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A minimal atlas for the rotation group $SO(3)$

Erik W. Grafarend & Wolfgang Kühnel

We describe explicitly an atlas for the rotation group $SO(3)$ consisting of four charts where each chart is defined by Euler angles or each chart is defined by Cardan angles. This is best possible since it is well known that three charts do not suffice.

It is our daily experience that the Earth rotates, and it is our yearly experience that the Earth revolves around the Sun. The rotation of the Earth is nowadays descibed by a rotation matrix, an element of the three-dimensional rotation group $SO(3)$. The rotation group is presented in various monographs, for instance in [5] and [7]. The rotation matrix of the Earth is given by the International Earth Rotation and Reference Service (IERS) in terms of daily, monthly and yearly data, namely for precession/nutation versus polar motion/length of day variations (POM/LOD).

The problem we are discussing here originates in the various parameter systems of the rotation of rigid or deformable bodies. The characteristic equations are the kinematic Euler equations and the dynamic Euler equations parameterized in terms of Euler or Cardan angles. For a deformable body the dynamic Euler equations are generalized into Euler-Liouville equations. As another parameter system Hamilton's unit quaternions are used. General references are [3], [4], [10], [18] and, in particular, the previous article [8] by the first author on the same problem and the references quoted there.

The Earth has to be considered as a gyroscope with exotic movements like precession and nutation in an inertial frame of reference or polar motion and length of day variation in an Earth-fixed frame of reference. These movements are described by elements of the *rotation group* $SO(3)$ which is defined as the set of all real (3×3)-matrices with $\det A = 1$ and with three orthonormal rows and columns. It is a compact three-dimensional subgroup of the 9-dimensional group $GL(3, \mathbb{R})$. Compare [22] for the reduction of 9 parameters to 3 parameters. In particular $SO(3)$ is a connected Lie group and an analytic 3-manifold. Its universal 2-sheeted covering is the group $Spin(3)$ which can be identified with the set of all unit quaternions $\mathbb{H}_1 = Sp(1) = \{q \in \mathbb{H} \bigm| ||q|| = 1\}$ which, as a manifold, is diffeomorphic with the 3-dimensional standard sphere $S^3 \subset \mathbb{R}^4$. It is also isomorphic with the group $SU(2)$. See [2] for the details.

It is necessary to have a parameterization of the rotation group by three independent parameters. In particular it is a very natural goal to define an atlas for the rotation group by charts. A *chart* is an injective and differentiable map $\Phi\colon U \to SO(3)$ of maximal rank where U is an open subset of $\mathbb{R}^3$ diffeomorphic with an open 3-dimensional ball. In other words: A chart Φ is a diffeomorphic map from $U \subset \mathbb{R}^3$ onto its image $\Phi(U) \subset SO(3)$. An *atlas* is a set of charts covering the entire group $SO(3)$ such that all coordinate changes are differentiable maps. For a Lie group one may require that all coordinate changes are real analytic.

Our contribution deals with the problem how to find such an atlas explicitly, especially one with a minimum number of maps. Here we can take advantage of the Lusternik-Schnirelmann category cat [12], [13], [14], [6]. The *category* of a manifold is defined as the minimum number of subsets of a covering such that each subset has a contractible neighborhood in the manifold. The number of critical points of a real function on M cannot be smaller than $\mathrm{cat}(M)$, see [23]. In particular the well known equation $\mathrm{cat}(SO(3)) = 4$ indicates that at least four charts should be needed to cover the group $SO(3)$ completely. This holds for any type of charts, not only for Euler charts or Cardan charts. From the practical point of view, the construction of a minimum atlas on $SO(3)$ runs into the same problem as the construction of a minimum atlas on the unit sphere S^2. For the unit sphere the equation $CAT(S^2) = 2$ holds because the sphere is not contractible itself, and because we can cover it by two overlapping 2-discs around north and south pole. So in any case we need at least two distinct charts to cover the sphere completely. The standard spherical coordinates (spherical longitude and spherical latitude as a parameter system) become singular at the two poles. A well defined *transverse coordinate system*, also called meta-longitude and meta-latitude, as a parameter system has to be constructed. The union of the charts longitude/latitude and meta-longitude/meta-latitude cover the unit sphere completely in the sense of a minimum atlas, see [9, Sect.3-3].

For the three-dimensional rotation group the singularities of the parameter system Euler angles, Cardan angles and Hamilton unit quaternions are well known. They have been already analysed in the monographs [16], [24] and [19]. For spaceborne gyroscopes, deforming in time, we refer to [20], [21]. For Euler angles and Cardan angles Rimrott [19] found six different charts which constitute a complete atlas. Thus it remained an open question to find four charts of Euler or Cardan type which constitute a minimum atlas for $SO(3)$.

Definition (Euler angles, Cardan angles)
We say that two charts $F_1\colon U_1 \to SO(3), F_2\colon U_2 \to SO(3)$ are *of the same type* if there is a translation T in the parameter domain $\mathbb{R}^3$ and a rotation $R \in SO(3)$ such that $F_2 = R \circ F_1 \circ T$.

1. By the *Cayley parameters* we mean the Cayley map $C\colon \mathfrak{so}(3) \to SO(3)$ defined by

$$C(A) = (1 + A)(1 - A)^{-1}.$$

The exceptional locus $SO(3) \setminus C(\mathfrak{so}(3))$ consists of all rotation matrices with an angle π. The inverse map is explicitly given by $C^{-1}(B) = (B + 1)^{-1}(B - 1)$.

2. The *Geodesic polar coordinates* are given by the exponential map

$$\exp\colon \{A \in \mathfrak{so}(3) \bigm| ||A|| < \pi\} \to SO(3)$$

with

$$\exp(tA) = \sum_{n\geq 0} \frac{1}{n!}(tA)^n = 1 + (\sin t)A + (1 - \cos t)A^2$$

whenever $||A|| = 1$. Here $||\ ||$ denotes the operator norm of a matrix. Again the exceptional locus consists of all rotation matrices with an angle π. If $B \in SO(3)$ is a matrix an angle distinct from π then we have $\mathrm{tr}(B) \neq -1$, and B is not symmetric unless it is the identity matrix $1 = \exp(0)$. Hence we can recover the corresponding A from the skew-symmetric part of B and its operator norm, where the scalar factor $\sin t$ uniquely determines the parameter t in the interval $0 < t < \pi$.

3. The standard *Euler chart* is the following map $F\colon U \to SO(3)$ for a suitable domain $U \subset \mathbb{R}^3$ (for the precise choice of U see below)

$$F(\gamma, \alpha, \gamma^*) = \mathbf{R}_z(\gamma) \cdot \mathbf{R}_x(\alpha) \cdot \mathbf{R}_z(\gamma^*)$$

$$= \begin{pmatrix} \cos\gamma & -\sin\gamma & 0 \\ \sin\gamma & \cos\gamma & 0 \\ 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\alpha & -\sin\alpha \\ 0 & \sin\alpha & \cos\alpha \end{pmatrix} \cdot \begin{pmatrix} \cos\gamma^* & -\sin\gamma^* & 0 \\ \sin\gamma^* & \cos\gamma^* & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} \cos\gamma\cos\gamma^* - \sin\gamma\cos\alpha\sin\gamma^* & -\cos\gamma\sin\gamma^* - \sin\gamma\cos\alpha\cos\gamma^* & \sin\gamma\sin\alpha \\ \sin\gamma\cos\gamma^* + \cos\gamma\cos\alpha\sin\gamma^* & -\sin\gamma\sin\gamma^* + \cos\gamma\cos\alpha\cos\gamma^* & -\cos\gamma\sin\alpha \\ \sin\alpha\sin\gamma^* & \sin\alpha\cos\gamma^* & \cos\alpha \end{pmatrix}$$

The angles γ, α, γ^* are called the *Euler angles*[1] associated with the matrix on the right hand side. It is well known that the map $F\colon \mathbb{R}^3 \to SO(3)$ is surjective, compare [17] for a theoretical foundation in terms of generators and relations of groups.

[1] named after Leonhard Euler (1707–1783)

4. The standard *Cardan chart* $G\colon V \to SO(3)$ is the following map for a suitable domain $V \subset \mathbb{R}^3$ (for the precise choice of V see below)

$$G(\alpha,\beta,\gamma) = \mathbf{R}_x(\alpha)\cdot\mathbf{R}_y(\beta)\cdot\mathbf{R}_z(\gamma)$$

$$= \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\alpha & -\sin\alpha \\ 0 & \sin\alpha & \cos\alpha \end{pmatrix} \cdot \begin{pmatrix} \cos\beta & 0 & -\sin\beta \\ 0 & 1 & 0 \\ \sin\beta & 0 & \cos\beta \end{pmatrix} \cdot \begin{pmatrix} \cos\gamma & -\sin\gamma & 0 \\ \sin\gamma & \cos\gamma & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} \cos\beta\cos\gamma & -\cos\beta\sin\gamma & -\sin\beta \\ \cos\alpha\sin\gamma - \sin\alpha\sin\beta\cos\gamma & \cos\alpha\cos\gamma + \sin\alpha\sin\beta\sin\gamma & -\sin\alpha\cos\beta \\ \sin\alpha\sin\gamma + \cos\alpha\sin\beta\cos\gamma & \sin\alpha\cos\gamma - \cos\alpha\sin\beta\sin\gamma & \cos\alpha\cos\beta \end{pmatrix}$$

Accordingly, the angles α,β,γ are called the *Cardan angles*[2] associated with the matrix on the right hand side, sometimes also called *Tait-Bryan angles* or *yaw, pitch* and *roll*. The map G is surjective since the three standard rotations generated the entire rotation group. The exceptional sets of matrices where the Euler angles or the Cardan angles fail to be unique will be determined in the proof of our theorem below.

Theorem (four versions of a minimum atlas)

(i) Any atlas of the rotation group $SO(3)$ consists of at least four charts.

(ii) For each of the following four types there is an atlas with four charts consisting only of charts of this type:

1. *Cayley parameters,*
2. *geodesic polar coordinates,*
3. *Euler angles,*
4. *Cardan angles.*

Proof. Part (i) is direct application of the Lusternik-Schnirelmann category: The group $SO(3)$ is diffeomorphic with the real projective 3-space $\mathbb{R}P^3$, and it is well known that the category of $\mathbb{R}P^3$ equals 4, compare [6], [8]. In general we have $\mathrm{cat}(\mathbb{R}P^n) = n+1$. Therefore $SO(3)$ cannot be covered by three embedded open 3-balls. Roughly this can be seen as follows: In $\mathbb{R}P^3$ any 3-ball leaves some projective plane $\mathbb{R}P^2$ in its complement (up to homeomorphism). A second 3-ball leaves a projective line $\mathbb{R}P^1$ in its complement in this projective plane $\mathbb{R}P^2$. Finally the remaining projective line requires two more 3-balls since a closed curve in one 3-ball is always contractible but a projective line is not a contractible curve.

For Part (ii) we explicitly define such an atlas as follows:

(1) For an atlas consisting of modified Cayley charts (all of the same type) we define

$$C_1 = C, \quad C_2 = \mathbf{R}_x(\pi)\circ C, \quad C_3 = \mathbf{R}_y(\pi)\circ C, \quad C_4 = \mathbf{R}_z(\pi)\circ C.$$

For any given matrix $B \in SO(3)$ it is impossible that all four matrices

$$A,\ \mathbf{R}_x(\pi)B,\ \mathbf{R}_y(\pi)B,\ \mathbf{R}_z(\pi)B$$

[2] named after Girolamo Cardano (1501–1576)

are rotations with an angle π. Therefore B lies in the image of at least one of the four Cayley charts. Compare the exposition in the elementary textbook [11, Sect.5A], compare also [8].

(2) For the case of geodesic polar coordinates we proceed similarly and define

$$\exp_1 = \exp, \quad \exp_2 = \mathbf{R}_x(\pi) \circ \exp, \quad \exp_3 = \mathbf{R}_y(\pi) \circ \exp, \quad \exp_4 = \mathbf{R}_z(\pi) \circ \exp\,.$$

Again it is impossible that for any given matrix $B \in SO(3)$ all four matrices $B, \mathbf{R}_x(\pi)B,\ \mathbf{R}_y(\pi)B,\ \mathbf{R}_z(\pi)B$ are rotations with an angle π. Therefore B lies in the image of at least one of the four charts in geodesic polar coordinates.

In terms of the 2-sheeted covering $\mathbf{R}\colon \mathbb{H}_1 \to SO(3)$ this atlas is induced by the eight open half-spheres outside the four coordinate hyperplanes. The four exceptional loci are the intersections with these four coordinate hyperplanes. In $\mathbb{H}_1$ the image of the original map exp corresponds to the non-vanishing of the real part of a unit quaternion since precisely for those $q \in \mathbb{H}_1$ with a vanishing real part the matrix $\mathbf{R}(q)$ is a rotation with an angle π (see below).

(3) Since we have $F(0,0,0) = F(\gamma, 0, -\gamma)$ for any γ the map F is not injective in any neighborhood of the origin. In particular this means that the identity matrix has no uniquely defined Euler angles. We define the open set

$$U = (-\pi, \pi) \times (0, \pi) \times (-\pi, \pi)$$

in the domain of all triples $(\gamma, \alpha, \gamma^*)$ of possible Euler angles. The identity matrix is not contained in $F(U)$ since it would require an angle α with $\cos\alpha = 1$.

We claim that the restriction of F to U is injective. For a proof let

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$$

be a given element of $SO(3)$. By the condition $a_{33} = \cos\alpha$ the angle α is uniquely defined whenever such an $\alpha \in (0, \pi)$ exists, i.e., if $a_{33} \neq \pm 1$. By the condition $\sin\alpha > 0$ the angle γ is uniquely defined by a_{13} and a_{23} together, and γ^* is uniquely defined by a_{31} and a_{32} together. Therefore the equation $F(\gamma, \alpha, \gamma^*) = A$ has at most one solution in U. The differential of F has maximal rank in U since there exists a differentiable inverse map on $F(U)$.

Moreover, this domain U is maximal with this property: One cannot go further in direction of γ or γ^* since sin und cos are 2π-periodic. One cannot go further in direction of α because of $F(\gamma + \pi, \pi + \alpha, \gamma^* + \pi) = F(\gamma, \pi - \alpha, \gamma^*)$. In addition the differential of F degenerates for $\cos\alpha = \pm 1$.

The exceptional locus: The set $SO(3) \setminus F(U)$ consist of all matrices A with $a_{33} = \pm 1$ on the one hand and of those matrices which are covered by F only using angles $\gamma = \pm\pi$ or $\gamma^* = \pm\pi$ on the other hand. The special case $a_{33} = \pm 1$ corresponds to $\alpha = 0$ oder $\alpha = \pi$, in der literature known as the “gimbal lock”. Topologically the complement of the “gimbal lock” corresponds to the cartesian product of the α-interval $(0, \pi)$ and the (γ, γ^*)-torus $S^1 \times S^1$.

Now we have to find three other charts of the Euler type covering the exceptional locus. Here we use the well known description in terms of unit quaternions $q = a + bi + cj + dk \in \mathbb{H}_1$ (where $a^2 + b^2 + c^2 + d^2 = 1$) with the 2-sheeted covering $\mathbf{R}\colon \mathbb{H}_1 \to SO(3)$ defined by

$$\mathbf{R}(q) = \begin{pmatrix} 1 - 2(c^2 + d^2) & -2ad + 2bc & 2ac + 2bd \\ 2ad + 2bc & 1 - 2(b^2 + d^2) & -2ab + 2cd \\ -2ac + 2bd & 2ab + 2cd & 1 - 2(b^2 + c^2) \end{pmatrix}.$$

Here we have $\mathbf{R}(q)(x) = qxq^{-1}$ and $\mathbf{R}(qr) = \mathbf{R}(q) \cdot \mathbf{R}(r)$. The trace of the matrix $\mathbf{R}(q)$ equals $3 - 4(b^2 + c^2 + d^2)$. On the other hand the trace of a rotation matrix equals $1 + 2\cos\varphi$ if φ denotes the angle of the rotation. Consequently $\varphi = \pi$ if and only if $a = 0$. Furthermore this implies that the set of all $\mathbf{R}(q)$ with $\cos\alpha = 1$ is induced by the set of all q with $b = c = 0$, and that the set of $\mathbf{R}(q)$ with $\cos\alpha = -1$ is induced by the set of all q with $a = d = 0$. Therefore in $\mathbb{H}_1$ the "gimbal lock" appears as the union of two opposite great circles. In terms of the standard rotations $\mathbf{R}_x(\pi), \mathbf{R}_y(\pi), \mathbf{R}_z(\pi)$ above we can write

$$\mathbf{R}_x(\pi) = \mathbf{R}(i),\ \mathbf{R}_y(\pi) = \mathbf{R}(j),\ \mathbf{R}_z(\pi) = \mathbf{R}(k).$$

Now we define two charts (of the same type) $F_1, F_2 \colon U \to SO(3)$ by

$$F_1 = F, \quad F_2(\gamma, \alpha, \gamma^*) = F(\gamma + \pi, \alpha, \gamma^* + \pi)$$

and calculate the remaining exceptional locus $SO(3) \setminus \big(F_1(U) \cup F_2(U)\big)$. Except for the "gimbal lock" the exceptional locus consists of those matrices $F(\gamma, \alpha, \gamma^*)$ with $(\gamma, \alpha, \gamma^*) = (0, \alpha, \pi)$ or $(\gamma, \alpha, \gamma^*) = (\pi, \alpha, 0)$. By the equation $\cos\gamma \cdot \cos\gamma^* = -1$ these matrices are of the form

$$\begin{pmatrix} -1 & 0 & 0 \\ 0 & -\cos\alpha & \pm\sin\alpha \\ 0 & \pm\sin\alpha & \cos\alpha \end{pmatrix}.$$

The fct that the trace equals -1 tells us that each such matrix represents a rotation by the angle π, independently of α. In terms of unit quaternions these matrices are characterized by $a = b = 0$. Therefore the remaining exceptional locus is the union of the three great circles $d = a = 0,\ a = b = 0,\ b = c = 0$.

Now we are going to cover this exceptional locus by a third and a fourth chart. We define

$$F_3(\gamma, \alpha, \gamma^*) = \Omega \cdot F_1(\gamma, \alpha, \gamma^*), \quad F_4(\gamma, \alpha, \gamma^*) = \Omega \cdot F_2(\gamma, \alpha, \gamma^*)$$

with the matrix

$$\Omega = \frac{1}{3}\begin{pmatrix} -1 & 2 & 2 \\ 2 & -1 & 2 \\ 2 & 2 & -1 \end{pmatrix}$$

which also represents a rotation by an angle π. In terms of unit quaternions it corresponds to

$$\omega = \frac{1}{\sqrt{3}}\big(i + j + k\big)$$

in the sense that $\Omega = \mathbf{R}(\omega)$. In some sense F_3, F_4 together can be called a *transverse pair of Euler charts* compared to F_1, F_2, by analogy with [9, Sect.3-3]. The remaining exceptional set $SO(3) \setminus \big(F_3(U) \cup F_4(U)\big)$ is nothing but the other one $SO(3) \setminus \big(F_1(U) \cup F_2(U)\big)$ after rotation by the action of Ω. If these two exceptional sets are disjoint then we are done since the four charts F_1, F_2, F_3, F_4 cover the entire rotation group $SO(3)$. And in fact they are disjoint, as follows from the calculation of the products

$$\begin{array}{rcll} (i+j+k)(a+dk) & = & -d + (a+d)i + (a-d)j + ak, & \text{with } a^2 + d^2 = 1 \\ (i+j+k)(bi+cj) & = & -(b+c) - ci + bj + (c-b)k, & \text{with } b^2 + c^2 = 1 \\ (i+j+k)(cj+dk) & = & -(c+d) + (d-c)i - dj + ck, & \text{with } c^2 + d^2 = 1. \end{array}$$

An analogous construction is well known for the ordinary spherical coordinates where the exceptional locus of one standard chart is half of a great circle from north pole to south pole. The "gimbal lock" corresponds to the two poles. By a transversal map we obtain an atlas consisting of two standard charts. We just have to make sure that the two exceptional loci are disjoint.

(4) For the Cardan angles we simiarly define a set

$$V = (-\pi, \pi) \times (-\pi/2, \pi/2) \times (-\pi, \pi)$$

in the domain of all possible triples of angles (α, β, γ). In this case the unit matrix occurs as $G(0, 0, 0)$.

We claim that the restriction of G to V is injective. For a proof we proceed as in case (3) above. A given rotation matrix

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$$

determines the angle $\beta \in (-\pi/2, \pi/2)$ uniquely by the condition $a_{13} = -\sin\beta$ whenever such a β exists, i.e., if $a_{13} \neq \pm 1$). Bebause of $\cos\beta > 0$ the angle α is then uniquely determined by a_{23} and a_{33}, and γ is uniquely determined by a_{11} and a_{12}. Therefore the equation $G(\alpha, \beta, \gamma) = A$ has at most one solution in V. The differential of G has maximal rank in V since there exists a differentiable inverse map on $G(V)$.

Again V is maximal with this property, by analogy with the case of the Euler angles above: In α-direction or in γ-direction we cannot go further. In β-direction this is impossible because of $G(\alpha, \pi/2 + \beta, \gamma) = G(\alpha + \pi, \pi/2 - \beta, \gamma + \pi)$.

The exceptional locus: The set $SO(3) \setminus G(V)$ consists of all matrices A with $a_{13} = \pm 1$ and of those matrices, which can only be represented by $\alpha = \pm\pi$ or $\gamma = \pm\pi$. The special case $a_{13} = \pm 1$ corresponds to $\beta = \mp\pi/2$. The rest ist nothing but the cartesian product of the β-interval $(-\pi/2, \pi/2)$ and the (α, γ)-torus $S^1 \times S^1$.

In terms of unit quaternions $q = a + bi + cj + dk \in \mathbb{H}_1$ the set of all $\mathbf{R}(q)$ with $\sin\beta = \pm 1$ concides with the set of all q with

$$ad = bc,\ ab = cd,\ b^2 = d^2,\ b^2 + c^2 = \tfrac{1}{2},\ ac + bd = \pm\tfrac{1}{2}.$$

These equations imply $a^2 = c^2$ with the sign restriction $ad = bc$. Therefore this part of the exceptional locus lies in the union of the 2-dimensionalen linear subspaces $c = a, d = b$ und $c = -a, d = -b$.

Now we introduce two maps $G_1, G_2 \colon V \to SO(3)$ by

$$G_1 = G, \quad G_2(\alpha, \beta, \gamma) = G(\alpha + \pi, \beta, \gamma + \pi)$$

and consider the remaining exceptional locus $SO(3) \setminus \big(G_1(V) \cup G_2(V)\big)$. Besides the case $\sin\beta = \pm 1$ described above the exceptional locus consists of the matrices $G(\alpha, \beta, \gamma)$ with $(\alpha, \beta, \gamma) = (0, \beta, \pi)$ or with $(\alpha, \beta, \gamma) = (\pi, \beta, 0)$. Because of $\cos\alpha \cdot \cos\gamma = -1$ these have the form

$$\begin{pmatrix} \mp\cos\beta & 0 & -\sin\beta \\ 0 & -1 & 0 \\ -\sin\beta & 0 & \pm\cos\beta \end{pmatrix}.$$

Again the trace tells us that each such matrix represents a rotation by π, independently of β. In terms of quaternions these matrices are characterized by $a = c = 0$. It follows that the entire exceptional locus can be described by linear equations between a, b, c, d.

Finally we have to cover the exceptional locus by a third and a fouth map. We choose

$$G_3(\alpha, \beta, \gamma) = \Omega \cdot G_1(\alpha, \beta, \gamma), \quad G_4(\alpha, \beta, \gamma) = \Omega \cdot G_2(\alpha, \beta, \gamma)$$

with the same matrix Ω as above. The exceptional locus $SO(3) \setminus \big(G_3(V) \cup G_4(V)\big)$ is the same as the other one, just rotated by Ω. These two exceptional sets are disjoint. This follows by analogy with the case of the Euler angles above. As an example we have

$$(i + j + k)(a + bi + aj + bk) = -(2b + a) + bi + aj + (2a - b)k.$$

$\square$

How to switch between the four charts. In the case of Cayley parameters or geodesic polar coordinates the four charts are transformed into each other by the subgroup $\{\mathbf{1}, \mathbf{R}_x(\pi), \mathbf{R}_y(\pi), \mathbf{R}_z(\pi)\}$ acting on $SO(3)$ from the left. This group is isomorphic with the additive group $\mathbb{Z}_2 \times \mathbb{Z}_2$. From the practical point of view we have the following procedure: For a given matrix B one has to check which of the four matrices

$$B,\ \mathbf{R}_x(\pi)B,\ \mathbf{R}_y(\pi)B,\ \mathbf{R}_z(\pi)B$$

has an angle distinct from π or, equivalently, a trace distinct from -1. The corresponding chart can be used. It follows that all parameter transformations between the charts are real analytic.

In the case of Euler angles or Cardan angles it seems that such a group argument can not work for any such atlas of four charts. Instead, switching between F_1 and F_2 [or G_1 and G_2] means that each of the quantities $\sin\gamma, \cos\gamma, \sin\gamma^*, \cos\gamma^*$ [or $\sin\alpha, \cos\alpha, \sin\gamma, \cos\gamma$] is replaced by its negative. This procedure is nothing but conjugation in $O(3)$ by the reflection matrix

$$T = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}.$$

By definition switching between F_1 and F_3 or F_2 and F_4 means multiplication by Ω from the left, similarly for G_1, G_2, G_3, G_4. It follows that switching from F_3 to F_4 means multiplication by Ω, followed by conjugation by T, and again followed by multiplication by Ω. In particular the "gimbal lock" can never occur for A and ΩA simulaneously. So from the practical point of view for a given matrix $A \in SO(3)$ one has to check the four matrices

$$A,\ TAT,\ \Omega A,\ \Omega TAT.$$

By the proof above at least one of them lies in the image of the standard Euler chart or standard Cardan chart, respectively, with the domain U or V as above. The inverse map leads to the corresponding Euler angles in U or Cardan angles in V. For the practical calculation compare [1]. Again all parameter transformations are real analytic. For numerical stability one can avoid a certain "dangerous region" around the boundaries of the charts, due to a large overlap of the four charts. In particular one can always stay away from the "gimbal lock".

Remark. Higher dimensional analogues of the Euler angles are studied in [15].

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