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Fachbereich Mathematik
Fakultät Mathematik und Physik
Universität Stuttgart
Pfaffenwaldring 57
D-70 569 Stuttgart

**E-Mail:** preprints@mathematik.uni-stuttgart.de
**WWW:** http://www.mathematik.uni-stuttgart.de/preprints

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# Non-Existence of Isomorphisms Between Certain Unitals

Theo Grundhöfer, Boris Krinn, Markus Stroppel

**Abstract**

We show that the Ree-Tits unitals are neither classical nor isomorphic to the polar unitals found in the Coulter-Matthews planes. To this end, we determine the full automorphism groups of the (finite) Ree unitals.

## Introduction

A finite incidence structure $(P, L)$ is called a *unital* of order $q$ if any two points are joined by a unique block in $L$, each block in $L$ has exactly $q + 1$ points, and $|P| = q^3 + 1$; this implies that each point is on exactly $q^2$ blocks. Unitals of order $q$ occur with $P$ the set of absolute points of a suitable polarity of a projective plane of order $q^2$; the blocks are the traces of secants. Classical examples are defined by a hermitian form of Witt index $1$ on a three-dimensional vector space over a commutative field. However, unitals are also studied in their own right, without any embedding into a projective plane (and, a fortiori, no connection with a polarity). Notably, no embedding of any of the finite Ree unitals $\mathrm{RTU}(q)$ into a projective plane of order $q^2$ is known. In fact, there are several results that exclude an embedding of $\mathrm{RTU}(q)$ into a finite projective plane such that the action of the group $\mathrm{R}(q)$ extends; see [10], [14], [15]. Note also that examples are known of unitals of order $q$ that do not admit any embeddings into projective planes of order $q^2$, see [1, A.2].

Very recently, a new class of finite unitals has been discovered in [12]. These come from polarities of the Coulter–Matthews planes, the order coincides with the order of a Ree unital. It has already been proved in [12, 6.8] that this new series contains many non-classical unitals. In the present paper, we also show that they are not isomorphic to Ree unitals. To this end, we determine the full group of automorphisms for each finite Ree unital, using the classification of finite simple groups.

We also give new proofs of the fact that a (possibly infinite) Ree-Tits unital is never isomorphic to a classical unital defined by a hermitian form of Witt index $1$ on a three-dimensional vector space over a commutative field. This follows from the known fact that every automorphism of a classical unital over a commutative field extends to an automorphism of the ambient plane ([16], cf. [18]) together with a result by Lüneburg [14].

We present two different approaches because both of them allow generalizations. The first one uses the classification of finite simple groups (via the classification of the two-transitive actions) to determine the full group of automorphisms of $\mathrm{RTU}(q)$. This method allows to see that the class of Ree unitals is disjoint from the class of polar unitals found in Coulter–Matthews planes (see Section 6 below). The second approach uses only partial knowledge of the full group of automorphisms. It thus avoids the heavy machinery and at the same time also includes the Ree-Tits unitals obtained by generalizing the construction of $\mathrm{RTU}(q)$ to suitable infinite fields.

In both approaches, the crucial point is that we know the full group of automorphisms of one of the unitals in question, and an interesting part of the automorphism group of the other one.

## 1 Tits endomorphisms

In order to construct the Ree group $\mathrm{R}(\theta, \mathbb{K})$ we need a field $\mathbb{K}$ of characteristic $3$ and a *Tits endomorphism* $\theta$: i.e., an endomorphism $\theta$ of $\mathbb{K}$ with $\theta^2 = \phi$, where $\phi\colon x \mapsto x^3$ denotes the Frobenius endomorphism.

**1.1 Lemma.** *If $\mathbb{K}$ is any field of characteristic $3$ admitting a Tits endomorphism $\theta$ then $-1$ is not a square in $\mathbb{K}$. For a finite field $\mathbb{K}$ of characteristic $3$ a Tits endomorphism exists if, and only if, there is no square root of $-1$ in $\mathbb{K}$; i.e., precisely if the order of $\mathbb{K}$ is $3^k$ with an odd integer $k$.*

*Proof.* The existence of a square root $i$ of $-1$ entails that $\mathbb{K}$ contains $\mathbb{F}_3(i) \cong \mathbb{F}_9$. Every endomorphism of $\mathbb{K}$ would leave $\mathbb{F}_3(i)$ invariant, and $\theta$ would induce a Tits endomorphism of $\mathbb{F}_3(i)$. This contradicts the fact that the Frobenius endomorphism induces the generator of $\mathrm{Aut}(\mathbb{F}_3(i)) \cong \mathbb{Z}/2\mathbb{Z}$.

If $\mathbb{K}$ is a finite field of order $3^k$ then $\mathrm{Aut}(\mathbb{K})$ is a cyclic group of order $k$. Thus the generator $\phi$ of $\mathrm{Aut}(\mathbb{K})$ possesses a square root precisely if $k$ is odd. □

**1.2 Remarks.** In the finite case, the Tits endomorphism is unique (if it exists at all) because $\mathrm{Aut}(\mathbb{F}_q)$ is cyclic. We write $\mathrm{R}(q) := \mathrm{R}(\theta, \mathbb{F}_q)$.

If the field $\mathbb{K}$ is infinite, there may exist different Tits endomorphisms, or none at all (even if there is no square root of $-1$ in $\mathbb{K}$). Cf. Section 4 of [19], where Tits automorphisms of fields of characteristic $2$ are discussed.

**1.3 Lemma.** *Let $\mathbb{K}$ be a field of characteristic $3$ with Tits endomorphism $\theta$.*

- **a.** *The endomorphism $\theta + 2 := (x \mapsto x^\theta x^2)$ of the multiplicative group $\mathbb{K}^*$ is bijective.*
- **b.** *The endomorphisms $\theta + 1$ and $\theta - 1$ of $\mathbb{K}^*$ are neither injective nor surjective. However, each one of the sets $\left\{x^{\theta+1} \,\middle|\, x \in \mathbb{K}^*\right\}$ and $\left\{x^{\theta-1} \,\middle|\, x \in \mathbb{K}^*\right\}$ additively generates $\mathbb{K}$.*

*Proof.* A straightforward computation in the endomorphism ring of $\mathbb{K}^*$ shows that $2 - \theta$ is the inverse of $\theta + 2$.

Clearly $-1$ lies in the kernel of both $\theta + 1$ and $\theta - 1$. From $x^{\theta-1} = -1$ we infer $x^\theta = -x$ and then $x^\phi = x^{\theta^2} = -x^\theta = x$. Thus $x \in \{1, -1\}$ and $x^\theta = 1$; a contradiction. The case $\theta + 1$ is treated analogously.

In any field $\mathbb{K}$ the group $S$ additively generated by the squares contains the subgroup $\{(x+1)^2 - (x-1)^2 \mid x \in \mathbb{K}^*\} = \{4x \mid x \in \mathbb{K}^*\}$. This shows $S = \mathbb{K}$ if $\operatorname{char} \mathbb{K} \neq 2$.

We compute $(\theta - 1)(\theta + 1) = (\theta + 1)(\theta - 1) = \theta^2 - 1 = 2$. Thus $\left\{x^{\theta+1} \,\middle|\, x \in \mathbb{K}^*\right\}$ and $\left\{x^{\theta+1} \,\middle|\, x \in \mathbb{K}^*\right\}$ both contain all squares, and generate $\mathbb{K}$. □

## 2 The point stabilizer and its regular normal subgroup

From now on, we consider a field $\mathbb{K}$ with a Tits endomorphisms $\theta$. Note that $\theta = \mathrm{id}$ occurs precisely if $|\mathbb{K}| = 3$.

We write $(a, b, c)^\intercal$ for the column with entries $a$, $b$, $c$, and define a group operation $*$ on the set $\mathrm{U}(\theta, \mathbb{K}) := \{(x, y, z)^\intercal \mid x, y, z \in \mathbb{K}\}$ of columns by

$$\begin{pmatrix} a \\ b \\ c \end{pmatrix} * \begin{pmatrix} x \\ y \\ z \end{pmatrix} := \begin{pmatrix} a + x \\ b + y + ax^\theta \\ c + z + ay - bx - ax^{\theta+1} \end{pmatrix}.$$

The third power of $(a, b, c)^\intercal$ is $\left(0, 0, -a^{\theta+2}\right)^\intercal$; in particular, our group has exponent $3^2$. The inverse of $(a, b, c)^\intercal$ is $\left(-a, a^{\theta+1} - b, -c\right)^\intercal$.

**2.1 Lemma.** *The multiplicative group $\mathbb{K}^*$ acts on $\mathrm{U}(\theta, \mathbb{K})$ via the group homomorphism*

$$h\colon \mathbb{K}^* \to \mathrm{Aut}(\mathrm{U}(\theta, \mathbb{K}))\colon f \mapsto h_f := \left( \begin{pmatrix} a \\ b \\ c \end{pmatrix} \mapsto \begin{pmatrix} fa \\ f^{\theta+1}b \\ f^{\theta+2}c \end{pmatrix} \right).$$

This gives rise to a semidirect product $G_\infty := \mathbb{K}^* \ltimes_h \mathrm{U}(\theta, \mathbb{K})$. We still extend this product by the centralizer $\mathrm{C}_{\mathrm{Aut}(\mathbb{K})}(\theta)$ of $\theta$ in $\mathrm{Aut}(\mathbb{K})$ stipulating that $\alpha \in \mathrm{C}_{\mathrm{Aut}(\mathbb{K})}(\theta)$ acts as $\hat{\alpha}\colon\ (a, b, c)^\intercal \mapsto (a^\alpha, b^\alpha, c^\alpha)^\intercal$. Note that $\hat{\alpha}^{-1} h_f \hat{\alpha} = h_{f^\alpha}$. The corresponding semidirect product will be denoted $\Gamma_\infty := \mathrm{C}_{\mathrm{Aut}(\mathbb{K})}(\theta) \ltimes G_\infty$.

**2.2 Definition.** In order to generate the full group $\mathrm{R}(\theta, \mathbb{K})$ we need a single transformation (taken from [20], with a correction in the definition of $N$, cf. [5]) that interchanges $(0, 0, 0)$ with $\infty$:

$$\omega\colon \begin{pmatrix} a \\ b \\ c \end{pmatrix} \mapsto \frac{-1}{N(a, b, c)} \begin{pmatrix} a^\theta b^\theta - c^\theta + ab^2 + bc - a^{2\theta+3} \\ a^2 b - ac + b^\theta - a^{\theta+3} \\ c \end{pmatrix}$$

where $N(a, b, c) := -ac^\theta + a^{\theta+1}b^\theta - a^{\theta+3}b - a^2b^2 + b^{\theta+1} + c^2 - a^{2\theta+4}$. In order to make sure that $\omega$ is well defined (and injective) for points different from $(0, 0, 0)$ one has to check that $N(a, b, c) = 0 \iff a = b = c = 0$; this is done in [5]. (Both in [20, Sect. 5] and in [21, 7.7.15] one has to correct a misprint and replace the summand $a^{\theta+1}b$ by $a^{\theta+1}b^\theta$.)

The group $\mathrm{R}(\theta, \mathbb{K})$ of bijections of $\mathrm{U}(\theta, \mathbb{K})$ generated by $G_\infty$ and $\omega$ is called the *Ree-Tits group*.

**2.3 Remark.** The groups $G_\infty$ and $\Gamma_\infty$ are the stabilizers of a point $\infty$ in the Ree group $G = \mathrm{R}(\theta, \mathbb{K})$ and its holomorph $\Gamma\mathrm{R}(\theta, \mathbb{K}) := \mathrm{C}_{\mathrm{Aut}(\mathbb{K})}(\theta) \ltimes \mathrm{R}(\theta, \mathbb{K})$, respectively.

For our present purposes, it suffices to know $G_\infty$ if one is willing to accept that $G$ acts (two-transitively) on $\mathrm{U}(\theta, \mathbb{K}) \cup \{\infty\}$ in such a way that the stabilizer of $\infty$ is $G_\infty$, and that this stabilizer acts regularly on the complement of $\infty$ (i.e. on itself, by multiplication from the right). See [21, Sect. 7] for a discussion of $G$ and its action as a subgroup of the centralizer of a polarity of a Moufang hexagon.

**2.4 Remarks.** The Tits endomorphism of $\mathbb{F}_3$ is the identity. This case needs separate treatment, and we will exclude it in 4.2 and 4.4 below. The group $\mathrm{R}(3)$ is isomorphic to $\mathrm{P\Gamma L}(2, 8)$, and not simple. For fields with more than 3 elements, the group $\mathrm{R}(\theta,\mathbb{K})$ is simple (cf. [5]). Some authors (e.g., see [21, 7.7.19]) study a larger group which coincides with $\mathrm{R}(\theta,\mathbb{K})$ only if $\mathbb{K}^*$ is generated by $-1$ together with the group of squares in $\mathbb{K}^*$. This is satisfied for finite fields admitting a Tits endomorphism because the existence of that endomorphism secures that $-1$ is not a square, cf. 1.1.

**2.5 Lemma.** *The first term in the descending central series (i.e., the commutator subgroup) of* $\mathrm{U}(\theta,\mathbb{K})$ *is* $\zeta^1(\mathrm{U}(\theta,\mathbb{K})) = \{(0,y,z)^\intercal \mid y \in C, z \in \mathbb{K}\}$ *where* $C := \left\langle ax^\theta - xa^\theta) \,\middle|\, a,x \in \mathbb{K}\right\rangle$. *The second term in the descending central series is* $\zeta^2(\mathrm{U}(\theta,\mathbb{K})) = [\mathrm{U}(\theta,\mathbb{K}),\zeta^1(\mathrm{U}(\theta,\mathbb{K}))] = \{(0,0,z)^\intercal \mid z \in \mathbb{K}\} = \mathrm{Z}(\mathrm{U}(\theta,\mathbb{K}))$. *The set* $C$ *degenerates to* $\{0\}$ *precisely if* $|\mathbb{K}| = 3$. *In all other cases we have* $C = \mathbb{K}$.

*Proof.* Computing commutators

$$\left[\begin{pmatrix} a \\ b \\ c \end{pmatrix}, \begin{pmatrix} x \\ y \\ z \end{pmatrix}\right] = \begin{pmatrix} a \\ b \\ c \end{pmatrix}^{-1} * \begin{pmatrix} x \\ y \\ z \end{pmatrix}^{-1} * \begin{pmatrix} a \\ b \\ c \end{pmatrix} * \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$
$$= \begin{pmatrix} 0 \\ ax^\theta - xa^\theta \\ bx - ay + x^2a^\theta - a^2x^\theta + ax^{\theta+1} - xa^{\theta+1} \end{pmatrix}$$

we see that the center $\mathrm{Z}(\mathrm{U}(\theta,\mathbb{K}))$ of $\mathrm{U}(\theta,\mathbb{K})$ is $\{(0,0,z)^\intercal \mid z \in \mathbb{K}\}$, the commutator subgroup is $\{(0,y,z)^\intercal \mid (y,z) \in C \times \mathbb{K}\}$, and $\zeta^2(\mathrm{U}(\theta,\mathbb{K})) = \mathrm{Z}(\mathrm{U}(\theta,\mathbb{K}))$.

If $\mathbb{K}$ has more than 3 elements then $\theta \neq \mathrm{id}$ and $C$ contains some element $c \neq 0$. From 2.1 we infer that $C$ contains the additive closure of $\left\{f^{\theta+1}c \,\middle|\, f \in \mathbb{K}^*\right\}$. Thus $C = \mathbb{K}$ by 1.3. □

**2.6 Corollary.** *The group* $\mathrm{U}(\theta,\mathbb{K})$ *is nilpotent of class* 3 *unless* $|\mathbb{K}| = 3$. *In the latter case, the group is nilpotent of class* 2. □

## 3 The unital

We abbreviate $P := \mathrm{U}(\theta,\mathbb{K}) \cup \{\infty\}$.

**3.1 Lemma** ([6], see [21, 7.7.18]). *The stabilizer* $\mathrm{R}(\theta,\mathbb{K})_{a,b}$ *of any two points* $a,b \in P$ *contains a unique involution* $\rho_{a,b}$. *Every involution in* $\mathrm{R}(\theta,\mathbb{K})$ *either is a conjugate of* $\rho_{0,\infty} = h_{-1}$ *or acts without any fixed points.* □

The set of fixed points of $\rho_{0,\infty}$ is $\ell_{0,\infty} := \{\infty\} \cup \{(0,y,0)^\intercal \mid y \in \mathbb{K}\}$. Let $L$ denote the orbit of $\ell_{0,\infty}$ under $\mathrm{R}(\theta,\mathbb{K})$; this is also the orbit under $\Gamma\mathrm{R}(\theta,\mathbb{K})$ because $\hat{\alpha}$ fixes $\ell_{0,\infty}$ for each $\alpha \in \mathrm{C}_{\mathrm{Aut}(\mathbb{K})}(\theta)$. Note that $\omega$ interchanges $\infty$ with 0, and also fixes $\ell_{0,\infty}$. We define the Ree-Tits unital as the incidence structure $\mathrm{RTU}(\theta,\mathbb{K}) := (P,L)$; in the finite case, we abbreviate $\mathrm{RTU}(q) := \mathrm{RTU}(\theta,\mathbb{K})$ with $q := |\mathbb{K}|$.

In the finite case, one may describe $P$ as the set of Sylow 3-subgroups of $\mathrm{R}(\theta,\mathbb{K})$ and $L$ as the set of involutions with fixed points, such an involution being incident with each Sylow 3-subgroup that is normalized by the involution. See [14].

The following is an immediate consequence of 3.1.

**3.2 Proposition.** *The incidence structure* $\mathrm{RTU}(\theta,\mathbb{K})$ *is a linear space.* □

**3.3 Examples.** The blocks through $\infty$ form an orbit under $\mathrm{U}(\theta,\mathbb{K})$; we have

$$\ell_{\infty,0} * (a,b,c)^\intercal = \left\{(a,y,c-ya)^\intercal \mid y \in \mathbb{K}\right\} = \ell_{\infty,(a,0,c)^\intercal} .$$

The stabilizer of $\ell_{\infty,0}$ in $\mathrm{U}(\theta,\mathbb{K})$ is $B := \left\{(0,b,0)^\intercal \mid b \in \mathbb{K}\right\}$; this is not a normal subgroup of $\mathrm{U}(\theta,\mathbb{K})$. Each coset $(a,b,c)^\intercal \zeta^1(\mathrm{U}(\theta,\mathbb{K}))$ is the union of the orbit of $\ell_{\infty,(a,0,c)^\intercal}$ under the center $\mathrm{Z}(\mathrm{U}(\theta,\mathbb{K}))$.

# 4 The finite case

In this section we collect some information about the action of a Ree group over a finite field on the corresponding Ree unital. We are going to use that information to determine the full group of automorphisms of the Ree unital, see 4.4. The proof will use the classification of finite simple groups (via the classification of two-transitive groups).

Before we know the full group $\mathrm{Aut}(\mathrm{RTU}(q))$ we will prove that $\mathrm{RTU}(q)$ is not classical. The proof is a simplified version of the proof of 5.3. Note, however, that our treatment of the general case also covers the finite case, including the case $q = 3$ which plays a special role and has to be excluded in 4.2 below. We present this separate argument explicitly because the finite situation allows a more direct approach. This is mainly due to the fact that the automorphism group of a finite field is cyclic.

**4.1 Lemma.** *No Sylow 3-subgroup of $\mathrm{R}(q)$ is isomorphic to any subgroup of $\mathrm{PU}(3,q)$.*

*Proof.* If the Sylow 3-subgroup $\mathrm{U}(q)$ of $\mathrm{R}(q)$ were isomorphic to some subgroup $\mathrm{PU}(3,q)$ that subgroup would lie in a conjugate of the subgroup $H$ of $\mathrm{P\Gamma U}(3,q)$ induced by

$$\left\{ \begin{pmatrix} 1 & 0 & 0 \\ x & 1 & 0 \\ z & -x^\kappa & 1 \end{pmatrix} \;\middle|\; \begin{array}{c} x,z \in \mathbb{F}_q \\ z + z^\kappa = -x^\kappa x \end{array} \right\} .$$

But $H$ is a group of exponent 3 while $\mathrm{U}(q)$ has exponent $3^2$. □

**4.2 Theorem.** *Let $q = 3^{2k+1}$ for some integer $k \geq 1$. Then $\mathrm{RTU}(q)$ is not isomorphic to the classical (hermitian) unital $\mathrm{H}(q)$ of order $q$.*

*Proof.* The automorphism group of the classical unital is $\mathrm{P\Gamma U}(3,q)$; see [16], cf. [18]. Thus every isomorphism from $\mathrm{RTU}(q)$ onto the classical unital would induce an embedding of $\Gamma\mathrm{R}(q)$ into $\mathrm{P\Gamma U}(3,q)$. Since $\mathrm{U}(q)$ is a group of order $q^3 = 3^{6k+3}$ contained in the commutator group of $\Gamma\mathrm{R}(q)$, its image in $\mathrm{P\Gamma U}(3,q)$ is contained in a Sylow 3-subgroup of the commutator group of $\mathrm{P\Gamma U}(3,q)$. This is impossible by 4.1. □

**4.3 Lemma.** *For any $k \in \mathbb{N}$ the number $3^{3(2k+1)} + 1$ is divisible by $4 \cdot 7$ but not by 8.*

*Proof.* We abbreviate $q := 3^{2n+1}$. Now $-1 \equiv 3^3 \equiv q^3 \pmod{28}$ yields $q^3 + 1 \in (4 \cdot 7)\mathbb{Z}$. On the other hand, we have $1 \equiv 3^2 \equiv q/3 \pmod 8$ and thus $4 \equiv q + 1 \pmod 8$. □

Note that $4 \cdot 7 = 28 = 3^3 + 1$ is the order of the smallest Ree unital $\mathrm{RTU}(3)$. This case plays a special role because the corresponding Ree group is not simple.

We are now going to use the classification of finite simple groups in order to determine the full group $\mathrm{Aut}(\mathrm{RTU}(q))$. Thus we re-prove a special case of Kantor's results [11]; we include the proof for the convenience of the reader.

**4.4 Theorem.** *Let* $q = 3^{2k+1}$ *for some integer* $k \geq 1$*. Then the Ree group* $\mathrm{R}(q)$ *is normal in* $\mathrm{Aut}(\mathrm{RTU}(q))$ *and* $\mathrm{Aut}(\mathrm{RTU}(q)) \cong \mathrm{Aut}(\mathrm{R}(q)) \cong \mathrm{Aut}(\mathbb{F}_q) \ltimes \mathrm{R}(q)$*.*

*Proof.* The group $A := \mathrm{Aut}(\mathrm{RTU}(q))$ acts 2-transitively on $\mathrm{RTU}(q)$ because its subgroup $\mathrm{R}(q)$ does. According to a theorem of Burnside's ([2, § 154, Thm. XIII, p. 202], cf. [3, 4.3] or [8, Thm. 4.1B]), the group $A$ contains a transitive normal subgroup $S$ which is either simple or elementary abelian. The latter case is ruled out by 4.3. Thus $S$ occurs in the list of almost simple two-transitive groups, cf. [3, 7.4] where $\mathrm{R}(q)$ is denoted by $R_1(q)$.

For most of the entries in the list it is easy to see that the degree $n$ does not meet the requirements of 4.3. After that observation, it remains to exclude the following candidates for the action of $S$.

- Actions of alternating groups:
  these are 3-transitive and will not respect the system of blocks.
- Actions of $\mathrm{PSL}(d, p^e)$ of degree $(p^{de} - 1)/(p^e - 1) = \sum_{j=0}^{d-1} p^{je}$ with $d \geq 2$:
  then $p^e \sum_{j=0}^{d-2} p^{je} = 3^{6k+3}$ implies $p = 3$ and then $d = 2$. Thus $\mathrm{R}(q)$ would be contained in $\mathrm{PSL}(2, q^3)$, contradicting the fact that the latter group has abelian Sylow 3-subgroups.
- Actions of $\mathrm{Sp}(2d, 2)$ of degree $n = 2^{2d-1} \pm 2^{d-1} = 2^{d-1}(2^d \pm 1)$ with $d \geq 3$:
  here $d = 3$ by 4.3 and the remaining possibilities for $n \in \{2^2 \cdot 7, 2^2 \cdot 9\}$ are smaller than $q^3 + 1 = 3^{6k+3} + 1$ because $k \geq 1$.
- Actions of $\mathrm{PSU}(3, q)$:
  such an action would imply $\mathrm{R}(q) \leq \mathrm{PSU}(3, q)$, contradicting 4.1.

Thus we have verified that $A$ is contained in $\mathrm{Aut}(\mathrm{R}(q))$. The automorphisms of the Ree groups have been determined in [17], cf. also [9, 2.5.12].

Conversely, every automorphism of $\mathrm{R}(q)$ preserves the unique conjugacy class of involutions and the conjugacy class of normalizers of Sylow 3-subgroups. Thus $\mathrm{Aut}(\mathrm{R}(q))$ is contained in $A$, and we have equality. □

# 5 The general case

In the general case of a possibly infinite field $\mathbb{K}$ it is not obvious that the groups $\mathrm{R}(\theta, \mathbb{K})$ and $\mathrm{U}(\theta, \mathbb{K})$ have to be mapped into the group of linearly induced automorphisms of the classical unital. The commutator argument used in 4.2 only works if $\mathrm{Aut}(\mathbb{K})$ is abelian. Moreover, it is not clear a priori which field should be used for the definition of the classical unital. We even have to deal with the possibility that the characteristic of this field may be different from 3.

An infinite field may have more than one involutory automorphism (or none at all). We consider a commutative field $\mathbb{E}$ with an involution $\kappa \in \mathrm{Aut}(\mathbb{E})$. Then we raise the question whether or not the Ree-Tits unital $\mathrm{RTU}(\theta, \mathbb{K})$ is isomorphic to the classical unital defined by a $\kappa$-hermitian form $f_\kappa$ of Witt index 1 on $\mathbb{E}^3$. Without loss of generality, we may replace the form $f_\kappa$ by a scalar multiple, and assume that it is given by $f_\kappa(x, y) = x_1 y_3^\kappa + x_2 y_2^\kappa + x_3 y_1^\kappa$.

According to [18], the group

$$\Gamma\mathrm{U}(3, \mathbb{E}, \kappa) = \left\{ \mu \in \Gamma\mathrm{L}(3, \mathbb{E}) \;\middle|\; \begin{array}{r} \exists\, \sigma \in \mathrm{Aut}(\mathbb{E})\, \exists\, s \in \mathbb{E}^* \,\forall\, x, y \in \mathbb{E}^3 : \\ f_\kappa(x^\mu, y^\mu) = s f_\kappa(x, y)^\sigma \end{array} \right\}$$

of semi-similitudes (cf. [7, §10]) induces the full automorphism group $\mathrm{P}\Gamma\mathrm{U}(3, \mathbb{E}, \kappa)$ of the classical unital $\mathrm{H}(\kappa, \mathbb{E})$ whose point set consists of the isotropic one-dimensional subspaces with respect to $f_\kappa$.

**5.1 Lemma.** *An automorphism $\alpha \in \mathrm{Aut}(\mathbb{E})$ occurs as the companion automorphism of an element of $\Gamma\mathrm{U}(3,\mathbb{E},\kappa)$ if, and only if, it centralizes $\kappa$.*

*Proof.* Consider $\mu \in \Gamma\mathrm{U}(3,\mathbb{E},\kappa)$ with companion automorphism $\alpha$; i.e., $(au)^\mu = a^\alpha u^\mu$ for all $a \in \mathbb{E}$ and $u \in \mathbb{E}^3$. Evaluating $f_\kappa(x^\mu, y^\mu) = sf_\kappa(x,y)^\sigma$ at $x = au$ and $y = bv$ with $a, b \in \mathbb{E}$ and $u, v \in \mathbb{E}^3$ and using commutativity of $\mathbb{E}$ we find $\alpha = \sigma = \kappa\alpha\kappa$. Thus $\alpha$ centralizes $\kappa$. Conversely, assume that $\alpha$ centralizes $\kappa$. Then $(x_1,x_2,x_3) \mapsto (x_1^\alpha, x_2^\alpha, x_3^\alpha)$ is a semi-similitude with companion $\alpha$. □

**5.2 Remark.** Our proof of 5.1 makes essential use of the commutativity of the multiplication. If one drops this assumption, not much can be said about the subgroup of $\mathrm{Aut}(\mathbb{E})$ consisting of those automorphisms that occur as companions of semi-similitudes, cf. [7, §10].

**5.3 Theorem.** *Let $\mathbb{K}$ be any field of characteristic $3$ with a Tits endomorphism $\theta$. Then the incidence structure $\mathrm{RTU}(\theta,\mathbb{K})$ is not isomorphic to any classical (hermitian) unital over a commutative field.*

*Proof.* Assume, to the contrary, that there exists a commutative field $\mathbb{E}$ with involution $\kappa$ such that $\mathrm{RTU}(\theta,\mathbb{K})$ is isomorphic to $\mathrm{H}(\kappa,\mathbb{E})$. According to [18] any isomorphism from $\mathrm{RTU}(\theta,\mathbb{K})$ onto $\mathrm{H}(\kappa,\mathbb{E})$ induces an embedding of the action of $\mathrm{R}(\theta,\mathbb{K})$ into the action of $\mathrm{P}\Gamma\mathrm{U}(3,\mathbb{E},\kappa)$. In particular, we obtain an embedding of the group $\mathrm{U}(\theta,\mathbb{K})$ as a subgroup of a point stabilizer in $\mathrm{P}\Gamma\mathrm{U}(3,\mathbb{E},\kappa)$. We may (and will) assume that the fixed point is $\infty_\mathrm{c} := \mathbb{E}(1,0,0)$.

The group $\zeta^1(\mathrm{U}(\theta,\mathbb{K}))$ centralizes the stabilizer $\Lambda := \mathrm{U}(\theta,\mathbb{K})_{\ell_{\infty,0}}$. Thus $\Lambda$ fixes each of the blocks $\ell_{\infty,(0,b,c)^\intercal} = \ell_{\infty,(0,0,c)^\intercal}$. However, it does not fix *all* the blocks through $\infty$ because it is not a normal subgroup of $\mathrm{U}(\theta,\mathbb{K})$.

Straightforward computations show that $\mathrm{PU}(3,\mathbb{E},\kappa)_{\infty_\mathrm{c}}$ acts two-transitively on the set of blocks through $\infty_\mathrm{c}$. Thus we may assume that $\Lambda$ is mapped into the stabilizer of the two lines joining $\infty_\mathrm{c}$ with $\mathbb{E}(0,0,1)$ and with $\mathbb{E}(0,1,1)$, respectively. This stabilizer is the product of the centralizer $\mathrm{C}_{\mathrm{Aut}(\mathbb{E})}(\kappa)$ of $\kappa$ in the group of field automorphisms and the group $\mathrm{T}$ consisting of all those elements of $\mathrm{P}\Gamma\mathrm{U}(3,\mathbb{E},\kappa)$ that fix *all* the lines through $\infty$. The group $\mathrm{T}$ is induced by the subgroup

$$\left\{ \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ c & 0 & 1 \end{pmatrix} \,\middle|\, c \in \mathbb{E}, c^\kappa = -c \right\}.$$

For $\alpha \in \mathrm{C}_{\mathrm{Aut}(\mathbb{E})}(\kappa)$ and $c \in \{x \in \mathbb{E} \mid x^\kappa = -x\}$ we assume that the semilinearly induced map $\lambda\colon \mathbb{E}(x,y,z) \mapsto \mathbb{E}(x^\alpha + c, y^\alpha, z^\alpha)$ belongs to the image of $\Lambda$. Since $\Lambda$ has exponent $3$, we know $\alpha^3 = \mathrm{id}$, and $c^{\alpha^2} + c^\alpha + c = 0$.

If $\alpha \neq \mathrm{id}$ we know by the additive version of Hilbert's Theorem 90 (cf. [13, VI, 6.3]) that there exists $d \in \mathbb{E}$ with $d - d^\alpha = c$. If $\operatorname{char}\mathbb{E} \neq 2$ we put $x := \frac{1}{2}(d - d^\kappa)$ and find $x^\kappa = -x$ and $x - x^\alpha = c$ (here we use that $\alpha$ commutes with $\kappa$, cf. 5.1). If $\operatorname{char}\mathbb{E} = 2$ we apply Hilbert's Theorem 90 again to the restriction of $\alpha$ to $\mathrm{Fix}(\kappa)$ in order to find $x$ with these properties.

In any case, the point $\mathbb{E}(x,0,1) \neq \infty_\mathrm{c}$ on $\mathrm{H}(\kappa,\mathbb{E})$ is fixed by $\lambda$. This contradiction shows that $\alpha = \mathrm{id}$, and $\lambda \in \mathrm{T}$ fixes each block through $\infty_\mathrm{c}$. Again, we reach a contradiction. Therefore we have no embedding of $\Lambda$ as required, and thus there cannot be any isomorphism from $\mathrm{RTU}(\theta,\mathbb{K})$ onto $\mathrm{H}(\kappa,\mathbb{E})$. □

# 6 Polar unitals in Coulter–Matthews planes

The Coulter–Matthews planes introduced in [4] are finite projective planes of order $3^e$, constructed from suitable "planar" functions of the form $x \mapsto x^n$. The number of isomorphism classes of Coulter-Matthews planes (including the desarguesian one which is obtained if $n = 2$) of order $3^e$ equals $\phi(e)$ if $e$ is even and $\frac{1}{2}\phi(e)$ if $e$ is odd, see [12, 3.5].

The polarities of these planes have been determined completely, see [12, 5.2]. It turns out that unitary polarities (where the absolute points form a unital) exist precisely if $e$ is even. In that case there is just one conjugacy class of such polarities. Deviating from the notation in [12] we denote the square root of $3^e$ by $q$. The corresponding unital will be denoted by $\mathrm{KSU}(n, q)$. Some of these unitals will be classical ones but there are infinite series of non-classical examples, see [12, 6.8].

Each unital $\mathrm{KSU}(n, q)$ has order $q = 3^e$ and thus $q^3 + 1$ points. Thus the question arises naturally whether $\mathrm{KSU}(n, q)$ and $\mathrm{RTU}(q)$ might be isomorphic for some values of $q$ and $n$. We are going to show that this is not the case. Since $\mathrm{KSU}(2, q)$ is the classical unital of order $q$ our results in this section include another alternative proof of the fact that the Ree unitals are not classical. However, this variant uses the classification of finite simple groups (via 4.4).

**6.1 Proposition** ([12, 5.6]). *The commutator group of* $\mathrm{Aut}(\mathrm{KSU}(q))$ *contains an elementary abelian group* $\Xi$ *of order* $q^2$ *such that* $\Xi$ *fixes precisely one point* $\infty$ *of* $\mathrm{KSU}(q)$ *and contains a subgroup* $\mathrm{T}$ *of order* $q$ *that fixes each block through* $\infty$.

**6.2 Theorem.** *Choose* $q = 3^{2k+1}$ *and* $n \in \mathbb{N}$ *such that* $x \mapsto x^n$ *yields a Coulter–Matthews plane of order* $q^2$*. Then the unitals* $\mathrm{KSU}(n, q)$ *and* $\mathrm{RTU}(q)$ *are not isomorphic.*

*Proof.* If $\eta\colon \mathrm{KSU}(n, q) \to \mathrm{RTU}(q)$ were an isomorphism then $\eta \circ \Xi \circ \eta^{-1}$ would be contained in the commutator group of $\mathrm{Aut}(\mathrm{RTU}(q))$ and thus in a Sylow 3-subgroup of $\mathrm{R}(q)$. Without loss, we may then assume $\eta \circ \Xi \circ \eta^{-1} \leq \mathrm{U}(q)$, and $\eta \circ \mathrm{T} \circ \eta^{-1}$ would induce a subgroup of $\mathrm{U}(q)$ fixing each block through $\infty$. This is impossible because $\mathrm{U}(q)$ acts faithfully on the pencil of blocks, see 3.3. □

**6.3 Remark.** Our result 6.2 cannot be reduced to Lüneburg's non-embeddability theorem [14] because we do not know whether each automorphism of $\mathrm{KSU}(n, q)$ extends to an automorphism of an ambient projective plane.

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Theo Grundhöfer
Institut für Mathematik
Universität Würzburg
Am Hubland
D-97074 Würzburg
Germany

Boris Krinn
Institut für Geometrie und Topologie
Fakultät für Mathematik und Physik
Universität Stuttgart
70550 Stuttgart
Germany

Markus Stroppel
Fachbereich Mathematik
Fakultät für Mathematik und Physik
Universität Stuttgart
70550 Stuttgart
Germany

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