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Polarities of Schellhammer Planes

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Abstract

We construct polarities for projective planes admitting a group of automorphisms that acts regularly on the complements of an anti-flag in the point and line set. In particular, we determine all polarities of the compact connected planes studied by I. Schellhammer and by P. Sperner. For both classes of planes we solve the conjugacy problem, and determine the set of absolute points for each one of the polarities. Among these polarities we find the first examples of elliptic polarities in non-Moufang compact projective planes.

1 Schellhammer planes

Let $\mathcal{P} = (P, \mathcal{L})$ be a projective plane. For each point $p \in P$ and each line $L \in \mathcal{L}$ let $\mathcal{L}_p$ and P_L denote the set of lines through p and the set of points on L, respectively. Incidence will be *denoted* by "$\in$". In the explicit examples, the incidence relation between affine points and lines will really *be* the element relation.

1.1 Definition. The plane $\mathcal{P}$ is called a *Schellhammer plane* if there exists a group $S \leq \mathrm{Aut}(\mathcal{P})$ such that S fixes an anti-flag (o, ∞) and acts regularly (i.e., sharply transitively) both on $P \setminus (\{o\} \cup P_\infty)$ and on $\mathcal{L} \setminus (\mathcal{L}_o \cup \{\infty\})$. Then S is called a *Schellhammer group* on $\mathcal{P}$.

The present study was initiated by the observation that (similar as in the case of shift planes, cf. [13], [14]) the existence of a commutative Schellhammer group secures the existence of polarities (cf. [4, 1.2.13, 1.2.14]). Our aim is to classify the latter. A more general approach (using non-commutative groups) is needed for the treatment of an interesting class of compact connected planes (see 3.6 below).

We will adopt an affine point of view, taking ∞ as the line at infinity and identifying affine lines with their affine point rows. Choosing $a \in P \setminus (\{o\} \cup P_\infty)$ and $A \in \mathcal{L}_a \setminus \mathcal{L}_o$ we obtain a labelling of the lines in the regular orbit by $L(s) := s(A)$ for $s \in S$. We will identify $s \in S$ with $s(a)$, the affine point row of A then becomes $L(1) = \{t \in S \mid t(a) \in A\}$. After this identification, the action of S on the affine point set appears as multiplication from the left.

1.2 Lemma. *Let $\mathcal{P} = (P, \mathcal{L})$ and $\mathcal{P}' = (P', \mathcal{L}')$ be projective planes with non-incident point-line pairs (o, ∞) and (o', ∞'), respectively. Then every isomorphism between the incidence structures $(P \setminus (\{o\} \cup P_\infty), \mathcal{L} \setminus (\mathcal{L}_o \cup \{\infty\}))$ and $\big(P' \setminus (\{o'\} \cup P'_{\infty'}), \mathcal{L}' \setminus (\mathcal{L}'_{o'} \cup \{\infty'\})\big)$ extends uniquely to an isomorphism from $\mathcal{P}$ onto $\mathcal{P}'$.*

Proof. The affine plane $(P \setminus P_\infty, \mathcal{L} \setminus \{\infty\})$ can be reconstructed uniquely from the structure $(P \setminus (\{o\} \cup P_\infty), \mathcal{L} \setminus (\mathcal{L}_o \cup \{\infty\}))$ because the lines through o are the classes of the equivalence relation "not joined", and extension to the projective plane is standard. □

2 Schellhammer groups on compact connected planes

In this section let $\mathcal{P} = (P, \mathcal{L})$ be a compact connected projective plane. The study of such planes was initiated by Salzmann. An overview of results and techniques is given in [20]. The class of compact connected Schellhammer planes is of particular interest: on the one hand, there are huge classes of (non-classical) examples in this realm while, on the other hand, the fact that only two isomorphism types of locally compact connected Schellhammer groups are possible leads to a coherent theory.

Before we give examples of Schellhammer planes in Section 3 we reduce the list of candidates for the groups. It appears reasonable to impose topological assumptions on the Schellhammer group S, as well: we stipulate that S is a locally compact, σ-compact topological group and that the action $\omega\colon S \times P \to P\colon (s,p) \to s(p)$ on the point space is continuous. The compact–open topology (with respect to its action on P) turns $\operatorname{Aut}(\mathcal{P})$ into a locally compact topological group, cf. [20, 44.3]. The action ω of S induces an injective continuous group homomorphism $\delta\colon S \to \operatorname{Aut}(\mathcal{P})$, see [26, 10.4 (a), 9.14].

We pick a point $o \in P$ and a line $\infty \in \mathcal{L} \setminus \mathcal{L}_o$.

2.1 Lemma. *Let S be a locally compact, σ-compact topological group and assume that there exists an injective continuous group homomorphism $\delta\colon S \to \operatorname{Aut}(\mathcal{P})$ such that $\delta(S)$ acts regularly on $P \setminus (\{o\} \cup P_\infty)$. Then δ is a homeomorphism from S onto its image. In particular, the subgroup $\delta(S)$ is closed in $\operatorname{Aut}(\mathcal{P})$ and S is connected and homeomorphic to $\mathbb{R}^{2d} \setminus \{0\}$ where $2d = \dim P$.*

Proof. The orbit of a under $\operatorname{Aut}(\mathcal{P})$ is open in P because it contains $P \setminus (\{o\} \cup P_\infty)$. This implies that $\operatorname{Aut}(\mathcal{P})$ is a Lie group, see [20, 53.1]. Thus $\operatorname{Aut}(\mathcal{P})$ has no small subgroups. Now injectivity and continuity of δ yield that S has no small subgroups. This means that S is a Lie group, see [12]. The stabilizer of ∞ in $\operatorname{Aut}(\mathcal{P})$ acts transitively on the line ∞ because it contains $\delta(S)$. From [20, 52.3] it now follows that $P_\infty \approx \mathbb{S}_d$ and $P \setminus P_\infty \approx \mathbb{R}^{2d}$.

We pick $a \in P \setminus (\{o\} \cup P_\infty)$, then $\omega_a\colon S \to P\colon s \mapsto s(a)$ is a continuous injection. Our next aim is to show that $\dim S = \dim P$. For every neighborhood W in S we have $\dim W = \dim S$. If W is compact then ω_a induces a homeomorphism from W onto $\omega_a(W)$, and $\dim S = \dim W = \dim \omega_a(W) \le \dim P$ follows. Since S is σ-compact and locally compact, there exists a sequence $(W_n)_{n\in\mathbb{N}}$ of compact neighborhoods in S with $S = \bigcup_{n\in\mathbb{N}} W_n$. The Sum Theorem of topological dimension (cf. [20, 92.9]) says $\dim P \le \dim S$, and we obtain $\dim S = \dim P$.

Thus S and P are manifolds of the same dimension. Domain invariance (cf. [20, 51.19]) yields that δ induces a homeomorphism from S onto $\delta(S)$, as claimed. In particular, the group $\delta(S)$ is locally compact and thus closed in $\operatorname{Aut}(\mathcal{P})$ (cf. [26, 4.7]). The corestriction $\omega_a\colon S \to P \setminus (\{o\} \cup P_\infty) \approx \mathbb{R}^{2d} \setminus \{0\}$ of the evaluation map is open by a Baire category argument; see [26, 10.10(c)] or [20, 96.8]. □

2.2 Proposition. *If S is a topological group homeomorphic to $\mathbb{R}^n \setminus \{0\}$ with $n > 1$ then S is a Lie group isomorphic either to $\mathbb{C}^\times$ or to $\mathbb{H}^\times$.*

Proof. Assume that S is homeomorphic to $\mathbb{R}^n \setminus \{0\}$ for some $n > 1$. Then S has two ends (in the sense of Freudenthal [5]) and is thus isomorphic to the direct product $\mathbb{R} \times K$ with a compact connected group K by [11, Thm. 5], [6]. We note that K is a quotient of the locally euclidean group S, and thus a Lie group. Moreover, the group K is homotopy equivalent to S and thus to the sphere of dimension $n - 1$.

The Hopf–Samelson Theorem for connected compact Lie groups (see [8, 6.88]) is used in [8, 6.95] to show that a topological (Lie) group homeomorphic to a sphere is isomorphic either to $\mathbb{S}_{\mathbb{C}}$ or to $\mathbb{S}_{\mathbb{H}}$. Since the proof uses only cohomological invariants, it remains valid under the weaker hypothesis that the group in question is a compact connected group homotopic to a sphere. □

A much more elementary proof for the case $n = 2$ can be found in [21, 14.4] .

2.3 Lemma. *Assume that a locally compact, σ-compact topological group S acts continuously on $\mathcal{P}$, fixing o and ∞ such that the induced action on $P \smallsetminus (\{o\} \cup P_\infty)$ is regular. Then S also acts regularly on $\mathcal{L} \smallsetminus (\mathcal{L}_o \cup \{\infty\})$.*

Proof. Let $L \in \mathcal{L} \smallsetminus (\mathcal{L}_o \cup \{\infty\})$. The groups in question are known from 2.1 and 2.2. We study $S = \mathbb{C}^\times$ first. The stabilizer S_L fixes $L \wedge \infty$ and thus a line through o. Since S_L is centralized by the transitive group S on $\mathcal{L}_o$ we infer that S_L fixes each line through o. Now S_L is trivial because the two lines L and ∞ outside $\mathcal{L}_o$ are also fixed. The orbit of L has full dimension in P and is thus open, see [20, 53.1]. Since this applies to any L we infer that S acts transitively on the connected set $\mathcal{L} \smallsetminus (\mathcal{L}_o \cup \{\infty\})$.

It only remains to consider $S = \mathbb{H}^\times$. As before, we infer that the center Z of S has trivial intersection with S_L. Therefore, the quotient map q from S onto $S/Z \cong \mathrm{SO}(3, \mathbb{R})$ induces an embedding of the Lie algebra of S_L into the Lie algebra of $\mathrm{SO}(3, \mathbb{R})$. The proper subalgebras of the latter have dimension at most one. This implies that the connected component of $q(S_L)$ is closed, and S_L contains elements of prime order if $\dim S_L > 0$. Any such element would fix more than one line through o (cf. [20, 55.25]) and then also a point in $P \smallsetminus (\{o\} \cup P_\infty)$. This contradiction shows that the stabilizer of any L is discrete. Again, every orbit in $\mathcal{L} \smallsetminus (\mathcal{L}_o \cup \{\infty\})$ is open, and transitivity of S follows.

Both the group $S = \mathbb{H}^\times$ and its orbit $\mathcal{L} \smallsetminus (\mathcal{L}_o \cup \{\infty\})$ are simply connected. The discrete stabilizer is trivial because it coincides with the fundamental group of the orbit. □

2.4 Lemma. *Let $\mathcal{P}$ be a 2-dimensional compact Schellhammer plane.*

1. *If $\dim \mathrm{Aut}(\mathcal{P}) > 2$ then $\mathcal{P}$ is a Moulton plane (possibly the classical real plane).*
2. *If $\dim \mathrm{Aut}(\mathcal{P}) \le 2$ then the connected component of $\mathrm{Aut}(\mathcal{P})$ is the (unique) Schellhammer group.*

Proof. If $\dim \mathrm{Aut}(\mathcal{P}) \ge 4$ then the assertion follows from Salzmann's characterization of the Moulton planes [19, 4.8], see [20, 38.1]. Thus it remains to investigate planes $\mathcal{P}$ with $\dim \mathrm{Aut}(\mathcal{P}) = 3$. These planes and their automorphism groups have been determined, see [20, Sect. 34–37]; it turns out that none of these contains a Schellhammer group.

The assertion for $\dim \mathrm{Aut}(\mathcal{P}) \le 2$ follows from the fact that every closed subgroup of full dimension contains the connected component. □

2.5 Theorem. *Every polarity of a non-desarguesian Moulton plane normalizes a Schellhammer group. The Schellhammer groups form a single conjugacy class.*

Proof. One knows that the automorphism group A of any non-desarguesian Moulton plane is a Lie group with Lie algebra (isomorphic to) $\mathfrak{gl}(2, \mathbb{R})$, cf. [20, 34.8]. Moreover, the commutator group A' is closed [20, 34.4 6)] and isomorphic to the simply connected covering of $\mathrm{SL}(2, \mathbb{R})$. In particular, the (infinite cyclic) center Z of A' is closed in A, and $A/Z \cong \mathbb{R}/\mathbb{Z} \times \mathrm{PSL}(2, \mathbb{R})$.

Each polarity of the plane induces an involutory automorphism of the group A and thus also an involution on the Lie algebra $\mathfrak{sl}(2,\mathbb{R}) = \{X \in \mathbb{R}^{2\times 2} \bigm| \operatorname{tr} X = 0\}$ of traceless matrices. Like every automorphism of $\mathfrak{sl}(2,\mathbb{R})$ the polarity preserves Killing's quadratic form $\kappa(X) := -\det X$ for $X \in \mathfrak{sl}(2,\mathbb{R})$. This form is extended to $\mathfrak{gl}(2,\mathbb{R}) = \mathbb{R}\,\mathrm{id} \oplus \mathfrak{sl}(2,\mathbb{R})$ by $\kappa(s\,\mathrm{id} + X) := \kappa(X)$ for $s \in \mathbb{R}$ and $X \in \mathfrak{sl}(2,\mathbb{R})$.

By 2.1 every Schellhammer group $S \le A$ is closed in A and isomorphic to $\mathbb{C}^\times$; its Lie algebra $\mathfrak{s}$ is 2-dimensional and commutative. As $\mathfrak{sl}(2,\mathbb{R})$ has no two-dimensional abelian subalgebras we find $\mathfrak{s} = \mathbb{R}\,\mathrm{id} \oplus \mathbb{R}\,X$ for some $X \in \mathfrak{sl}(2,\mathbb{R})$. The restriction $\kappa_{\mathfrak{s}}$ of the Killing form κ to $\mathfrak{s}$ is degenerate because $\mathfrak{s}$ contains the radical $\mathbb{R}\,\mathrm{id}$ of κ. The value $\kappa(X)$ decides whether $\kappa_{\mathfrak{s}}$ is positive semidefinite, negative semidefinite, or zero.

The maximal compact subgroup $U \cong \mathbb{R}/\mathbb{Z}$ of S is not central in A because the center of A is isomorphic to $\mathbb{R} \times \mathbb{Z}$. Thus S/Z is compact, and we infer that $\kappa_{\mathfrak{s}}$ is negative semidefinite. This means that the Schellhammer groups in A form a single conjugacy class, see [7, Prop. 1.2].

The automorphism ψ induced by a polarity on the Lie algebra $\mathfrak{gl}(2,\mathbb{R})$ leaves the derived algebra $\mathfrak{sl}(2,\mathbb{R})$ invariant and satisfies $\psi^2 = \mathrm{id}$. We claim that ψ fixes $\mathbb{R}\,X$ for some $X \in \mathfrak{sl}(2,\mathbb{R})$ with $\kappa(X) < 0$ and thus the Lie algebra $\mathbb{R}\,\mathrm{id} \oplus \mathbb{R}\,X$ of some Schellhammer group.

Since $\psi^2 = \mathrm{id}$ we have $\mathfrak{sl}(2,\mathbb{R}) = E_+ \oplus E_-$ where $E_\sigma := \{Y \in \mathfrak{sl}(2,\mathbb{R}) \mid \psi(Y) = \sigma Y\}$ and $E_+ \perp E_-$ with respect to κ. Note that $E_+ \neq \{0\}$ because $-\mathrm{id}$ is not an automorphism of $\mathfrak{sl}(2,\mathbb{R})$. If E_+ does not contain X as required then the restriction of κ to E_+ is positive definite. Then the restriction of κ to the orthogonal complement E_- is negative definite, and we find the required subalgebra there. □

2.6 Remark. An alternative, less direct but more elegant argument uses deeper structure theory of Lie groups in order to prove conjugacy of Schellhammer groups of Moulton planes in the following way. Each Schellhammer group on a Moulton plane is isomorphic to $\mathbb{C}^\times \cong \mathbb{R}/\mathbb{Z} \times \mathbb{R}$ and thus (contained in) the centralizer of some maximal compact subgroup. The latter are in a single conjugacy class by the Mal′tsev–Iwasawa Theorem (see [10, Thm. 13]).

3 Compact connected Schellhammer planes

3.1 Construction. Schellhammer ([22, § 7], cf. [20, 34.1]) has shown that many examples of Schellhammer planes can be obtained from the action of the multiplicative group $\mathbb{C}^\times$ of complex numbers on $\mathbb{C} \cong \mathbb{R}^2$, as follows. The lines through 0 remain the usual ones (these are of the form $c\mathbb{R}$ with $c \in \mathbb{C}^\times$). In order to describe the representative $L(1)$ for the regular orbit of lines, we use a continuous function $g\colon \left(-\frac{\pi}{2}, \frac{\pi}{2}\right) \to (0,\infty)$ with $\lim_{\varphi\to\pm\pi/2} g(\varphi) = 0$ such that $\log(g)$ is a strictly concave function. Now we set $L_g(1) := \{e^{i\varphi}/g(\varphi) \bigm| -\frac{\pi}{2} < \varphi < \frac{\pi}{2}\}$ and $L_g(c) := cL_g(1)$. Then $\mathcal{A}_g := (\mathbb{C}, \{L_g(c) \mid c \in \mathbb{C}^\times\} \cup \{c\mathbb{R} \mid c \in \mathbb{C}^\times\})$ is an affine plane, and $\mathbb{C}^\times$ is a Schellhammer group on $\mathcal{A}_g$. In the sequel we will assume $1 \in L(1)$; this just means $g(0) = 1$. Multiplying with a suitable real number (i.e., applying an element of the Schellhammer group) we may always achieve this.

It has been shown in [22] that *every* two-dimensional compact projective plane admitting a group isomorphic to $\mathbb{C}^\times$ is obtained by the construction in 3.1. This is of importance in view of the fact that $\mathbb{R}^2$ and $\mathbb{C}^\times \cong \mathbb{R} \times \mathbb{R}/\mathbb{Z}$ are the only two-dimensional commutative locally compact connected groups acting faithfully on two-dimensional compact projective planes, cf. [20, 32.18]. The third candidate from the list of two-dimensional commutative connected Lie groups, namely the compact group $(\mathbb{R}/\mathbb{Z})^2$ has been excluded by Salzmann in [16, Hauptsatz 4.3].

3.2 Remarks. If $g\colon \left(-\frac{\pi}{2},\frac{\pi}{2}\right) \to (0,+\infty)$ is differentiable then $\log(g)$ is strictly concave if, and only if, the derivative $(\log(g))' = g'/g$ is a decreasing homeomorphism from $\left(-\frac{\pi}{2},\frac{\pi}{2}\right)$ onto $\mathbb{R}$. Thus a differentiable function $g\colon \left(-\frac{\pi}{2},\frac{\pi}{2}\right) \to (0,+\infty)$ satisfies the conditions in 3.1 if, and only if, its reciprocal $f := 1/g$ satisfies $\lim_{\varphi\to\pm\pi/2} f(\varphi) = +\infty$ and the logarithmic derivative $f'/f = (\log(f))' = -(\log(g))' = -g'/g$ is an increasing homeomorphism from $\left(-\frac{\pi}{2},\frac{\pi}{2}\right)$ onto $\mathbb{R}$. The treatment in [20, 34.1] takes this point of view.

3.3 Moulton planes. Fix a real number $s > 0$ and define $g_s^M\colon \left(-\frac{\pi}{2},\frac{\pi}{2}\right) \to (0,+\infty)$ by $g_s^M(\varphi) := \cos(\varphi)/e^{s\varphi}$. Then g_s^M satisfies the assumptions of 3.1. The projective hull of the resulting plane $\mathcal{A}_{g_s^M}$ is isomorphic to the projective hull of a Moulton plane. This model of the Moulton plane is due to Betten [2], see [20, 34.2] (where the reciprocal $f_s = 1/g_s^M$ is used, cf. 3.2). For some values of s, the standard lines defined by these functions are shown in Fig. 1. For $s = 0$ the construction also works, yielding $L(1) = 1 + i\mathbb{R}$: thus $\mathcal{A}_{\cos}$ is the affine plane over $\mathbb{R}$, described as the classical Schellhammer plane for $\mathbb{C}/\mathbb{R}$.

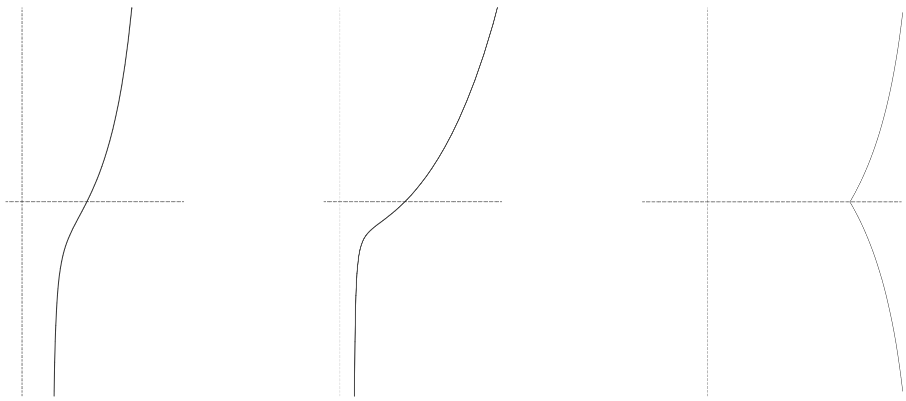

Figure 1: The lines $L_{g_s^M}(1)$ for $s \in \{0.5, 1\}$, and the line L_{h_m} for $m = 1.1$ (from left to right)

Note that the Schellhammer group on the Moulton plane is not uniquely determined. However, all the Schellhammer groups on a given Moulton plane (i.e., for given $s > 0$) fix the same anti-flag, and they form a single conjugacy class in the group of all automorphisms, cf. 2.5.

3.4 Examples. As another explicit family of examples of functions with the required properties, we mention $g_m\colon x \mapsto \frac{\cos(x)}{m+(1-m)\cos(x)}$, for $m \in (0,2]$. These functions are obtained by translating Sperner's examples f_m from [23] to our present setting. For some values of m, the standard lines defined by these functions are shown in Fig. 2.

The function g used to define the standard line $L_g(1)$ need not be differentiable. For instance, the map (see Fig. 1) $h_m\colon \left(-\frac{\pi}{2},\frac{\pi}{2}\right) \to (0,+\infty)\colon \varphi \mapsto \cos\left(\frac{\pi-2m}{\pi}|\varphi| + m\right)/\cos(m)$ is admissible for any $m \in [0, \frac{\pi}{2} - 1)$ but not differentiable at $\varphi = 0$.

3.5 Notation. We are going to use Hamilton's quaternions $\mathbb{H} = \mathbb{C} + j\mathbb{C}$: the multiplication rule is $(u+jv)(x+jy) = (ux-\bar{v}y)+j(vx+\bar{u}y)$. Identifying $\mathbb{H}$ with the space $\mathbb{C}^{2\times 1}$ of columns (i.e., using coordinates with respect to the basis $1, j$ for the right vector space $\mathbb{H}$ over $\mathbb{C}$), we

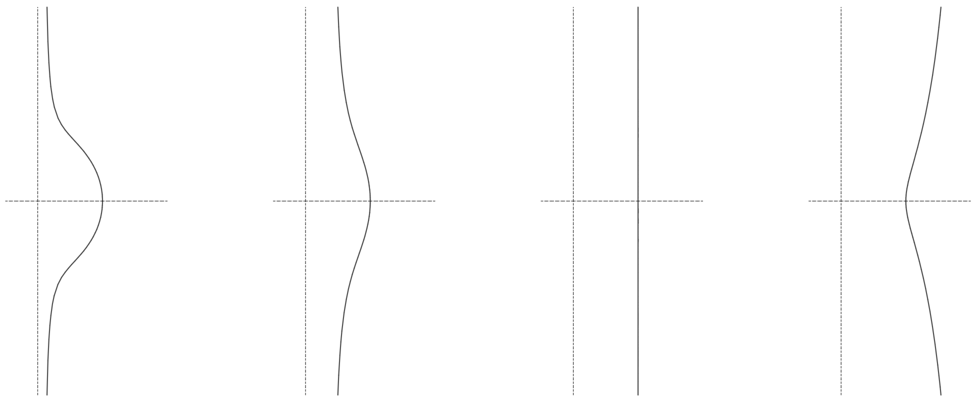

Figure 2: The lines $L_{g_m}(1)$ for $m \in \{0.1,\ 0.4,\ 1,\ 2\}$ (from left to right)

describe multiplication with $u + jv$ from the left by the matrix $\left(\begin{smallmatrix} u & -\bar{v} \\ v & \bar{u} \end{smallmatrix}\right) \in \mathrm{SU}(2,\mathbb{C}) \cdot \mathbb{R}^{>}$, where $\mathbb{R}^{>} := (0, +\infty)$ is the multiplicative group of positive real numbers.

Recall that the map $\kappa\colon \mathbb{H} \to \mathbb{H}\colon u + jv \mapsto \overline{u + jv} := \bar{u} - jv$ is an anti-automorphism of the field $\mathbb{H}$ that extends complex conjugation. For any subset $U \subseteq \mathbb{H}$, let $\mathbb{S}_U := \{x \in U \mid x\bar{x} = 1\}$.

3.6 Construction. The following class of planes was introduced by Sperner [23] (who used the function $f(r) := \frac{\pi}{2} - g^{-1}(\frac{1}{r})$ in his description[1] — we modify the description to show the similitude to Schellhammer's examples 3.1, and to adapt to our general naming conventions):

Let $g\colon [0, \frac{\pi}{2}) \to (0, 1]$ be a differentiable bijection such that $\log(g)$ is strictly concave, and choose a real constant c.

In our notation $K_{g,c} := \left\{ \left(g(\varphi)^{-1} t \sin(\varphi), g(\varphi)^{-1-ic} \cos(\varphi) \right) \,\middle|\, \varphi \in [0, \frac{\pi}{2}),\ t \in \mathbb{S}_{\mathbb{C}} \right\}$ replaces Sperner's set $L_{f,c}$. The group $G := \mathrm{SU}(2,\mathbb{C}) \times \mathbb{C}^{\times}$ acts by linear transformations on $\mathbb{C}^{2\times 1}$ via $(x, y)' \mapsto X(x, y)'z$; here $(X, z) \in G$ and $(x, y)'$ is the transpose of (x, y). Let $\mathcal{L}_0 := \{ v\mathbb{C} \mid v \in \mathbb{C}^{2\times 1} \smallsetminus \{(0,0)'\} \}$ be the set of complex lines through $o := (0,0)'$, and let $\mathcal{L}_1$ be the orbit of $K_{g,c}$ under G. Sperner shows that the group $S := \mathrm{SU}(2,\mathbb{C}) \times \mathbb{R}^{>}$ acts transitively on $\mathcal{L}_1$, and that $\mathcal{E}_{g,c} := (\mathbb{C}^2, \mathcal{L}_0 \cup \mathcal{L}_1)$ is an affine plane (in fact, the projective hull $\overline{\mathcal{E}_{g,c}}$ is a four-dimensional compact projective plane). Moreover, he notes that every plane $\mathcal{E}_{g,c}$ is isomorphic to $\mathcal{E}_{\tilde{g},0}$ for some admissible function $\tilde{g}$. Therefore, we will only consider the case $c = 0$ in the sequel.

Note that the group S is nothing but the group of left multiplications by elements from $\mathbb{H}^{\times}$. The function $g_1 = \cos$ yields $L(1) = 1 + j\mathbb{C}$; the plane $\mathcal{E}_{\cos,0}$ is the affine plane over $\mathbb{C}$.

3.7 Proposition. *The group $S = \mathbb{H}^{\times}$ is a Schellhammer group on $\overline{\mathcal{E}_{g,0}}$. The point $a := 1$ lies on*

$$L(1) := j^{-1} K_{g,0} = \left\{ \left(\cos(\varphi) + \sin(\varphi) j t \right) g(\varphi)^{-1} \,\middle|\, \varphi \in [0, \tfrac{\pi}{2}), t \in \mathbb{S}_{\mathbb{C}} \right\} .$$

The composition $\alpha := \kappa\iota$ with the inversion map $\iota\colon u + jv \mapsto (\bar{u}u + \bar{v}v)^{-1}(\bar{u} - jv)$ is an automorphism of S and $\alpha\iota = \kappa$ leaves the set $L(1)$ invariant.

For $z \in \mathbb{C}^{\times} \cup j\mathbb{C}^{\times}$ the inner automorphism $\zeta_z\colon h \mapsto zhz^{-1}$ of $\mathbb{H}$ extends to an automorphism of $\overline{\mathcal{E}_{g,0}}$: the line map is given by $zL(x)z^{-1} = L(zxz^{-1})$. The following special cases will be of particular importance.

- *$\beta\colon \mathbb{H} \to \mathbb{H}\colon h = u + jv \mapsto -jhj = \bar{u} + j\bar{v}$ is a Baer involution, the set $\mathbb{R} + j\mathbb{R}$ of affine fixed points of β carries the Schellhammer plane $\mathcal{A}_g$.*

[1] Our concavity condition on $\log(g)$ is Sperner's convexity condition on $\log(f^{-1}(\arcsin))$.

- $\rho\colon \mathbb{H} \to \mathbb{H}\colon h = u + jv \mapsto -ihi = u - jv$ *is a reflection (i.e., an involutory perspectivity) with axis* $\mathbb{C}$*, the center is the point at infinity for* $L(1)$ *and* $j\mathbb{C}$.
- $\beta\rho = \rho\beta\colon \mathbb{H} \to \mathbb{H}\colon h = u + jv \mapsto -ijhij = \bar{u} - j\bar{v}$ *is another Baer involution: it is the conjugate of* β *under the automorphism* $\lambda\colon h = u + jv \mapsto \frac{1}{2}(1+i)h(1-i) = u - jiv$.

The group generated by $\{\alpha, \beta, \rho\}$ *in* $\mathrm{Aut}(\mathbb{H}^\times) \cong \mathbb{R}^\times \times \mathrm{SO}(3, \mathbb{R})$ *is elementary abelian of order* 2^3.

Proof. It is clear that S is a group of automorphisms of $\mathcal{E}_{g,0}$ acting regularly on $\mathbb{H} \smallsetminus \{0\}$. Regularity of the action on $\mathcal{L} \smallsetminus \mathcal{L}_0$ follows from 2.3.

For $z \in \mathbb{C}^\times \cup j\mathbb{C}^\times$ it is easy to see that the inner automorphism $h \mapsto zhz^{-1}$ fixes 1 and leaves $L(1)$ invariant. Thus $zL(x)z^{-1} = zxL(1)z^{-1} = zxz^{-1}L(1) = L(zxz^{-1})$, as claimed. Invariance of $\mathcal{L}_o$ is clear if $z \in \mathbb{C}^\times$, and easy to check for $z \in j\mathbb{C}^\times$. Thus we have an automorphism of the affine plane $\mathcal{E}_{g,0}$ which extends to the projective hull.

The fixed points for β, ρ and $\rho\beta$ are easy to see. It remains to note that the intersection of $L(1)$ with $\mathbb{R} + j\mathbb{R}$ equals the standard line in $\mathcal{A}_g$ if we use the model $\mathbb{R} + j\mathbb{R}$ for $\mathbb{C}$. □

4 The normalizer of the Schellhammer group

We consider a Schellhammer plane $\mathcal{P}$ with Schellhammer group S. The normalizer of S in $\mathrm{Aut}(\mathcal{P})$ will be denoted by N. The unique fixed elements o and ∞ of S are also fixed by N; thus N acts on $P \smallsetminus (\{o\} \cup P_\infty)$. Since S acts transitively on that set the normalizer is the product of S and the stabilizer N_a, for any $a \in P \smallsetminus (\{o\} \cup P_\infty)$.

4.1 Lemma. *Via conjugation, the stabilizer* N_a *acts faithfully on the group* S.

Proof. The kernel of the action of N on S is the centralizer $C := \mathrm{C}_N(S)$. For $c \in C_a$ and $s \in S$ we have $cs(a) = sc(a) = s(a)$. Thus c fixes each point in the orbit $S(a)$, and is trivial. □

We return to our identification of S with its regular orbit $S(a)$; then $a = 1$ and $N_a = N_1$.

4.2 Lemma. *An automorphism* μ *of* S *extends to an element of* N_1 *if, and only if, there exists* $m \in S$ *such that* $\mu(L(1)) = L(m)$.
If this is the case, the action of μ *on the lines is given by* $\mu(L(s)) = L(\mu(s)\, m)$.

Proof. In order to show that μ extends to an automorphism of $\mathcal{P}$ it suffices by 1.2 to show that the set $\mathcal{L} \smallsetminus (\mathcal{L}_o \cup \{\infty\})$ is invariant under μ. We compute $\mu(L(s)) = \mu(\{sx \mid x \in L(1)\}) = \{\mu(sx) \mid x \in L(1)\} = \mu(s)\mu(L(1)) = \mu(s)L(m)$, and the assertion follows. □

4.3 Proposition. *Consider* $\mathcal{A}_g$ *for a function* g *as in 3.1, with* $g(0) = 1$*. Then either the group* N_1 *is trivial, or it has order* 2*. In the latter case, the involution* $\kappa\colon \mathbb{C} \to \mathbb{C}\colon x \mapsto \bar{x}$ *generates* N_1. *The corresponding automorphism of* $\overline{\mathcal{A}_g}$ *is a reflection with axis* $\mathbb{R}$*. The center is the point at infinity for* $i\mathbb{R}$ *and* $L(1)$.

The involution κ *belongs to* N_1 *if, and only if, the function* g *is even, i.e., if* $g(-\varphi) = g(\varphi)$ *holds for all* $\varphi \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$.

Proof. This is shown in [22, 7.8], we give a proof for the reader's convenience. The plane $\mathcal{A}_g$ is an affine $\mathbb{R}^2$-plane, its projective hull is a two-dimensional compact projective plane, and all its collineations are continuous (cf. [20, 32.9]). Thus any element of N_1 induces a continuous automorphism of $\mathbb{C}^\times \cong \mathbb{R} \times \mathbb{R}/\mathbb{Z}$. The automorphisms of the additive group $\mathbb{R} \times \mathbb{R}/\mathbb{Z}$ are the maps of the form $(x, y+\mathbb{Z}) \mapsto (rx, \pm y + xt + \mathbb{Z})$ with $r \in \mathbb{R}^\times$ and $t \in \mathbb{R}$ (e.g., see [26, 25.8]). The corresponding map $e^{x+2\pi i y} \mapsto e^{rx+2\pi i(xt\pm y)}$ will be denoted by $\alpha^{\pm}_{r,t}$.

The square of any such automorphism fixes each element of $\{0\} \times \mathbb{R}/\mathbb{Z}$: this subgroup corresponds to $\mathbb{S}_\mathbb{C}$ and contains two points from each line through o. Therefore, the squares of elements of N_1 lie in the stabilizer of a quadrangle in a two-dimensional compact projective plane. This stabilizer is trivial, cf. [20, 32.10], and $\rho^2 = \mathrm{id}$ follows for each $\rho \in N_1$.

The elements $\alpha^+_{r,t}$ fix $\mathbb{S}_\mathbb{C}$ point-wise, and do never belong to $N_1 \setminus \{\mathrm{id}\}$. If $\rho := \alpha^-_{r,t}$ belongs to N_1 then ρ fixes at least two affine points on the line $o \vee 1 = \mathbb{R}$. Thus $\rho^2 = \mathrm{id}$ implies that ρ is a reflection at that line (cf. [20, 32.12]), and fixes each point in $\mathbb{R}$. This means $\rho = \alpha^-_{1,0} = \kappa$. Since $\alpha^-_{1,0}$ fixes the line $i\mathbb{R}$ it also fixes the parallel $L_g(1)$ through 1. Thus g is even. □

5 Polarities

Polarities of compact connected planes have been investigated by Salzmann [17, p. 260], [18], Bedürftig [1], Polster [15], Immervoll [9] and the second author, see [25], [24]. If the plane is topological, we will tacitly assume that the polarity is continuous.

Let $\iota\colon S \to S\colon s \mapsto s^{-1}$ denote the inversion map on a Schellhammer group S.

5.1 Theorem. *Let γ be an automorphism of S such that $\gamma^2 = \mathrm{id}$ and such that $\gamma\iota(L(1)) = L(1)$. Then mapping s to $L(\gamma(s))$ extends to a polarity J_γ of $\mathcal{P}$. This polarity swaps o with ∞, and the line $o \vee s \in \mathcal{L}_o$ with the point at infinity for $L(\gamma(s))$.*

Proof. By 1.2 it suffices to show that interchanging s with $L(\gamma(s))$ gives an isomorphism of incidence structures from $(P \setminus (\{o\} \cup P_\infty), \mathcal{L} \setminus (\mathcal{L}_o \cup \{\infty\}))$ onto the dual incidence structure, namely $(\mathcal{L} \setminus (\mathcal{L}_o \cup \{\infty\}), P \setminus (\{o\} \cup P_\infty))$. To this end, we note $s \in L(t) \iff t^{-1}s \in L(1) \iff \gamma\iota(t^{-1}s) \in L(1) \iff \gamma(t) \in L(\gamma(s))$. □

If we discuss J_γ in the sequel, we will always tacitly assume that γ is an automorphism of S such that $\gamma^2 = \mathrm{id}$ and such that $L(1)$ is left invariant by $\gamma\iota$. Note that γ will in general not be an automorphism of $\mathcal{P}$.

5.2 Example. If the Schellhammer group S is commutative then we may use the inversion map ι for γ. The map $\gamma\iota$ is then just the identity, and imposes no restriction on the shape of $L(1)$. We call J_ι the *standard polarity* of the Schellhammer plane.

Once we have found one polarity J_γ of $\mathcal{P}$, we know that each other polarity of $\mathcal{P}$ belongs to the coset $\mathrm{Aut}(\mathcal{P})J_\gamma$. In each one of the known cases every polarity of $\mathcal{P}$ normalizes some Schellhammer group S in $\mathrm{Aut}(\mathcal{P})$. It is worth the effort to study the polarities in the coset $NJ_\gamma = SN_1J_\gamma$ in detail.

5.3 Theorem. *Let γ be an automorphism of S such that J_γ is a polarity. Consider $\mu \in N_1$ as in 4.2 and let m denote the unique element $m \in S$ with $\mu(L(1)) = L(m)$.*

1. *For $s \in S$ the composition $s\mu J_\gamma$ is a polarity precisely if $(\mu\gamma)^2$ is the inner automorphism $x \mapsto mxm^{-1}$ of S and $s\,\mu\gamma(s) = m^{-1}$. We also have $\mu\gamma(m) = m$ in that case.*
   If μ fixes $L(1)$ these conditions simplify to $(\mu\gamma)^2 = \mathrm{id}$ and $\mu\gamma(s) = s^{-1}$.
2. *In particular, the composition μJ_γ is a polarity exactly if $(\mu\gamma)^2 = \mathrm{id}$ and $m = 1$. In other words: $\mu\gamma$ satisfies the requirements in 6.1 such that $J_{\mu\gamma}$ is a polarity, and $\mu J_\gamma = J_{\mu\gamma}$.*
3. *If S is commutative, we may further specialize these assertions: for $\mu \in N_1$ and $s \in S$ the composition $s\mu J_\iota$ is a polarity exactly if $\mu^2 = \mathrm{id}$ and μ fixes $L(s^{-1})$.*

In particular, the coset $\{sJ_\iota \mid s \in S\}$ *consists of polarities if* S *is commutative.*

Proof. Using $\mu(L(y)) = L(\mu(y)m)$ we compute the image of $L(x)$ under the square of $s\mu J_\gamma$ as $L\left(s\,\mu\gamma(s)\,(\mu\gamma)^2(x)\,m\right)$. Thus $s\mu J_\gamma$ is a polarity precisely if $x = s\,\mu\gamma(s)\,(\mu\gamma)^2(x)\,m$ holds for all $x \in S$. Specializing this for $x = 1$ we find $s\,\mu\gamma(s) = m^{-1}$, and $(\mu\gamma)^2(x) = mxm^{-1}$ follows. Evaluating the square of $s\mu J_\gamma$ at the point 1 we also obtain $s\,\mu\gamma(s) = \mu\gamma(m^{-1})$, and thus $\mu\gamma(m) = m$.

For $s = 1$, we find $1 = s\,\mu\gamma(s) = m^{-1}$ and $(\mu\gamma)^2 = \mathrm{id}$ follows. Finally, if S is commutative and $\gamma = \iota$, the condition $s\,\mu\iota(s) = m^{-1}$ means $\mu(L(s^{-1})) = L(s^{-1})$. □

Together with 2.4, 2.5 and 4.3 our discussion so far yields:

5.4 Theorem. *Let* g *be a function as in 3.1 with* $g(0) = 1$.

1. *Every polarity of* $\overline{\mathcal{A}_g}$ *normalizes a Schellhammer group.*
2. *In any case we have the standard polarity* J_ι.
3. *Apart from the conjugates of* J_ι *further polarities exist precisely if the plane admits a reflection at an axis through* o. *In that case we may assume that* g *is even. Then* $\kappa \in N_1$ *leads to polarities* $sJ_{\kappa\iota}$ *with* $s \in \mathbb{R}^\times$. *Every polarity of* $\overline{\mathcal{A}_g}$ *is conjugate to* J_ι *or to one of these.* □

See 6.3 for conjugacy of these polarities, and 7.4 for centralizers and sets of absolute points.

5.5 Examples. Consider one of Sperner's planes $\overline{\mathcal{E}_{g,0}}$, cf. 3.6 and 3.7. If such a plane is not isomorphic to $\mathcal{P}_2\mathbb{C}$ then we know from [23] that the Schellhammer group $\mathbb{H}^\times$ is normal in $\mathrm{Aut}(\overline{\mathcal{E}_{g,0}})$, and the stabilizer N_1 is the semidirect product of $\langle\beta\rangle$ and the group $\mathrm{Z} := \{\zeta_z \mid z \in \mathbb{S}_\mathbb{C}\}$ of inner automorphisms $\zeta_z\colon h \mapsto zh\bar{z}$. Consider $\alpha = \kappa\iota$ as in 3.7, and recall that each element of $\langle\beta\rangle\mathrm{Z}$ fixes the line $L(1)$.

According to 5.3, we thus know that the duality $s\mu J_\alpha$ with $s \in S$ and $\mu \in \langle\beta\rangle\mathrm{Z}$ is a polarity precisely if $\mu\alpha$ is an involution and $\mu\alpha(s) = s^{-1}$. We determine the involutions in $\langle\beta\rangle\mathrm{Z}\alpha$ first.

As α acts trivially on $\mathbb{S}_\mathbb{H}$, it centralizes $\langle\beta\rangle\mathrm{Z}\alpha$. This means that, apart from the identity, we have to find the involutions in $\langle\beta\rangle\mathrm{Z}$. On one hand, the inner automorphism $\beta\zeta_z$ is induced by $jz \in j\mathbb{C}$, and $(jz)^2 \in \mathbb{R}$ implies that $\beta\mathrm{Z}$ consists of involutions. On the other hand, the only involution in Z is $\zeta_i = \rho$.

Thus we have determined the set $\left\{\mu \in N_1 \,\middle|\, (\mu\alpha)^2 = \mathrm{id}\right\} = \beta\mathrm{Z} \cup \{\mathrm{id}, \rho\}$. For each one of these automorphisms, the set $F_{\mu\kappa}$ of fixed points of $\mu\alpha\iota = \mu\kappa$ in $\mathbb{H}$ describes the set $\{s \in S \mid sJ_{\mu\alpha} \text{ is a polarity}\} = \left\{s \in S \,\middle|\, \mu\alpha(s) = s^{-1}\right\} = \{s \in F_{\mu\kappa} \mid s \neq 0\}$.

While it is easy to see that $F_\kappa = \mathbb{R}$ and $F_{\rho\kappa} = \mathbb{R} + \mathbb{C}j$, the case $\mu \in \beta\mathrm{Z}$ requires a little more effort: for $u, v \in \mathbb{C}$ and $z \in \mathbb{S}_\mathbb{C}$ we compute $\beta\zeta_z\kappa(u + jv) = -jz(\bar{u} - jv)\bar{z}j = u - jz^2\bar{v}$, and obtain the condition $-z\bar{v} = \bar{z}v$. This means $\bar{z}v \in i\mathbb{R}$, and $F_{\beta\zeta_z\kappa} = \mathbb{C} + jzi\mathbb{R}$.

We have thus found all polarities for Sperner's planes (conjugacy of these polarities will be discussed in 6.5 below, the centralizers and the sets of absolute points in 7.6):

5.6 Theorem. *Let* $\overline{\mathcal{E}_{g,0}}$ *be one of Sperner's planes, with* $\overline{\mathcal{E}_{g,0}} \not\cong \mathcal{P}_2\mathbb{C}$, *and let* $\alpha = \kappa\iota$, β, *and* ρ *be as in 3.7. Then the polarities of* $\overline{\mathcal{E}_{g,0}}$ *are the following:*

1. sJ_α *with* $s \in \mathbb{R} \setminus \{0\}$,
2. $sJ_{\rho\alpha}$ *with* $s \in (\mathbb{R} + j\mathbb{C}) \setminus \{0\}$,
3. $sJ_{\beta\zeta_z\alpha}$ *with* $s \in (\mathbb{C} + jzi\mathbb{R}) \setminus \{0\}$ *and* $\zeta_z(h) = zh\bar{z}$ *for* $z \in \mathbb{S}_\mathbb{C}$. □

6 Conjugacy Classes of Polarities

6.1 Theorem. *For $s,t \in S$, the conjugate of sJ_γ under t is $ts\,\gamma\iota(t)J_\gamma$. Thus polarities sJ_γ and $uJ_{\gamma'}$ are conjugates under S exactly if $\gamma=\gamma'$ and there exists $t\in S$ with $ts\,\gamma\iota(t)=u$.*

Proof. Computing the image of $L(x)$ under $tsJ_\gamma t^{-1}$ as $ts\,\gamma(t^{-1})\,\gamma(x)$ we obtain the first assertion. Now $tsJ_\gamma t^{-1}=uJ_{\gamma'}$ means $ts\,\gamma(t^{-1})\,\gamma(x)=u\,\gamma'(x)$ for all $x\in S$. Specializing $x=1$ we find $ts\,\gamma(t^{-1})=u$, and $\gamma=\gamma'$ follows. □

6.2 Corollary. *If S is commutative then the conjugacy classes under S in the set $\{sJ_\iota \mid s\in S\}$ of polarities (cf. 5.3) are the orbits under the group $S^\square := \{t^2 \mid t\in S\}$ of squares.* □

6.3 Examples. Let g be a function as in 3.1. We have $\mathbb{C}^\times = S = S^\square$, and the polarities sJ_ι and uJ_ι are conjugates for any choice of $s,u\in S$.

If g is even and $s,u\in\mathbb{R}^\times$ then $sJ_{\kappa\iota}$ and $uJ_{\kappa\iota}$ are polarities. They are conjugates under S precisely if there exists $t\in S$ with $t\bar{t}s=u$: this means that s and u have the same sign.

The Moulton planes do not admit a reflection in N_1. It has been proved by Salzmann [18] that the polarities of a non-desarguesian Moulton plane form a single conjugacy class; cf. also 2.5. Thus every polarity of such a plane is found by our present methods.

The full group $\mathrm{Aut}(\overline{\mathcal{A}_g})$ coincides with the normalizer N of S unless $\mathcal{A}_g$ is isomorphic to a Moulton plane $\mathcal{A}_{g^M_s}$ (including $\mathcal{A}_{g_0}=\mathcal{A}_2\mathbb{R}$), cf. 3.3. If $\mathcal{A}_g\not\cong\mathcal{A}_2\mathbb{R}$ admits the reflection κ, we thus obtain $\mathrm{Aut}(\overline{\mathcal{A}_g})=\langle\kappa\rangle S$, and there are three conjugacy classes of polarities, represented by J_ι, $J_{\kappa\iota}$, and $-J_{\kappa\iota}$.

6.4 Examples. We determine the conjugacy classes of polarities for a plane $\overline{\mathcal{E}_{g,0}}$, cf. 3.6 and 5.6. First of all, for any $s\in(\mathbb{C}+j i\mathbb{R})\smallsetminus\{0\}$ the conjugate of $sJ_{\beta\alpha}$ under an inner automorphism ζ_z with $z\in\mathbb{S}_\mathbb{C}$ equals $\zeta_z(s)J_{\beta\zeta_{\bar z}^2\alpha}=\zeta_z(s)J_{\beta\zeta_z^{-2}\alpha}$. As each element of $\mathbb{S}_\mathbb{C}$ possesses a square root in $\mathbb{S}_\mathbb{C}$, this yields that we have to search for representatives for the conjugacy classes among the elements sJ_μ with $\mu\in\{\alpha,\rho\alpha,\beta\alpha\}$ and suitable s.

For $\mu=\alpha$, we have $s\in\mathbb{R}^\times$, and $tJ_\alpha t^{-1}=t\bar{t}J_\alpha$ shows that $\{sJ_\alpha \mid s\in\mathbb{R}^\times\}$ is the union of two conjugacy classes, represented by J_α and $-J_\alpha$, respectively.

If $\mu\in\{\rho\alpha,\beta\alpha\}$ then our task is to understand the actions of $\mathbb{H}^\times$ on the real vector space $F_{\mu\iota}$ via $(x,t)\mapsto tx\,\mu\iota(t)$. These actions are linear representations[2] of $\mathbb{H}^\times$ on real vector spaces of dimension 3. The compact group $\mathbb{S}_\mathbb{H}$ does not act trivially, and thus induces a conjugate of the group of rotations. This group acts transitively on the set of rays in the space $F_{\mu\iota}$. It remains to understand the action for $t\in\mathbb{R}^\times$: since $\mu\iota\kappa=\mu\alpha\in\{\rho,\beta\}$ fixes each element of $\mathbb{R}$, the real factor acts via multiplication by t^2 on $F_{\mu\iota}$. We have shown that the given action of $\mathbb{H}^\times$ is transitive on $F_{\mu\iota}$. This means that each one of the sets $\{sJ_{\rho\alpha} \mid s\in F_{\rho\kappa}\}$ and $\{sJ_{\beta\alpha} \mid s\in F_{\beta\kappa}\}$ forms a conjugacy class under S, represented by $J_{\rho\alpha}$ and $J_{\beta\alpha}$, respectively.

6.5 Theorem. *Assume that $\overline{\mathcal{E}_{g,o}}$ is not isomorphic to $\mathcal{P}_2\mathbb{C}$. Then every polarity of $\overline{\mathcal{E}_{g,o}}$ is conjugate to one of J_α, $-J_\alpha$, $J_{\rho\alpha}$, and $J_{\beta\alpha}$.* □

[2] The linear map $\lambda\colon F_{\rho\kappa}\to F_{\beta\kappa}\colon x\mapsto jxi$ and the group automorphism ρ form a quasi-equivalence (λ,ρ) between these representations.

7 Centralizers of Polarities, and Absolute Points

In this section we will determine the absolute points (i.e., the points incident with their image) for representatives of conjugacy classes of polarities of the known Schellhammer planes. We concentrate on the non-classical examples from 3.1, 3.3 and 3.6. In these cases, the full group of automorphisms fixes the line ∞, and an affine point of view is reasonable.

7.1 Definitions. Let γ be an automorphism of S with $\gamma^2 = \mathrm{id}$ and such that $L(1)$ is left invariant by $\gamma\iota$ (i.e., such that J_γ is a polarity, cf. 5.1). We write $C_\gamma := \{s \in S \mid sJ_\gamma = J_\gamma s\}$ for the centralizer of the polarity in S. The set $A_\gamma := \{s \in S \mid s \in L(\gamma(s))\}$ consists of all affine absolute points of J_γ.

7.2 Remarks. A straightforward computation yields that $C_\gamma = \{s \in S \mid \gamma(s) = s\}$ is contained in $A_\gamma = \{s \in S \mid \gamma\iota(s)\, s \in L(1)\}$. The subgroup C_γ of S acts on A_γ via multiplication from the left. Thus A_γ is the union of a collection of right cosets $C_\gamma b$.

The absolute points at infinity (if any) are the parallel classes of lines $o \vee s$ such that $o \vee \gamma\iota(s)\, s \parallel L(1)$.

We did not introduce names for the set of affine points of polarities of the form tJ_γ. This is justified by the fact (cf. 6.3, 6.5) that $-J_{\kappa\iota}$ (on $\overline{\mathcal{A}_g}$) and $-J_\alpha$ (on $\overline{\mathcal{E}_{g,0}}$) represent all the polarities that we leave out: these polarities have no absolute points at all (and appear to be quite interesting exactly for that reason).

The centralizer of the standard polarity is almost trivial, and does not help much if we study the set of absolute points:

7.3 Proposition. *If S is commutative then the centralizer of the standard polarity in S is $C_\iota = \{s \in S \mid s^2 = 1\}$, while $A_\iota = \{s \in S \mid s^2 \in L(1)\}$.* □

7.4 Examples. For the Schellhammer planes $\mathcal{A}_g$ we find $C_\iota = \{1, -1\}$, while the defining equation for A_ι can be solved using polar coordinates: for $\varphi \in (-\frac{\pi}{2}, \frac{\pi}{2})$, put $s_\varphi^\pm := \pm e^{i\varphi/2}/\sqrt{g(\varphi)}$. Then $A_\iota = \{s_\varphi^+ \mid \varphi \in (-\frac{\pi}{2}, \frac{\pi}{2})\} \cup \{s_\varphi^- \mid \varphi \in (-\frac{\pi}{2}, \frac{\pi}{2})\}$. The two components of A_ι correspond to the two branches of the hyperbola formed by the affine absolute points in the classical case (where the defining condition reduces to $s^2 \in 1 + i\mathbb{R}$). We find that the absolute points at infinity belong to the two lines $(1 \pm i)\mathbb{R}$.

The situation is quite different if N_1 contains an involution fixing $L(1)$: according to 4.3, we may assume that g is an even function, and that the involution is κ. Then $J_{\kappa\iota}$ is a polarity. We find $C_{\kappa\iota} = \{s \in \mathbb{C} \mid s\bar{s} = 1\} = \mathbb{S}_\mathbb{C}$ and $A_{\kappa\iota} = \{s \in \mathbb{C}^\times \mid s\bar{s} \in L(1)\}$. Now $s\bar{s} \in \mathbb{R}$ yields that $s\bar{s} \in L(1)$ is equivalent to $s\bar{s} \in \mathbb{R} \cap L(1) = \{(o \vee 1) \wedge L(1)\} = \{1\}$, and $A_{\kappa\iota} = \mathbb{S}_\mathbb{C}$ follows. There are no absolute points at infinity.

It remains to discuss the polarities $rJ_{\kappa\iota}$ with $r \in \mathbb{R}^\times$, cf. 5.4. In fact, it suffices to consider $r \in \{1, -1\}$ by 6.2. The affine absolute points satisfy the condition $s\bar{s}/r \in L(1)$: this means $s\bar{s}/r \in (o \vee 1) \cap L(1) = \{1\}$. This equation means $s \in r\mathbb{S}_\mathbb{C}$ if $r > 0$, and has no solution if $r < 0$. In any case, there are no absolute points at infinity.

Thus $C_{\kappa\iota}$ acts regularly on the set $A_{\kappa\iota}$ of absolute points of $J_{\kappa\iota}$. As a marked contrast, the centralizer C_ι cannot act transitively on A_ι.

We note that the absolute points (if they exist) of a polarity of any two-dimensional compact plane form an oval (homeomorphic to $\mathbb{S}_\mathbb{C}$), cf. [16, Hilfssatz 5.8] and [1]. These ovals have very good topological intersection properties, see [3, 2.5]. Apart from secants and tangents, there also exist passing lines for each one of these ovals, see [3, 3.1, 3.7].

7.5 Examples. The sets $\{rJ_{\kappa\iota} \mid r > 0\}$ and $\{sJ_\iota \mid s \in S\}$ from 6.3 are fused into a single conjugacy class if $\mathcal{A}_g$ is the classical plane. However, these sets form two different conjugacy classes under the automorphism group of any plane $\mathcal{A}_g \not\cong \mathcal{A}_2\mathbb{R}$ with an even function g. For instance, we may choose g as $g_m \colon x \mapsto \frac{\cos(x)}{m+(1-m)\cos(x)}$, for $1 \neq m \in (0,2]$, cf. 3.4.

In any case, the members of $\{tJ_{\kappa\iota} \mid t < 0\}$ are not conjugates of J_ι because they are elliptic polarities (i.e., they do not have any absolute points), see 7.4. The fact that J_ι and $J_{\kappa\iota}$ are not conjugates under the normalizer of the Schellhammer group (which fixes the line at infinity) corresponds to the fact that J_ι has absolute points at infinity while $J_{\kappa\iota}$ has none.

7.6 Examples. Sperner's planes $\overline{\mathcal{E}_{g,0}}$ (see 3.6) are Schellhammer planes with Schellhammer group $S = \mathbb{H}^\times$. Since $\mathbb{H}^\times$ is not commutative there are no standard polarities. We concentrate on the case where the plane is not isomorphic to $\mathcal{P}_2\mathbb{C}$: then every continuous polarity is a conjugate of J_α, $-J_\alpha$, $J_{\rho\alpha}$, or $J_{\beta\alpha}$, see 6.5.

We compute C_γ for $\gamma \in \{\alpha, \rho\alpha, \beta\alpha\}$ first. One has $C_\gamma \subseteq \mathbb{S}_\mathbb{H}$ because α inverts the absolute value while ρ and β preserve it. Now α fixes each element of $\mathbb{S}_\mathbb{H}$, and we find $C_\gamma = \mathbb{S}_{F_\gamma}$. Explicitly, this gives $C_\alpha = \mathbb{S}_\mathbb{H}$, $C_{\rho\alpha} = \mathbb{S}_\mathbb{C}$, and $C_{\beta\alpha} = \mathbb{S}_{\mathbb{R}+j\mathbb{R}}$.

The affine absolute points of J_γ are those in $A_\gamma = \{s \in \mathbb{H}^\times \mid \gamma\iota(s)\, s \in L(1)\}$. For $s \in A_\alpha$ the real number $\bar{s}s$ lies in $\mathbb{C} \cap L(1)$. The only point in that intersection is the point 1 where the lines $\mathbb{C}$ and $L(1)$ meet. Thus $A_\alpha = \mathbb{S}_\mathbb{H}$.

For $s = r(u + jv) \in A_{\rho\alpha}$ with $r > 0$ and $u + jv \in \mathbb{S}_\mathbb{H}$ we compute $\rho\alpha\iota(s)\, s = (\bar{u}u - \bar{v}v) + j(uv - vu) = \bar{u}u - \bar{v}v + 2juv$. This quaternion belongs to $L(1)$ only if there exist $\varphi \in [0, \frac{\pi}{2})$ and $t \in \mathbb{S}_\mathbb{C}$ such that

$$r^2 = \frac{1}{g(\varphi)}\,, \quad \bar{u}u - \bar{v}v = \cos(\varphi)\,, \quad \text{and} \quad 2uv = t\sin(\varphi)\,.$$

We write $u = r_u s_u$ and $v = r_v s_v$ with $r_u, r_v > 0$ and $s_u, s_v \in \mathbb{S}_\mathbb{C}$. Then our conditions read $r_u^2 - r_v^2 = \cos(\varphi)$, $2r_u r_v = \varepsilon\sin(\varphi)$, and $t = \varepsilon s_u s_v$, where the sign $\varepsilon \in \{1, -1\}$ is chosen suitably. For $\psi := \varepsilon\varphi$ this becomes $r_u^2 - r_v^2 = \cos(\psi)$, $2r_u r_v = \sin(\psi)$, and there is $\vartheta \in (-\frac{\pi}{4}, \frac{\pi}{4}) \cup (\frac{3\pi}{4}, \frac{5\pi}{4})$ such that $r_u = \cos(\vartheta)$ and $r_v = \sin(\vartheta)$. Since we are free to replace s_u by $-s_u$ or s_v by $-s_v$, it suffices to use $\vartheta \in [0, \frac{\pi}{4})$. We thus have found

$$A_{\rho\alpha} = \left\{ \frac{1}{\sqrt{g(2\vartheta)}} \left(\cos(\vartheta)s_u + j\sin(\vartheta)s_v\right) \,\middle|\, \vartheta \in [0, \tfrac{\pi}{4}), \quad s_u, s_v \in \mathbb{S}_\mathbb{C} \right\}.$$

The case $\gamma = \beta\alpha$ involves the most complicated arguments. For $r > 0$ and $w \in \mathbb{S}_\mathbb{H}$ we have $rw \in A_{\beta\alpha}$ precisely if there exist $\varphi \in [0, \frac{\pi}{2})$ and $t \in \mathbb{S}_\mathbb{C}$ such that $r^2 = 1/g(\varphi)$ and $-j\bar{w}jw = \cos(\varphi) + j\sin(\varphi)t$. If we write $w = u + jv$ with $u, v \in \mathbb{C}$, the latter condition becomes $u^2 + v^2 = \cos(\varphi)$ and $(\bar{u}v - \bar{v}u)i = \varepsilon\sin(\varphi)$, with $\varepsilon \in \{1, -1\}$. We write $\psi := \varepsilon\varphi$ again, then our conditions are

$$u^2 + v^2 = \cos(\psi)\,, \quad (\bar{u}v - \bar{v}u)i = \sin(\psi)\,, \quad \bar{u}u + \bar{v}v = 1\,.$$

The first of these three conditions implies that there exists $f(\psi) > 0$ such that $(u + vi)(u - vi) = u^2 + v^2 = f(\psi)(u + vi)(\overline{u + vi})$, and $u - vi = f(\psi)(\bar{u} - \bar{v}i)$ follows. Comparing real and imaginary parts, we find $u = x + F(\psi)yi$ and $v = y - F(\psi)xi$ with $x, y \in \mathbb{R}$ and $F(\psi) := \frac{1-f(\psi)}{1+f(\psi)}$. Using $1 = \bar{u}u + \bar{v}v = (1 + F(\psi)^2)(x^2 + y^2)$ we obtain $\cos(\psi) = u^2 + v^2 = (1-F(\psi)^2)(x^2+y^2) = \frac{1-F(\psi)^2}{1+F(\psi)^2}$ and $\sin(\psi) = \frac{2F(\psi)}{1+F(\psi)^2}$. We find $F(\psi)^2 = \frac{1-\cos(\psi)}{1+\cos(\psi)}$, plug this into the second condition and obtain $F(\psi) = \frac{\sin(\psi)}{1+\cos(\psi)} = \tan(\frac{\psi}{2})$. From $1 = (1 + F(\psi)^2)(x^2 + y^2)$ we now infer $x^2 + y^2 = (1 + \tan(\frac{\psi}{2})^2)^{-1} = \cos(\frac{\psi}{2})^2$. These considerations show

$$A_{\beta\alpha} = \left\{ \frac{1}{\sqrt{g(|\psi|)}} \Big(\big(x + \tan(\tfrac{\psi}{2})yi\big) + j\big(y - \tan(\tfrac{\psi}{2})xi\big) \Big) \;\middle|\; \begin{array}{c} \psi \in (-\frac{\pi}{2}, \frac{\pi}{2}) \\ x, y \in \mathbb{R} \\ x^2 + y^2 = \cos(\frac{\psi}{2})^2 \end{array} \right\} .$$

Note that F does not depend on the choice of g: the set $A_{\beta\alpha}$ is a distorted version of the hyperbola $\{a + jb \in \mathbb{H} \mid a^2 + b^2 = 1\}$ that is obtained as $A_{\beta\alpha}$ if $g(\psi) = \cos(\psi)$.

Absolute points of J_γ at infinity correspond to lines $s\mathbb{C} \in \mathcal{L}_o$ with $s \in \mathbb{H}^\times$ such that $\gamma\iota(s)\, s\,\mathbb{C} \parallel L(1)$. This means $\gamma\iota(s)\, s \in j\mathbb{C}$. There are no solutions if $\gamma = \alpha$. For $\gamma = \rho\alpha$ the condition $\rho\alpha\iota(s)\, s \in j\mathbb{C}$ has the solutions $s = u + jv$ with $\bar{u}u - \bar{v}v = 0$. As $u = 0$ would imply $s = 0$, the absolute lines through o are those of the form $(1 + jt)\mathbb{C}$ with $t \in \mathbb{S}_\mathbb{C}$. The absolute points at infinity form a set homeomorphic to a sphere of dimension 1. If $\gamma = \beta\alpha$ then $(u+jv)\mathbb{C}$ is absolute precisely if $u^2+v^2 = 0$: this gives the two lines $(i+j)\mathbb{C}$ and $(i-j)\mathbb{C}$.

It remains to discuss the polarity $-J_\alpha$: the centralizer is $\mathbb{S}_\mathbb{H}$ as for J_α, but the condition for an affine absolute point s becomes $-s\bar{s} = 1$. This equation has no solutions, and there are also no solutions for $-s\bar{s} \in j\mathbb{C}$. Thus the polarity $-J_\alpha$ has no absolute points at all.

7.7 Remark. It appears that the polarities $-J_{\kappa\iota}$ on Schellhammer's planes $\overline{\mathcal{A}_g}$ with even functions g and the polarities $-J_\alpha$ on Sperner's planes are the only known examples of *elliptic* polarities (i.e., polarities without absolute points) for non-Moufang compact projective planes.

7.8 Remarks. The classical plane $\mathcal{A}_2\mathbb{C}$ is obtained as $\mathcal{E}_{\cos,0}$, we have $L(1) = 1 + \mathbb{C}j$ there. Using inhomogeneous coordinates we find that the polarities sJ_α, $J_{\rho\alpha}$ and $J_{\beta\alpha}$ and their sets of absolute points are described in the usual way by the hermitian forms

$$\begin{aligned} h^s_\alpha(u,v,w) &:= \bar{u}u + \bar{v}v - s\bar{w}w\,, \\ h_{\alpha\rho}(u,v,w) &:= \bar{u}u - \bar{v}v - \bar{w}w\,, \\ h_{\alpha\beta}(u,v,w) &:= u^2 + v^2 - w^2\,. \end{aligned}$$

This collection contains a set of representatives for the three conjugacy classes under the full group of continuous automorphisms of $\mathcal{P}_2\mathbb{C}$: the polarities J_α and $J_{\rho\alpha}$ become conjugates here. In the non-desarguesian planes they cannot be conjugates because then the line ∞ is fixed by the full group of automorphisms but has different intersection with the sets of absolute points. The complicated description for the elements of $A_{\beta\alpha}$ in 7.6 just reduces to $A_{\beta\alpha} = \{u + jv \in \mathbb{H} \mid u^2 + v^2 = 1\}$ in the classical case.

7.9 Theorem. *For each continuous polarity of a Sperner plane $\overline{\mathcal{E}_{g,0}}$ the set of absolute points is either empty or homeomorphic to a sphere. Explicitly, we have:*

1. *The polarity J_α has no absolute points at infinity; the affine absolute points form the set $A_\alpha = \mathbb{S}_\mathbb{H} \approx \mathbb{S}_3$. The centralizer $C_\alpha = \mathbb{S}_\mathbb{H}$ acts regularly on A_α.*
2. *The polarity $-J_\alpha$ is elliptic, it has no absolute points at all.*
3. *The polarity $J_{\rho\alpha}$ has affine absolute points and absolute points at infinity; together these points form a set homeomorphic to $\mathbb{S}_3$ while the set of absolute points at infinity is homeomorphic to $\mathbb{S}_\mathbb{C} \approx \mathbb{S}_1$. We have $C_{\rho\alpha} = \mathbb{S}_\mathbb{C}$.*
4. *The polarity $J_{\beta\alpha}$ has affine absolute points and two absolute points at infinity; together these points form a set homeomorphic to $\mathbb{S}_{\mathrm{Pu}(\mathbb{H})} \approx \mathbb{S}_2$ while the absolute points at infinity form a sphere $\mathbb{S}_\mathbb{R} \approx \mathbb{S}_0$. We have $C_{\beta\alpha} = \mathbb{S}_{\mathbb{R}+j\mathbb{R}}$.*

Proof. Every continuous polarity is a conjugate of J_α, $-J_\alpha$, $J_{\rho\alpha}$, or $J_{\beta\alpha}$, see 6.5. Thus it suffices to discuss these polarities. The sets of absolute points have been determined in 7.6. For J_α and $-J_\alpha$ there is nothing left to be done but the remaining two cases present difficulties because we have to glue in the absolute points at infinity.

For each Sperner plane the polarity $J_{\rho\alpha}$ has the same absolute points at infinity, namely the parallel classes of the lines of the form $(1+jt)\mathbb{C}$ with $t \in \mathbb{S}_\mathbb{C}$. The set $A_{\rho\alpha}$ of affine absolute points, however, depends on the function g used to construct $\mathcal{E}_{g,0}$; we have $A_{\rho\alpha} = \{\Phi_g(\vartheta, w, z) \mid w, z \in \mathbb{S}_\mathbb{C}, \vartheta \in [0, \frac{\pi}{4})\}$, where Φ_g is the continuous surjection

$$\Phi_g \colon [0, \tfrac{\pi}{4}) \times \mathbb{S}^2_\mathbb{C} \to A_{\rho\alpha} \colon (\vartheta, w, z) \mapsto \frac{1}{\sqrt{g(2\vartheta)}} \left(\cos(\vartheta) w + j \sin(\vartheta) z \right) .$$

Note that Φ_g induces a homeomorphism $\widehat{\Phi_g}$ from $([0, \frac{\pi}{4}) \times \mathbb{S}^2_\mathbb{C})/{\sim}$ onto $A_{\rho\alpha}$, where $(\vartheta, w, z) \sim (\vartheta', w', z')$ holds if the two triplets are equal or $\vartheta = 0 = \vartheta'$ and $w = w'$. It remains to describe how the points at infinity are glued to $A_{\rho\alpha}$. Convergence of a sequence $\Phi_g(\vartheta_n, w_n, z_n)$ to the parallel class of $(1+jt)\mathbb{C}$ means that ϑ_n converges to $\frac{\pi}{4}$ (such that $\sqrt{g(2\vartheta_n)} \to 0$ and the absolute value of $\Phi_g(\vartheta_n, w_n, z_n)$ tends to ∞) and $z_n w_n^{-1} \to t$. In other words: the homeomorphism type of the set of absolute points does not depend on g. For $g(\vartheta) = \cos(\vartheta)$ we obtain the plane $\overline{\mathcal{E}_{g,0}}$ is isomorphic to the projective plane over $\mathbb{C}$; in homogeneous coordinates the set of absolute points $J_{\rho\alpha}$ is described by the equation $\bar{u}u - \bar{v}v - \bar{w}w = 0$: this is a set homeomorphic to the sphere $\mathbb{S}_3$ of dimension 3.

It remains to discuss $J_{\beta\alpha}$. In this case we have two absolute points at infinity, namely $(i+j)\mathbb{C}$ and $(i-j)\mathbb{C}$. The set of affine absolute points is $A_{\beta\alpha} = \left\{\Psi_g(\psi, z) \mid (\psi, z) \in (-\frac{\pi}{2}, \frac{\pi}{2}) \times \mathbb{S}_\mathbb{C}\right\}$ where $\Psi_g(\psi, x+yi) := \sqrt{\cos(\frac{\psi}{2})/g(|\psi|)} \left(\left(x + \tan(\frac{\psi}{2}) yi\right) + j\left(y - \tan(\frac{\psi}{2}) xi\right)\right)$ gives a continuous surjection Ψ_g from $(-\frac{\pi}{2}, \frac{\pi}{2}) \times \mathbb{S}_\mathbb{C}$ onto $A_{\beta\alpha}$. Convergence of $\Psi_g(\psi_n, z_n)$ to one of the points at infinity means that ψ_n converges to $\frac{\pi}{2}$ or to $-\frac{\pi}{2}$. In the first of these cases the sequence of lines $\Psi(\psi_n, z_n)\mathbb{C} = \left(\cos(\frac{\psi}{2})x + \sin(\frac{\psi}{2})yi\right) + j\left(\cos(\frac{\psi}{2})y - \sin(\frac{\psi}{2})xi\right)\mathbb{C}$ converges to $\sqrt{2}(x+yi, j(y-xi))\mathbb{C} = (i+j)\mathbb{C}$ because $\cos(\psi_n)$ and $\sin(\psi_n)$ both converge to $1/\sqrt{2}$. Thus our points converge to the parallel class of $(i+j)\mathbb{C}$. In the second case, we find that the sequence of points converges to the parallel class of $(i-j)\mathbb{C}$. Again, we obtain that the topological type of the set of absolute points does not depend on the function g, and we may recognize the topology by looking at the classical complex plane. In homogeneous coordinates $(u, v, w)\mathbb{C}$, the absolute points are then described by the equation $u^2 + v^2 - w^2 = 0$. Thus they form a conic, homeomorphic to $\mathbb{S}_\mathbb{C}$. □

7.10 Remark. The traces of lines on the sphere U of absolute points of a polarity J of $\overline{\mathcal{E}_{g,0}}$ have the same homeomorphism type as in the classical counterparts, see [9, 2.3]:

1. If $U \approx \mathbb{S}_3$ (i.e. if J is a conjugate of J_α or of $J_{\rho\alpha}$) then the intersection of any secant with U is homeomorphic to $\mathbb{S}_1$. Apart from secants and tangents, there are also lines in $\overline{\mathcal{E}_{g,0}}$ that do not meet U at all (for instance, take the line at infinity if $J = J_\alpha$, and take the line $j\mathbb{C}$ if $J = J_{\rho\alpha}$).
2. If $U \approx \mathbb{S}_2$ (i.e., if the polarity is a conjugate of $J_{\beta\alpha}$) then every secant meets U in precisely two points and U is a topological oval in $\overline{\mathcal{E}_{g,0}}$. Every line in $\overline{\mathcal{E}_{g,0}}$ meets U in at least one point, cf. [3, 3.1, 3.7]. Thus every line is either a secant or a tangent.

In any case, there is precisely one tangent line to U in any absolute point, cf. [9, 2.2].

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