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On integral-like units of modular group rings

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Abstract

In this note[1] we study units of modular group algebras over a prime field F which have similar properties as units of integral group rings. Nevertheless we demonstrate in specific examples that subgroups consisting of such units behave totally different as in the integral group ring case. Towards the construction of possible counterexamples to the modular isomorphism problem of p-groups we show that the normalized unit group $V(FG)$ of the modular group algebra of a finite p-group G may possess linearly independent subgroups non-isomorphic to a subgroup of G. In particular, a normalized monomorphism of group rings $FH \longrightarrow FG$ does not imply that H is isomorphic to a subgroup G. This stands in a strong contrast to the integral case where in the case when G is a p-group by [11, 13] a normalized monomorphism $\mathbb{Z}H \longrightarrow \mathbb{Z}G$ implies that H is a isomorphic to a subgroup of G. Even the p-rank of H may be bigger than that one of G, while H consists of elements with integral-like partial augmentations.

The object of this note are special investigations of the unit group of a modular group algebra. Let G be a finite group and let F be a prime field of characteristic dividing $|G|$. Denote the group algebra of G over F by FG and the integral group ring of G by $\mathbb{Z}G$.

A well known conjecture on torsion units of $\mathbb{Z}G$ due to H.Zassenhaus states that each torsion unit of augmentation 1 is conjugate to a unit of G within the group algebra $\mathbb{Q}G$.

Let R be a commutative ring and let $u = \sum_{g \in G} r_g g$ be an element of its group ring RG. The partial augmentation of u with respect to a conjugacy class C of G is defined as $\sum_{g \in C} r_g$ and denoted by $\varepsilon_C(u)$.
We say that a unit v of FG has *integral-like* partial augmentations if $\varepsilon_{C_i}(v) = 1$ for precisely one conjugacy class C_i, and on all other classes partial augmentations are equal to zero.

A torsion unit $u \in \mathbb{Z}G$ of order k is rationally conjugate to an element of G if, and only if, for every divisor m of k partial augmentations of u^m are integral-like [9, 10].

It is a natural question to ask whether a similar statement holds also in the modular group algebra FG. Clearly, if a unit u is conjugate within FG to an element $g \in G$, then the partial augmentations of each power of u will have this property. Our first example shows that the reverse statement is not true.

[1] revised on March 11th, 2011

Example 1.
Let $Q_8 = \langle a, b \mid a^4 = b^4 = 1, a^4 = b^2, b^{-1}ab = a^{-1} \rangle$. Then the normalised unit group of $\mathbb{F}_2Q_8$ contains three conjugacy classes of elements which are not conjugate to elements of G and have integral-like partial augmentations:

$$C_1 = \{b^{-1} + a^{-1}b^{-1} + a^{-1}b, b + a^{-1}b^{-1} + a^{-1}b, a + a^{-1} + b^{-1}, a + a^{-1} + b\},$$

$$C_2 = \{a^{-1} + a^{-1}b^{-1} + a^{-1}b, a + a^{-1}b^{-1} + a^{-1}b, b + a^{-1} + b^{-1}, a + b + b^{-1}\},$$

$$C_3 = \{a + a^{-1}b^{-1} + a^{-1}, b + a^{-1}b^{-1} + b^{-1}, a + a^{-1} + a^{-1}b, b + b^{-1} + a^{-1}b\}.$$

Clearly, the center of the unit group of FG may be much bigger than the torsion center of an integral group ring. However, if a unit has integral-like partial augmentations and is in $FZ(G)$, then it is obviously a central element of G. The well known theorem of S.Berman and G.Higman [1] states that central torsion units in integral group rings of finite groups are trivial. This certainly motivates to study in modular group algebras units with integral-like partial augmentations.

Proposition. Let G be a finite group and let F a finite field. Then the following are equivalent.

(i) Units of $V(FG)$ with integral-like partial augmentations are trivial, i.e. they are elements of $G \subset V(FG)$.

(ii) G is abelian.

Proof. The reduction map $\mathbb{Z}G \longrightarrow FG$ is injective on finite torsion subgroups by a result of Cohn-Livingstone [2]. Therefore if $V(\mathbb{Z}G)$ has non-trivial torsion units u with $\varepsilon_C(u) = 1$ for precisely one class C and $\varepsilon_K(u) = 0$ for all other conjugacy classes K of G, the same holds for FG.
Consequently FG has non-trivial units with integral-like partial augmentations provided G is not normal in $V(\mathbb{Z}G)$. Assume that $G \lhd V(\mathbb{Z}G)$ and let $u \in V(\mathbb{Z}G)$ be of finite order. Then $\langle G, u \rangle$ is a finite group. But torsion subgroups of $V(\mathbb{Z}G)$ divide $|G|$. Thus, if $G \lhd V(\mathbb{Z}G)$, then torsion units of G are trivial.
By the classification of finite groups G such that $V(\mathbb{Z}G)$ has only trivial units due to G.Higman [4] it follows that a counterexample to the Proposition has a subgroup isomorphic to Q_8. Now Example 1 completes the proof. q.e.d.

Torsion subgroups in integral group rings have the property that they consist of linearly independent elements over $\mathbb{Z}$. This is obviously not the case in modular group rings. Therefore it is reasonable to consider such subgroups of $V(FG)$ whose elements are linearly independent over F.

Example 2. Examples of a subgroup consisting of linearly independent elements being non-isomorphic to a subgroup of G. Using the GAP implementation reported in [7], in a project [3] under the supervision of the 2nd author it was verified that:

- the normalised unit group of the modular group algebra $\mathbb{F}_2Q_{16}$ contains linearly independent copy of $C_2 \times C_2$;
- the normalised unit group of the other two modular group algebras of 2-groups of maximal class of order 16 contains linearly independent copy of $C_4 \times C_2$.

Obviously we may combine both approaches. A subgroup $V \subseteq V(FG)$ is called *integral-like* if all elements of V have integral-like partial augmentations and the set of its elements is linearly independent over F.

[2] This is also a consequence of an old result of Minkowski

On the ICRA satellite conference in Granada in 2006 Z. Marciniak posed the question whether the finite group G has a subgroup isomorphic to the Klein four-group provided $V(\mathbb{Z}G)$ has such a subgroup. A positive answer to this question using the Brauer-Suzuki theorem has been given in [6]. The question whether the same is valid in the situation of a modular group ring FG under the assumption that the Klein four-group is integral-like is obvious. This question looks even more promising in the case when G is a p-group and F is the field of p elements. Note that for p-groups by results of Weiss [13] and Roggenkamp-Scott [11] each torsion subgroup of $V(\mathbb{Z}_pG)$ is conjugate to a subgroup of G.

However the answer is negative and surprisingly easy to find with the aid of the package LAGUNA [8] for the computational algebra system GAP [2]. We give a detailed description of the GAP session which serves as a tutorial on the LAGUNA package.

Example 3. We assume assume that the LAGUNA package is already loaded. First we create an auxiliary function to check that an element of a group ring has integral-like partial augmentations :

```
gap> IsIntegralLike := function( KG, u )
> local e,o,pa,i,ne,no;
> e := One(UnderlyingRing(KG));
> o := Zero(UnderlyingRing(KG));
> pa := PartialAugmentations(KG,u)[1];
> ne:=0; no:=0;
> for i in pa do
>   if i=e then ne:=ne+1;
>   elif i=o then no:=no+1;
>   else break;
>   fi;
> od;
> return ne=1 and no+1=Length(pa);
> end;
function( KG, u ) ... end
```

Now we retrieve from the GAP Small Groups Library the generalised quaternion group `G` of order 16 and create its modular group algebra `KG` over the field of two elements:

```
gap> G:=SmallGroup(16,9); StructureDescription(G);
<pc group of size 16 with 4 generators>
"Q16"
gap> KG:=GroupRing(GF(2),G);
<algebra-with-one over GF(2), with 4 generators>
```

LAGUNA package computes the group `V` – the normalised unit group of `KG` in the very efficient pc-presentation. Thus, it's very fast to list all representatives of conjugacy classes of elements of order 2 in `V`:

```
gap> V:=PcNormalizedUnitGroup(KG);
<pc group of size 32768 with 15 generators>
gap> cc:=Filtered(ConjugacyClasses(V),c->Order(Representative(c))=2);;
gap> reps:=List(cc,c->Representative(c));;
gap> Length(reps);
119
```

Now we will create another auxiliary function that will enumerate pairs of representatives of conjugacy classes of order two to check that they generate a Klein four-group consisting of linearly independent elements with integral-like partial augmentations:

```
gap> FindIntegralLikeKleinFourGroup:=function( KG )
> local V, cc, reps, f, i, j, x, y, H, t, elts;
> V:=PcNormalizedUnitGroup(KG);
> cc:=Filtered(ConjugacyClasses(V),c->Order(Representative(c))=2);;
> reps:=List(cc,c->Representative(c));;
> f:=NaturalBijectionToNormalizedUnitGroup(KG);
> for i in [1..Length(reps)-1] do
>  for j in [i+1..Length(reps)] do
>    x:=reps[i]; y:=reps[j]; H:=Group(x,y);
>    if IdGroup(H) = [4,2] then
>      elts := List( H, t -> t^f );
>      if ForAll( elts, t -> IsIntegralLike( KG, t ) ) then
>        if Dimension( Subspace( KG, elts ) ) = Size(H) then
>          return [x,y];
>        fi;
>      fi;
>    fi;
>  od;
> od;
> return fail;
> end;
function( KG ) ... end
```

This function now returns us a list of elements of `V` generating its integral-like subgroup isomorphic to $C_2 \times C_2$. We may check these properties directly in the GAP session:

```
gap> x:=FindIntegralLikeKleinFourGroup(KG);
[ f10*f11*f13*f14, f5*f7*f8*f10*f11 ]
gap> H:=Group(x); StructureDescription(H);
<pc group with 2 generators>
"C2 x C2"
gap> elts:=List(H,t->t^NaturalBijectionToNormalizedUnitGroup(KG));;
gap> List(elts,t->PartialAugmentations(KG,t)[1]);
[ [ Z(2)^0 ], [ Z(2)^0, 0*Z(2), 0*Z(2), 0*Z(2) ],
  [ 0*Z(2), Z(2)^0, 0*Z(2), 0*Z(2) ],
  [ Z(2)^0, 0*Z(2), 0*Z(2), 0*Z(2) ] ]
gap> Dimension(Subspace(KG,elts));
4
```

In the GAP notation, `0*Z(2)` is the zero element of F_2 and `Z(2)^0` is the identity of F_2, so it is easy to see that partial augmentations are integral-like for all elements of `H`.

Still the result may be not very readable since it refers to the generators of the group `V` in the pc-presentation, and even if we will map it back to the group algebra, the group algebra elements will be written in terms of the polycyclic generating system of the group `G` from the GAP Small Groups Library. It is possible to run the same computations slightly slower, creating Q_{16} in GAP as a finitely presented group $\langle a, b \mid a^8 = b^4 = 1, a^4 = b^2, b^{-1}ab = a^{-1} \rangle$ and find the following example of a generating set for the integral-like Klein four-group:

$$x_1 = 1 + a + ab + a^{-1}b^2 + a^{-3} + a^{-2}b + ab^{-1} + a^{-1} + a^{-2}b^{-1},$$

$$x_2 = a^2 + b^2 + ab + a^{-1}b^2 + a^{-3} + a^{-2} + a^{-1}b.$$

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