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Preprint 2010/004

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ISSN **1613-8309**

Abstract: In Differential Geometry an immersed smooth surface in 3-space is called *tight* if it has the minimum total absolute curvature among all immersed surfaces with the same topology. Tight surfaces without boundary were extensively studied, and tight surfaces with at least two boundary components are known to exist. An announcement of J.H.White in 1974 [19] stated that there are no smooth tight orientable surfaces of genus at least one with exactly one boundary component. Here we disprove this statement by a family of counterexamples, starting with a smooth tight torus with one hole. The non-orientable case is also studied. We show that there is no smooth tight Möbius band in 3-space and that there are smooth tight non-orientable surfaces of higher genus with exactly one boundary component. Three non-orientable cases of low genus remain open.

Introduction and Result

A smooth immersion $M \to \mathbb{E}^3$ of a compact and connected surface with or without boundary is called *tight* if the total absolute curvature attains its minimum. More precisely this means that it satisfies equality in the inequality

$$\int_{M \setminus \partial M} |K| do + \int_{\partial M} |\kappa| ds \geq 2\pi \sum_i \beta_i(M) \tag{1}$$

where K denotes the Gaussian curvature, $|\kappa|$ denotes the curvature of the boundary curve regarded as a space curve, and $\beta_i(M)$ denotes the rank of the homology $H_i(M;\mathbb{F})$ with coefficients in a field $\mathbb{F}$ (for nonorientable surfaces we take $\mathbb{F} = \mathbb{Z}_2$). The boundary ∂M is assumed to be smooth. The left hand side of the inequality is called the *total absolute curvature* of M. Inequality 1 holds for any compact C^2-surface in 3-space. This follows from the Morse inequalities for the linear height functions restricted to M [3], [21]. If $\partial M = \emptyset$ then tightness is equivalent to the equality

$$\int_M |K| do = 2\pi\big(4 - \chi(M)\big). \tag{2}$$

Compare the elementary discussion in the textbook [9, 4G]. If $\partial M \neq \emptyset$ then tightness is equivalent to the equality

$$\int_{M \setminus \partial M} |K| do + \int_{\partial M} |\kappa| ds = 2\pi\big(2 - \chi(M)\big). \tag{3}$$

In more generality a topological embedding $M \to \mathbb{E}^N$ of a compact manifold into euclidean space is called *tight*, if for any open half space $E_+ \subset \mathbb{E}^N$ the induced homomorphism

$$H_*(M \cap E_+;\mathbb{F}) \longrightarrow H_*(M;\mathbb{F})$$

is injective for some field $\mathbb{F}$.

For compact 2-manifolds without boundary (smooth or not, but always connected) tightness is equivalent to the two-piece property (TPP) which states that the intersection of M with any (open or closed) halfspace is connected. This is different in the case of $\partial M \neq \emptyset$. Smooth tight surfaces were investigated by N.H.Kuiper [11], [12] and others, compare [20], [21] [3], [18] for a survey. The study of tight polyhedral surfaces was initiated by T.F.Banchoff, for a survey see [1]. One of the results states that any given compact surface (with or without boundary) admits a tight polyhedral embedding into some euclidean space [8, 2C]. In particular any compact surface with boundary admits a tight polyhedral embedding into 3-space [8, 2.24]: One can start with polyhedral examples of a tight Möbius band, a tight torus with a hole, and a tight Klein bottle with a hole. By attaching polyhedral handles tightly and by cutting out convex holes one can obtain any compact surface with boundary.

For compact surfaces with r boundary components it is easy to construct smooth tight embeddings for genus zero and any r and for an orientable surface of genus $g \geq 1$ and any $r \geq 2$. Compare Proposition 2 below. The case of smooth tight surfaces with $g \geq 1$ and $r = 1$ still seems to be open. There are some related results in [6] but not on this case of a single boundary component. It was announced by J.H.White [19] that no such smooth immersion exists but the proof was not completely given: The proof of Lemma 3 in White's paper was missing. In fact, our key example below, depicted in Figure 2, shows that this lemma does not hold. It seems that within the 35 subsequent years neither another proof nor any further discussion of the problem appeared in the literature. Therefore, it was called *White's conjecture* in [1, 1.5.6].

What makes the situation more delicate is that there are well known polyhedral examples of this kind. A tight polyhedral torus with one hole is depicted in [1, Fig.7] and in [8, Fig.5]. The boundary is a non-convex polygon with 8 vertices which is affinely equivalent with a Hamiltonian cycle in the edge graph of an ordinary cube. Another tight polyhedral torus with one hole is obtained from Császár's torus [1, Fig.1] by cutting out the open star of one of the vertices in the interior of the convex hull. In this case the boundary is a non-convex hexagon. One can ask whether any of these examples (possibly with modifications) can be smoothed out tightly. No smooth tight Möbius band exists [13]. For nonorientable surfaces of higher genus with one boundary component existence or non-existence still seems to be open in the smooth case. Most of the cases will be decided by our main theorem below. In particular we disprove White's conjecture for smooth surfaces of class C^∞, in contradiction with the announcement in [19]. However, the conjecture holds for surfaces with an analytic boundary curve by Corollary 5.

Main Theorem

1. *There is a smooth tight immersion of any compact orientable surface with exactly one boundary component into* 3*-space (not contained in a plane) except for the disc. The boundary can be chosen as a closed geodesic.*
2. (folklore [1, 1.5.4]) *Any tightly immersed* 2*-disc is planar and convex, and the immersion is an embedding in this case.*
3. (N.H.Kuiper [13]) *There is no smooth tight Möbius band in any euclidean space.*
4. *There is a smooth tight immersion of any compact non-orientable surface M with exactly one boundary component into* 3*-space provided that $\chi(M) = -3$ or $\chi(M) \leq -5$.*

The key example for the proof of Part 1 of the theorem is the construction of a smooth tight torus with one disc removed, see Figure 2. In Part 4 the non-orientable cases of $\chi(M) = -1$ (Klein bottle with one hole), $\chi(M) = -2$ (Möbius band with one handle) and $\chi(M) = -4$ (Möbius band with two handles) remain open. A polyhedral tight Klein bottle with one hole exists [16], [8, Fig.5]. It is quite possible that the case $\chi = -2$ is as subtle as the case $\chi = -1$ for closed surfaces, possibly with a similar non-smoothability result. Here a tight polyhedral closed surface was obtained in [4] which cannot be smoothed tightly by the results in [7]. Furthermore, there is no smooth tight immersion of any surface with boundary which is substantial in d-space for $d \geq 4$ by [17, Thm.3].

Basic Results and Examples

As a prerequisite for the proof of the main theorem we first discuss the following lemma which is - more or less - well known. Certain parts of Lemma 1 can be found in [17] and [19]. Other parts are folklore results.

Lemma 1 *Let $M \to \mathbb{E}^3$ be a smooth tight immersion of a compact surface with nonempty boundary such that the image is not contained in any plane. Then the following hold:*

1. *$K \leq 0$ everywhere*
2. *$\mathcal{H}(M) = \mathcal{H}(\partial M)$ where $\mathcal{H}$ denotes the convex hull. It follows that if there is exactly one boundary component then the boundary curve is not planar.*
3. *$K(p) = 0$ for any point $p \in (M \setminus \partial M) \cap \partial\mathcal{H}M$*

4. $\int_{\partial M}(|\kappa| + \kappa_g)ds = 4\pi$ where κ_g denotes the geodesic curvature of the boundary for an appropriate orientation (a neighbourhood of ∂M is always orientable).

5. The ε-tube M_ε of M is a tight immersion of a closed orientable surface $\overline{M}$ with Euler characteristic $\chi(\overline{M}) = 2\chi(M)$ which is of class C^1 and which is smooth on an open and dense subset.

6. There is a unique smooth Darboux frame $c', \nu, N = c' \times \nu$ along the boundary curve $c(s)$, parametrised by arc length, where ν denotes the inner normal of the curve (tangent to M). Moreover, c'' is a nonpositive multiple of ν at every boundary point which lies in the interior of $\mathcal{H}M$. In particular, if $c'' \neq 0$ at such a point then the osculating plane of the boundary curve coincides with the tangent plane of the surface. This holds also for nonorientable surfaces even though N does not extend to a global normal vector field on the surface. However, ν and N always exist as vector fields along the boundary.

PROOF. 1. Let $p \in M \setminus \partial M$ be a point with $K(p) > 0$. Then we can cut out a small disc around p with total curvature $\delta > 0$ and a convex planar boundary and obtain a surface M_* of the same genus with one more boundary component. For the total absolute curvature we have

$$\int_{M_*\setminus\partial M_*} |K| do + \int_{\partial M_*} |\kappa| ds = \int_{M\setminus\partial M} |K| do - \delta + \int_{\partial M} |\kappa| ds + 2\pi = 2\pi - \delta + 2\pi(1 + \beta_1(M)).$$

This contradicts Inequality 1 for the total absolute curvature of M_* since $\beta_1(M_*) = \beta_1(M) + 1$.

2. If the two sides do not coincide then the convex hull of M is larger than the convex hull of its boundary. This is only possible by a point in $(M \setminus \partial M) \cap \partial\mathcal{H}M$ with positive curvature. This contradicts Part 1. It is well known that (1) $\Rightarrow$ (2) holds for any compact surface with boundary.

3. Such a point $p \in M \setminus \partial M$ lies in the boundary of a convex body. Therefore we have $K \geq 0$. Together with Part 1 we obtain $K = 0$.

4. For one of the two orientations of the boundary we have the Gauss-Bonnet equation $\int_{M\setminus\partial M} K do + \int_{\partial M} \kappa_g ds = 2\pi(1 - \beta_1(M))$. Together with Equation 3 which by Part 1 takes the form

$$-\int_{M\setminus\partial M} K do + \int_{\partial M} |\kappa| ds = 2\pi(1 + \beta_1(M))$$

this implies $\int_{\partial M}(|\kappa| + \kappa_g)ds = 4\pi$.

5. For sufficiently small ε the ε-tube M_ε around a compact smooth immersed manifold M without boundary is an immersion of the unit normal bundle $\perp(f)$ by $f_\varepsilon(p, e) = f(p) + \varepsilon e$ if f denotes the immersion of M. In the case of a non-empty boundary we have the same formula for the modified unit normal bundle $\perp_+(f)$ where on the boundary only those normals are considered which point away from the surface, that is, which have a nonpositive inner product with the inner normal ν which is tangent to M and normal to ∂M. By construction f_ε is of class C^1 everywhere and smooth for $p \in M \setminus \partial M$ and for all (p, e) with $p \in \partial M$ such that e has a negative inner product with ν. By construction the total absolute curvature of M_ε is the sum of $2\int_{M\setminus\partial M} |K| do$ and half of the total absolute curvature of the boundary ∂M defined on the unit normal bundle. So we have $\int_{M_\varepsilon} |K| do = 2\int_{M\setminus\partial M} |K| do + 2\int_{\partial M} |\kappa| ds$ which is twice the total absolute curvature of M. On the other hand it is well known that $\chi(M_\varepsilon) = 2\chi(M)$. By the tightness of M we have

$$\int_{M_\varepsilon} |K| do = 4\pi(1 + \beta_1(M)) = 2\pi(4 - 2\chi(M)) = 2\pi(4 - \chi(M_\varepsilon)).$$

Therefore M_ε is tight.

6. By the tightness of M_ε the positive curvature of M_ε is concentrated on the boundary of its convex hull. Let $p \in \partial M$ be a point in the interior of the convex hull of M. Then for sufficiently small ε the resulting points of M_ε are also in the interior of the convex hull of M_ε. Therefore there

is no positive curvature in the part of M_ε resulting from p. On the other hand the curvature is positive at (p, e) if and only if $\langle e, c''\rangle < 0$ because the Gaussian curvature of the ε-tube at $p + \varepsilon e$ is given [2, p.400] by the equation

$$K\big(p + \varepsilon e\big) = -\frac{\langle e, c''\rangle}{\varepsilon\big(1 - \varepsilon\langle e, c''\rangle\big)}. \tag{4}$$

Since any unit normal at p can be written as $e = \sin\theta \cdot N + \cos\theta \cdot (-\nu)$ for some $\theta \in [-\pi/2, \pi/2]$, the curvature is nonpositive for every such e if and only if $\langle c'', N\rangle = 0$ and $\langle c'', \nu\rangle \leq 0$. The assertion follows. In other words: At such a point we either have $c'' = 0$, or the osculating plane of the boundary coincides with the tangent plane of the surface. □

Corollary 1 *For any smooth immersion of a compact surface M with nonempty boundary the inequality $2\int_{K>0} K do + \int_{\partial M}(|\kappa| + \kappa_g)ds \geq 4\pi$ holds with equality if and only if the immersion is tight (which by Lemma 1 implies that the first integral vanishes).*

In particular, the two conditions $K \leq 0$ and $\int_{\partial M}(|\kappa| + \kappa_g)ds = 4\pi$ together imply the tightness of the immersed M.

This follows from Inequality 1 and the Gauss-Bonnet equation by the same argument as used in the proof of Part 4 of Lemma 1.

Lemma 2 (L.Rodríguez [17])
If M has the TPP then the following equality holds: $\int_{K>0} K do + \int_{\partial M}(|\kappa| + \kappa_g)ds = 4\pi$ Conversely, this equation implies the TPP provided that the boundary curves consist of pieces which are either planar or asymptotic.

Corollary 2 *For a compact surface M with non-empty boundary the following are equivalent:*

1. *M is tight.*
2. *M has the TPP and $K \leq 0$ everywhere.*
3. *M has the TPP and $\mathcal{H}(M) = \mathcal{H}(\partial M)$.*
4. *M_ε is tight.*
5. *M_ε has the TPP.*

Proof. (1) implies any of the other conditions by Lemma 1.
(2) ⇒ (3) holds by the proof of Part 2 in Lemma 1.
(3) ⇒ (1) follows by Morse theory: Every nondegenerate height function has exactly one minimum and no critical point of index 2. This implies equality in the Morse inequality, hence equality in Inequality 1.
(4) ⇒ (1) is well known by the same calculation as in Part 5 of Lemma 1.
(4) ⇔ (5) holds for closed surfaces in general. □

Corollary 3
Let M be a tight compact surface with non-empty boundary.

1. *If each boundary component is a closed geodesic in M then the number of components is either one or two. If there are two components then both are planar and convex. If there is exactly one component then $\int_{\partial M}|\kappa|ds = 4\pi$ (and ∂M is not planar by Lemma 1).*
2. *If there are exactly two geodesic boundary components then each of the other components satisfies $\kappa_g = -|\kappa|$. Hence each of them consists of pieces which are either planar and convex (but concave in the surface) or asymptotic.*

3. *If one cuts an additional hole into M then the resulting surface is tight if and only if the additional boundary component satisfies $\kappa_g = -|\kappa|$, which means that it consists of pieces which are either planar and concave or asymptotic.*

This follows from Part 4 of Lemma 1 in connection with $|\kappa| + \kappa_g \geq 0$ and with Fenchel's inequality

$$\int_c |\kappa| ds \geq 2\pi \tag{5}$$

which holds for any closed space curve c, with equality precisely for planar and convex curves.

Example (L.Rodríguez [16])

There is a smooth TPP torus with one hole satisfying $\int_{K>0} K do = 4\pi$. It can be constructed from a smooth tight torus of revolution by cutting out a hole from the "inner" part with $K \leq 0$ such that the boundary consists only of one closed asymptotic curve, see Figure 1. In this case we have $\kappa_g = -|\kappa|$ along the boundary curve, in accordance with Lemma 2. This example cannot be tight by Lemma 1. The meridian curve of the particular example depicted in Figure 1 is the so-called *bean curve*, a quartic plane curve defined by the equation $x^4 + x^2y^2 + y^4 = x(x^2 + y^2)$.

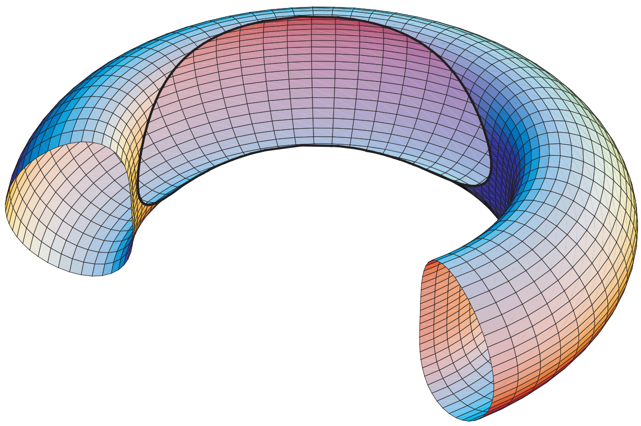

Figure 1: L.Rodríguez's TPP torus with one non-planar disc removed

Proposition 1 (following L.Rodríguez [17, Thm.1])

Any tight immersion of a sphere with $r \geq 1$ discs removed is an embedding. The image is either planar or it is contained in the boundary of a convex body such that each boundary component is planar and convex. Moreover the surface is locally a developable ruled surface (but not necessarily with a C^2-ruling). For $r \geq 3$ there is no such real analytic surface which is not contained in a plane.

Proof. Theorem 1 of [17] is the same statement for TPP immersions except that in this case the surface may contain parts with positive Gaussian curvature and except that it may be analytic. In our case we have $K = 0$ by Lemma 1. Moreover the surface is contained in the boundary of the convex hull of the convex boundary curves. A ruling is given by straight line intervals from a point of a boundary component to a point of another boundary component, defined by 1-dimensional intersections with supporting planes. If there are at least three boundary components then there is a supporting plane osculating the boundary at three or more points in distinct boundary components. Therefore the surface contains a planar part with inner points. Since it is not entirely planar it cannot be analytic. It seems that a generalised cylinder is the only analytic example with $r = 2$. □

Example From Proposition 1 we can derive a construction principle for such tight surfaces as follows: Start with a number $r \geq 2$ of planar convex curves in 3-space such that each one is contained in the boundary of the convex hull of the union of all of them. Then the boundary of this convex hull minus the interiors of all these curves is a tight C^1 surface. One can obtain examples which are C^2 or C^∞ by choosing the convex curves appropriately. If one wishes to prescribe a combinatorial structure on the surface then one can start with a convex 3-polytope (a compact set which is the convex hull of finitely many points in 3-space), at least for $r \geq 4$. Then one truncates each of the vertices by introducing a planar polygon nearby (representing the vertex figure). Furthermore in each of these polygons one constructs a smooth inscribed convex curve touching each of the edges of infinite order. Finally one defines the surface as the convex hull of their union (minus the interiors of the smooth curves). In this way one can construct surfaces with tetrahedral, octahedral or icosahedral symmetry (with at least 4, 6 or 12 boundary components, respectively). In addition one may add handles with nonpositive curvature between planar parts of the surface through the interior of the convex hull.

One can simplify this construction for higher genus as follows:

Proposition 2 (Tight surfaces with two or more boundary components)

1. *For any given genus $g \geq 1$ and any number $r \geq 2$ there is a smooth tight orientable surface of genus g with r boundary components in* 3*-space.*
2. *For any given $g \geq 2$ and any number $r \geq 2$ there is a smooth tight non-orientable surface with r boundary components and $\chi = 2 - 2g - r$ in* 3*-space.*

PROOF. There is a fairly simple construction as follows: We start with an orthogonal cylinder over a planar convex curve such that the curve contains certain straight line intervals. This provides a tightly embedded cylinder in 3-space. It corresponds to the case $g = 0$ and $r = 2$. Note that each of the two boundary components is a closed geodesic. In the planar parts one can easily attach handles with nonpositive curvature and, independently, cut out additional holes. These holes can be chosen as planar and convex (but concave in the oriented surface) in a planar region or they can consist of asymptotic curves according to Rodríguez's example above. This requires that the handles are modelled after the one in Rodríguez's example, up to affine transformations. The tightness of the resulting surface follows from Corollary 2 and Corollary 3 (or directly from the definition). This covers the orientable case. In the non-orientable case we start with an orientable tight surface of genus $g-1$, as described above. Then we add one non-orientable handle (producing a self-intersection) as in the standard example of a tight Klein bottle with one handle [14, Fig.12]. □

The case of a closed surface with an odd Euler characteristic minus a certain number of discs removed is not covered by the proposition. Here it seems to be easier to start with Part 4 of our main theorem and to cut out additional holes in planar regions. Tight immersions (smooth or polyhedral) into $\mathbb{R}^2$ with $r \geq 2$ exist also, see [14, Thm.1.6].

Corollary 4
For any given genus $g \geq 1$ and any number $r \geq 2$ and any s with $2 \leq s \leq r \leq 2g + s$ there is a smooth tight orientable surface of genus g with r boundary components in 3*-space such that precisely s boundary components are planar and convex (or concave).*

PROOF. We use the same construction as in Proposition 2 with the following modification for obtaining $r - s \leq 2g$ non-planar boundary curves: From each handle we cut out one or two discs in the region $K \leq 0$ bounded by a closed asymptotic curve, as in Rodríguez's example above. Note that the example in Figure 1 admits two disjoint and congruent holes simultaneously. The original cylinder has two planar and convex boundary components. In addition we can cut out $s - 2$ planar and convex holes from some planar region. We do not claim that this bound $r - s \leq 2g$ is optimal. For the same statement in the case $0 \leq s \leq 1$ see Remark 1 at the end of the paper. □

Definition (introduced by N.H.Kuiper [11, p.11])

A *topset* is the intersection of $M \cap \partial\mathcal{H}M$ with a supporting plane of the convex hull of M. We talk about a k-*topset* if the affine span of the intersection of $\partial\mathcal{H}M$ with this supporting plane is k-dimensional.

The theory of tight surfaces in 3-space (smooth or not) depends very much on a discussion of the possible topsets. We repeat some some well known facts: Every topset of a tight surface in 3-space is also tight. It is either a point or a straight line segment or a 2-dimensional convex set, possibly with a finite number of convex holes [3, 7.17,7.18]. In the latter case we call it *essential*, and a 1-cycle in it is called a *top-cycle* if it is not homologous to zero. If a 2-topset T of a tight surface with boundary contains no points of $M \setminus \partial M$ then the entire boundary of T must be contained in ∂M. In particular in this case there is one planar boundary component. For a tight surface with one boundary component this is impossible unless it is a disc itself. Any 0-topset of a tight surface with boundary is just one point of ∂M by Part 1 of Lemma 1: Otherwise there would be points of positive Gaussian curvature nearby. Furthermore the boundary of the convex hull of any 2-topset T must be entirely contained in M, and the convex hull is spanned by points in the boundary, i.e., $\mathcal{H}T = \mathcal{H}(T \cap \partial M)$. This implies that $\partial\mathcal{H}T \setminus \partial M$ consists of straight line segments (or it is empty).

Examples: From the example after Proposition 1 we see that a 2-topset of a smooth tight surface with at least three boundary components can be a planar polygon together with its relative interior. Each of the four essential 2-topsets of our key example in Figure 2 is a square with two corners rounded off.

Lemma 3 (compare [17, Cor.7])
Let M be a tight surface with boundary in 3-space. Then any arc of ∂M which is contained in $\partial\mathcal{H}M$ is a union of subarcs which are planar.

Proof. Let $c(t)$ be such an arc in ∂M, parametrised by arc length. Then it can be decomposed into subarcs where either $c''(t) \equiv 0$ or $c''(t) \neq 0$ along any entire subarc. An arc with $c''(t) = 0$ is a straight line segment and thus planar. Now we consider an arc with $c''(t) \neq 0$. Let $B(t) = \frac{c'(t)\times c''(t)}{\|c''(t)\|}$ be the binormal vector. Assume first that $c''(t)$ is transverse to M along some subarc I, and consider the orthogonal projections $\tilde{B}(t) \neq 0$ of $B(t)$ to $T_{c(t)}M$. We can orient ∂M so that $\tilde{B}(t)$ is everywhere exterior to M at $c(t)$. Then, $p(t) = c(t) + \varepsilon B(t) \in M_\varepsilon$. In fact, it is interior to the surface $U = M_\varepsilon \cap (\partial M)_\varepsilon = \{x \in M_\varepsilon \mid d(x, \partial M) = \varepsilon\}$. By Equation 4 the Gaussian curvature of the ε-tube at a point $c(t) + \varepsilon e$ is given by $K\big(c(t) + \varepsilon e\big) = -\frac{\langle e, c''(t)\rangle}{\varepsilon\big(1-\varepsilon\langle e, c''(t)\rangle\big)}$. Hence $p(t) = c(t) + \varepsilon B(t)$ lies in the boundaries (relative to U) of both regions $U_- = \{x \in U \mid K(x) < 0\}$ and $U_+ = \{x \in U \mid K(x) > 0\}$. Let $H = \partial\mathcal{H}(M_\varepsilon)$ denote the boundary of the convex hull of M_ε, and let us consider the splitting $H = (H \cap M_\epsilon) \cup (H \setminus M_\varepsilon)$. By the previous remarks, $p(t) \in H \cap M_\varepsilon$, and $p(t)$ is not in the interior of $H \cap M_\varepsilon$ relative to M_ε. Then, $p(t)$ is not in the interior of $H \cap M_\varepsilon$ relative to H. Therefore $p(t)$ is in the closure (relative to H) of the region $H \setminus M_\varepsilon$. Since M_ε is tight and H is convex, $H \setminus M_\varepsilon$ is a finite union of planar convex sets (cf. [3, 7.18]). By continuity, the curve $c(t) + \varepsilon B(t)$ with $t \in I$ lies in some fixed plane which supports M_ε. Hence, $c(I)$ is at constant distance ε from this plane, and is thus planar. Assume now that $c''(t)$ is tangent to M along some arc I (i.e., $c(t)$ is an asymptotic curve). Assume $c''(t)$ is interior to M at some point $c(t)$, i.e., $c''(t) = |\kappa|\nu$. By Equation 4 the Gaussian curvature is positive for all $c(t) + \varepsilon e$ with $\langle e, c''\rangle < 0$. By the tightness of M_ε this part is contained in $\partial\mathcal{H}M_\varepsilon$. In particular it follows that $c(t) + \epsilon B(t), c(t) - \epsilon B(t) \in M_\epsilon \cap \partial\mathcal{H}(M_\epsilon)$. Hence, there are two parallel planes at distance 2ϵ which support $\partial\mathcal{H}(M_\epsilon)$. Therefore M is planar. We are left with the case of an asymptotic curve $c(t)$, $t \in I$, such that $c''(t)$ is everywhere exterior to M. For each $t \in I$, let P be a supporting plane of $\mathcal{H}(M)$ at $c(t)$. Since P supports ∂M, the tangent vector $c'(t)$ must be included in P. Since $c''(t)$ is a negative multiple of $\nu(t)$, the supporting plane P must contain $c''(t)$ (otherwise it would split M in two components). Hence, P coincides with the osculating plane of ∂M at $c(t)$. In particular, ∂M is locally on one side of its osculating plane. Therefore, $c(t)$ has vanishing torsion for all $t \in I$. $\square$

Corollary 5 *Let M be a smooth tight surface which is not contained in any plane. Assume further that the boundary of M consists of exactly one component, then the boundary curve is not analytic.*

Proof. Since M and its boundary ∂M both are smooth, the convex hull $\mathcal{H}M$ cannot have a vertex, that is, a point p whose neighbourhood looks like a proper cone with an isolated apex at p. We know that $\mathcal{H}M$ is the convex hull of points in ∂M. This implies that $\partial\mathcal{H}M$ must contain an arc of ∂M. Then Lemma 3 implies that this boundary component is a planar curve. In the case of exactly one analytic boundary component this contradicts Part 2 of Lemma 1. □

Corollary 6 *Let M be a smooth tight surface which is not contained in any plane. Assume further that ∂M consists of exactly one boundary component, and that this is contained in $\partial\mathcal{H}M$. Then the boundary curve ∂M is not a Frenet curve, i.e., it has inflection points.*

Proof. This follows from Lemma 3 by the same argument as used in Corollary 5: The boundary curve is locally planar but not globally planar. The torsion vanishes whenever the torsion is defined. This is impossible unless there are points with vanishing curvature. An example is part of the proof of our Main Theorem below. From Corollary 4 we see that in general the boundary of a tight surface M does not have to be contained in $\partial\mathcal{H}M$. However, it can be conjectured that this is indeed the case if ∂M is connected. Then one of the assumptions in Corollary 6 would be superfluous. □

Proof of the Main Theorem.

Part 1: Let M be orientable. The case of genus zero is discussed in Part 2. So we assume that the genus is at least 1, hence the surface cannot be contained in a 2-dimensional plane by [14, Thm.1.6]. Thus $\mathcal{H}M$ will have to be a 3-dimensional convex body with a nonempty interior. In this case ∂M cannot be planar by Part 2 of Lemma 1. The key example is a surface of genus one with a connected boundary. A polyhedral example is the following: Start with the polyhedral cylinder obtained from the boundary of an ordinary cube by removing two opposite squares. Then take two translational copies of it and define ∂M to be the union of these two, glued together along a common square, but after rotating one copy against the other by a right angle. Then the boundary is a connected polygon which appears as a Hamiltonian cycle in the edge graph of the convex hull. For a picture see [1, Fig.7].

This polyhedral surface cannot be smoothed tightly by the procedure described in [10]. However, there is a smooth analogue depicted in Figure 2. It is based on a building block of Scherk's minimal surface $(x, y) \mapsto (x, y, \log \frac{\cos x}{\cos y})$ which is defined on the open square $-\pi/2 < x, y < \pi/2$. The function $z \mapsto \arctan z$ leads to a compactified version

$$F(x, y) = (x, y, \arctan \log \tfrac{\cos x}{\cos y})$$

which is defined on the compact square $-\pi/2 \le x, y \le \pi/2$ minus the four corners. At the four corners of the square we insert four vertical straight line segments. Therefore the surface appears as the graph of a smooth function on the interior of the square with a continuous extension of the surface to the boundary. The following calculation shows that the Gaussian curvature is strictly negative over the interior of the square, and that at the boundary the Gaussian curvature tends to zero in such a way that the tangent planes tend to vertical planes. A sufficient condition for negative Gaussian curvature is that the tangent plane at each point p intersects the surface locally around p. For the graph of a function it is sufficient that the Hessian of this function is indefinite. If we define $z = f(x, y) = \log \cos x - \log \cos y$ and $h(z) = \arctan z$ then F is the graph of the function $h \circ f$. The determinant of the Hessian of f is $-(1 + \tan^2 x)(1 + \tan^2 y)$. Furthermore we have $d(h \circ f) = (h' \circ f) \cdot df$ and

$$\text{hess}(h \circ f) = (h'' \circ f) \cdot df \otimes df + (h' \circ f) \cdot \text{hess}(f).$$

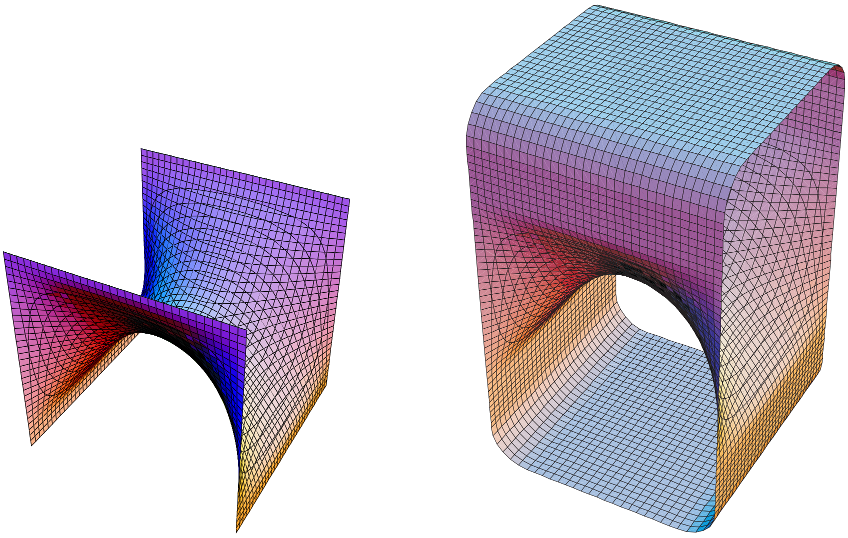

Figure 2: A tight torus with one disc removed (right), based on a compactified Scherk block (left)

It follows that the determinant of the Hessian of $h \circ f$ equals

$$-(1+z^2)^{-3}\Big((1+z)^2\tan^2 x + (1-z)^2\tan^2 y + (1+z^2)(1+\tan^2 x\tan^2 y)\Big)$$

which is strictly negative everywhere in the interior of the square. Therefore the Hessian is indefinite there, and we have $K \leq 0$ on the compactified Scherk block.

This block is inscribed inside a cube of side length π in 3-space, and the boundary forms a Hamiltonian cycle in the edge graph of the cube, as in the case of the polyhedral example mentioned above. It contains two horizontal straight line segments $z = \pm\pi/2$ on top and two others on bottom. We have to show that the surface is smooth even at the boundary and that the tangent plane tends smoothly to a vertical position. It is smooth at any point (x, y, z) with $|z| < \pi/2$ (including the vertical straight line segments) because it is the set of (x, y, z) satisfying the equation

$$F(x, y, z) = \cos y \cdot \mathrm{e}^{\tan z} - \cos x = 0.$$

The function F is smooth and regular with a non-vanishing gradient. The implicit function theorem implies that the block is smooth outside the horizontal straight line segments.

Next we consider a point p of the surface with $z = \pm\pi/2$. By symmetry we can assume $p = (\pi/2, y, -\pi/2)$ for some $y \in [-\pi/2, \pi/2]$. In a neighbourhood of p the surface is the graph over the (y, z)-plane of the function

$$x(y, z) = \arccos(\cos y \cdot \mathrm{e}^{\tan z})$$

where $\arccos\colon [-1, 1] \to [0, \pi]$. When z tends to $-\pi/2$, all partial derivatives (of any order) of $x(y, z)$ converge to 0. Indeed, by the chain rule it is enough to check this for

$$f(y, z) = \cos y \cdot \mathrm{e}^{\tan z}.$$

Note that

$$\left|\frac{\partial^{i+j}}{\partial y^i \partial z^j} f(y, z)\right| \leq \frac{\partial^j}{\partial z^j}\mathrm{e}^{\tan z} = \mathrm{e}^{\tan z} p_j(\tan z)$$

for a certain polynomial p_j. Thus we have

$$\lim_{z\to-\pi/2} \frac{\partial^{i+j}}{\partial y^i \partial z^j} f(y, z) = 0.$$

It follows that the block is smooth along the entire horizontal segments $z = \pm\pi/2$, and can be smoothly extended there by vertical halfplanes.

Therefore by attaching two cylindrical parts on top and bottom we obtain a smooth embedded torus with one disc removed, see Figure 2. The boundary is a (non-planar but locally planar) closed geodesic consisting of four vertical straight line segments and the four boundary components of the cylindrical parts. By construction this surface is isotopic (via tight surfaces) to the polyhedral tight torus with one hole mentioned above. Since Scherk's surface (and its compactified companion) admits an isometric symmetry by interchanging x, y and $z, -z$ we can arrange that our surface is invariant under the same symmetry. The equations

$$\int_{\partial M} |(\kappa| + \kappa_g) ds = 4\pi$$

$$\int_{M \setminus \partial M} |K| do + \int_{\partial M} |\kappa| ds = 2\pi(2 - \chi(M)) = 6\pi$$

are obviously satisfied since $K \leq 0$ everywhere (with the same total curvature as the Scherk block) and $\kappa_g \equiv 0$ along ∂M. The tightness follows from the first equation in connection with Corollary 1. It is quite obvious how to attach handles tightly along the cylindrical parts on top and bottom if these cylindrical parts are assumed to contain planar regions. This proves that any genus $g \geq 1$ can be realised by a smooth tight surface with one boundary component. Since this example contains relatively open parts with $K = 0$ but also parts with $K < 0$, it is clearly not analytic. The boundary curve is not analytic either. In fact it cannot be analytic by Corollary 5.

Part 2: Let M be a smooth tight immersion of a 2-disc into 3-space. From Equation 3 we see that $K = 0$ everywhere and $\int_{\partial M} |\kappa| = 2\pi$. Then Fenchel's inequality 5 implies that the boundary curve is planar and convex. From Part 2 of Lemma 1 we see that M lies in the same plane. Moreover, it coincides with the convex hull of the boundary curve. In this case no selfintersection can occur, so the surface is embedded. By a similar argument any tightly immersed 2-disc in d-space for $d \geq 4$ would have a tight boundary, that is, a planar and convex boundary, and again the disc would have to be contained in the convex hull of the boundary.

Part 3: Let M be a smooth tight Möbius band in 3-space. This cannot be contained in a 2-plane, so $\mathcal{H}M$ is a 3-dimensional convex body. The boundary of $\mathcal{H}M$ cannot be contained in M, therefore there are essential 2-topsets. If there were exactly one essential 2-topset then it would decompose M into two components on the same side of one plane. This contradicts the TPP. Therefore there are at least two essential 2-topsets. Moreover it follows that there are exactly two essential 2-topsets [13, Lemma 5]. We denote them by T_1 and T_2. Each of them is a homotopy circle by the tightness and $\mathrm{rk}H_1(M) = 1$. Let $C_1 = \partial\mathcal{H}T_1$ and $C_2 = \partial\mathcal{H}T_2$ denote the outer top-cycles in T_1 and T_2, respectively. It follows that the convex hull of M coincides with the convex hull of $C_1 \cup C_2$ and that $\partial\mathcal{H}M \setminus (\mathcal{H}C_1 \cup \mathcal{H}C_2)$ is a developable surface which is contained in M. Moreover $C_i \setminus \partial M$ consists of straight line segments, $i = 1, 2$. By the intersection form of the Möbius band the two essential 2-topsets must intersect. This intersection $T_1 \cap T_2 = C_1 \cap C_2$ is an interval I which possibly degenerates to just one point. It follows that the endpoints of I must be contained in ∂M because this holds in general for 1-topsets and 0-topsets. If this interval contains an inner point $p \in M \setminus \partial M$, then the position of the tangent plane at p leads to a contradiction since M is contained in the convex sector defined by the two intersecting support planes. If this interval $I = T_1 \cap T_2 = C_1 \cap C_2$ is entirely contained in ∂M then beyond an endpoint of I at most one of the curves C_1, C_2 can be contained in ∂M. The other one contains a straight line segment with points $q \in M \setminus \partial M$ in any neighbourhood of the endpoint of I. Considering the tangent plane at q leads to a contradiction since it has to contain one of the straight line segments between a point of C_1 and a point of C_2 in the developable surface $\partial\mathcal{H}M \setminus (\mathcal{H}C_1 \cup \mathcal{H}C_2)$. This straight line segment cannot continue smoothly beyond q because it has a nontrivial angle with the support plane spanned by the essential 2-topset containing q. It follows that there is no possibility left for a smooth tight Möbius band. Therefore it cannot exist. Note that there are several distinct types of nonsmooth tight Möbius bands in 3-space, and that a tight Möbius band in 4-space is essentially unique and polyhedral [13]. There is no polyhedral tight Möbius band which is substantial in d-space for $d \geq 5$ [8, 2.25], not even a topological tight embedding of the Möbius band [13, p.281].

Part 4: By attaching a cylindrical part to the smooth tight torus with one hole in Part 1 from top to bottom (with a selfintersection in between, where it meets the Scherk block) we obtain a smooth tight surface with $\chi(M) = -3$, which is a Klein bottle with one handle and one hole. This is similar to the standard construction of a tight Klein bottle with one handle from a tight torus [12], [14, Fig.12]. Similarly one can attach further handles. This covers the case of an odd Euler characteristic. For the case of an even Euler characteristic we go back to the construction described in [10]. We glue a block of type *projective plane with three holes* with nonpositive curvature together with our smooth key example with three additional planar and convex holes. The only requirement for this glueing procedure in [10] is that our starting surface has three planar pieces in general position for cutting out three additional holes which can be guaranteed by a slight modification of the example above. This leads to a surface with Euler characteristic $\chi = -6$. By attaching further handles we can cover all cases of $\chi \leq -5$ (even or odd). □

REMARKS: 1. From the construction in Part 1 of our main theorem it follows that Corollary 4 remains true under the weaker assumption $r \geq 1$ and $0 \leq s \leq r \leq 2g + s$ but not $r = s = 1$. In the cases of $s = 1$ and $s = 0$ we start with our key example and then add handles and cut holes as usual.

2. By a suitable projective transformation one can modify our key example in such a way, that one can draw the cone from a certain point in space to the entire boundary without creating self-intersections. By adding this cone one obtains a tight torus.

3. There are smooth tight immersions of compact surfaces with any number of boundary components into a flat 3-torus or other euclidean space forms. Starting with the examples in [15] one can cut convex holes into planar regions. In particular there is a flat and tight torus with any number of holes in a flat 3-torus.

4. Moreover, there are smooth tight immersions of noncompact surfaces with one boundary component. In this case one has to consider a proper immersion of a noncompact surface M of finite topological type into 3-space. In particular the image must be complete, and the ends go to infinity. Particular examples are a plane with a convex hole and a cylinder $S^1 \times [0, \infty)$, each with exactly one boundary component and with $K = 0$ in the interior. The total absolute curvature is 2π in these cases, thus realizing equality in the inequality $\int_{M \setminus \partial M} |K| do + \int_{\partial M} |\kappa| ds \geq 2\pi\big(\sum_i \beta_i(M) - 1\big)$ which replaces Inequality 1 in the case of one end [5, Thm.3.3]. Similarly, one can obtain orientable examples of higher genus by attaching standard handles to the cylinder in a version with planar pieces.

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