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Control Synthesis using Dynamic D-Scales: Part I – Robust Control

Carsten W. Scherer and I. Emre Köse

Abstract

We consider uncertain dynamical systems described in the standard LFT form. Following the methods familiar from μ-theory, we use dynamic (*i.e.*, frequency-dependent) D-scales for verifying robust stability of the system. The main result of the paper gives necessary conditions for the existence of robustly stabilizing controllers using parametrized dynamic D-scales which are sufficient for robust stability in a certain sense. Based on these conditions, we propose a primal/dual D-scale iteration for the design of robust controllers as an alternative to the well-known D/K-iteration. A numerical example illustrates the advantages of the proposed iteration. The results of this paper lead to a solution of the gain-scheduled control problem as reported in the sequel of this paper.

I. Introduction

The robust control synthesis problem can be summarized as one of finding a robustly stabilizing K in Figure 1, where G is the nominal plant, Δ represents the uncertainties/nonlinearities involved in the system model and $\mathcal{G}_{cl}$ stands for the lower LFT of G with respect to K, which gives the nominal closed-loop system.

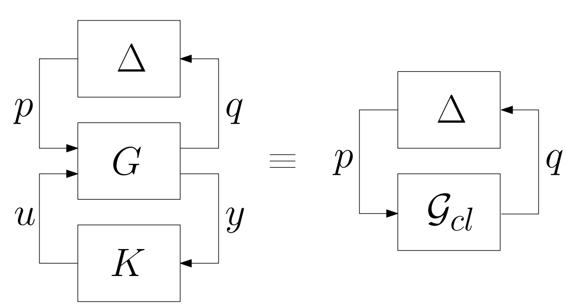

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Fig. 1. The closed-loop system.

If Δ is linear time-invariant, stable, norm-bounded by unity and possesses a given structure, robust stability is guaranteed iff K is nominally stabilizing and the structured singular value, μ, of $\mathcal{G}_{cl}$ with respect to the uncertainty structure remains below 1 for all frequencies [16]. Since the computation of μ is non-convex in general, we resort to verifying that an upper bound of μ is less than 1 in order to guarantee robust stability. This upper bound is given in terms of so-called D-scales, which are frequency-dependent (*i.e.*, *dynamic*) positive definite matrices that commute with the structure of Δ.

Except for some special cases such as robust output estimation [13] and robust disturbance feedforward [6], the joint search for such a D-scale and a nominally stabilizing K is a non-convex problem in general. The most common procedure for overcoming this non-convexity issue is the D/K-iteration [3]. The iteration is initiated with $D = I$ and at each following step, one of D and K is sought with the other one fixed from the previous step. When D is fixed, the search for K can be cast as a nominal $\mathcal{H}_\infty$ synthesis problem. When K is fixed, the search for the D-scale is carried out at discrete frequency points first and the overall expression for $D(j\omega)$ is then obtained through curve fitting. Although each step in the D/K-iteration is convex, the overall procedure is not. Hence, convergence to the global minimum is not guaranteed.

In this paper, in contrast to standard μ-synthesis, we concentrate on the *existence conditions* for a robustly stabilizing controller. In particular, we obtain existence conditions for a robustly stabilizing K while parametrizing the D-scales in a numerically useful fashion. To that end, we begin by factorizing the dynamic D-scale as $D = \psi^*\psi$, where ψ is frequency-dependent. Using the state-space realization of ψ, we apply the Kalman-Yakubovich-Popov (KYP) Lemma to obtain LMI conditions for the stability of the closed-loop condition. Elimination of K from these LMIs yields necessary and sufficient existence conditions for K using *non-parametrized* D-scales. Through a sequence of non-trivial manipulations on the LMIs, we can substitute the realization of ψ and its inverse with sufficiently close approximations obtained from appropriate basis functions. What we thus obtain is necessary LMI conditions for the existence of a robustly stabilizing controller using *parametrized* frequency-dependent D-scales. *Disregarding approximate inverse relations* in the resulting LMIs, these conditions are jointly convex.

In the reverse direction, when the parametrized LMIs are satisfied, we can guarantee the existence of a controller that robustly stabilizes the system against uncertainties with a quantifiable norm bound that is necessarily less than 1.

We can utilize the findings of this paper in two ways. First, the main result lays the foundation of the solution of the gain-scheduled control problem where it is assumed that the uncertainty Δ can be reproduced on-line. With this assumption, the approximate inverse relations in the solvability conditions are relaxed and we obtain a convex solution of the gain-scheduled control synthesis problem using dynamic D-scales. This solution is given in full detail in the sequel to the present paper [14].

Second, the main result allows us to formulate a novel iterative solution to the existence conditions that avoids the difficulties encountered in the D/K-iteration. This solution is based on the maximization of the uncertainty norm bound against which the designed controller is guaranteed to robustly stabilize the system. Since bounded from above by 1 and non-decreasing at each step, this sequence of guaranteed uncertainty bounds converges. Moreover, the procedure we propose avoids curve fitting or loop transformations completely [3]. However, due to the non-convex nature of the underlying problem, a general comparison with the D/K-iteration is not possible. Still, the application of the proposed solution to a mechanical system demonstrates better behavior than the D/K-iteration that improves even further when higher dynamics in the D-scales are allowed.

The paper is organized as follows: In Section II, we introduce a parametrization of suitable D-scales that provides arbitrary accuracy in approximating any given stable transfer function. Also in this section, we give two different nominal stability characterizations that are duals of each other in a certain sense. Our main result, namely a new set of conditions for the existence of a robustly stabilizing controller, is stated in Section III. In Section IV, we propose an alternative to the D/K-iteration that does not involve obtaining the controller itself at any step. In Section V, we apply the main result and the related iterative solution to the model of a mechanical system. We give a summary and a brief discussion in Section VII. Technical results and the proof of the main theorem are given in the Appendix.

Notation and conventions for realizations. $\mathbb{C}^0$ denotes the extended imaginary axis. For the adjoint of a transfer matrix G with realization (A, B, C, D) we use the notation $G^*(s) = G(-s)^T$ and the realization $G^* = \left[\begin{array}{c|c} -A^T & C^T \\ \hline -B^T & D^T \end{array}\right]$. If D is non-singular we use $G^{-1} =$

$\left[\begin{array}{c|c} A - BD^{-1}C & BD^{-1} \\ \hline -D^{-1}C & D^{-1} \end{array}\right]$. If A has no eigenvalues in $\mathbb{C}^0$ and $M = M^T$,

$$G^* M G \prec 0 \tag{1}$$

is read as $G(j\omega)^* M G(j\omega) \prec 0$ for all $\omega \in \mathbb{R} \cup \{\infty\}$ and called frequency-domain inequality (FDI). By the KYP-Lemma, it is equivalent to feasibility of the LMI

$$\begin{pmatrix} I & 0 \\ A & B \\ C & D \end{pmatrix}^T \underbrace{\begin{pmatrix} 0 & X & 0 \\ X & 0 & 0 \\ 0 & 0 & M \end{pmatrix}}_{=:\mathcal{M}(X,M)} \begin{pmatrix} I & 0 \\ A & B \\ C & D \end{pmatrix} \prec 0, \tag{2}$$

for some $X = X^T$. It is convenient to say that (2) certifies (1) or that X is a certificate for the FDI (1). In expressions like $G^* M G$ we address M as middle term and G as outer term/factor (not to be confused with outer transfer matrices), and use such a convention also for LMIs like (2). If required by space-limitations, we abbreviate blocks that can be inferred by symmetry (such as the left outer-factor in (2)) by $\star$. Lastly, we use $\mathrm{He}(M) := M + M^T$ and $J(M) := \mathbf{diag}\,(M, -M)$.

II. Preliminaries

A. The closed-loop interconnection

Let the interconnection in Figure 1 be described as

$$\begin{pmatrix} q \\ y \end{pmatrix} = \left[\begin{array}{c|cc} A & B_p & B_u \\ \hline C_q & D_{qp} & D_{qu} \\ C_y & D_{yp} & 0 \end{array}\right] \begin{pmatrix} p \\ u \end{pmatrix}, \quad \text{and} \quad u = \left[\begin{array}{c|c} A_c & B_c \\ \hline C_c & D_c \end{array}\right] y \tag{3}$$

which is affected by the uncertainty $p = \Delta q$. For notational simplicity we consider full-block-structured dynamic uncertainties only. Hence Δ can be any proper and stable transfer matrix which admits the structure

$$\Delta = \mathop{\mathbf{diag}}_{i=1}^{m}\,(\Delta_i) \quad \text{and satisfies} \quad \|\Delta\|_\infty \leq 1.$$

The uncertain closed-loop system is described by $z = \mathcal{G}_{cl} w$, $w = \Delta z$ with $\mathcal{G}_{cl}$ having the realization

$$\left[\begin{array}{c|c} \mathcal{A} & \mathcal{B} \\ \hline \mathcal{C} & \mathcal{D} \end{array}\right] = \left[\begin{array}{cc|c} A + B_u D_c C_y & B_u C_c & B_p + B_u D_c D_{yp} \\ B_c C_y & A_c & B_c D_{yp} \\ \hline C_q + D_{qu} D_c C_y & D_{qu} C_c & D_{qp} + D_{qu} D_c D_{yp} \end{array}\right].$$

Let us recall the so-called D-scalings stability test from structured singular value theory [9]. For this purpose consider the set

$$\mathcal{Q} := \left\{ \operatorname*{\mathbf{diag}}_{k=1}^{m} (I_{n_k} \otimes q_k) : \; q_k \in R\mathcal{L}_\infty, \; q_k > 0 \right\}$$

in correspondence with the structure of Δ. Robust stability of the controlled closed-loop system is then guaranteed if there exists some multiplier $Q \in \mathcal{Q}$ with

$$\begin{pmatrix} \mathcal{G}_{cl} \\ I \end{pmatrix}^* \begin{pmatrix} Q & 0 \\ 0 & -Q \end{pmatrix} \begin{pmatrix} \mathcal{G}_{cl} \\ I \end{pmatrix} \prec 0. \tag{4}$$

B. *Parametrization of D-scales*

If Q satisfies (4) we can determine, for $k = 1, \dots, m$, a spectral factorization $q_k = \hat{q}_k^* \hat{q}_k$ where $\hat{q}_k$ is stable and has a stable inverse. This motivates to parametrize the multipliers Q by the stable factors $\hat{q}_k$ in such a description. For this purpose we choose a pole-location $p > 0$ and introduce the transfer function basis vector

$$b_\nu(s) = \begin{pmatrix} 1 & \frac{s-p}{s+p} & \frac{(s-p)^2}{(s+p)^2} & \cdots & \frac{(s-p)^\nu}{(s+p)^\nu} \end{pmatrix}^T \tag{5}$$

for $\nu \in \mathbb{N}$. Then any proper and stable transfer function $\hat{q}$ can be uniformly approximated on $\mathbb{C}^0$ with arbitrary quality by $c^T b_\nu$ for a suitable real-valued column vector c and sufficiently large ν [10], [4], [11]. In particular, for sufficiently large ν there exist $c_1, \dots, c_m$ such that

$$Q = \operatorname*{\mathbf{diag}}_{i=1}^{m} \left(I \otimes (c_i^T b_\nu)^* (c_i^T b_\nu) \right)$$

still satisfies (4). Now observe that $I \otimes b_\nu^* (c_i c_i^T) b_\nu = (I \otimes b_\nu)^* (I \otimes M_i)(I \otimes b_\nu)$ for $M_i := c_i c_i^T$. With

$$\psi_\nu := \operatorname*{\mathbf{diag}}_{i=1}^{m} (I \otimes b_\nu) \quad \text{and} \quad M := \operatorname*{\mathbf{diag}}_{i=1}^{m} (I \otimes M_i),$$

this leads to the parametrization

$$Q = \psi_\nu^* M \psi_\nu \quad \text{with} \quad M \in \mathcal{M}_\nu \tag{6}$$

where

$$\mathcal{M}_\nu := \left\{ \operatorname*{\mathbf{diag}}_{i=1}^{m} (I \otimes M_i) : \;\; M_i = M_i^T \;\; \forall i = 1 : m \right\}.$$

Clearly $\mathcal{M}_\nu$ admits an LMI description while the dependence on ν reflects the dependence of the dimensions $\nu+1$ of the diagonal blocks on this integer. We have proved the following fact: There exists $Q \in \mathcal{Q}$ with (4) iff there exists some ν and $M \in \mathcal{M}_\nu$ with

$$\psi_\nu^* M \psi_\nu \succ 0, \tag{7a}$$

$$\begin{pmatrix} \psi_\nu \mathcal{G}_{cl} \\ \psi_\nu \end{pmatrix}^* \begin{pmatrix} M & 0 \\ 0 & -M \end{pmatrix} \begin{pmatrix} \psi_\nu \mathcal{G}_{cl} \\ \psi_\nu \end{pmatrix} \prec 0. \tag{7b}$$

C. Primal State-Space Conditions for Robust Stability

Now choose the input-balanced (minimal) realization $\psi_\nu = \left[\begin{array}{c|c} A_{\psi_\nu} & B_{\psi_\nu} \\ \hline C_{\psi_\nu} & D_{\psi_\nu} \end{array}\right]$ such that A_{ψ_ν} is Hurwitz. It is then easy to translate (7) into LMIs. For the purpose of synthesis it is also required to guarantee that $\mathcal{A}$ is Hurwitz. The following analysis result incorporates this stability property as a suitable constraint on the solutions of the respective LMIs. Note that, for this purpose, $\mathcal{X}$ is partitioned into three blocks in a natural fashion.

Lemma 1: $\mathcal{A}$ is Hurwitz and (4) holds for some $Q \in \mathcal{Q}$ iff there exist ν and $M \in \mathcal{M}_\nu$ such that the following LMIs are feasible:

$$\star^T \mathcal{M}\left(\mathcal{X}, J(M)\right) \left(\begin{array}{ccc:c} I & 0 & 0 & 0 \\ 0 & I & 0 & 0 \\ 0 & 0 & I & 0 \\ \hdashline A_{\psi_\nu} & 0 & B_{\psi_\nu}\mathcal{C} & B_{\psi_\nu}\mathcal{D} \\ 0 & A_{\psi_\nu} & 0 & B_{\psi_\nu} \\ 0 & 0 & \mathcal{A} & \mathcal{B} \\ \hdashline C_{\psi_\nu} & 0 & D_{\psi_\nu}\mathcal{C} & D_{\psi_\nu}\mathcal{D} \\ 0 & C_{\psi_\nu} & 0 & D_{\psi_\nu} \end{array}\right) \prec 0, \tag{8}$$

$$\star^T \mathcal{M}\left(Z, M\right) \begin{pmatrix} I & 0 \\ A_{\psi_\nu} & B_{\psi_\nu} \\ C_{\psi_\nu} & D_{\psi_\nu} \end{pmatrix} \succ 0, \tag{9}$$

$$\mathcal{X} + \mathbf{diag}\,(-Z, Z, 0) \succ 0. \tag{10}$$

Proof: Assume that (9) is feasible. Then, $\hat{M} := D_{\psi_\nu}^T M D_{\psi_\nu} \succ 0$. Hence, there exists some square and non-singular $D_{\hat{\psi}_\nu}$ such that $D_{\hat{\psi}_\nu}^T D_{\hat{\psi}_\nu} = \hat{M}$. Moreover, since $(A_{\psi_\nu}, B_{\psi_\nu})$ is controllable, the related ARE

$$A_{\psi_\nu}^T \hat{Z} + \hat{Z} A_{\psi_\nu} + C_{\psi_\nu}^T M C_{\psi_\nu} - (\hat{Z} B_{\psi_\nu} + C_{\psi_\nu}^T M D_{\psi_\nu}) \hat{M}^{-1} \left(\star\right)^T = 0 \tag{11}$$

has a stabilizing (largest) solution $\hat{Z}$ [16]. If defining $C_{\hat{\psi}_\nu} := D_{\hat{\psi}_\nu}^{-T}(B_{\psi_\nu}^T \hat{Z} + D_{\psi_\nu}^T M C_{\psi_\nu})$ this means that $A_{\hat{\psi}_\nu^i} := A_{\psi_\nu} - B_{\psi_\nu} D_{\hat{\psi}_\nu}^{-1} C_{\hat{\psi}_\nu}$ is Hurwitz. With $A_{\hat{\psi}_\nu} := A_{\psi_\nu}$ and $B_{\hat{\psi}_\nu} := B_{\psi_\nu}$ notice that (11) can be rewritten as

$$\star^T \mathcal{M}\left(\hat{Z}, \mathbf{diag}\,(-I, M)\right) \begin{pmatrix} I & 0 \\ A_{\hat{\psi}_\nu} & B_{\hat{\psi}_\nu} \\ \hdashline C_{\hat{\psi}_\nu} & D_{\hat{\psi}_\nu} \\ C_{\psi_\nu} & D_{\psi_\nu} \end{pmatrix} = 0 \tag{12}$$

which certifies the spectral factorization

$$\left[\begin{array}{c|c} A_{\hat{\psi}_\nu} & B_{\hat{\psi}_\nu} \\ \hline C_{\hat{\psi}_\nu} & D_{\hat{\psi}_\nu} \end{array}\right]^* \left[\begin{array}{c|c} A_{\hat{\psi}_\nu} & B_{\hat{\psi}_\nu} \\ \hline C_{\hat{\psi}_\nu} & D_{\hat{\psi}_\nu} \end{array}\right] = \left[\begin{array}{c|c} A_{\psi_\nu} & B_{\psi_\nu} \\ \hline C_{\psi_\nu} & D_{\psi_\nu} \end{array}\right]^* M \left[\begin{array}{c|c} A_{\psi_\nu} & B_{\psi_\nu} \\ \hline C_{\psi_\nu} & D_{\psi_\nu} \end{array}\right]. \tag{13}$$

If we diagonally combine (12) with the negative of (9), we get

$$\star^T \mathcal{M}\left(\begin{pmatrix} -Z & 0 \\ 0 & \hat{Z} \end{pmatrix}, \mathbf{diag}\,(-I, -J(M))\right) \left(\begin{array}{cc:cc} I & 0 & 0 & 0 \\ 0 & I & 0 & 0 \\ \hdashline A_{\psi_\nu} & 0 & B_{\psi_\nu} & 0 \\ 0 & A_{\hat{\psi}_\nu} & 0 & B_{\hat{\psi}_\nu} \\ \hdashline 0 & C_{\hat{\psi}_\nu} & 0 & D_{\hat{\psi}_\nu} \\ \hdashline C_{\psi_\nu} & 0 & D_{\psi_\nu} & 0 \\ 0 & C_{\psi_\nu} & 0 & D_{\psi_\nu} \end{array}\right) \preceq 0. \tag{14}$$

Note that the left-upper block of (14) is negative definite. As one of the key technical ingredients introduced in this paper, let us now systematically merge the LMIs (8) and (14) by using the instrumental Gluing Lemma (Section A). In fact, Lemma 6 a) and c) imply that

$$\star^T \mathcal{M}\left(\hat{\mathcal{X}}, -I\right) \left(\begin{array}{ccc:c} I & 0 & 0 & 0 \\ 0 & I & 0 & 0 \\ 0 & 0 & I & 0 \\ \hdashline A_{\psi_\nu} & 0 & B_{\psi_\nu}\mathcal{C} & B_{\psi_\nu}\mathcal{D} \\ 0 & A_{\hat{\psi}_\nu} & 0 & B_{\hat{\psi}_\nu} \\ 0 & 0 & \mathcal{A} & \mathcal{B} \\ \hdashline 0 & C_{\hat{\psi}_\nu} & 0 & D_{\hat{\psi}_\nu} \end{array}\right) \prec 0,$$

where $\hat{\mathcal{X}} := \mathcal{X} + \mathbf{diag}\left(-Z, \hat{Z}, 0\right)$. By an elementary operation (congruence) to eliminate $C_{\hat{\psi}_\nu}$, this implies

$$\star^T \mathcal{M}\left(\hat{\mathcal{X}}, -I\right) \left(\begin{array}{ccc:c} I & 0 & 0 & 0 \\ 0 & I & 0 & 0 \\ 0 & 0 & I & 0 \\ \hdashline A_{\psi_\nu} & * & B_{\psi_\nu}\mathcal{C} & B_{\psi_\nu}\mathcal{D} \\ 0 & A_{\hat{\psi}^i_\nu} & 0 & B_{\hat{\psi}_\nu} \\ 0 & * & \mathcal{A} & \mathcal{B} \\ \hdashline 0 & 0 & 0 & D_{\hat{\psi}_\nu} \end{array}\right) \prec 0$$

so that $\mathrm{He}\left(\hat{\mathcal{X}}\begin{pmatrix} A_{\psi_\nu} & * & B_{\psi_\nu}\mathcal{C} \\ 0 & A_{\hat{\psi}^i_\nu} & 0 \\ 0 & * & \mathcal{A} \end{pmatrix}\right) \prec 0$. Since A_{ψ_ν} and $A_{\hat{\psi}^i_\nu}$ are Hurwitz, stability of $\mathcal{A}$ is hence equivalent to $\hat{\mathcal{X}} \succ 0$.

Now suppose that $\mathcal{A}$ is Hurwitz and $Q \in \mathcal{Q}$ satisfies (4). Then there exists a sufficiently large ν and some $M \in \mathcal{M}_\nu$ such that (7) holds. Let us fix M and apply the KYP Lemma in order to infer that (8) and (9) have solutions X and Z. For *any* Z we can now exploit the preparation in order to see that $\hat{\mathcal{X}} \succ 0$. Since Z can be chosen arbitrarily closely to $\hat{Z}$, we arrive at the validity of (10) for some particular Z.

Conversely, suppose that (8)-(10) are feasible for some $M \in \mathcal{M}_\nu$. Since $\hat{Z} \succ Z$ we infer that $\hat{\mathcal{X}} \succ 0$ holds as well. Therefore $\mathcal{A}$ is Hurwitz. Then (7) follows from (8) and (9) by applying the KYP Lemma. Therefore we have found some $Q \in \mathcal{Q}$, namely $Q = \psi_\nu^* M \psi_\nu \succ 0$, for which (4) is valid. ■

D. Dual State-Space Conditions for Robust Stability

Due to the dualization lemma [12], (4) is equivalent to

$$\begin{pmatrix} I \\ -\mathcal{G}_{cl}^* \end{pmatrix}^* \begin{pmatrix} Q^{-1} & 0 \\ 0 & -Q^{-1} \end{pmatrix} \begin{pmatrix} I \\ -\mathcal{G}_{cl}^* \end{pmatrix} \succ 0. \tag{15}$$

Let us now introduce the stable (typically wide) transfer matrix $\phi_\nu := \psi_\nu^T$ as well as $\mathcal{N}_\nu := \mathcal{M}_\nu$. Moreover let us parameterize Q^{-1} as

$$Q^{-1} = \phi_\nu N \phi_\nu^* \quad \text{with} \quad N \in \mathcal{N}_\nu. \tag{16}$$

Choose the natural realization of ϕ_ν as

$$\phi_\nu = \left[\begin{array}{c|c} A_{\phi_\nu} & B_{\phi_\nu} \\ \hline C_{\phi_\nu} & D_{\phi_\nu} \end{array}\right] = \left[\begin{array}{c|c} A_{\psi_\nu}^T & C_{\psi_\nu}^T \\ \hline B_{\psi_\nu}^T & D_{\psi_\nu}^T \end{array}\right] \tag{17}$$

which is minimal and output-balanced. It is then not difficult to formulate a dual version of Theorem 1.

Lemma 2: $\mathcal{A}$ is Hurwitz and (15) holds for some $Q \in \mathcal{Q}$ iff there exist ν and $N \in \mathcal{N}_\nu$ such that

$$\star \mathcal{M}\left(\mathcal{Y}, J(N)\right) \left(\begin{array}{ccc|c} -A_{\phi_\nu}^T & 0 & 0 & C_{\phi_\nu}^T \\ 0 & -A_{\phi_\nu}^T & -C_{\phi_\nu}^T \mathcal{B}^T & -C_{\phi_\nu}^T \mathcal{D}^T \\ 0 & 0 & -\mathcal{A}^T & -\mathcal{C}^T \\ \hline I & 0 & 0 & 0 \\ 0 & I & 0 & 0 \\ 0 & 0 & I & 0 \\ \hline -B_{\phi_\nu}^T & 0 & 0 & D_{\phi_\nu}^T \\ 0 & -B_{\phi_\nu}^T & -D_{\phi_\nu}^T \mathcal{B}^T & -D_{\phi_\nu}^T \mathcal{D}^T \end{array}\right) \succ 0, \tag{18}$$

$$\star^T \mathcal{M}\left(W, N\right) \left(\begin{array}{cc} -A_{\phi_\nu}^T & C_{\phi_\nu}^T \\ I & 0 \\ \hline -B_{\phi_\nu}^T & D_{\phi_\nu}^T \end{array}\right) \succ 0, \tag{19}$$

$$\mathcal{Y} + \mathbf{diag}\left(-W, W, 0\right) \succ 0. \tag{20}$$

The proof follows the same steps as for Theorem 1. For future reference, we note that it involves certifying the factorization $\phi_\nu N \phi_\nu^* = \hat{\phi}_\nu \hat{\phi}_\nu^*$ (in which $\hat{\phi}_\nu$ and $\hat{\phi}_\nu^{-1}$ are stable) with the smallest solution $\hat{W}$ of the ARE

$$\star^T \mathcal{M}\left(\hat{W}, \mathbf{diag}\left(-I, N\right)\right) \begin{pmatrix} I & 0 \\ \hline -A_{\hat{\phi}_\nu}^T & C_{\hat{\phi}_\nu}^T \\ \hline -B_{\hat{\phi}_\nu}^T & D_{\hat{\phi}_\nu}^T \\ -B_{\phi_\nu}^T & D_{\phi_\nu}^T \end{pmatrix} = 0 \tag{21}$$

where $A_{\hat{\phi}_\nu} := A_{\phi_\nu}$ and $C_{\hat{\phi}_\nu} := C_{\phi_\nu}$.

III. MAIN RESULT

For system (3), introduce the annihilators $U = \begin{pmatrix} 0 \\ 0 \\ C_y^T \\ D_{yp}^T \end{pmatrix}_\perp$ and $V = \begin{pmatrix} 0 \\ 0 \\ B_u \\ D_{qu} \end{pmatrix}_\perp$ where $M_\perp$ denotes a basis matrix of the null-space of M^T and the zero blocks are chosen compatibly with the dimension of A_{ψ_ν} (for U) and A_{ϕ_ν} (for V) respectively.

Theorem 3: Consider the system in Figure 1 with G realized as in (3).

(i) *Suppose that the inequalities*

$$\star^T \mathcal{M}\left(X, J(M)\right) \left(\begin{array}{ccc:c} I & 0 & 0 & 0 \\ 0 & I & 0 & 0 \\ 0 & 0 & I & 0 \\ \hdashline A_{\psi_\nu} & 0 & B_{\psi_\nu} C_q & B_{\psi_\nu} D_{qp} \\ 0 & A_{\psi_\nu} & 0 & B_{\psi_\nu} \\ 0 & 0 & A & B_p \\ \hdashline C_{\psi_\nu} & 0 & D_{\psi_\nu} C_q & D_{\psi_\nu} D_{qp} \\ 0 & C_{\psi_\nu} & 0 & D_{\psi_\nu} \end{array}\right) U \prec 0, \tag{22}$$

$$\star^T \mathcal{M}\left(Y, J(N)\right) \left(\begin{array}{ccc:c} -A_{\phi_\nu}^T & 0 & 0 & C_{\phi_\nu}^T \\ 0 & -A_{\phi_\nu}^T & -C_{\phi_\nu}^T B_p^T & -C_{\phi_\nu}^T D_{qp}^T \\ 0 & 0 & -A^T & -C_q^T \\ \hdashline I & 0 & 0 & 0 \\ 0 & I & 0 & 0 \\ 0 & 0 & I & 0 \\ \hdashline -B_{\phi_\nu}^T & 0 & 0 & D_{\phi_\nu}^T \\ 0 & -B_{\phi_\nu}^T & -D_{\phi_\nu}^T B_p^T & -D_{\phi_\nu}^T D_{qp}^T \end{array}\right) V \succ 0, \tag{23}$$

$$\left(\begin{array}{ccc:ccc} X_{11} - R_{11} & X_{12} & X_{13} & -R_{12} & 0 & 0 \\ X_{21} & X_{22} + R_{11} & X_{23} & 0 & -R_{12} & 0 \\ X_{31} & X_{32} & X_{33} & 0 & 0 & I \\ \hdashline -R_{21} & 0 & 0 & Y_{11} - R_{22} & Y_{12} & Y_{13} \\ 0 & -R_{21} & 0 & Y_{21} & Y_{22} + R_{22} & Y_{23} \\ 0 & 0 & I & Y_{31} & Y_{32} & Y_{33} \end{array}\right) \succ 0, \tag{24}$$

$$\star^T \mathcal{M}\left(R, \mathbf{diag}\left(M, N, \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}\right)\right) \left(\begin{array}{cc:cc} I & 0 & 0 & 0 \\ 0 & I & 0 & 0 \\ \hdashline A_{\psi_\nu} & 0 & B_{\psi_\nu} & 0 \\ 0 & -A_{\phi_\nu}^T & 0 & C_{\phi_\nu}^T \\ \hdashline C_{\psi_\nu} & 0 & D_{\psi_\nu} & 0 \\ 0 & -B_{\phi_\nu}^T & 0 & D_{\phi_\nu}^T \\ \hdashline 0 & 0 & 0 & I \\ 0 & 0 & I & 0 \end{array}\right) \succ 0. \tag{25}$$

are feasible for some ν and $M \in \mathcal{M}_\nu$, $N \in \mathcal{N}_\nu$. Then, there exists a controller rendering $\mathcal{A}$ Hurwitz and for which

$$\star^* \begin{pmatrix} (\phi_\nu N \phi_\nu^*)^{-1} & 0 \\ 0 & -\psi_\nu^* M \psi_\nu \end{pmatrix} \begin{pmatrix} \mathcal{G}_{cl} \\ I \end{pmatrix} \prec 0. \tag{26}$$

(ii) *Suppose there exists a controller which renders $\mathcal{A}$ Hurwitz and a $Q \in \mathcal{Q}$ for which (4) holds. Then there exist ν and $M \in \mathcal{M}_\nu$, $N \in \mathcal{N}_\nu$ for which the LMIs (22)-(25) are* *feasible.*

The controller in (i) guarantees (4) for $Q = \psi_\nu^* M \psi_\nu$ in case that $(\phi_\nu N \phi_\nu^*)^{-1} = \psi_\nu^* M \psi_\nu$. This non-convex constraint on M and N forces us to rely on a heuristic iteration for robust controller synthesis as discussed in the next section.

Remark 4: When external disturbances (w) and controlled outputs (z) are present in the system, the problem of designing robustly stabilizing controllers that achieve a closed-loop $\mathcal{H}_\infty$-gain less than γ can be solved by replacing the plant by

$$\left[\begin{array}{c|ccc} A & B_p & B_w & B_u \\ \hline C_q & D_{qp} & D_{qw} & D_{qu} \\ C_z & D_{zp} & D_{zw} & D_{zu} \\ C_y & D_{yp} & D_{yw} & 0 \end{array}\right],$$

the multiplier $\mathbf{diag}\,(\psi_\nu^* M \psi_\nu, -\psi_\nu^* M \psi_\nu)$ by $\mathbf{diag}\,(\psi_\nu^* M \psi_\nu, \gamma^{-1} I, -\psi_\nu^* M \psi_\nu, -\gamma I)$ and $\mathbf{diag}\,(\phi_\nu N \phi_\nu^*, -\phi_\nu N \phi_\nu^*)$ by $\mathbf{diag}\,(\phi_\nu N \phi_\nu^*, \gamma I, -\phi_\nu N \phi_\nu^*, -\gamma^{-1} I)$. In this formulation, γ can be treated as a variable which, after taking the Schur-complement, enters the solvability conditions linearly.

Remark 5: Note that Theorem 3 comprises various well-known specializations. For example, the LMIs (22)-(25) for $M = N = I$ and $\psi_\nu = \phi_\nu = I$ (with empty A_{ψ_ν} and A_{ϕ_ν}) are identical to those appearing in standard H_∞-synthesis [1], [5]. In general, the additional LMI (25) certifies the multiplier coupling

$$\begin{pmatrix} \psi_\nu^* M \psi_\nu & I \\ I & \phi_\nu N \phi_\nu^* \end{pmatrix} \succ 0. \tag{27}$$

If the multiplies are non-dynamic ($\psi_\nu = \phi_\nu = I$) then (22)-(25) are identical to those in [8], [2] for the gain-scheduling synthesis problem with static D-scalings. In fact, the main motivation for this work is to use Theorem 3 in order to arrive at a solution for gain-scheduling synthesis with dynamic D-scalings as described in [14].

IV. A Primal Multiplier/Dual Multiplier Iteration

Due to [7], the FDI (26) implies robust stability for all proper and stable uncertainties structured Δ with

$$\begin{pmatrix} I \\ \Delta \end{pmatrix}^* \begin{pmatrix} (\phi_\nu N \phi_\nu^*)^{-1} & 0 \\ 0 & -\psi_\nu^* M \psi_\nu \end{pmatrix} \begin{pmatrix} I \\ \Delta \end{pmatrix} \succeq 0$$

$$\iff \Delta^*(\psi_\nu^* M \psi_\nu)\Delta \preceq (\phi_\nu N \phi_\nu^*)^{-1}$$

$$\iff \Delta^*\Delta \preceq (\psi_\nu^* M \psi_\nu)^{-\frac{1}{2}}(\phi_\nu N \phi_\nu^*)^{-1}(\psi_\nu^* M \psi_\nu)^{-\frac{1}{2}}$$

since $(\psi_\nu^* M \psi_\nu)\Delta = \Delta(\psi_\nu^* M \psi_\nu)$. This leads to a frequency-dependent norm-bound on the individual blocks of Δ for which robust stabilization of the controller is guaranteed. Due to (27), note that the right-hand side is bounded from above by I. Hence, it is desired to push this matrix as close as possible to I uniformly on $\mathbb{C}^0$, by minimizing $\eta \in (1, \infty)$ such that

$$\frac{1}{\eta} I \prec (\psi_\nu^* M \psi_\nu)^{-\frac{1}{2}}(\phi_\nu N \phi_\nu^*)^{-1}(\psi_\nu^* M \psi_\nu)^{-\frac{1}{2}} \prec I$$

$$\iff \psi_\nu^* M \psi_\nu \prec \eta(\phi_\nu N \phi_\nu^*)^{-1} \ \ \& \ \ (27) \tag{28}$$

$$\iff \phi_\nu N \phi_\nu^* \prec \eta(\psi_\nu^* M \psi_\nu)^{-1} \ \ \& \ \ (27). \tag{29}$$

This leads us to the following iteration for robust controller synthesis:

Initialization: Fix some ν for which (22)-(25) are feasible.

Repeat until convergence:

Step k**:** Fix N and minimize η over (22)-(25) and the LMI corresponding to (28).

Step $k+1$**:** Fix M and minimize η over (22)-(25) and the LMI corresponding to (29).

For fixed ν, the initialization amounts to a convex feasibility problem. If no suitable ν exists, it is assured by Theorem 3 that no controller and $Q \in \mathcal{Q}$ can render (4) satisfied. The iterations between steps k and $k+1$ serve to minimize η. Since (28) and (29) can be turned into LMIs in M and N respectively, both steps just require to solve standard LMI problems. In each step the achieved level η implies that robust stability against structured uncertainties with a norm-bound $\frac{1}{\sqrt{\eta}}$ can be assured.

Note that steps k and $k+1$ are more powerful when compared to a completely separated iteration between the search for a multiplier for a fixed controller and controller synthesis for fixed multipliers as in the standard D/K-iteration [3]. As the essential novel features, our robust synthesis result is formulated directly in terms of the original description of the uncontrolled system and for multipliers that are parameterized with general tall outer factors without any further technical restrictions, such that the suggested iteration allows to completely avoid frequency gridding or frequency domain multiplier fitting.

V. Numerical Example

Consider the mechanical system shown in Figure 2.

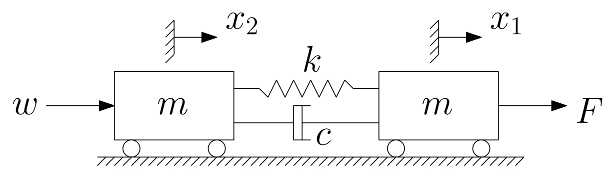

Fig. 2. Mechanical system with uncertain spring and damper.

We assume that the values of k and c are constant, but that they vary around their nominal values, k_0 and c_0, as $k = k_0(1 + k^*\delta_k)$ and $c = c_0(1 + c^*\delta_c)$, where $|\delta_k| \leq 1$ and $|\delta_c| \leq 1$. We use the numerical values $m_0 = 10$ kg, $k_0 = 10$ N/m, $c_0 = 10$ Ns/m and $k^* = c^* = 0.5$. Take x_1 as the measured output and x_2 as the controlled output. We can now express the system as

$$
\begin{pmatrix} q \\ z \\ y \end{pmatrix} = \left[\begin{array}{cccc|cc:c:c}
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
-\frac{k_0}{m} & \frac{k_0}{m} & -\frac{c_0}{m} & \frac{c_0}{m} & -\sqrt{\frac{k_0 k^*}{m}} & -\sqrt{\frac{c_0 c^*}{m}} & 0 & \frac{1}{m} \\
\frac{k_0}{m} & -\frac{k_0}{m} & \frac{c_0}{m} & -\frac{c_0}{m} & \sqrt{\frac{k_0 k^*}{m}} & \sqrt{\frac{c_0 c^*}{m}} & \frac{1}{m} & 0 \\
\hline
\sqrt{\frac{k_0 k^*}{m}} & -\sqrt{\frac{k_0 k^*}{m}} & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & \sqrt{\frac{c_0 c^*}{m}} & -\sqrt{\frac{c_0 c^*}{m}} & 0 & 0 & 0 & 0 \\
\hdashline
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
\hdashline
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{array}\right] \begin{pmatrix} p \\ w \\ u \end{pmatrix}
$$

and $p = \begin{pmatrix} \delta_k & 0 \\ 0 & \delta_c \end{pmatrix} q$. Our goal is to obtain robust controllers that yield the minimum achievable performance level, γ, from the disturbance to x_2 for different values of ν. Note that the algorithm described in Section IV yields η values larger than 1, implying that the performance guarantees are valid only for uncertainties with bound $1/\sqrt{\eta}$. Since we want guaranteed performance over the whole range of parameters (*i.e.*, $k^* = c^* = 0.5$), we run the algorithm for $k^* = c^* = 0.75$ and find the smallest value of γ that yields $1/\sqrt{\eta} = 2/3$, or, $\eta = 2.25$.

We then solve for the resulting controller, form the closed-loop system and compute the closed-loop $\mathcal{H}_\infty$-norms for frozen values of the parameters corresponding to $k^* = 0.5$, $c^* = 0.5$. The results we obtain are listed in the table below. Note that due to non-convexity, the γ value is not guaranteed to be monotonically decreasing with increasing ν. For each value of ν, the worst value of the frozen $\mathcal{H}_\infty$-norm is given in the third row (labeled "$\gamma_{achieved}$").

ν	0	1	2	3	4
γ	4.98	1.55	1.49	1.46	1.45
$\gamma_{achieved}$	0.56	0.42	0.43	0.51	0.44

The $D/G - K$ iteration as implemented in [3] yields a worst value of 1.08 for the frozen $\mathcal{H}_\infty$-norm computed in the same manner as the last row in the table above. (Note that since neither one of the parameters is repeated, there is no material difference between D-scales and D/G-scales in this problem.) For 25 samples of possible k and c values, the responses to a unit step disturbance for the cases $\nu = 0, 2, 4$ and the D/K-controller are given in Figure 3. These plots indicate better behavior than the D/K-controller even for the case $\nu = 0$ and further improvement when ν is increased.

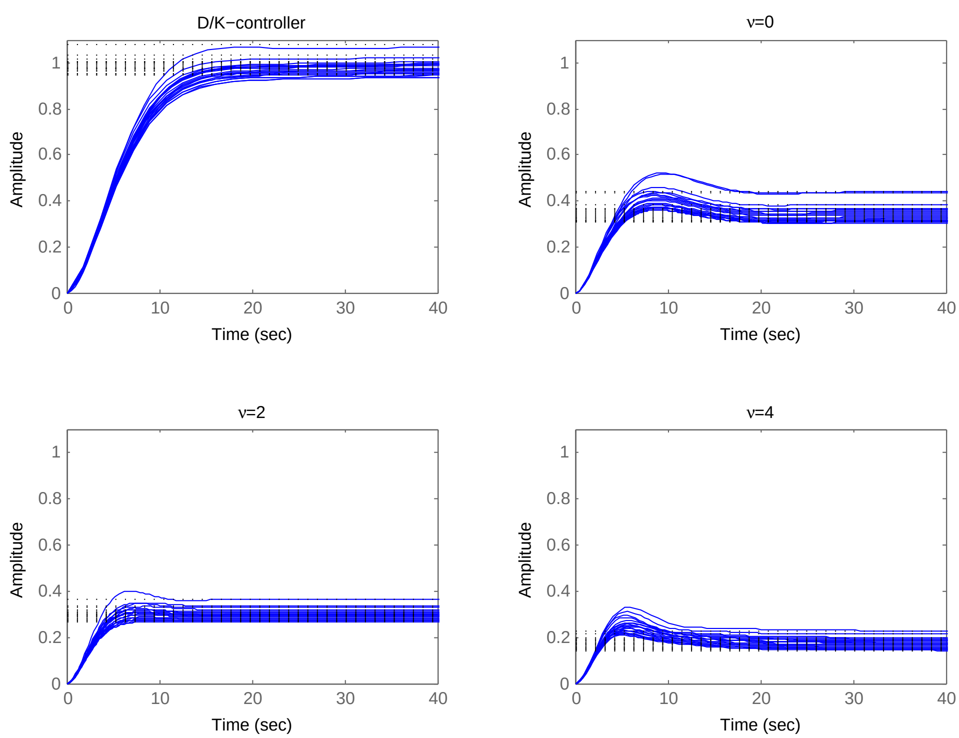

Fig. 3. Sampled responses to a unit step disturbance of the controlled systems obtained from the D/K-iteration and the multiplier iteration for different ν values.

VI. Conclusions

Using parametrized dynamic D-scales, we have given necessary existence conditions for a controller that robustly stabilizes a system against uncertainties bounded in norm by 1. These conditions are shown to be sufficient for robust stability against uncertainties with a norm bound demonstrably less than 1. We have also proposed an iterative procedure for the maximization of this guaranteed allowable norm bound. Unlike the conventional D/K-iteration, this procedure does not necessitate the computation of the controller and involves basis functions for approximating D-scales only. The application of the proposed iterative solution to a mechanical system yields better results than the conventional D/K-iteration. The main result of the paper is essential for the solution of the gain-scheduled control problem using dynamic D-scales as reported in [14].

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APPENDIX

A - OPERATIONS ON FDIS AND CORRESPONDING LMIS

The FDI $G^*(\psi_o^* M_o \psi_o)G \prec 0$ for the 'old' multiplier $\psi_o^* M_o \psi_o$ persists to hold for the 'new' multiplier $\psi_n^* M_n \psi_n$ as $G^*(\psi_n^* M_n \psi_n)G \prec 0$ in case that $\psi_n^* M_n \psi_n - \psi_o^* M_o \psi_o \preceq 0$. With natural notations for the corresponding realizations, the following gluing lemma reveals a relation of suitable KYP certificates.

Lemma 6: (Gluing) Suppose that D_o is invertible and that $A_o - B_o D_o^{-1} C_o$, A_n have no eigenvalues in $\mathbb{C}^0$. Then there exist R_o, R_n with $(A_o - B_o D_o^{-1} C_o)^T R_o + R_o(A_o - B_o D_o^{-1} C_o) \prec 0$ and $A_n^T R_n + R_n A_n \prec 0$. Let X and R satisfy

$$\star^T \mathcal{M}\left(X, M_o\right) \left(\begin{array}{cc:c} I & 0 & 0 \\ 0 & I & 0 \\ \hdashline A_o & B_o C & B_o D \\ 0 & A & B \\ \hdashline C_o & D_o C & D_o D \end{array}\right) \prec 0, \tag{30}$$

$$\star^T \mathcal{M}\left(R, \mathbf{diag}\left(M_n, -M_o\right)\right) \left(\begin{array}{cc:c} I & 0 & 0 \\ 0 & I & 0 \\ \hdashline A_n & A_{no} & B_n \\ 0 & A_o & B_o \\ \hdashline C_n & C_{no} & D_n \\ 0 & C_o & D_o \end{array}\right) \preceq 0. \tag{31}$$

a) Then there exist $\epsilon > 0$ and $\delta > 0$ (that can be taken arbitrarily small) such that

$$\star^T \mathcal{M}\left(X_{no}, M_n\right) \left(\begin{array}{ccc:c} I & 0 & 0 & 0 \\ 0 & I & 0 & 0 \\ 0 & 0 & I & 0 \\ \hdashline A_n & A_{no} & B_n C & B_n D \\ 0 & A_o & B_o C & B_o D \\ 0 & 0 & A & B \\ \hdashline C_n & C_{no} & D_n C & D_n D \end{array}\right) \prec 0 \tag{32}$$

holds for $X_{no} = \begin{pmatrix} R_{11} + \epsilon R_n & R_{12} & 0 \\ R_{12}^T & X_{11} + R_{22} + \delta R_o & X_{12} \\ 0 & X_{21} & X_{22} \end{pmatrix}$. *b) If $A_{no} = 0$ and $C_{no} = 0$ then the middle block of X_{no} is non-singular and its Schur complement, denoted as X_n, satisfies*

$$\star^T \mathcal{M}\left(X_n, M_n\right) \left(\begin{array}{cc:c} I & 0 & 0 \\ 0 & I & 0 \\ \hdashline A_n & B_n C & B_n D \\ 0 & A & B \\ \hdashline C_n & C & D \end{array}\right) \prec 0. \tag{33}$$

c) If the left-upper block of (31) is negative definite then a) and b) remain true for $\delta = 0$ and $\epsilon = 0$.

If $\phi^i = \phi^{-1}$ exists we require to relate certificates for the following, obviously equivalent, FDIs:

$$\begin{pmatrix} \psi^* \psi & I \\ I & \phi \phi^* \end{pmatrix} \succ 0, \tag{34}$$

$$\begin{pmatrix} \psi \\ \phi^{-1} \end{pmatrix}^* \begin{pmatrix} -I & 0 \\ 0 & I \end{pmatrix} \begin{pmatrix} \psi \\ \phi^{-1} \end{pmatrix} \prec 0. \tag{35}$$

Lemma 7: *Let D_ϕ be non-singular and suppose that A_ϕ, $A_\phi - B_\phi D_\phi^{-1} C_\phi$ have no eigenvalues in $\mathbb{C}^0$.*

a) Suppose R certifies (34) as

$$\star^T \mathcal{M}\left(R, \left(\begin{array}{cc:cc} I & 0 & 0 & 0 \\ 0 & I & 0 & 0 \\ \hdashline 0 & 0 & 0 & I \\ 0 & 0 & I & 0 \end{array}\right)\right) \left(\begin{array}{cc:cc} I & 0 & 0 & 0 \\ 0 & I & 0 & 0 \\ \hdashline A_\psi & 0 & B_\psi & 0 \\ 0 & -A_\phi^T & 0 & C_\phi^T \\ \hdashline C_\psi & 0 & D_\psi & 0 \\ 0 & -B_\phi^T & 0 & D_\phi^T \\ \hdashline 0 & 0 & I & 0 \\ 0 & 0 & 0 & I \end{array}\right) \succ 0. \tag{36}$$

Then R_{22} is non-singular and Γ which can be taken arbitrarily closely to

$$\begin{pmatrix} R_{12}R_{22}^{-1}R_{21} - R_{11} & R_{12}R_{22}^{-1} \\ R_{22}^{-1}R_{21} & R_{22}^{-1} \end{pmatrix}$$

certifies (35) as

$$\star^T \mathcal{M}\left(\Gamma, -J(I)\right) \left(\begin{array}{cc:c} I & 0 & 0 \\ 0 & I & 0 \\ \hdashline A_\psi & 0 & B_\psi \\ 0 & A_{\phi^i} & B_{\phi^i} \\ \hdashline C_\psi & 0 & D_\psi \\ 0 & C_{\phi^i} & D_{\phi^i} \end{array}\right) \prec 0. \tag{37}$$

b) If Γ is a certificate for (35) as in (37) then Γ_{22} is non-singular and

$$\begin{pmatrix} \Gamma_{12}\Gamma_{22}^{-1}\Gamma_{21} - \Gamma_{11} & \Gamma_{12}\Gamma_{22}^{-1} \\ \Gamma_{22}^{-1}\Gamma_{21} & \Gamma_{22}^{-1} \end{pmatrix}$$

satisfies the non-strict version of (36), which certifies the non-strict version of the FDI (34).

C - Proof of Theorem 3 - Statement (i)

We begin by noting that, through the same arguments as for the proofs of Lemma 1 and Lemma 2, we can rewrite (22)-(25) as

$$\star^T \mathcal{M}\left(\hat{X}, J(I)\right) \left(\begin{array}{ccc|c} I & 0 & 0 & 0 \\ 0 & I & 0 & 0 \\ 0 & 0 & I & 0 \\ \hline A_{\hat{\psi}_\nu} & 0 & B_{\hat{\psi}_\nu} C_q & B_{\hat{\psi}_\nu} D_{qp} \\ 0 & A_{\hat{\psi}_\nu} & 0 & B_{\hat{\psi}_\nu} \\ 0 & 0 & A & B_p \\ \hline C_{\hat{\psi}_\nu} & 0 & D_{\hat{\psi}_\nu} C_q & D_{\hat{\psi}_\nu} D_{qp} \\ 0 & C_{\hat{\psi}_\nu} & 0 & D_{\hat{\psi}_\nu} \end{array}\right) U \prec 0 \tag{38}$$

$$\star^T \mathcal{M}\left(\hat{Y}, J(I)\right) \left(\begin{array}{ccc|c} -A_{\hat{\phi}_\nu}^T & 0 & 0 & C_{\hat{\phi}_\nu}^T \\ 0 & -A_{\hat{\phi}_\nu}^T & -C_{\hat{\phi}_\nu}^T B_p^T & -C_{\hat{\phi}_\nu}^T D_{qp}^T \\ 0 & 0 & -A^T & -C_q^T \\ \hline I & 0 & 0 & 0 \\ 0 & I & 0 & 0 \\ 0 & 0 & I & 0 \\ \hline -B_{\hat{\phi}_\nu}^T & 0 & 0 & D_{\hat{\phi}_\nu}^T \\ 0 & -B_{\hat{\phi}_\nu}^T & -D_{\hat{\phi}_\nu}^T B_p^T & -D_{\hat{\phi}_\nu}^T D_{qp}^T \end{array}\right) V \succ 0, \tag{39}$$

$$\left(\begin{array}{ccc|ccc} \hat{X}_{11} - \hat{R}_{11} & \hat{X}_{12} & \hat{X}_{13} & -\hat{R}_{12} & 0 & 0 \\ \hat{X}_{21} & \hat{X}_{22} + \hat{R}_{11} & \hat{X}_{23} & 0 & -\hat{R}_{12} & 0 \\ \hat{X}_{31} & \hat{X}_{32} & \hat{X}_{33} & 0 & 0 & I \\ \hline -\hat{R}_{21} & 0 & 0 & \hat{Y}_{11} - \hat{R}_{22} & \hat{Y}_{12} & \hat{Y}_{13} \\ 0 & -\hat{R}_{21} & 0 & \hat{Y}_{21} & \hat{Y}_{22} + \hat{R}_{22} & \hat{Y}_{23} \\ 0 & 0 & I & \hat{Y}_{31} & \hat{Y}_{32} & \hat{Y}_{33} \end{array}\right) \succ 0, \tag{40}$$

$$\star^T \mathcal{M}\left(\hat{R}, \mathbf{diag}\left(\left(\begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix}, \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}\right)\right)\right) \left(\begin{array}{cc:cc} I & 0 & 0 & 0 \\ 0 & I & 0 & 0 \\ \hdashline A_{\hat{\psi}_\nu} & 0 & B_{\hat{\psi}_\nu} & 0 \\ 0 & -A^T_{\hat{\phi}_\nu} & 0 & C^T_{\hat{\phi}_\nu} \\ \hdashline C_{\hat{\psi}_\nu} & 0 & D_{\hat{\psi}_\nu} & 0 \\ 0 & -B^T_{\hat{\phi}_\nu} & 0 & D^T_{\hat{\phi}_\nu} \\ \hdashline 0 & 0 & 0 & I \\ 0 & 0 & I & 0 \end{array}\right) \succ 0 \tag{41}$$

with the definitions

$$\hat{X} := X + \mathbf{diag}(-\hat{Z}, \hat{Z}, 0), \quad \hat{Y} := Y + \mathbf{diag}\left(-\hat{W}, \hat{W}, 0\right), \quad \hat{R} := R + \mathbf{diag}\left(-\hat{Z}, -\hat{W}\right).$$

The key ingredient of the proof is to use Lemmas 6 and 7 in order to reduces these coupled LMIs to standard H_∞-synthesis LMIs.

Step 1. From (41), we infer

$$\mathrm{He}\left(\left(-(A_{\hat{\phi}_\nu} - B_{\hat{\phi}_\nu} D^{-1}_{\hat{\phi}_\nu} C_{\hat{\phi}_\nu})\right) \hat{R}_{22}\right) \succ 0 \quad \text{and} \quad \mathrm{He}\left(\left(A_{\hat{\psi}_\nu} - B_{\hat{\psi}_\nu} D^{-1}_{\hat{\psi}_\nu} C_{\hat{\psi}_\nu}\right) \hat{R}_{11}\right) \succ 0.$$

Since $-(A_{\hat{\phi}_\nu} - B_{\hat{\phi}_\nu} D^{-1}_{\hat{\phi}_\nu} C_{\hat{\phi}_\nu}) = -A_{\hat{\phi}^i_\nu}$ is anti-Hurwitz, we have $\hat{R}_{22} \succ 0$. Similarly, since $A_{\hat{\psi}_\nu} - B_{\hat{\psi}_\nu} D^{-1}_{\hat{\psi}_\nu} C_{\hat{\psi}_\nu} = A_{\hat{\psi}^i_\nu}$ is Hurwitz, $\hat{R}_{11} \prec 0$. By the Schur complement formula, we infer that (40) is equivalent to

$$\left(\begin{array}{cccc:cccc} \hat{R}_{22}^{-1} & 0 & 0 & 0 & I & 0 & 0 & 0 \\ 0 & \hat{X}_{11} - \hat{R}_{11} & \hat{X}_{12} & \hat{X}_{13} & -\hat{R}_{12} & 0 & 0 & 0 \\ 0 & \hat{X}_{21} & \hat{X}_{22} & \hat{X}_{23} & 0 & I & -\hat{R}_{12} & 0 \\ 0 & \hat{X}_{31} & \hat{X}_{32} & \hat{X}_{33} & 0 & 0 & 0 & I \\ \hdashline I & -\hat{R}_{21} & 0 & 0 & \hat{Y}_{11} & 0 & \hat{Y}_{12} & \hat{Y}_{13} \\ 0 & 0 & I & 0 & 0 & -\hat{R}_{11}^{-1} & 0 & 0 \\ 0 & 0 & -\hat{R}_{21} & 0 & \hat{Y}_{21} & 0 & \hat{Y}_{22} + \hat{R}_{22} & \hat{Y}_{23} \\ 0 & 0 & 0 & I & \hat{Y}_{31} & 0 & \hat{Y}_{32} & \hat{Y}_{33} \end{array}\right) \succ 0$$

By elementary operations we can eliminate $-\hat{R}_{12}$ which leads to

$$\left(\begin{array}{cccc:cccc}
\hat{R}_{22}^{-1} & \hat{R}_{22}^{-1}\hat{R}_{21} & 0 & 0 & I & 0 & 0 & 0\\
\hat{R}_{12}\hat{R}_{22}^{-1} & \hat{R}_{12}\hat{R}_{22}^{-1}\hat{R}_{21}-\hat{R}_{11}+\hat{X}_{11} & \hat{X}_{12} & \hat{X}_{13} & 0 & 0 & 0 & 0\\
0 & \hat{X}_{21} & \hat{X}_{22} & \hat{X}_{23} & 0 & I & 0 & 0\\
0 & \hat{X}_{31} & \hat{X}_{32} & \hat{X}_{33} & 0 & 0 & 0 & I\\
\hdashline
I & 0 & 0 & 0 & \hat{Y}_{11} & 0 & \hat{Y}_{12} & \hat{Y}_{13}\\
0 & 0 & I & 0 & 0 & -\hat{R}_{11}^{-1} & -\hat{R}_{11}^{-1}\hat{R}_{12} & 0\\
0 & 0 & 0 & 0 & \hat{Y}_{21} & -\hat{R}_{21}\hat{R}_{11}^{-1} & \hat{Y}_{22}+\hat{R}_{22}-\hat{R}_{12}\hat{R}_{11}^{-1}\hat{R}_{12} & \hat{Y}_{23}\\
0 & 0 & 0 & I & \hat{Y}_{31} & 0 & \hat{Y}_{32} & \hat{Y}_{33}
\end{array}\right) \succ 0. \tag{42}$$

Step 2. If we apply Lemma 7 a) to (41) and permute we find $\hat{\Gamma}$ that can be taken arbitrarily close to $\begin{pmatrix} \hat{R}_{22}^{-1} & \hat{R}_{22}^{-1}\hat{R}_{21} \\ \hat{R}_{12}\hat{R}_{22}^{-1} & \hat{R}_{12}\hat{R}_{22}^{-1}\hat{R}_{21}-\hat{R}_{11} \end{pmatrix}$ and that satisfies

$$\star^T \mathcal{M}\left(\hat{\Gamma}, J(I)\right) \left(\begin{array}{cc:c}
I & 0 & 0\\
0 & I & 0\\
\hdashline
A_{\hat{\phi}_\nu}-B_{\hat{\phi}_\nu}D_{\hat{\phi}_\nu}^{-1}C_{\hat{\phi}_\nu} & 0 & B_{\hat{\phi}_\nu}D_{\hat{\phi}_\nu}^{-1}\\
0 & A_{\hat{\psi}_\nu} & B_{\hat{\psi}_\nu}\\
\hdashline
-D_{\hat{\phi}_\nu}^{-1}C_{\hat{\phi}_\nu} & 0 & D_{\hat{\phi}_\nu}^{-1}\\
0 & C_{\hat{\psi}_\nu} & \hat{D}_{\hat{\psi}_\nu}
\end{array}\right) \prec 0. \tag{43}$$

Similarly, performing a permutation in (41), applying Lemma 7 a), and permuting back, one shows that there exists some $\tilde{\Gamma}$ arbitrarily close to $\begin{pmatrix} \hat{R}_{11}^{-1} & \hat{R}_{11}^{-1}\hat{R}_{12} \\ \hat{R}_{21}\hat{R}_{11}^{-1} & \hat{R}_{21}\hat{R}_{11}^{-1}\hat{R}_{12}-\hat{R}_{22} \end{pmatrix}$ which satisfies

$$\star^T \mathcal{M}\left(\tilde{\Gamma}, J(I)\right) \left(\begin{array}{cc:c}
I & 0 & 0\\
0 & I & 0\\
\hdashline
-A_{\hat{\psi}_\nu^i}^T & 0 & C_{\hat{\psi}_\nu^i}^T\\
0 & -A_{\hat{\phi}_\nu}^T & C_{\hat{\phi}_\nu}^T\\
\hdashline
-B_{\hat{\psi}_\nu^i}^T & 0 & D_{\hat{\psi}_\nu^i}^T\\
0 & -B_{\hat{\phi}_\nu}^T & D_{\hat{\phi}_\nu}^T
\end{array}\right) \prec 0. \tag{44}$$

In view of (42) we can hence make sure in addition that

$$\left(\begin{array}{cccc:cccc} \hat{\Gamma}_{11} & \hat{\Gamma}_{12} & 0 & 0 & I & 0 & 0 & 0 \\ \hat{\Gamma}_{21} & \hat{\Gamma}_{22}+\hat{X}_{11} & \hat{X}_{12} & \hat{X}_{13} & 0 & 0 & 0 & 0 \\ 0 & \hat{X}_{21} & \hat{X}_{22} & \hat{X}_{23} & 0 & I & 0 & 0 \\ 0 & \hat{X}_{31} & \hat{X}_{32} & \hat{X}_{33} & 0 & 0 & 0 & I \\ \hdashline I & 0 & 0 & 0 & \hat{Y}_{11} & 0 & \hat{Y}_{12} & \hat{Y}_{13} \\ 0 & 0 & I & 0 & 0 & -\tilde{\Gamma}_{11} & -\tilde{\Gamma}_{12} & 0 \\ 0 & 0 & 0 & 0 & \hat{Y}_{21} & -\tilde{\Gamma}_{21} & \hat{Y}_{22}-\tilde{\Gamma}_{22} & \hat{Y}_{23} \\ 0 & 0 & 0 & I & \hat{Y}_{31} & 0 & \hat{Y}_{32} & \hat{Y}_{33} \end{array}\right) \succ 0. \tag{45}$$

Step 3. Let us now expand (43) by a last zero block row and column which then leads to

$$\star^T \mathcal{M}\left(\hat{\Gamma}, \mathbf{diag}\,(J(I), J(-I))\right) \left(\begin{array}{cc:cc} I & 0 & 0 & 0 \\ 0 & I & 0 & 0 \\ \hdashline A_{\hat{\phi}^i_\nu} & 0 & B_{\hat{\phi}^i_\nu} & 0 \\ 0 & A_{\hat{\psi}_\nu} & B_{\hat{\psi}_\nu} & 0 \\ \hdashline C_{\hat{\phi}^i_\nu} & 0 & D_{\hat{\phi}^i_\nu} & 0 \\ 0 & 0 & 0 & I \\ \hdashline 0 & C_{\hat{\psi}_\nu} & D_{\hat{\psi}_\nu} & 0 \\ 0 & 0 & 0 & I \end{array}\right) \preceq 0. \tag{46}$$

Note that the left-upper block of this LMI is still negative definite. We can thus apply Lemma 6 c) (which persists to be true despite the annihilator U) to infer from (38) that

$$\star^T \mathcal{M}\left(\tilde{X}, J(I)\right) \left(\begin{array}{ccc:c} I & 0 & 0 & 0 \\ 0 & I & 0 & 0 \\ 0 & 0 & I & 0 \\ \hdashline A_{\hat{\phi}^i_\nu} & 0 & B_{\hat{\phi}^i_\nu} C_q & B_{\hat{\phi}^i_\nu} D_{qp} \\ 0 & A_{\hat{\psi}_\nu} & 0 & B_{\hat{\psi}_\nu} \\ 0 & 0 & A & B_p \\ \hdashline C_{\hat{\phi}^i_\nu} & 0 & D_{\hat{\phi}^i_\nu} C_q & D_{\hat{\phi}^i_\nu} D_{qp} \\ 0 & C_{\hat{\psi}_\nu} & 0 & D_{\hat{\psi}_\nu} \end{array}\right) U \prec 0 \tag{47}$$

is satisfied by $\tilde{X}$ given as $\begin{pmatrix} \hat{\Gamma}_{11} & 0 & 0 \\ 0 & \hat{X}_{22} & \hat{X}_{23} \\ 0 & \hat{X}_{32} & \hat{X}_{33} \end{pmatrix} - \begin{pmatrix} \hat{\Gamma}_{12} \\ \hat{X}_{21} \\ \hat{X}_{31} \end{pmatrix} (\hat{\Gamma}_{22} + X_{11})^{-1}(\star)^T$. Dually, we can expand (44) to

$$\star^T \left(\tilde{\Gamma}, \mathbf{diag}\left(J(-I), J(I)\right)\right) \left(\begin{array}{cc:cc} I & 0 & 0 & 0 \\ 0 & I & 0 & 0 \\ \hdashline -A^T_{\psi^i_\nu} & 0 & 0 & C^T_{\hat{\psi}^i_\nu} \\ 0 & -A^T_{\hat{\phi}_\nu} & 0 & -C^T_{\hat{\phi}_\nu} \\ \hdashline 0 & 0 & I & 0 \\ -B^T_{\psi^i_\nu} & 0 & 0 & D^T_{\hat{\psi}^i_\nu} \\ \hdashline 0 & 0 & I & 0 \\ 0 & B^T_{\hat{\phi}_\nu} & 0 & D^T_{\hat{\phi}_\nu} \end{array}\right) \preceq 0$$

and glue it with (the negative of) (39) to infer

$$\star^T \mathcal{M}\left(\tilde{Y}, J(I)\right) \left(\begin{array}{ccc:c} -A^T_{\hat{\phi}_\nu} & 0 & 0 & C^T_{\hat{\phi}_\nu} \\ 0 & -A^T_{\hat{\psi}^i_\nu} & -C^T_{\hat{\psi}^i_\nu} B^T_p & -C^T_{\hat{\psi}^i_\nu} D^T_{qp} \\ 0 & 0 & -A^T & -C^T_q \\ \hdashline I & 0 & 0 & 0 \\ 0 & I & 0 & 0 \\ 0 & 0 & I & 0 \\ \hdashline -B^T_{\hat{\phi}_\nu} & 0 & 0 & D^T_{\hat{\phi}_\nu} \\ 0 & -B^T_{\hat{\psi}^i_\nu} & -D^T_{\hat{\psi}^i_\nu} B^T_p & -D^T_{\hat{\psi}^i_\nu} D^T_{qp} \end{array}\right) V \succ 0 \tag{48}$$

for $\tilde{Y}$ given by $\begin{pmatrix} \hat{Y}_{11} & 0 & \hat{Y}_{13} \\ 0 & -\tilde{\Gamma}_{11} & 0 \\ \hat{Y}_{31} & 0 & \hat{Y}_{33} \end{pmatrix} - \begin{pmatrix} \hat{Y}_{12} \\ -\tilde{\Gamma}_{12} \\ \hat{Y}_{32} \end{pmatrix} (\hat{Y}_{22} - \tilde{\Gamma}_{22})^{-1}(\star)^T$. By taking Schur-complements in (45) we finally get

$$\begin{pmatrix} \tilde{X} & I \\ I & \tilde{Y} \end{pmatrix} \succ 0. \tag{49}$$

By standard $\mathcal{H}_\infty$ theory, (47), (48) and (49) imply that there exists a stabilizing controller such that $\|\hat{\phi}_\nu^{-1}\mathcal{G}_{cl}\hat{\psi}_\nu^{-1}\|_\infty < 1$ or $\begin{pmatrix} \mathcal{G}_{cl} \\ I \end{pmatrix}^* \begin{pmatrix} (\hat{\phi}_\nu\hat{\phi}_\nu^*)^{-1} & 0 \\ 0 & -\hat{\psi}_\nu^*\hat{\psi}_\nu \end{pmatrix} \begin{pmatrix} \mathcal{G}_{cl} \\ I \end{pmatrix} \prec 0$. Since $\hat{\phi}_\nu\hat{\phi}_\nu^* = \phi_\nu N \phi_\nu^*$, $\hat{\psi}_\nu^*\hat{\psi}_\nu^* = \psi_\nu^* M \psi_\nu$, this is (26).

B - Proof of Theorem 3 - Statement (ii)

Step 1. Suppose that there exists a stabilizing controller which renders (4) satisfied with $Q = \psi^*\psi$, where ψ is minimum-phase and has the same diagonal structure as Q. Then there is some $\delta \in (0,1)$, close to one, with

$$\star^* \begin{pmatrix} \frac{1}{\delta^2}\psi^*\psi & 0 \\ 0 & -\delta^2\psi^*\psi \end{pmatrix} \begin{pmatrix} \mathcal{G}_{cl} \\ I \end{pmatrix} \prec 0. \tag{50}$$

For sufficiently large ν_0 we can make sure that $\hat{\phi} := \phi_{\nu_0}\hat{N}_{\nu_0}$ with $\hat{N}_{\nu_0}\hat{N}_{\nu_0}^T =: N_{\nu_0} \in \mathcal{N}_{\nu_0}$ is so close to ψ^{-1} such that it is minimum-phase and (50) persists to hold when ψ is replaced by $\hat{\phi}^i = \hat{\phi}^{-1}$. Then

$$\star^* \begin{pmatrix} \frac{1}{\delta^2}I & 0 \\ 0 & -\delta^2 I \end{pmatrix} \begin{pmatrix} \hat{\phi}^i\mathcal{G}_{cl}\hat{\phi} \\ I \end{pmatrix} \prec 0.$$

Standard LMI controller synthesis techniques now imply that there exist solutions $\tilde{X}$ and $\tilde{Y}$ of the LMIs,

$$\begin{pmatrix} \tilde{X} & I \\ I & \tilde{Y} \end{pmatrix} \succ 0, \tag{51}$$

$$\star^T \mathcal{M}\left(\tilde{X}, \mathbf{diag}\left(\frac{1}{\delta^2}I, -\delta^2 I\right)\right) \left(\begin{array}{ccc|c} I & 0 & 0 & 0 \\ 0 & I & 0 & 0 \\ 0 & 0 & I & 0 \\ \hline A_{\hat{\phi}^i} & 0 & B_{\hat{\phi}^i}C_q & B_{\hat{\phi}^i}D_{qp} \\ 0 & A_{\hat{\phi}^i} & 0 & B_{\hat{\phi}^i} \\ 0 & 0 & A & B_p \\ \hline C_{\hat{\phi}^i} & 0 & D_{\hat{\phi}^i}C_q & D_{\hat{\phi}^i}D_{qp} \\ 0 & C_{\hat{\phi}^i} & 0 & D_{\hat{\phi}^i} \end{array}\right) U \prec 0, \tag{52}$$

$$\star^T \mathcal{M}\left(\tilde{Y}, \mathbf{diag}\left(\delta^2 I, -\frac{1}{\delta^2} I\right)\right) \left(\begin{array}{ccc|c} -A_{\hat{\phi}}^T & 0 & 0 & C_{\hat{\phi}}^T \\ 0 & -A_{\hat{\phi}}^T & -C_{\hat{\phi}}^T B_p^T & -C_{\hat{\phi}}^T D_{qp}^T \\ 0 & 0 & -A^T & -C_q^T \\ \hline I & 0 & 0 & 0 \\ 0 & I & 0 & 0 \\ 0 & 0 & I & 0 \\ \hline -B_{\hat{\phi}}^T & 0 & 0 & D_{\hat{\phi}}^T \\ 0 & -B_{\hat{\phi}}^T & -D_{\hat{\phi}}^T B_p^T & -D_{\hat{\phi}}^T D_{qp}^T \end{array}\right) V \succ 0. \quad (53)$$

Note that (53) still holds if the last two rows of the outer factor are multiplied by $\frac{1}{\delta}$ and the inner matrix is replaced by $\mathcal{M}\left(\tilde{Y}, \mathbf{diag}\left(\delta^4 I, -I\right)\right)$. Since $\delta \in (0,1)$, we obtain the following LMI which is of the format as required in (39):

$$\star^T \mathcal{M}\left(\tilde{Y}, J(I)\right) \left(\begin{array}{ccc|c} -A_{\hat{\phi}}^T & 0 & 0 & C_{\hat{\phi}}^T \\ 0 & -A_{\hat{\phi}}^T & -C_{\hat{\phi}}^T B_p^T & -C_{\hat{\phi}}^T D_{qp}^T \\ 0 & 0 & -A^T & -C_q^T \\ \hline I & 0 & 0 & 0 \\ 0 & I & 0 & 0 \\ 0 & 0 & I & 0 \\ \hline -\frac{1}{\delta} B_{\hat{\phi}}^T & 0 & 0 & \frac{1}{\delta} D_{\hat{\phi}}^T \\ 0 & -\frac{1}{\delta} B_{\hat{\phi}}^T & -\frac{1}{\delta} D_{\hat{\phi}}^T B_p^T & -\frac{1}{\delta} D_{\hat{\phi}}^T D_{qp}^T \end{array}\right) V \succ 0. \quad (54)$$

Step 2. In order to arrive at (38) we bring $\hat{\psi}_\nu$ into (52) by gluing. For this purpose we choose a sequence of coefficient matrices $\hat{M}_\nu$ with $\hat{M}_\nu^T \hat{M}_\nu =: M_\nu \in \mathcal{M}_\nu$ and such that

$$\hat{\psi}_\nu := \hat{M}_\nu \psi_\nu \overset{\nu\to\infty}{\longrightarrow} \hat{\phi}^{-1}$$

exponentially in the $\mathcal{H}_\infty$-norm. The existence of such a sequence is guaranteed by our choice of the basis functions in the multiplier parametrization. For some sufficiently large $\nu > \nu_0$ it is clearly assured that $\hat{\psi}_\nu^* \hat{\psi}_\nu \prec \frac{1}{\delta^2}(\hat{\phi}^i)^* \hat{\phi}^i$ and $-\hat{\psi}_\nu^* \hat{\psi}_\nu \prec -\delta^2 (\hat{\phi}^i)^* \hat{\phi}^i$. As proved in [15], one can even certify both FDIs as

$$\star^T \mathcal{M}\left(\hat{\Gamma}^{\nu,\mu}, \mathbf{diag}\left(\delta_1 I, \delta_2 I\right)\right)\left(\begin{array}{cc:c} I & 0 & 0 \\ 0 & I & 0 \\ \hdashline A_{\hat{\psi}_\nu} & 0 & B_{\hat{\psi}_\nu} \\ 0 & A_{\hat{\phi}^i} & B_{\hat{\phi}^i} \\ \hdashline C_{\hat{\psi}_\nu} & 0 & D_{\hat{\psi}_\nu} \\ 0 & C_{\hat{\phi}^i} & D_{\hat{\phi}^i} \end{array}\right) \prec 0 \tag{55}$$

for $(\delta_1, \delta_2) = (1, -1/\delta^2)$ and $(\delta_1, \delta_2) = (-1, \delta^2)$ by

$$\hat{\Gamma}^{\nu,\mu} = \begin{pmatrix} \hat{\Gamma}^{\nu,\mu}_{11} & \hat{\Gamma}^{\mu}_{12} \\ \hat{\Gamma}^{\mu}_{21} & \hat{\Gamma}^{\mu}_{22} \end{pmatrix} := \left(\begin{array}{cc:cc} \mu K + \beta L & 0 & -\mu K & 0 \\ 0 & K_\nu & 0 & 0 \\ \hdashline -\mu K & 0 & \mu K & 0 \\ 0 & 0 & 0 & \mu \tilde{K} \end{array}\right). \tag{56}$$

Here $K \succ 0$, $\tilde{K} \succ 0$, $L \succ 0$, $\beta > 0$ are fixed and the sequence $K_\nu \succ 0$ satisfies $K_\nu \to 0$ for $\nu \to \infty$. Precisely, for all sufficiently large μ there exists some $\nu(\mu)$ such that (55) holds for all $\nu \geq \nu(\mu)$ and for both choices of (δ_1, δ_2). We can combine the two LMIs (55) and obtain $\begin{pmatrix} \hat{\Gamma}^{\nu,\mu}_{11} & 0 & \hat{\Gamma}^{\mu}_{12} & 0 \\ 0 & \hat{\Gamma}^{\nu,\mu}_{11} & 0 & \hat{\Gamma}^{\mu}_{12} \\ \hat{\Gamma}^{\mu}_{21} & 0 & \hat{\Gamma}^{\mu}_{22} & 0 \\ 0 & \hat{\Gamma}^{\mu}_{21} & 0 & \hat{\Gamma}^{\mu}_{22} \end{pmatrix}$ as a certificate for the inequality

$$(\star)^* \begin{pmatrix} I & 0 & 0 & 0 \\ 0 & -I & 0 & 0 \\ 0 & 0 & -\frac{1}{\delta^2} I & 0 \\ 0 & 0 & 0 & \delta^2 I \end{pmatrix} \begin{pmatrix} \hat{\psi}_\nu & 0 \\ 0 & \hat{\psi}_\nu \\ \hat{\phi}^{-1} & 0 \\ 0 & \hat{\phi}^{-1} \end{pmatrix} \prec 0.$$

Let us glue the corresponding LMI with (52) by Lemma 6 c). This implies that $\hat{X}$, defined as the Schur-complement of $\left(\begin{array}{cc:cc:c} \hat{\Gamma}^{\nu,\mu}_{11} & 0 & \hat{\Gamma}^{\mu}_{12} & 0 & 0 \\ 0 & \hat{\Gamma}^{\nu,\mu}_{11} & 0 & \hat{\Gamma}^{\mu}_{12} & 0 \\ \hdashline \hat{\Gamma}^{\mu}_{21} & 0 & \tilde{X}_{11} + \hat{\Gamma}^{\mu}_{22} & \tilde{X}_{12} & \tilde{X}_{13} \\ 0 & \hat{\Gamma}^{\nu,\mu}_{21} & \tilde{X}_{21} & \tilde{X}_{22} + \hat{\Gamma}^{\mu}_{22} & \tilde{X}_{23} \\ \hdashline 0 & 0 & \tilde{X}_{31} & \tilde{X}_{32} & \tilde{X}_{33} \end{array}\right)$ with respect to the middle block, satisfies (38).

Step 3. Now consider (41). We apply Lemma 7 b) to (55) for $(\delta_1, \delta_2) = (-1, \delta^2)$ to infer that

$$\tilde{R} := \begin{pmatrix} \hat{\Gamma}^{\mu}_{12}(\hat{\Gamma}^{\mu}_{22})^{-1}\hat{\Gamma}^{\mu}_{21} - \hat{\Gamma}^{\nu,\mu}_{11} & \hat{\Gamma}^{\mu}_{12}(\hat{\Gamma}^{\mu}_{22})^{-1} \\ (\hat{\Gamma}^{\mu}_{22})^{-1}\hat{\Gamma}^{\mu}_{21} & (\hat{\Gamma}^{\mu}_{22})^{-1} \end{pmatrix} \tag{57}$$

satisfies

$$\star^T \mathcal{M}\left(\begin{pmatrix} \tilde{R}_{11} & \tilde{R}_{12} \\ \tilde{R}_{21} & \tilde{R}_{22} \end{pmatrix}, \left(\begin{array}{cc|cc} I & 0 & 0 & 0 \\ 0 & I & 0 & 0 \\ \hline 0 & 0 & 0 & I \\ 0 & 0 & I & 0 \end{array}\right)\right) \left(\begin{array}{cc|cc} I & 0 & 0 & 0 \\ 0 & I & 0 & 0 \\ \hline A_{\hat{\psi}_\nu} & 0 & 0 & B_{\hat{\psi}_\nu} \\ 0 & -A^T_{\hat{\phi}} & C^T_{\hat{\phi}} & 0 \\ \hline C_{\hat{\psi}_\nu} & 0 & 0 & D_{\hat{\psi}_\nu} \\ 0 & -\frac{1}{\delta}B^T_{\hat{\phi}} & \frac{1}{\delta}D^T_{\hat{\phi}} & 0 \\ \hline 0 & 0 & I & 0 \\ 0 & 0 & 0 & I \end{array}\right) \succeq 0. \tag{58}$$

Step 4. We arrive at (40) by exploiting the structure of the sequence $\hat{\Gamma}^{\nu,\mu}$ in order to establish the asymptotic behavior of $\hat{X}$ and $\tilde{R}$ for $\mu \to \infty$. For this purpose let $E := \begin{pmatrix} I \\ 0 \end{pmatrix}$ and $E_\perp = \begin{pmatrix} 0 \\ I \end{pmatrix}$ in the row partition of $\hat{\Gamma}^{\nu,\mu}_{11}$. Then pre- and post-multiply (51) with $\mathbf{diag}\,(E, E, I, I, I, I)^T$ and its transpose to obtain

$$\left(\begin{array}{ccc|ccc} 2\beta L + E^T\tilde{X}_{11}E & E^T\tilde{X}_{12}E & E^T\tilde{X}_{13} & E^T & 0 & 0 \\ E^T\tilde{X}_{21}E & E^T\tilde{X}_{22}E & E^T\tilde{X}_{23} & 0 & E^T & 0 \\ \tilde{X}_{31} & \tilde{X}_{32}\tilde{E} & \tilde{X}_{33} & 0 & 0 & I \\ \hline E & 0 & 0 & \tilde{Y}_{11} & \tilde{Y}_{12} & \tilde{Y}_{13} \\ 0 & E & 0 & \tilde{Y}_{21} & \tilde{Y}_{22} & \tilde{Y}_{23} \\ 0 & 0 & I & \tilde{Y}_{31} & \tilde{Y}_{32} & \tilde{Y}_{33} \end{array}\right) \succ 0. \tag{59}$$

(since $2\beta L \succ 0$). As shown in Section D we have

$$\mathbf{diag}\,(E, E, I)^T\, \hat{X}\, \mathbf{diag}\,(E, E, I) \overset{\mu\to\infty}{\longrightarrow} \begin{pmatrix} \beta L + E^T\tilde{X}_{11}E & E^T\tilde{X}_{12}E & E^T\tilde{X}_{13} \\ E^T\tilde{X}_{21}E & \beta L + E^T\tilde{X}_{22}E & E^T\tilde{X}_{23} \\ \tilde{X}_{31}E & \tilde{X}_{32}E & \tilde{X}_{33} \end{pmatrix} \tag{60}$$

and

$$\begin{pmatrix} E^T\tilde{R}_{11}E & E^T\tilde{R}_{12} \\ \tilde{R}_{21}E & \tilde{R}_{22} \end{pmatrix} \stackrel{\mu\to\infty}{\longrightarrow} \begin{pmatrix} -\beta L & -E^T \\ -E & 0 \end{pmatrix}. \tag{61}$$

Hence, for sufficiently large μ_0 (fixed from now on) we have

$$\left(\begin{array}{ccc|ccc} E^T(\hat{X}_{11}-\tilde{R}_{11})E & E^T\hat{X}_{12}E & E^T\hat{X}_{13} & -E^T\tilde{R}_{12} & 0 & 0 \\ E^T\hat{X}_{21}E & E^T(\hat{X}_{22}+\tilde{R}_{11})E & E^T\hat{X}_{23} & 0 & -E^T\tilde{R}_{12} & 0 \\ \hat{X}_{31}E & \hat{X}_{32}E & \hat{X}_{33} & 0 & 0 & I \\ \hline -\tilde{R}_{21} & 0 & 0 & \tilde{Y}_{11}-\tilde{R}_{22} & \tilde{Y}_{12} & \tilde{Y}_{13} \\ 0 & -\tilde{R}_{21}E & 0 & \tilde{Y}_{21} & \tilde{Y}_{22}+\tilde{R}_{22} & \tilde{Y}_{23} \\ 0 & 0 & I & \tilde{Y}_{31} & \tilde{Y}_{32} & \tilde{Y}_{33} \end{array}\right) \succ 0. \tag{62}$$

We can then increase and fix ν to a sufficiently large ν_1 with $\nu_1 \geq \nu_0$ such that (55) and, hence, also (38) and (58) hold for $\nu = \nu_1$. Recall $K_{\nu_1} \succ 0$ which guarantees

$$\left(\begin{array}{ccccc|ccc} E^T(\hat{X}_{11}-\tilde{R}_{11})E & 0 & E^T\hat{X}_{12}E & 0 & E^T\hat{X}_{13} & -E^T\tilde{R}_{12} & 0 & 0 \\ 0 & 2K_{\nu_1} & 0 & 0 & 0 & 0 & 0 & 0 \\ E^T\hat{X}_{21}E & 0 & E^T(\hat{X}_{22}+\tilde{R}_{11})E & 0 & E^T\hat{X}_{23} & 0 & -E^T\tilde{R}_{12} & 0 \\ 0 & 0 & 0 & \epsilon I & 0 & 0 & 0 & 0 \\ \hat{X}_{31}E & 0 & \hat{X}_{32}E & 0 & \hat{X}_{33} & 0 & I & 0 \\ \hline -\tilde{R}_{21} & 0 & 0 & 0 & 0 & \tilde{Y}_{11}-\tilde{R}_{22} & \tilde{Y}_{12} & \tilde{Y}_{13} \\ 0 & 0 & -\tilde{R}_{21}E & 0 & 0 & \tilde{Y}_{21} & \tilde{Y}_{22}+\tilde{R}_{22} & \tilde{Y}_{23} \\ 0 & 0 & 0 & 0 & I & \tilde{Y}_{31} & \tilde{Y}_{32} & \tilde{Y}_{33} \end{array}\right) \succ 0. \tag{63}$$

for any $\epsilon > 0$. If we add $\mathbf{diag}(0, \epsilon I)$ to $\hat{X}_{22}$ for some small $\epsilon > 0$, the modified $\hat{X}$ still satisfies (38) for $\nu = \nu_1$ and (64) is assured since its left-hand side is identical to that of (63).

Step 5. Recall that $\begin{pmatrix} \hat{\psi}_{\nu_1}^*\hat{\psi}_{\nu_1} & I \\ I & (\frac{1}{\delta}\hat{\phi})(\frac{1}{\delta}\hat{\phi})^* \end{pmatrix} \succ 0$. Due to (58) we can thus construct a certificate of the corresponding strict inequality for $\nu = \nu_1$ that is so close to $\tilde{R}$, and still denoted by $\tilde{R}$, such that

$$\left(\begin{array}{ccc:ccc} \hat{X}_{11}-\tilde{R}_{11} & \hat{X}_{12} & \hat{X}_{13} & -\tilde{R}_{12} & 0 & 0 \\ \hat{X}_{21} & \hat{X}_{22}+\tilde{R}_{11} & \hat{X}_{23} & 0 & -\tilde{R}_{12} & 0 \\ \hat{X}_{31} & \hat{X}_{32} & \hat{X}_{33} & 0 & 0 & I \\ \hdashline -\tilde{R}_{21} & 0 & 0 & \hat{Y}_{11}-\tilde{R}_{22} & \hat{Y}_{12} & \hat{Y}_{13} \\ 0 & -\tilde{R}_{21} & 0 & \hat{Y}_{21} & \hat{Y}_{22}+\tilde{R}_{22} & \hat{Y}_{23} \\ 0 & 0 & I & \hat{Y}_{31} & \hat{Y}_{32} & \hat{Y}_{33} \end{array}\right) \succ 0 \tag{64}$$

persists.

Step 6. So far $\tilde{Y}$ satisfies (54) for $\hat{\phi}$ of the form $\phi_{\nu_0}\hat{N}_{\nu_0}$. The last step consists of expanding $\tilde{Y}$ into $\hat{Y}$ in order to arrive at (39) for $\nu = \nu_1$ which was taken with $\nu_1 \geq \nu_0$. In fact, by vertically concatenating $\frac{1}{\delta}\tilde{N}_{\nu_0}$ with a zero block column of suitable length we obtain a coefficient matrix $\hat{N}_{\nu_1}$ with $N_{\nu_1} := \hat{N}_{\nu_1}\hat{N}_{\nu_1}^T \in \mathcal{N}_{\nu_1}$ and such that

$$\left[\begin{array}{c|c} A_{\hat{\phi}} & \frac{1}{\delta}B_{\hat{\phi}} \\ \hline C_{\hat{\phi}} & \frac{1}{\delta}D_{\hat{\phi}} \end{array}\right] = \left[\begin{array}{c|c} A_{\phi_{\nu_0}} & \frac{1}{\delta}B_{\phi_{\nu_0}}\tilde{N}_{\nu_0} \\ \hline C_{\phi_{\nu_0}} & \frac{1}{\delta}D_{\phi_{\nu_0}}\tilde{N}_{\nu_0} \end{array}\right] = \left[\begin{array}{c|c} A_{\phi_{\nu_1}} & B_{\phi_{\nu_1}}\tilde{N}_{\nu_1} \\ \hline C_{\phi_{\nu_1}} & D_{\phi_{\nu_1}}\tilde{N}_{\nu_1} \end{array}\right] = \left[\begin{array}{c|c} A_{\hat{\phi}_{\nu_1}} & B_{\hat{\phi}_{\nu_1}} \\ \hline C_{\hat{\phi}_{\nu_1}} & D_{\hat{\phi}_{\nu_1}} \end{array}\right].$$

Since both realizations are observable, the larger one can be adjusted by a state-coordinate change (without loss of generality) such that $\left(\begin{array}{c:c} A_{\hat{\phi}_{\nu_1}} & B_{\hat{\phi}_{\nu_1}} \\ \hdashline C_{\hat{\phi}_{\nu_1}} & D_{\hat{\phi}_{\nu_1}} \end{array}\right) = \left(\begin{array}{cc:c} A_0 & 0 & 0 \\ * & A_{\hat{\phi}} & \frac{1}{\delta}B_{\hat{\phi}} \\ \hdashline * & C_{\hat{\phi}} & \frac{1}{\delta}D_{\hat{\phi}1} \end{array}\right)$. Since A_0 is stable we can choose $\tilde{Y}_0 \succ 0$ with $-A_0\tilde{Y}_0 - \tilde{Y}_0A_0^T \succ 0$. Next to (54) let us now consider the corresponding inequality with the new realization of the outer factor. Due to the particular realization structure it is assured that $\hat{Y} := \left(\begin{array}{cc:cc:c} \beta_1\tilde{Y}_0 & 0 & 0 & 0 & 0 \\ 0 & \tilde{Y}_{11} & 0 & \tilde{Y}_{12} & \tilde{Y}_{13} \\ \hdashline 0 & 0 & \beta_1\tilde{Y}_0 & 0 & 0 \\ 0 & \tilde{Y}_{21} & 0 & \tilde{Y}_{22} & \tilde{Y}_{23} \\ \hdashline 0 & \tilde{Y}_{31} & 0 & \tilde{Y}_{32} & \tilde{Y}_{33} \end{array}\right)$ renders the extended counterpart of (54) feasible for all sufficiently small $\beta_1 > 0$. Note that this is just (39) for $\nu = \nu_1$. Similarly we can consider the strict version of (58) for the extended realization. Expanding $\tilde{R}$ as $\left(\begin{array}{c:cc} \tilde{R}_{11} & 0 & \tilde{R}_{12} \\ \hdashline 0 & \beta_2\tilde{Y}_0 & 0 \\ \tilde{R}_{21} & 0 & \tilde{R}_{22} \end{array}\right)$ generates a (strict) solution for the extended realization if we choose $\beta_2 > 0$ sufficiently small. This leads to satisfaction of (39) and (41) for $\nu = \nu_1$ as desired. Finally, (64) implies (40) if we assure that the newly introduced diagonal blocks in $\hat{Y}_{11} - \hat{R}_{22}$ and $\hat{Y}_{22} + \hat{R}_{22}$, which read as $(\beta_1 - \beta_2)\tilde{Y}_0$ and $(\beta_1 + \beta_2)\tilde{Y}_0$, are positive definite; this is achieved by taking $\beta_1 - \beta_2 > 0$.

Since, by construction, $\hat{\phi}_{\nu_1}\hat{\phi}^*_{\nu_1} = \phi_{\nu_1} N_{\nu_1} \phi^*_{\nu_1}$ and $\hat{\psi}^*_{\nu_1}\hat{\psi}^*_{\nu_1} = \psi^*_{\nu_1} M_{\nu_1} \psi_{\nu_1}$, it is finally clear that (38)-(41) are identical to (22)-(25) for $\nu = \nu_1$, $M = M_{\nu_1}$ and $N = N_{\nu_1}$. This finishes the proof.

D - Proof of (60) and (61)

Recall the definition (56) and that $\hat{X}$ is given as

$$\hat{X} = \left(\begin{array}{cc|c} \hat{\Gamma}^{\nu,\mu}_{11} & 0 & 0 \\ 0 & \hat{\Gamma}^{\nu,\mu}_{11} & 0 \\ \hline 0 & 0 & \tilde{X}_{33} \end{array}\right) - \left(\begin{array}{cc} \hat{\Gamma}^{\nu,\mu}_{12} & 0 \\ 0 & \hat{\Gamma}^{\nu,\mu}_{12} \\ \hline \tilde{X}_{31} & \tilde{X}_{32} \end{array}\right) \begin{pmatrix} \tilde{X}_{11} + \hat{\Gamma}^{\mu}_{22} & \tilde{X}_{12} \\ \tilde{X}_{21} & \tilde{X}_{22} + \hat{\Gamma}^{\mu}_{22} \end{pmatrix}^{-1} \left(\begin{array}{cc|c} \hat{\Gamma}^{\nu,\mu}_{21} & 0 & \tilde{X}_{13} \\ 0 & \hat{\Gamma}^{\nu,\mu}_{21} & \tilde{X}_{23} \end{array}\right). \tag{65}$$

We clearly have for any symmetric matrix Z and for any $H_\mu \to 0$ that $(Z + \hat{\Gamma}^{\mu}_{22} + H_\mu)^{-1} \to 0$. Moreover,

$$\begin{aligned} E^T \hat{\Gamma}^{\mu}_{12}(Z + \hat{\Gamma}^{\mu}_{22} + H_\mu)^{-1} &= -E^T \hat{\Gamma}^{\mu}_{22}(Z + \hat{\Gamma}^{\mu}_{22} + H_\mu)^{-1} \\ &= -E^T \left(\left(Z + \hat{\Gamma}^{\mu}_{22} + H_\mu \right) \left(\hat{\Gamma}^{\mu}_{22} \right)^{-1} \right)^{-1} \\ &= -E^T \left(\underbrace{(Z + H_\mu)}_{\to Z} \underbrace{\left(\hat{\Gamma}^{\mu}_{22} \right)^{-1}}_{\to 0} + I \right)^{-1} \to -E^T. \end{aligned}$$

We then also get

$$\begin{aligned} & E^T [\hat{\Gamma}^{\nu,\mu}_{11} - \hat{\Gamma}^{\mu}_{12}(Z + \hat{\Gamma}^{\mu}_{22} + H_\mu)^{-1} \hat{\Gamma}^{\mu}_{21}] E \\ &= \beta L + \underbrace{\mu K}_{-E^T \hat{\Gamma}^{\mu}_{12} E} - E^T \hat{\Gamma}^{\mu}_{12}(Z + \hat{\Gamma}^{\mu}_{22} + H_\mu)^{-1} \hat{\Gamma}^{\mu}_{21} E \\ &= \beta L - E^T \hat{\Gamma}^{\mu}_{12} \left[I + (Z + \hat{\Gamma}^{\mu}_{22} + H_\mu)^{-1} \hat{\Gamma}^{\mu}_{21} \right] E \\ &= \beta L - E^T \hat{\Gamma}^{\mu}_{12}(Z + \hat{\Gamma}^{\mu}_{22} + H_\mu)^{-1} \left[(Z + \hat{\Gamma}^{\mu}_{22} + H_\mu) + \hat{\Gamma}^{\mu}_{21} \right] E \\ &= \beta L - \underbrace{E^T \Gamma^{\mu}_{12} \left(Z + H_\mu + \hat{\Gamma}^{\mu}_{22} \right)^{-1}}_{\to -E^T} \left[\underbrace{(Z + H_\mu)}_{\to Z} E + \underbrace{\hat{\Gamma}^{\mu}_{22} E + \hat{\Gamma}^{\mu}_{21} E}_{0} \right] \to \beta L + E^T Z E. \end{aligned} \tag{66}$$

If we hence define $G_\mu := (\tilde{X}_{11} + \hat{\Gamma}^\mu_{22})^{-1}$, and $H_\mu := [(\tilde{X}_{22} + \hat{\Gamma}^\mu_{22}) - \tilde{X}_{21} G_\mu \tilde{X}_{12}]^{-1}$, we infer

$$\begin{pmatrix} G_\mu \\ H_\mu \end{pmatrix} \to 0, \quad \begin{pmatrix} E^T \hat{\Gamma}^\mu_{12} G_\mu \\ E^T \hat{\Gamma}^\mu_{12} H_\mu \end{pmatrix} \to \begin{pmatrix} -E^T \\ -E^T \end{pmatrix}$$

and

$$\begin{pmatrix} E^T(\hat{\Gamma}^{\nu,\mu}_{11} - \hat{\Gamma}^\mu_{12} G_\mu \hat{\Gamma}^\mu_{21})E \\ E^T(\hat{\Gamma}^{\nu,\mu}_{11} - \hat{\Gamma}^\mu_{12} H_\mu \hat{\Gamma}^\mu_{21})E \end{pmatrix} \to \begin{pmatrix} \beta L + E^T \tilde{X}_{11} E \\ \beta L + E^T \tilde{X}_{22} E \end{pmatrix}.$$

Therefore, we have

$$\begin{pmatrix} \tilde{X}_{11} + \Gamma^\mu_{22} & \tilde{X}_{12} \\ \tilde{X}_{21} & \tilde{X}_{22} + \Gamma^\mu_{22} \end{pmatrix}^{-1} = \begin{pmatrix} G_\mu [I + \tilde{X}_{12} H_\mu \tilde{X}_{21} G_\mu] & -G_\mu \tilde{X}_{12} H_\mu \\ -H_\mu \tilde{X}_{21} G_\mu & H_\mu \end{pmatrix} \to 0, \tag{67}$$

$$\begin{aligned} &\begin{pmatrix} E^T \hat{\Gamma}^\mu_{12} & 0 \\ 0 & E^T \hat{\Gamma}^\mu_{12} \end{pmatrix} \begin{pmatrix} \tilde{X}_{11} + \hat{\Gamma}^\mu_{22} & \tilde{X}_{12} \\ \tilde{X}_{21} & \tilde{X}_{22} + \hat{\Gamma}^\mu_{22} \end{pmatrix}^{-1} \\ &\quad = \begin{pmatrix} E^T \hat{\Gamma}^\mu_{12} G_\mu [I + \tilde{X}_{12} H_\mu \tilde{X}_{21} G_\mu] & -E^T \hat{\Gamma}^\mu_{12} G_\mu \tilde{X}_{12} H_\mu \\ -E^T \hat{\Gamma}^\mu_{12} H_\mu \tilde{X}_{21} G_\mu & E^T \hat{\Gamma}^\mu_{12} H_\mu \end{pmatrix} \to \begin{pmatrix} -E^T & 0 \\ 0 & -E^T \end{pmatrix} \end{aligned} \tag{68}$$

and

$$\begin{aligned} &\begin{pmatrix} E^T \hat{\Gamma}^{\nu,\mu}_{11} E & 0 \\ 0 & E^T \hat{\Gamma}^{\nu,\mu}_{11} E \end{pmatrix} - \begin{pmatrix} E^T \hat{\Gamma}^\mu_{21} & 0 \\ 0 & E^T \hat{\Gamma}^\mu_{21} \end{pmatrix}^T \begin{pmatrix} \tilde{X}_{11} + \hat{\Gamma}^\mu_{22} & \tilde{X}_{12} \\ \tilde{X}_{21} & \tilde{X}_{22} + \hat{\Gamma}^\mu_{22} \end{pmatrix}^{-1} \begin{pmatrix} \hat{\Gamma}^\mu_{21} E & 0 \\ 0 & \hat{\Gamma}^\mu_{21} E \end{pmatrix} \\ &\quad = \begin{pmatrix} E^T(\hat{\Gamma}^{\nu,\mu}_{11} - \hat{\Gamma}^\mu_{12} G_\mu [I + \tilde{X}_{12} H_\mu \tilde{X}_{21} G_\mu] \hat{\Gamma}^\mu_{21})E & E^T \hat{\Gamma}^\mu_{12} G_\mu \tilde{X}_{12} H_\mu \hat{\Gamma}^\mu_{21} E \\ E^T \hat{\Gamma}^\mu_{12} H_\mu \tilde{X}_{21} G_\mu \hat{\Gamma}^\mu_{21} E & E^T(\hat{\Gamma}^\mu_{11} - \hat{\Gamma}^\mu_{12} H_\mu \hat{\Gamma}^\mu_{21})E \end{pmatrix} \\ &\qquad \to \begin{pmatrix} \beta L + E^T \tilde{X}_{11} E & E^T \tilde{X}_{12} E \\ E^T \tilde{X}_{21} E & \beta L + E^T \tilde{X}_{22} E \end{pmatrix}. \end{aligned} \tag{69}$$

Due to (65) and (57), these imply (60) and (61).

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