,k,r}^{(j)}(z_1,\ldots,z_{n-1}; u_1,\ldots,u_{n+j}) = \frac{w_{n,k,r}^{(j)}}{\sum_{s,t=1}^{\infty} w_{n,s,t}^{(j)}}.$$

The estimate $\hat{m}_j^{(n)}$ is defined as the convex combination of all estimates $\hat{m}_{j,n,(k,h_r)}$ using weights $v_{n,k,r}^{(j)}$, i.e., $\hat{m}_j^{(n)}$ is defined by

$$\hat{m}_j^{(n)}(z_1,\ldots,z_n; u_1,\ldots,u_{n+j}) := \sum_{k,r=1}^{\infty} v_{n,k,r}^{(j)}\cdot\hat{m}_{j,n,(k,h_r)}(z_1,\ldots,z_n; u_1,\ldots,u_{n+j}).$$

By using a backwards induction with respect to j starting with L it is easy to see that we have

$$\hat{q}_j^{(n)}(x_{-n},\ldots,x_j) = \hat{m}_j^{(n)}\left(\frac{x_{-n+1}}{x_{-n}},\ldots,\frac{x_0}{x_{-1}};\frac{x_{-n+1}}{x_{-n}},\ldots,\frac{x_j}{x_{j-1}}\right).$$

Using

$$\frac{X_i^{(l)}}{X_{i-1}^{(l)}} = Z_{l+i}$$

we get

$$\hat{q}_j^{(l)}(X_{-l}^{(l)}, \dots, X_{-1}^{(l)}, X_0^{(l)}, \dots, X_j^{(l)}) = \hat{m}_j^{(l)}(Z_1, \dots, Z_l; Z_1, \dots, Z_{l+j}) \tag{15}$$

for all $l \in \{1, \dots, n\}$. This together with (14), from which we get

$$q_j(\dots, X_{-1}^{(l)}, X_0^{(l)}, \dots, X_j^{(l)}) = m_j(\dots, Z_{-1}, Z_0, \dots, Z_{l+j}) \quad a.s.$$

(since $\dots, X_{-1}^{(l)}, X_0^{(l)}, \dots, X_j^{(l)}$ has corresponding returns $\dots, Z_{-1}, Z_0, \dots, Z_{l+j}$) for all $l \in \{1, \dots, n\}$, implies that we have with probability one

$$\begin{aligned}
&\frac{1}{n} \sum_{l=1}^{n} \left| \hat{q}_j^{(l)}(X_{-l}^{(l)}, \dots, X_{-1}^{(l)}, X_0^{(l)}, \dots, X_j^{(l)}) - q_j(\dots, X_{-1}^{(l)}, X_0^{(l)}, \dots, X_j^{(l)}) \right|^2 \\
&= \frac{1}{n} \sum_{l=1}^{n} \left| \hat{m}_j^{(l)}(Z_1, \dots, Z_l; Z_1, \dots, Z_{l+j}) - m_j(\dots, Z_{-1}, Z_0, \dots, Z_{l+j}) \right|^2.
\end{aligned}$$

So (12) in turn is implied by

$$\frac{1}{n} \sum_{l=1}^{n} \left| \hat{m}_t^{(l)}(Z_1, \dots, Z_l; Z_1, \dots, Z_{l+t}) - m_t(\dots, Z_{-1}, Z_0, \dots, Z_{l+t}) \right|^2 \to 0 \tag{16}$$

in probability for all $t \in \{0, 1, \dots, L\}$, which we show by backwards induction with respect to t.

We start with $t = L$ in which the assertion is trivial since

$$\hat{m}_L^{(l)} = 0 \quad \text{and} \quad m_L = 0$$

for all $l \in \mathbb{N}$.

Assume now that (16) holds for $t = j + 1$ for some $j \in \{0, 1, \dots, L - 1\}$. We have to show that in this case it is also valid for $t = j$.

Set

$$\begin{aligned}
L_n(\hat{m}_j) := \frac{1}{n} \sum_{l=1}^{n-1} \Bigg| &\hat{m}_j^{(l)}(Z_1, \dots, Z_l; Z_1, \dots, Z_{l+j}) \\
&- \max\left\{ e^{-r \cdot (j+1)} \cdot f(100 \cdot Z_{l+1} \cdot \ldots \cdot Z_{l+j+1}), m_{j+1}(\dots, Z_0, \dots, Z_{l+j+1}) \right\} \Bigg|^2,
\end{aligned}$$

$$\begin{aligned}
L_n(\hat{m}_{j,\cdot,(k,h)}) := \frac{1}{n} \sum_{l=1}^{n-1} \Bigg| &\hat{m}_{j,l,(k,h)}(Z_1, \dots, Z_l; Z_1, \dots, Z_{l+j}) \\
&- \max\left\{ e^{-r \cdot (j+1)} \cdot f(100 \cdot Z_{l+1} \cdot \ldots \cdot Z_{l+j+1}), m_{j+1}(\dots, Z_0, \dots, Z_{l+j+1}) \right\} \Bigg|^2,
\end{aligned}$$

$$\hat{L}_n(\hat{m}_j) := \frac{1}{n}\sum_{l=1}^{n-1}\Bigg|\hat{m}_j^{(l)}(Z_1,\ldots,Z_l;Z_1,\ldots,Z_{l+j})$$
$$-\max\left\{e^{-r\cdot(j+1)}\cdot f(100\cdot Z_{l+1}\cdot\ldots\cdot Z_{l+j+1}),\hat{m}_{j+1}^{(l)}(Z_1,\ldots,Z_l;Z_1,\ldots,Z_{l+j+1})\right\}\Bigg|^2,$$

and

$$\hat{L}_n(\hat{m}_{j,\cdot,(k,h)}) := \hat{L}_{n,j}(k,h;Z_1,\ldots,Z_{n-1};Z_1,\ldots,Z_{n+j})$$
$$:= \frac{1}{n}\sum_{l=1}^{n-1}\Bigg|\hat{m}_{j,l,(k,h)}(Z_1,\ldots,Z_l;Z_1,\ldots,Z_{l+j})$$
$$-\max\left\{e^{-r\cdot(j+1)}\cdot f(100\cdot Z_{l+1}\cdot\ldots\cdot Z_{l+j+1}),\hat{m}_{j+1}^{(l)}(Z_1,\ldots,Z_l;Z_1,\ldots,Z_{l+j+1})\right\}\Bigg|^2.$$

By Lemma 27.3 in Györfi et al. (2002) we get

$$\hat{L}_n(\hat{m}_j) \le \inf_{k,r\in\mathbb{N}}\left(\hat{L}_n(\hat{m}_{j,\cdot,(k,h_r)}) - c\cdot\frac{\ln p_{k,r}}{n}\right). \tag{17}$$

Set

$$L_j^* := \mathbf{E}\Bigg\{\Bigg|m_j(\ldots,Z_{j-1},Z_j)$$
$$-\max\left\{e^{-r\cdot(j+1)}\cdot f(100\cdot\prod_{r=1}^{j+1}Z_r),m_{j+1}(\ldots,Z_j,Z_{j+1})\right\}\Bigg|^2\Bigg\}.$$

In order to show (16) we show first

$$L_n(\hat{m}_j)\to L_j^* \quad \text{in probability.} \tag{18}$$

By $|\max\{a,b\}-\max\{a,c\}|\le|b-c|$ $(a,b,c\in\mathbb{R})$ and (16) for $t=j+1$ we get

$$\frac{1}{n}\sum_{l=1}^{n-1}\Bigg|\max\left\{e^{-r\cdot(j+1)}\cdot f(100\cdot Z_{l+1}\cdot\ldots\cdot Z_{l+j+1}),m_{j+1}(\ldots,Z_0,\ldots,Z_{l+j+1})\right\}$$
$$-\max\left\{e^{-r\cdot(j+1)}\cdot f(100\cdot Z_{l+1}\cdot\ldots\cdot Z_{l+j+1}),\hat{m}_{j+1}^{(l)}(Z_1,\ldots,Z_l;Z_1,\ldots,Z_{l+j+1})\right\}\Bigg|^2$$
$$\le \frac{1}{n}\sum_{l=1}^{n-1}\left|\hat{m}_{j+1}^{(l)}(Z_1,\ldots,Z_l;Z_1,\ldots,Z_{l+j+1})-m_{j+1}(\ldots,Z_0,\ldots,Z_{l+j+1})\right|^2$$
$$\to 0 \quad \text{in probability.}$$

Using

$$\frac{1}{n}\sum_{l=1}^{n-1}|a_l-b_l|^2-\frac{1}{n}\sum_{l=1}^{n-1}|a_l-c_l|^2=\frac{1}{n}\sum_{l=1}^{n-1}(a_l-b_l+a_l-c_l)\cdot(c_l-b_l)$$
$$\le\left(\frac{1}{n}\sum_{l=1}^{n-1}(a_l-b_l+a_l-c_l)^2\right)^{1/2}\cdot\left(\frac{1}{n}\sum_{l=1}^{n-1}(c_l-b_l)^2\right)^{1/2}$$

and the boundedness of the payoff function we see that this implies

$$L_n(\hat{m}_j) - \hat{L}_n(\hat{m}_j) \to 0 \quad \text{and} \quad L_n(\hat{m}_{j,\cdot,(k,h)}) - \hat{L}_n(\hat{m}_{j,\cdot,(k,h)}) \to 0 \tag{19}$$

in probability. Hence for an arbitrary subsequence $(n_l)_l$ of $(n)_n$ we find a subsubsequence $(n_{l_s})_s$ of $(n_l)_l$ such that we have with probability one

$$\begin{aligned}
\limsup_{s\to\infty} L_{n_{l_s}}(\hat{m}_j) &= \limsup_{s\to\infty} \hat{L}_{n_{l_s}}(\hat{m}_j) \\
&\overset{(17)}{\leq} \limsup_{s\to\infty} \inf_{k,r\in\mathbb{N}} \left(\hat{L}_{n_{l_s}}(\hat{m}_{j,\cdot,(k,h_r)}) - c \cdot \frac{\ln p_{k,r}}{n_{l_s}} \right) \\
&\leq \inf_{k,r\in\mathbb{N}} \limsup_{s\to\infty} \left(\hat{L}_{n_{l_s}}(\hat{m}_{j,\cdot,(k,h_r)}) - c \cdot \frac{\ln p_{k,r}}{n_{l_s}} \right) \\
&= \inf_{k,r\in\mathbb{N}} \limsup_{s\to\infty} L_{n_{l_s}}(\hat{m}_{j,\cdot,(k,h_r)}).
\end{aligned} \tag{20}$$

Of course, this relation also holds if we replace $(n_{l_s})_s$ by any of its subsequences (which we will do later in the proof).

Next we analyze $L_n(\hat{m}_{j,\cdot,(k,h_r)})$. We have

$$\begin{aligned}
&\hat{m}_{j,n,(k,h)}(Z_1,\ldots,Z_n;v_{-n+1},\ldots,v_j) \\
&= \frac{\sum_{i=k+1}^{n-j-1} \max\left\{e^{-r\cdot(j+1)} \cdot f(100 \cdot \prod_{r=i+1}^{i+j+1} Z_r), \hat{m}_{j+1}^{(i)}(Z_1^i; Z_1^{i+j+1})\right\} \cdot K\left(\frac{Z_{i-k}^{i+j} - v_{-k}^j}{h}\right)}{\sum_{i=k+1}^{n-j-1} K\left(\frac{Z_{i-k}^{i+j} - v_{-k}^j}{h}\right)} \\
&= \frac{A_n}{C_n} + \frac{B_n - A_n}{C_n}
\end{aligned}$$

where

$$Z_r^s := (Z_r, Z_{r+1}, \ldots, Z_s) \quad \text{and} \quad v_r^s := (v_r, v_{r+1}, \ldots, v_s) \quad \text{for } r \leq s,$$

$$\begin{aligned}
A_n \quad := \quad & \frac{1}{n-j-k-1} \sum_{i=k+1}^{n-j-1} \max\left\{e^{-r\cdot(j+1)} \cdot f(100 \cdot \prod_{r=i+1}^{i+j+1} Z_r), m_{j+1}(\ldots, Z_{i+j}, Z_{i+j+1})\right\} \\
& \cdot K\left(\frac{Z_{i-k}^{i+j} - v_{-k}^j}{h}\right),
\end{aligned}$$

$$\begin{aligned}
B_n \quad := \quad & \frac{1}{n-j-k-1} \sum_{i=k+1}^{n-j-1} \max\left\{e^{-r\cdot(j+1)} \cdot f(100 \cdot \prod_{r=i+1}^{i+j+1} Z_r), \hat{m}_{j+1}^{(i)}(Z_1^i; Z_1^{i+j+1})\right\} \\
& \cdot K\left(\frac{Z_{i-k}^{i+j} - v_{-k}^j}{h}\right)
\end{aligned}$$

and

$$C_n := \frac{1}{n-j-k-1} \sum_{i=k+1}^{n-j-1} K\left(\frac{Z_{i-k}^{i+j} - v_{-k}^j}{h}\right).$$

By the ergodic theorem we get

$$A_n \to \mathbf{E}\left\{\max\left\{e^{-r\cdot(j+1)} \cdot f(100 \cdot Z_1 \cdot \ldots \cdot Z_{j+1}), m_{j+1}(\ldots, Z_j, Z_{j+1})\right\} \cdot K\left(\frac{Z_{-k}^j - v_{-k}^j}{h}\right)\right\}$$

a.s. and

$$C_n \to \mathbf{E}\left\{K\left(\frac{Z_{-k}^j - v_{-k}^j}{h}\right)\right\} \quad a.s.$$

If we use the continuity of the kernel function we can even apply an ergodic theorem in the separable Banach space of continuous functions vanishing at infinity (with supremum norm) and get that the almost sure convergence of A_n and C_n is uniformly with respect to v_{-k}^j (cf., e.g., Krengel (1985), Chapter 4, Theorem 2.1).

Furthermore, using the the triangle inequality,

$$|\max\{a,b\} - \max\{a,c\}| \le |b-c| \quad (a,b,c \in \mathbb{R}),$$

and the Cauchy-Schwarz inequality we can conclude

$$\begin{aligned}
&|B_n - A_n| \\
&\le \frac{1}{n-j-k-1} \sum_{i=k+1}^{n-j-1} \left|\hat{m}_{j+1}^{(i)}(Z_1^i; Z_1^{i+j+1}) - m_{j+1}(\ldots, Z_{i+j}, Z_{i+j+1})\right| \cdot K\left(\frac{Z_{i-k}^{i+j} - v_{-k}^j}{h}\right) \\
&\le \sqrt{\frac{1}{n-j-k-1} \sum_{i=k+1}^{n-j-1} \left|\hat{m}_{j+1}^{(i)}(Z_1^i; Z_1^{i+j+1}) - m_{j+1}(\ldots, Z_{i+j}, Z_{i+j+1})\right|^2} \\
&\qquad \cdot \sqrt{\frac{1}{n-j-k-1} \sum_{i=k+1}^{n-j-1} K\left(\frac{Z_{i-k}^{i+j} - v_{-k}^j}{h}\right)^2}.
\end{aligned}$$

By the ergodic theorem the second factor on the right-hand side above converges to

$$\sqrt{\mathbf{E}\left\{K\left(\frac{Z_{-k}^j - v_{-k}^j}{h}\right)^2\right\}} < \infty$$

with probability one (where we have again uniform convergence with respect to v_{-k}^j), and the first factor converges in probability to zero by (16) for $t = j+1$. Because of $K \ge c \cdot I_{S_{0,r}}$

for suitable $c > 0$, $r > 0$, where $S_{0,r}$ is the ball in $\mathbb{R}^{j+k+1}$ centered at 0 with radius r, we have

$$\mathbf{E}\left\{K\left(\frac{Z_{-k}^j - v_{-k}^j}{h}\right)\right\} \geq c \cdot \mathbf{P}_{Z_{-k}^j}\left(v_{-k}^j + S_{0,r\cdot h}\right) > 0 \qquad \mathbf{P}_{Z_{-k}^j} - \text{almost everywhere} \quad (21)$$

(cf., e.g., Györfi et al. (2002), pp. 499, 500). Therefore

$$\frac{B_n - A_n}{C_n} \to 0 \quad \text{in probability } \mathbf{P}_{Z_{-k}^j} - \text{almost everywhere,}$$

from which we get

$$\hat{m}_{j,n,(k,h)}(Z_1, \ldots, Z_n; v_{-n+1}, \ldots, v_j) \to m_{j,(k,h)}(v_{-k}, \ldots, v_j) \quad \text{in probability}$$

$\mathbf{P}_{Z_{-k}^j}$-almost everywhere, where

$$m_{j,(k,h)}(v_{-k}, \ldots, v_j) = \frac{\mathbf{E}\left\{\max\left\{e^{-r\cdot(j+1)} \cdot f(100 \cdot Z_1 \cdot \ldots \cdot Z_{j+1}), m_{j+1}(\ldots, Z_j, Z_{j+1})\right\} \cdot K\left(\frac{Z_{-k}^j - v_{-k}^j}{h}\right)\right\}}{\mathbf{E}\left\{K\left(\frac{Z_{-k}^j - v_{-k}^j}{h}\right)\right\}}.$$

Let $\epsilon > 0$ be arbitrary and set

$$S_\epsilon = \left\{v_{-k}^j \in \mathbb{R}^{k+j+1} \quad : \quad \mathbf{E}\left\{K\left(\frac{Z_{-k}^j - v_{-k}^j}{h}\right)\right\} > \epsilon\right\}.$$

By (21) we know

$$\mathbf{P}_{Z_{-k}^j}(S_\epsilon) \to 1 \quad (\epsilon \to 0).$$

Since the nominators and the denominators above converge uniformly with respect to v_{-k}^j and since the limit of the denominators is greater than ϵ on S_ϵ we know in addition

$$\sup_{v_{-n+1}, \ldots, v_{-k-1} \in \mathbb{R}, v_{-k}^j \in S_\epsilon} \left|\hat{m}_{j,n,(k,h)}(Z_1, \ldots, Z_n; v_{-n+1}, \ldots, v_j) - m_{j,(k,h)}(v_{-k}, \ldots, v_j)\right| \to 0 \quad (22)$$

in probability. In the sequel we want to use this to show

$$\begin{aligned} & L_{n,j}(\hat{m}_{j,\cdot,(k,h)}) \\ &= \frac{1}{n}\sum_{i=1}^{n-1}\Bigg(\hat{m}_{j,i,(k,h)}(Z_1, \ldots, Z_i; Z_1, \ldots, Z_{i+j}) \qquad (23) \\ &\qquad - \max\left\{e^{-r\cdot(j+1)} \cdot f(100 \cdot \prod_{r=i+1}^{i+j+1} Z_r), m_{j+1}(\ldots, Z_{i+j}, Z_{i+j+1})\right\}\Bigg)^2 \\ &\to \mathbf{E}\left\{\left|m_{j,(k,h)}(Z_{-k}^j) - \max\left\{e^{-r\cdot(j+1)} \cdot f(100 \cdot \prod_{r=1}^{j} Z_r), m_{j+1}(\ldots, Z_j, Z_{j+1})\right\}\right|^2\right\} \end{aligned}$$

in probability. To do this, we observe first that the ergodic theorem implies

$$
\begin{aligned}
& L_{n,j}(m_{j,(k,h)}) \\
& = \frac{1}{n}\sum_{i=1}^{n-1}\Bigg(m_{j,(k,h)}(Z_{i-k},\dots,Z_{i+j}) \\
& \qquad\qquad - \max\left\{ e^{-r\cdot(j+1)}\cdot f(100\cdot\prod_{r=i+1}^{i+j+1} Z_r), m_{j+1}(\dots,Z_{i+j},Z_{i+j+1})\right\}\Bigg)^2 \\
& \to \mathbf{E}\left\{\left| m_{j,(k,h)}(Z_{-k}^j) - \max\left\{ e^{-r\cdot(j+1)}\cdot f(100\cdot\prod_{r=1}^{j} Z_r), m_{j+1}(\dots,Z_j,Z_{j+1})\right\}\right|^2\right\}
\end{aligned}
$$

almost surely. Because of boundedness of the payoff function we have in addition

$$
\begin{aligned}
& \left| L_{n,j}(\hat{m}_{j,\cdot,(k,h)}) - L_{n,j}(m_{j,(k,h)})\right| \\
& \le c_1\cdot\frac{1}{n}\sum_{i=1}^{n-1}\left|\hat{m}_{j,i,(k,h)}(Z_1^i;Z_1^{i+j}) - m_{j,(k,h)}(Z_{i-k}^{i+j})\right| \\
& \le c_2\cdot\frac{1}{n}\sum_{i=1}^{n-1} I_{S_\epsilon^c}(Z_{i-k}^{i+j}) \\
& \quad + c_1\cdot\frac{1}{n}\sum_{i=1}^{n-1}\sup_{v_{-i+1},\dots,v_{-k-1}\in\mathbb{R}, v_{-k}^j\in S_\epsilon}\left|\hat{m}_{j,i,(k,h)}(Z_1^i;v_{-i+1}^j) - m_{j,(k,h)}(v_{-k}^j)\right| \\
& \to c_2\cdot\mathbf{P}_{Z_{-k}^j}(S_\epsilon^c)
\end{aligned}
$$

in probability by (22) and by the ergodic theorem. By letting $\epsilon\to 0$ we get (23). And by replacing (n_{l_s}) by a suitable subsequence of $(n_{l_s})_s$ we can assume w.l.o.g. even that (23) holds for almost sure convergence if we replace n by n_{l_s} in (23).

Next we use Lemma 24.8 in Györfi et al. (2002) which implies

$$
m_{j,(k,h)}(z_{-k},\dots,z_j)\to m_{j,k}(z_{-k},\dots,z_j) \qquad \mathbf{P}_{Z_{-k}^j} - \text{almost everywhere}
$$

for $h\to 0$, where

$$
\begin{aligned}
& m_{j,k}(z_{-k},\dots,z_j) \\
& := \mathbf{E}\Big\{\max\left\{ e^{-r\cdot(j+1)}\cdot f(100\cdot\prod_{r=1}^{j+1} Z_r), m_{j+1}(\dots,Z_{-1},Z_0,\dots,Z_{j+1})\right\} \\
& \qquad\qquad\qquad\qquad \Big| Z_{-k}=z_{-k},\dots,Z_j=z_j\Big\}.
\end{aligned}
$$

And by the martingale convergence theorem we have

$$
m_{j,k}(Z_{-k},\dots,Z_j)\to m_j(\dots,Z_0,\dots,Z_j) \quad a.s.
$$

for $k \to \infty$ (since the almost sure limit X of the left-hand side satisfies

$$\int_A X \, dP = \int_A m_j(\ldots, Z_0, \ldots, Z_j) \, dP$$

for all $A \in \mathcal{F}(Z_{-k}, \ldots, Z_j)$ and all $k \in \mathbb{N}$, cf., e.g., Chapter 32.4A in Loève (1977) for more general results in this respect). From this we conclude by dominated convergence

$$\begin{aligned}
&\limsup_{s\to\infty} L_{n_{l_s}}(\hat{m}_j) \\
&\stackrel{(20)}{\leq} \inf_{k,r\in\mathbb{N}} \limsup_{s\to\infty} L_{n_{l_s}}(\hat{m}_{j,\cdot,(k,h_r)}) \\
&\stackrel{(23)}{=} \inf_{k,r\in\mathbb{N}} \mathbf{E}\left\{ \left| m_{j,(k,h_r)}(Z_{-k}^j) - \max\left\{ e^{-r\cdot(j+1)} \cdot f(100 \cdot \prod_{r=1}^{j+1} Z_r), m_{j+1}(\ldots, Z_j, Z_{j+1}) \right\} \right|^2 \right\} \\
&\leq L_j^* \quad a.s.
\end{aligned}$$

Because of

$$\liminf_{n\to\infty} L_n(\hat{m}_j) \geq L_j^* \quad a.s.$$

(cf., e.g., Section 27.5 in Györfi et al. (2002)) this completes the proof of (18).

Now we use (18) to show (16) for $t = j$. To do this we proceed as in the proof of Corollary 27.1 in Györfi et al. (2002). Consider the following decomposition:

$$\begin{aligned}
&\left(\hat{m}_j^{(l)}(Z_1^l; Z_1^{l+j}) - \max\left\{ e^{-r\cdot(j+1)} \cdot f(100 \cdot \prod_{r=l+1}^{l+j+1} Z_r), m_{j+1}(Z_{-\infty}^{l+j+1}) \right\} \right)^2 \\
&= \left(\hat{m}_j^{(l)}(Z_1^l; Z_1^{l+j}) - m_j(Z_{-\infty}^{l+j}) \right)^2 \\
&\quad + \left(m_j(Z_{-\infty}^{l+j}) - \max\left\{ e^{-r\cdot(j+1)} \cdot f(100 \cdot \prod_{r=l+1}^{l+j+1} Z_r), m_{j+1}(Z_{-\infty}^{l+j+1}) \right\} \right)^2 \\
&\quad +2 \cdot \left(\hat{m}_j^{(l)}(Z_1^l; Z_1^{l+j}) - m_j(Z_{-\infty}^{l+j}) \right) \\
&\qquad\qquad \cdot \left(m_j(Z_{-\infty}^{l+j}) - \max\left\{ e^{-r\cdot(j+1)} \cdot f(100 \cdot \prod_{r=l+1}^{l+j+1} Z_r), m_{j+1}(Z_{-\infty}^{l+j+1}) \right\} \right).
\end{aligned}$$

By (18) we know

$$\frac{1}{n} \sum_{l=1}^{n} \left(\hat{m}_j^{(l)}(Z_1^l; Z_1^{l+j}) - \max\left\{ e^{-r\cdot(j+1)} \cdot f(100 \cdot \prod_{r=l+1}^{l+j+1} Z_r), m_{j+1}(Z_{-\infty}^{l+j+1}) \right\} \right)^2 \to L_j^*$$

in probability. Furthermore, by the ergodic theorem we have

$$\frac{1}{n} \sum_{l=1}^{n} \left(m_j(Z_{-\infty}^{l+j}) - \max\left\{ e^{-r\cdot(j+1)} \cdot f(100 \cdot \prod_{r=l+1}^{l+j+1} Z_r), m_{j+1}(Z_{-\infty}^{l+j+1}) \right\} \right)^2 \to L_j^* \quad a.s.$$

Hence it suffices to show

$$\frac{1}{n}\sum_{l=1}^{n}\left(\hat{m}_j^{(l)}(Z_1^l;Z_1^{l+j}) - m_j(Z_{-\infty}^{l+j})\right)$$
$$\cdot\left(m_j(Z_{-\infty}^{l+j}) - \max\left\{e^{-r\cdot(j+1)}\cdot f(100\cdot\prod_{r=l+1}^{l+j+1} Z_r), m_{j+1}(Z_{-\infty}^{l+j+1})\right\}\right)$$
$$\to 0 \quad a.s.$$

But this is a consequence of Theorem A.6 in Györfi et al. (2002) (which we apply with $c_i = 1$) since the martingale differences

$$\left(\hat{m}_j^{(l)}(Z_1^l;Z_1^{l+j}) - m_j(Z_{-\infty}^{l+j})\right)$$
$$\cdot\left(m_j(Z_{-\infty}^{l+j}) - \max\left\{e^{-r\cdot(j+1)}\cdot f(100\cdot\prod_{r=l+1}^{l+j+1} Z_r), m_{j+1}(Z_{-\infty}^{l+j+1})\right\}\right)$$

(cf. (13)) are bounded by $4B^2$. □

A Appendix: Proofs of auxiliary results

A.1 Proof of Lemma 1

Set

$$f_j(x) = e^{-j\cdot r}\cdot f(x).$$

We prove (3) by induction. The assertion is trivial for $t = L$ because in this case we have

$$\tau_{L-1}^* = L \quad \text{and} \quad \tau = L$$

for any $\tau \in \mathcal{T}(L)$.

Let $t \in \{0, \dots, L-1\}$ and assume that

$$V_s(\dots, x_{-1}, x_0, \dots, x_s) = \mathbf{E}\left\{f_{\tau_{s-1}^*}(X_{\tau_{s-1}^*})|\dots, X_{-1} = x_{-1}, X_0 = x_0, \dots, X_s = x_s\right\}$$

holds for $s = t+1$. In the sequel we prove (3). To do this, let $\tau \in \mathcal{T}(t, \dots, L)$ be arbitrary. On $\{\tau > t\}$ we have $\tau = \max\{\tau, t+1\}$, hence

$$\begin{aligned} f_\tau(X_\tau) &= f_\tau(X_\tau)\cdot 1_{\{\tau=t\}} + f_\tau(X_\tau)\cdot 1_{\{\tau>t\}} \\ &= f_t(X_t)\cdot 1_{\{\tau=t\}} + f_{\max\{\tau,t+1\}}(X_{\max\{\tau,t+1\}})\cdot 1_{\{\tau>t\}}. \end{aligned}$$

Since $1_{\{\tau=t\}}$ and $1_{\{\tau>t\}} = 1 - 1_{\{\tau\le t\}}$ are measurable with respect to $\dots, X_{-1}, X_0, \dots, X_t$ we have

$$\begin{aligned}
&\mathbf{E}\{f_\tau(X_\tau)|\dots, X_{-1}, X_0, \dots, X_t\}\\
&= f_t(X_t)\cdot 1_{\{\tau=t\}} + 1_{\{\tau>t\}}\cdot \mathbf{E}\{f_{\max\{\tau,t+1\}}(X_{\max\{\tau,t+1\}})|\dots, X_{-1}, X_0, \dots, X_t\}.
\end{aligned}$$

Using the definition of V_{t+1} together with $\max\{\tau, t+1\} \in \mathcal{T}(t+1, \dots, L)$ we get

$$\begin{aligned}
&\mathbf{E}\{f_{\max\{\tau,t+1\}}(X_{\max\{\tau,t+1\}})|\dots, X_{-1}, X_0, \dots, X_t\}\\
&= \mathbf{E}\{\mathbf{E}\{f_{\max\{\tau,t+1\}}(X_{\max\{\tau,t+1\}})|\dots, X_{-1}, X_0, \dots, X_{t+1}\}|\dots, X_{-1}, X_0, \dots, X_t\}\\
&\le \mathbf{E}\{V_{t+1}(\dots, X_{-1}, X_0, \dots, X_{t+1})|\dots, X_{-1}, X_0, \dots, X_t\},
\end{aligned}$$

from which we can conclude

$$\begin{aligned}
&\mathbf{E}\{f_\tau(X_\tau)|\dots, X_{-1}, X_0, \dots, X_t\}\\
&\le f_t(X_t)\cdot 1_{\{\tau=t\}} + 1_{\{\tau>t\}}\cdot \mathbf{E}\{V_{t+1}(\dots, X_{-1}, X_0, \dots, X_{t+1})|\dots, X_{-1}, X_0, \dots, X_t\}\\
&\le \max\{f_t(X_t), \mathbf{E}\{V_{t+1}(\dots, X_{-1}, X_0, \dots, X_{t+1})|\dots, X_{-1}, X_0, \dots, X_t\}\}. \qquad (24)
\end{aligned}$$

Now we make the same calculations using $\tau = \tau^*_{t-1}$. We get

$$\begin{aligned}
&\mathbf{E}\{f_{\tau^*_{t-1}}(X_{\tau^*_{t-1}})|\dots, X_{-1}, X_0, \dots, X_t\}\\
&= f_t(X_t)\cdot 1_{\{\tau^*_{t-1}=t\}} + 1_{\{\tau^*_{t-1}>t\}}\cdot \mathbf{E}\{f_{\max\{\tau^*_{t-1},t+1\}}(X_{\max\{\tau^*_{t-1},t+1\}})|\dots, X_{-1}, X_0, \dots, X_t\}.
\end{aligned}$$

By definition of τ^*_t we have on $\{\tau^*_{t-1} > t\}$

$$\max\{\tau^*_{t-1}, t+1\} = \tau^*_t.$$

Using this and the induction hypothesis we can conclude

$$\begin{aligned}
&\mathbf{E}\{f_{\tau^*_{t-1}}(X_{\tau^*_{t-1}})|\dots, X_{-1}, X_0, \dots, X_t\}\\
&= f_t(X_t)\cdot 1_{\{\tau^*_{t-1}=t\}}\\
&\qquad +1_{\{\tau^*_{t-1}>t\}}\cdot \mathbf{E}\{\mathbf{E}\{f_{\tau^*_t}(X_{\tau^*_t})|\dots, X_{-1}, X_0, \dots, X_{t+1}\}|\dots, X_{-1}, X_0, \dots, X_t\}\\
&= f_t(X_t)\cdot 1_{\{\tau^*_{t-1}=t\}}\\
&\qquad +1_{\{\tau^*_{t-1}>t\}}\cdot \mathbf{E}\{V_{t+1}(\dots, X_{-1}, X_0, \dots, X_{t+1})|\dots, X_{-1}, X_0, \dots, X_t\}. \qquad (25)
\end{aligned}$$

Next we show

$$\mathbf{E}\{V_{t+1}(\ldots,X_{-1},X_0,\ldots,X_{t+1})|\ldots,X_{-1},X_0,\ldots,X_t\} = q_t(\ldots,X_{-1},X_0,\ldots,X_t). \quad (26)$$

To see this, we observe that by induction hypothesis and because of $\tau_t^* \in \mathcal{T}(t+1,\ldots,L)$ we have

$$\begin{aligned}
&\mathbf{E}\{V_{t+1}(\ldots,X_{-1},X_0,\ldots,X_{t+1})|\ldots,X_{-1},X_0,\ldots,X_t\} \\
&= \mathbf{E}\{\mathbf{E}\{f_{\tau_t^*}(X_{\tau_t^*})|\ldots,X_{-1},X_0,\ldots,X_{t+1}\}|\ldots,X_{-1},X_0,\ldots,X_t\} \\
&= \mathbf{E}\{f_{\tau_t^*}(X_{\tau_t^*})|\ldots,X_{-1},X_0,\ldots,X_t\} \\
&\leq \sup_{\tau\in\mathcal{T}(t+1,\ldots,L)} \mathbf{E}\left\{f_\tau(X_\tau)|\ldots,X_{-1},X_0,\ldots,X_t\right\} \\
&= q_t(\ldots,X_{-1},X_0,\ldots,X_t).
\end{aligned}$$

Furthermore the definition of V_{t+1} implies

$$\begin{aligned}
&\mathbf{E}\{V_{t+1}(\ldots,X_{-1},X_0,\ldots,X_{t+1})|\ldots,X_{-1},X_0,\ldots,X_t\} \\
&= \mathbf{E}\left\{\sup_{\tau\in\mathcal{T}(t+1,\ldots,L)} \mathbf{E}\left\{f_\tau(X_\tau)|\ldots,X_{-1},X_0,\ldots,X_{t+1}\right\}|\ldots,X_{-1},X_0,\ldots,X_t\right\} \\
&\geq \sup_{\tau\in\mathcal{T}(t+1,\ldots,L)} \mathbf{E}\left\{\mathbf{E}\left\{f_\tau(X_\tau)|\ldots,X_{-1},X_0,\ldots,X_{t+1}\right\}|\ldots,X_{-1},X_0,\ldots,X_t\right\} \\
&= q_t(\ldots,X_{-1},X_0,\ldots,X_t),
\end{aligned}$$

which concludes the proof of (26). Using the definition of τ_{t-1}^* we get

$$\begin{aligned}
&f_t(X_t)\cdot 1_{\{\tau_{t-1}^*=t\}} + 1_{\{\tau_{t-1}^*>t\}}\cdot\mathbf{E}\{V_{t+1}(\ldots,X_{-1},X_0,\ldots,X_{t+1})|\ldots,X_{-1},X_0,\ldots,X_t\} \\
&= f_t(X_t)\cdot 1_{\{\tau_{t-1}^*=t\}} + 1_{\{\tau_{t-1}^*>t\}}\cdot q_t(\ldots,X_{-1},X_0,\ldots,X_t) \\
&= \max\{f_t(X_t), q_t(\ldots,X_{-1},X_0,\ldots,X_t)\}. \qquad (27)
\end{aligned}$$

Summarizing the above results we have

$$\begin{aligned}
&V_t(\ldots,x_{-1},x_0,\ldots,x_t) \\
&:= \sup_{\tau\in\mathcal{T}(t,t+1,\ldots,L)} \mathbf{E}\left\{f_\tau(X_\tau)|\ldots,X_{-1}=x_{-1},X_0=x_0,\ldots,X_t=x_t\right\} \\
&\overset{(24)}{\leq} \max\{f_t(x_t),\mathbf{E}\{V_{t+1}(\ldots,X_{-1},X_0,\ldots,X_{t+1})|\ldots,X_{-1}=x_{-1},X_0=x_0,\ldots,X_t=x_t\}\} \\
&\overset{(26)}{=} \max\{f_t(x_t), q_t(\ldots,x_{-1},x_0,\ldots,x_t)\} \\
&\overset{(25),(27)}{=} \mathbf{E}\{f_{\tau_{t-1}^*}(X_{\tau_{t-1}^*})|\ldots,X_{-1}=x_{-1},X_0=x_0,\ldots,X_t=x_t\},
\end{aligned}$$

from which we conclude

$$
\begin{aligned}
&V_t(\ldots, x_{-1}, x_0, \ldots, x_t) \\
&= \max\{f_t(x_t), q_t(\ldots, x_{-1}, x_0, \ldots, x_t)\} \\
&= \mathbf{E}\{f_{\tau^*_{t-1}}(X_{\tau^*_{t-1}}) | \ldots, X_{-1} = x_{-1}, X_0 = x_0, \ldots, X_t = x_t\}, \qquad (28)
\end{aligned}
$$

which completes the proof of (3). In order to prove (4) we observe that by arguing as above we get

$$
\begin{aligned}
V_0 &:= \sup_{\tau \in \mathcal{T}(0,\ldots,L)} \mathbf{E}\left\{f_\tau(X_\tau)\right\} \\
&= \sup_{\tau \in \mathcal{T}(0,\ldots,L)} \mathbf{E}\left\{f_0(X_0) \cdot 1_{\{\tau=0\}} + f_{\max\{\tau,1\}}(X_{\max\{\tau,1\}}) \cdot 1_{\{\tau>0\}}\right\} \\
&= \mathbf{E}\left\{f_0(X_0) \cdot 1_{\{f_0(X_0) \geq q_0(\ldots,X_{-1},X_0)\}} + f_{\tau^*_0}(X_{\tau^*_0}) \cdot 1_{\{f_0(X_0) < q_0(\ldots,X_{-1},X_0)\}}\right\} \\
&= \mathbf{E}\Bigg\{ f_0(X_0) \cdot 1_{\{f_0(X_0) \geq q_0(\ldots,X_{-1},X_0)\}} \\
&\qquad + \mathbf{E}\{V_1(\ldots, X_{-1}, X_0, X_1) | \ldots, X_{-1}, X_0\} \cdot 1_{\{f_0(X_0) < q_0(\ldots,X_{-1},X_0)\}} \Bigg\} \\
&\overset{(26)}{=} \mathbf{E}\left\{f_0(X_0) \cdot 1_{\{f_0(X_0) \geq q_0(\ldots,X_{-1},X_0)\}} + q_0(\ldots, X_{-1}, X_0) \cdot 1_{\{f_0(X_0) < q_0(\ldots,X_{-1},X_0)\}}\right\} \\
&= \mathbf{E}\left\{\max\{f_0(X_0), q_0(\ldots, X_{-1}, X_0)\}\right\} \\
&\overset{(28)}{=} \mathbf{E}\left\{f_{\tau^*_{-1}}(X_{\tau^*_{-1}})\right\},
\end{aligned}
$$

which implies (4). □

A.2 Proof of Lemma 2

Set

$$
f_j(x) = e^{-j \cdot r} \cdot f(x).
$$

(5) is implied by (26) and (28). In order to prove (6) we observe that we have by (26) and Lemma 1

$$
\begin{aligned}
&q_t(\ldots, X_{-1}, X_0, \ldots, X_t) \\
&= \mathbf{E}\{V_{t+1}(\ldots, X_{-1}, X_0, \ldots, X_{t+1}) | \ldots, X_{-1}, X_0, \ldots, X_t\} \\
&= \mathbf{E}\{\mathbf{E}\{f_{\tau^*_t}(X_{\tau^*_t}) | \ldots, X_{-1}, X_0, \ldots, X_{t+1}\} | \ldots, X_{-1}, X_0, \ldots, X_t\} \\
&= \mathbf{E}\{f_{\tau^*_t}(X_{\tau^*_t}) | \ldots, X_{-1}, X_0, \ldots, X_t\}.
\end{aligned}
$$

□

A.3 Proof of Lemma 3.

Let

$$f_j(x) = e^{-j \cdot r} \cdot f(x).$$

Set

$$\hat{\tau}_t = \inf\{s \geq t+1 \, : \, \hat{q}_s(\dots, X_{-1}, X_0, \dots, X_s) \leq f_s(X_s)\},$$

and let $\mathcal{F}_t$ be the σ-algebra generated by $\dots, X_{-1}, X_0, \dots, X_t$. In the sequel we prove

$$\mathbf{E}\left\{ f_{\tau^*_{t-1}}(X_{\tau^*_{t-1}}) - f_{\hat{\tau}_{t-1}}(X_{\hat{\tau}_{t-1}}) | \mathcal{F}_{t-1} \right\}$$
$$\leq \sum_{k=t}^{L-1} \mathbf{E}\left\{ |\hat{q}_k(\dots, X_{-1}, X_0, \dots, X_k) - q_k(\dots, X_{-1}, X_0, \dots, X_k)| | \mathcal{F}_{t-1} \right\} \qquad (29)$$

for $t \in \{0, \dots, L\}$, from which we get the assertion of Lemma 3 by setting $t = 0$.

We prove (29) by induction. The assertion is trivial for $t = L$ (since $\tau^*_{L-1} = L = \hat{\tau}_{L-1}$). Assume that (29) holds for $t \in \{s+1, \dots, L\}$ for some $s \in \{0, 1, \dots, L-1\}$. In the sequel we prove that in this case it also holds for $t = s$. To do this, we use

$$\mathbf{E}\left\{ f_{\tau^*_{t-1}}(X_{\tau^*_{t-1}}) - f_{\hat{\tau}_{t-1}}(X_{\hat{\tau}_{t-1}}) | \mathcal{F}_{t-1} \right\}$$
$$= \sum_{k=t}^{L-1} \mathbf{E}\left\{ (f_{\tau^*_{t-1}}(X_{\tau^*_{t-1}}) - f_{\hat{\tau}_{t-1}}(X_{\hat{\tau}_{t-1}})) \cdot 1_{\{\hat{\tau}_{t-1}=k, \tau^*_{t-1}>k\}} | \mathcal{F}_{t-1} \right\}$$
$$+ \sum_{k=t}^{L-1} \mathbf{E}\left\{ (f_{\tau^*_{t-1}}(X_{\tau^*_{t-1}}) - f_{\hat{\tau}_{t-1}}(X_{\hat{\tau}_{t-1}})) \cdot 1_{\{\hat{\tau}_{t-1}>k, \tau^*_{t-1}=k\}} | \mathcal{F}_{t-1} \right\}$$
$$= \sum_{k=t}^{L-1} \mathbf{E}\left\{ (f_{\tau^*_k}(X_{\tau^*_k}) - f_k(X_k)) \cdot 1_{\{\hat{\tau}_{t-1}=k, \tau^*_{t-1}>k\}} | \mathcal{F}_{t-1} \right\}$$
$$+ \sum_{k=t}^{L-1} \mathbf{E}\left\{ (f_k(X_k) - q_k(\dots, X_{-1}, X_0, \dots, X_k)) \cdot 1_{\{\hat{\tau}_{t-1}>k, \tau^*_{t-1}=k\}} | \mathcal{F}_{t-1} \right\}$$
$$+ \sum_{k=t}^{L-1} \mathbf{E}\left\{ (q_k(\dots, X_{-1}, X_0, \dots, X_k) - f_{\hat{\tau}_k}(X_{\hat{\tau}_k})) \cdot 1_{\{\hat{\tau}_{t-1}>k, \tau^*_{t-1}=k\}} | \mathcal{F}_{t-1} \right\}$$
$$= T_1 + T_2 + T_3,$$

where we have used that $\hat{\tau}_{t-1} = \hat{\tau}_k$ on $\{\hat{\tau}_{t-1} > k\}$ and that $\tau^*_{t-1} = \tau^*_k$ on $\{\tau^*_{t-1} > k\}$. The random variables

$$1_{\{\hat{\tau}_{t-1}=k, \tau^*_{t-1}>k\}} \quad \text{and} \quad 1_{\{\hat{\tau}_{t-1}>k, \tau^*_{t-1}=k\}}$$

are $\mathcal{F}_k$-measurable, hence we get by Lemma 2

$$\begin{aligned}
&T_1\\
&= \sum_{k=t}^{L-1} \mathbf{E}\left\{ (\mathbf{E}\{f_{\tau_k^*}(X_{\tau_k^*}))|\mathcal{F}_k\} - f_k(X_k)) \cdot 1_{\{\hat{\tau}_{t-1}=k,\tau_{t-1}^*>k\}}|\mathcal{F}_{t-1}\right\}\\
&= \sum_{k=t}^{L-1} \mathbf{E}\left\{ (q_k(\ldots,X_{-1},X_0,\ldots,X_k) - f_k(X_k)) \cdot 1_{\{\hat{\tau}_{t-1}=k,\tau_{t-1}^*>k\}}|\mathcal{F}_{t-1}\right\}\\
&\leq \sum_{k=t}^{L-1} \mathbf{E}\left\{ (q_k(\ldots,X_{-1},X_0,\ldots,X_k) - \hat{q}_k(\ldots,X_{-1},X_0,\ldots,X_k)) \cdot 1_{\{\hat{\tau}_{t-1}=k,\tau_{t-1}^*>k\}}|\mathcal{F}_{t-1}\right\},
\end{aligned}$$

since $\hat{\tau}_{t-1} = k$ implies

$$f_k(X_k) \geq \hat{q}_k(\ldots,X_{-1},X_0,\ldots,X_k).$$

Similarly, $\hat{\tau}_{t-1} > k$ implies

$$f_k(X_k) < \hat{q}_k(\ldots,X_{-1},X_0,\ldots,X_k),$$

from which we can conclude

$$\begin{aligned}
T_2 \leq \sum_{k=t}^{L-1} \mathbf{E}\Bigg\{ &(\hat{q}_k(\ldots,X_{-1},X_0,\ldots,X_k)\\
&\qquad -q_k(\ldots,X_{-1},X_0,\ldots,X_k)) \cdot 1_{\{\hat{\tau}_{t-1}>k,\tau_{t-1}^*=k\}}|\mathcal{F}_{t-1}\Bigg\}.
\end{aligned}$$

Finally we have by Lemma 2

$$\begin{aligned}
T_3 &= \sum_{k=t}^{L-1} \mathbf{E}\left\{ \mathbf{E}\left\{q_k(\ldots,X_{-1},X_0,\ldots,X_k) - f_{\hat{\tau}_k}(X_{\hat{\tau}_k})|\mathcal{F}_k\right\} \cdot 1_{\{\hat{\tau}_{t-1}>k,\tau_{t-1}^*=k\}}|\mathcal{F}_{t-1}\right\}\\
&= \sum_{k=t}^{L-1} \mathbf{E}\left\{ \mathbf{E}\left\{ f_{\tau_k^*}(X_{\tau_k^*}) - f_{\hat{\tau}_k}(X_{\hat{\tau}_k})|\mathcal{F}_k\right\} \cdot 1_{\{\hat{\tau}_{t-1}>k,\tau_{t-1}^*=k\}}|\mathcal{F}_{t-1}\right\},
\end{aligned}$$

and by using the induction hypothesis we get

$$\begin{aligned}
T_3 &\leq \sum_{k=t}^{L-1} \mathbf{E}\Bigg\{ \sum_{j=k+1}^{L-1} \mathbf{E}\Big\{|\hat{q}_j(\ldots,X_{-1},X_0,\ldots,X_j) - q_j(\ldots,X_{-1},X_0,\ldots,X_j)||\mathcal{F}_k\Big\}\\
&\qquad \cdot 1_{\{\hat{\tau}_{t-1}>k,\tau_{t-1}^*=k\}}|\mathcal{F}_{t-1}\Bigg\}\\
&= \sum_{k=t}^{L-1} \mathbf{E}\Bigg\{ \sum_{j=k+1}^{L-1} |\hat{q}_j(\ldots,X_{-1},X_0,\ldots,X_j) - q_j(\ldots,X_{-1},X_0,\ldots,X_j)|
\end{aligned}$$

$$\cdot 1_{\{\hat{\tau}_{t-1}>k,\tau^*_{t-1}=k\}}|\mathcal{F}_{t-1}\Bigg\}$$

$$= \sum_{j=t+1}^{L-1} \mathbf{E}\Bigg\{|\hat{q}_j(\ldots,X_{-1},X_0,\ldots,X_j) - q_j(\ldots,X_{-1},X_0,\ldots,X_j)| \cdot \sum_{k=t}^{j-1} 1_{\{\hat{\tau}_{t-1}>k,\tau^*_{t-1}=k\}}|\mathcal{F}_{t-1}\Bigg\}$$

$$= \sum_{k=t+1}^{L-1} \mathbf{E}\Bigg\{|\hat{q}_k(\ldots,X_{-1},X_0,\ldots,X_k) - q_k(\ldots,X_{-1},X_0,\ldots,X_k)| \cdot \sum_{j=t}^{k-1} 1_{\{\hat{\tau}_{t-1}>j,\tau^*_{t-1}=j\}}|\mathcal{F}_{t-1}\Bigg\}.$$

Summarizing the above results, we get the assertion. □

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