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Preprint 2011/004

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ISSN **1613-8309**

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Control Synthesis using Dynamic D-Scales: Part II – Gain-Scheduled Control

Carsten W. Scherer and I. Emre Köse

Abstract

The gain-scheduled controller design problem for linear parameter-varying systems is considered. Parameter dependence in the plant is described in the standard linear fractional form familiar from robust control theory. It is assumed that the parameters take values within known bounds, but are constant in time. The controller reflects the structure of parametric dependence of the plant and thus has an LFT structure as well. In contrast to the existing results in the literature, dynamic (frequency-dependent) D-scales are used in obtaining sufficient conditions for robust stability of the closed-loop system in the form of frequency-dependent inequalities. Following the transformation to finite dimensions through the use of the Kalman-Yakubovich-Popov Lemma, the controller matrices are eliminated from the resulting matrix inequalities. The main result of the paper is given in terms of convex linear matrix inequalities for the existence of robustly stabilizing controllers. A numerical example highlights the advantages of frequency dependence in the D-scales.

I. Introduction

Gain-scheduled control synthesis has attracted considerable attention in the last two decades. Following a rigorous investigation of classical gain-scheduling techniques by Shamma and Athans [21], research efforts have concentrated on developing parameter-varying control synthesis methods for linear parameter-varying (LPV) systems. Two different approaches have become prominent in the literature. In the first approach, the system matrices are expressed in terms of parameters explicitly and Lyapunov techniques are used in synthesizing parameter-dependent controller matrices. The existence conditions for such controllers are commonly expressed as parameter-dependent LMIs, where the parameters are allowed to take values

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inside convex polytopes. Several relaxation methods have been proposed for reducing these problems to finite-dimensional LMIs, mostly involving solutions at extreme points of the parameter sets using multi-convexity arguments [6], [22], [23], [1].

A second line of research is based on the representation of parameter variations in the system through the feedback interconnection of the nominal system, G, and a perturbation operator, Δ, which represents parameter variations from their nominal values. In this setting, a controller is sought which has the same structure as that of the perturbed plant. The closed-loop system comprises the feedback interconnection of the nominal closed-loop system, namely $\mathcal{G}_{cl} := G \star K$, and a combined perturbation block, $\Delta_{cl} := \mathbf{diag}(\Delta, \Delta_K)$ as in Figure 1.

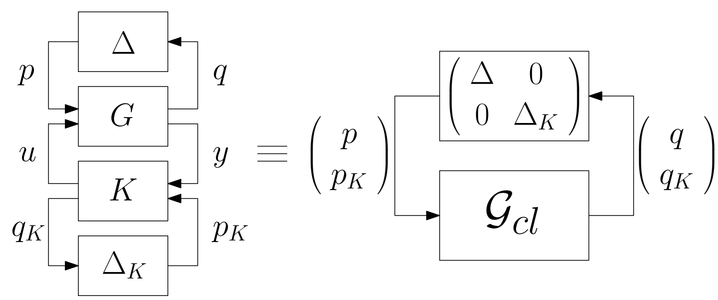

Fig. 1. Gain-scheduled control system.

The properties of Δ_{cl} (such as norm bounds, time-variations, *etc.*) are characterized by integral quadratic constraints (IQCs) defined through self-adjoint *multipliers* ([11]). The objective in this approach is to eliminate the controller matrices from the stability conditions for the closed-loop system and thus to obtain convex existence conditions for the controller. The first solutions to gain-scheduling problems in this framework were reported in [12] and [2] using static (*i.e.*, frequency-independent) multipliers of the form $\begin{pmatrix} D & 0 \\ 0 & -D \end{pmatrix}$ with $D = D^T \succ 0$. In [20], the multipliers were extended to $\begin{pmatrix} D & G \\ G^T & -D \end{pmatrix}$, where $G = -G^T$, thus reducing conservatism. A more general form $\begin{pmatrix} Q & R \\ R^T & S \end{pmatrix}$ was proposed in [15] with no positivity or skew-symmetry constraints on Q, R or S, resulting in even less conservatism. Yet none of these results take advantage of the reduction in conservatism offered by dynamic (*i.e.*, frequency-dependent) multipliers.

So far, very few convex synthesis results have appeared employing dynamic multipliers. The solutions of two types of problems are worth mentioning. First, solutions were given in [3] and [9] to the disturbance attenuation problem against uncertainties belonging to a class of signals described by dynamic IQCs. Secondly, the problems of robust estimator design and robust feedforward control were solved in [17] and [10] using general dynamic multipliers and in [19] using dynamic D/G-scales only. It was recently shown that these two types of problems can be cast as special cases of a single framework in [16].

The use of dynamic multipliers in the gain-scheduling problem poses some technical difficulties not encountered in the case of static multipliers. These difficulties are partly related to the fact that the multipliers have to be factorized as $\psi^* M \psi$ into a static core, M, and a dynamic outer factor, ψ, and its adjoint. In the search for a suitable overall multiplier, ψ is specified as a tall matrix consisting of basis functions and M is treated as a free variable. However, the nature of the gain-scheduling problem necessitates the use of the inverse of the multiplier $\psi^* M \psi$, which has the simple expression $\psi^{-1} M^{-1} \psi^{-*}$ only if M and ψ are both square and invertible. Hence, it is essential that one should be able to go back and forth between tall factorizations and square factorizations without losing equivalence.

An additional difficulty arises due to the elimination procedure that results in the disappearance of all portions of the multipliers related to the controller. Once the existence conditions for the controller are satisfied, the first step towards obtaining the controller is the construction of the full multiplier from portions of itself and its inverse. In the case of static multipliers, this procedure involves no difficulties. However, when the multipliers are dynamic, a straightforward application of the same procedure introduces additional dynamics not found in the solvability conditions. This makes it necessary to re-solve the existence conditions involving multipliers with new dynamics, leading to even more complications.

Here, we propose a solution to the gain-scheduled control design problem using dynamic D-scales. Our main result consists of convex conditions for the existence of a robustly stabilizing controller. In this setting, the difficulty with tall/square factorizations described above is circumvented in the proofs through the solution of AREs. The Lyapunov certificates in the existence conditions can be shifted back and forth, resulting in certificates for tall and square outer factors. Thus, we can take advantage of both tall (for basis functions) and square (for inverse operations) outer factors. Moreover, instead of extending the dynamic multipliers in a way similar to the static case, we propose a novel extension that precludes any additional dynamics. These findings can be seen as a preliminary step towards a general solution involving dynamic multipliers with no structural constraints. A numerical example highlights the application of our results to a mechanical control system.

The paper is organized as follows: In Section II, we introduce the problem setting and remind the reader of dynamic D-scales. The main result is given in Section III, and its proof follows in Section IV. The numerical example in Section V demonstrates the application of the findings of the paper. A summary of the main result and a discussion of possible future directions are given in Section VI. The Appendix contains some technical results used in the paper.

Notation: The space of matrix-valued functions with entries that are essentially bounded on the imaginary axis is denoted by $\mathcal{L}_\infty$. The symbol $\mathbb{C}^0$ is used for the extended imaginary axis $i\mathbb{R} \cup \{\infty\}$. The inertia of a Hermitian matrix M is $\mathbf{in}(M) = (n_+, n_-, n_0)$, where n_+, n_-, n_0 denote the number of positive, negative and zero eigenvalues of M. For any matrix A, we denote by $A_\perp$ a basis matrix of the orthogonal complement of the image of A. The Kronecker product of A and B is represented by $A \otimes B$. For a transfer matrix $G = \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right]$, we denote $G^*(s) = G(-s)^T$. We always use the realizations $G^* = \left[\begin{array}{c|c} -A^T & C^T \\ \hline -B^T & D^T \end{array}\right]$ and $G^{-1} = \left[\begin{array}{c|c} A - BD^{-1}C & BD^{-1} \\ \hline -D^{-1}C & D^{-1} \end{array}\right]$ if D is invertible. In expressions like G^*MG we address M as middle term and G as outer term/factor (not to be confused with outer transfer matrices), and we also use such a convention for LMIs like the one above. We represent the product A^*BA as $(\star)^*BA$ and the matrix $\begin{pmatrix} A & B \\ B^* & C \end{pmatrix}$ as $\begin{pmatrix} A & B \\ \star & C \end{pmatrix}$ whenever convenient. Lastly, we employ $J(M) := \mathbf{diag}(M, -M)$ and $\mathcal{M}(X, M) := \begin{pmatrix} 0 & X & 0 \\ X & 0 & 0 \\ 0 & 0 & M \end{pmatrix}$.

II. Gain-Scheduled Control Synthesis with Dynamic D-Scales

A. System Configuration

Consider the gain-scheduled system in Figure 1. Let the nominal plant and controller be realized as

$$G = \left[\begin{array}{c|c:c} A & B_p & B_u \\ \hline C_q & D_{qp} & D_{qu} \\ \hdashline C_y & D_{yp} & 0 \end{array}\right] \tag{1}$$

$$\text{and} \quad K = \left[\begin{array}{c|c:c} A_K & B_{K_y} & B_{K_{p_K}} \\ \hline C_{K_u} & D_{K_{uy}} & D_{K_{up_K}} \\ \hdashline C_{K_{q_K}} & D_{K_{q_K y}} & D_{K_{q_K p_K}} \end{array}\right]. \tag{2}$$

Then, the nominal closed-loop system is

$$\mathcal{G}_{cl} := \left[\begin{array}{c|c} A^a + B_u^a \mathbf{K} C_y^a & B_p^a + B_u^a \mathbf{K} D_{yp}^a \\ \hline C_q^a + D_{qu}^a \mathbf{K} C_y^a & D_{qp}^a + D_{qu}^a \mathbf{K} D_{yp}^a \end{array}\right],$$

where the superscript "a" stands for "augmented" and

$$\left(\begin{array}{c:c} A^a & B_p^a \\ \hdashline C_q^a & D_{qp}^a \end{array}\right) + \left(\begin{array}{c} B_u^a \\ \hdashline D_{qu}^a \end{array}\right) \mathbf{K} \left(\begin{array}{c:c} C_y^a & D_{yp}^a \end{array}\right)$$

$$:= \left(\begin{array}{cc:cc} A & 0 & B_p & 0 \\ 0 & 0 & 0 & 0 \\ \hdashline C_q & 0 & D_{qp} & 0 \\ 0 & 0 & 0 & 0 \end{array}\right) + \left(\begin{array}{ccc} B_u & 0 & 0 \\ 0 & 0 & I \\ \hdashline D_{qu} & 0 & 0 \\ 0 & I & 0 \end{array}\right) \left(\begin{array}{ccc} D_{K_{uy}} & D_{up_K} & C_{K_u} \\ D_{q_K y} & D_{q_K p_K} & C_{q_K} \\ B_{K_y} & B_{p_K} & A_K \end{array}\right) \left(\begin{array}{cc:cc} C_y & 0 & D_{yp} & 0 \\ 0 & 0 & 0 & I \\ 0 & I & 0 & 0 \end{array}\right).$$

The uncertainty blocks Δ and Δ_K are structured as

$$\Delta = \operatorname*{\mathbf{diag}}_{i=1}^{m} \left(\delta_i I_{d_i}\right) \quad \text{and} \quad \Delta_K = \operatorname*{\mathbf{diag}}_{i=1}^{m} \left(\delta_i I_{d_{K_i}}\right),$$

where $\|\delta_i\|_\infty \leq 1 \; \forall i = 1 : m$ and $\sum_{i=1}^m d_i =: d$ and $\sum_{i=1}^m d_{K_i} =: d_K$.

B. *D-scales and bases for $R\mathcal{H}_\infty$*

It is well-known [24] that robust stability of the interconnection in Figure 1 is guaranteed if $\mathcal{G}_{cl}$ is internally stable and if there exists a $\psi_{cl} \in R\mathcal{H}_\infty$ such that

$$\begin{pmatrix} \mathcal{G}_{cl} \\ I \end{pmatrix}^* \begin{pmatrix} \psi_{cl}^*\psi_{cl} & 0 \\ 0 & -\psi_{cl}^*\psi_{cl} \end{pmatrix} \begin{pmatrix} \mathcal{G}_{cl} \\ I \end{pmatrix} \prec 0 \tag{3}$$

on $\mathbb{C}^0$, where $\psi_{cl} = \begin{pmatrix} \psi_G & \psi_K \end{pmatrix}$,

$$\psi_G = \operatorname*{\mathbf{diag}}_{i=1}^{m}(\psi_{i_G}), \quad \text{and} \quad \psi_K = \operatorname*{\mathbf{diag}}_{i=1}^{m}(\psi_{i_K}).$$

The term $\psi_{cl}^*\psi_{cl}$ is commonly referred to as a *D-scale* in the robust control literature.

In searching for appropriate scalings, it is desirable to use a basis of suitable functions in $R\mathcal{H}_\infty$. Toward this end, we choose any $p > 0$ and introduce

$$b_\nu(s) := \begin{pmatrix} 1 & \frac{s-p}{s+p} & \left(\frac{s-p}{s+p}\right)^2 & \cdots & \left(\frac{s-p}{s+p}\right)^\nu \end{pmatrix}^T \tag{4}$$

with input-balanced (minimal) realization $b_\nu = \left[\begin{array}{c|c} A_{b_\nu} & B_{b_\nu} \\ \hline C_{b_\nu} & D_{b_\nu} \end{array}\right]$ for any $\nu \in \mathbb{N}$. Moreover we use the notation

$$\left[\begin{array}{c|c} A^d_{\psi_\nu} & B^d_{\psi_\nu} \\ \hline C^d_{\psi_\nu} & D^d_{\psi_\nu} \end{array}\right] := \left[\begin{array}{c|c} I_d \otimes A_{b_\nu} & I_d \otimes B_{b_\nu} \\ \hline I_d \otimes C_{b_\nu} & I_d \otimes D_{b_\nu} \end{array}\right] \quad \text{and} \quad \left[\begin{array}{c|c} A^d_{\phi_\nu} & B^d_{\phi_\nu} \\ \hline C^d_{\phi_\nu} & D^d_{\phi_\nu} \end{array}\right] := \left[\begin{array}{c|c} I_d \otimes A^T_{b_\nu} & I_d \otimes C^T_{b_\nu} \\ \hline I_d \otimes B^T_{b_\nu} & I_d \otimes D^T_{b_\nu} \end{array}\right].$$

Then, any element in $R\mathcal{H}_\infty^{d\times d}$ can be $\mathbb{C}^0$-uniformly approximated to arbitrary accuracy by

$$U^d_\nu \left[\begin{array}{c|c} A^d_{\psi_\nu} & B^d_{\psi_\nu} \\ \hline C^d_{\psi_\nu} & D^d_{\psi_\nu} \end{array}\right] \quad \text{or} \quad \left[\begin{array}{c|c} A^d_{\phi_\nu} & B^d_{\phi_\nu} \\ \hline C^d_{\phi_\nu} & D^d_{\phi_\nu} \end{array}\right] V^d_\nu,$$

respectively, through an appropriate choice of U^d_ν and V^d_ν and a sufficiently large ν [14].

III. MAIN RESULT

In what follows, we define

$$\mathcal{U} := \begin{pmatrix} 0 \\ 0 \\ C_y^T \\ D_{yp}^T \end{pmatrix}_\perp \quad \text{and} \quad \mathcal{V} := \begin{pmatrix} 0 \\ 0 \\ B_u \\ D_{qu} \end{pmatrix}_\perp$$

where the row-dimension of the zero blocks in the definitions of $\mathcal{U}$ and $\mathcal{V}$ are equal to those of $A^N_{\psi_\nu}$ and $A^N_{\phi_\nu}$ respectively.

Theorem 1: There exists a gain-scheduled controller as in (2) that renders $\mathcal{G}_{cl}$ internally stable and satisfies (3) for some structured $\psi_{cl} \in R\mathcal{H}_\infty$ if and only if there exist some positive integer ν and symmetric matrices X, Y, $R = \mathbf{diag}_{i=1}^m (R_i)$, $M = \mathbf{diag}_{i=1}^m (M_i)$ and $N = \mathbf{diag}_{i=1}^m (N_i)$ such that

$$\star^T \star^T \mathcal{M}(X, J(M)) \left(\begin{array}{ccc|c} I & 0 & 0 & 0 \\ 0 & I & 0 & 0 \\ 0 & 0 & I & 0 \\ \hline A^d_{\psi_\nu} & 0 & B^d_{\psi_\nu} C_q & B^d_{\psi_\nu} D_{qp} \\ 0 & A^d_{\psi_\nu} & 0 & B^d_{\psi_\nu} \\ 0 & 0 & A & B_p \\ \hline C^d_{\psi_\nu} & 0 & D^d_{\psi_\nu} C_q & D^d_{\psi_\nu} D_{qp} \\ 0 & C^d_{\psi_\nu} & 0 & D^d_{\psi_\nu} \end{array}\right) \mathcal{U} \prec 0, \tag{5}$$

$$\star^T \star^T \mathcal{M}(Y, J(N)) \left(\begin{array}{ccc|c} -\left(A^d_{\phi_\nu}\right)^T & 0 & 0 & \left(C^d_{\phi_\nu}\right)^T \\ 0 & -\left(A^d_{\phi_\nu}\right)^T & -\left(A^d_{\phi_\nu}\right)^T B_p^T & -\left(C^d_{\phi_\nu}\right)^T D_{qp}^T \\ 0 & 0 & -A^T & -C_q^T \\ \hline I & 0 & 0 & 0 \\ 0 & I & 0 & 0 \\ 0 & 0 & I & 0 \\ \hline -\left(B^d_{\phi_\nu}\right)^T & 0 & 0 & \left(D^d_{\phi_\nu}\right)^T \\ 0 & -\left(B^d_{\phi_\nu}\right)^T & -\left(D^d_{\phi_\nu}\right)^T B_p^T & -\left(D^d_{\phi_\nu}\right)^T D_{qp}^T \end{array}\right) \mathcal{V} \succ 0, \tag{6}$$

$$\star^T \mathcal{M}\left(R_i, \left(\begin{array}{cc|cc} M_i & 0 & 0 & 0 \\ 0 & N_i & 0 & 0 \\ \hline 0 & 0 & 0 & I \\ 0 & 0 & I & 0 \end{array}\right)\right)\left(\begin{array}{cc|cc} I & 0 & 0 & 0 \\ 0 & I & 0 & 0 \\ \hline A^{d_i}_{\psi_\nu} & 0 & B^{d_i}_{\psi_\nu} & 0 \\ 0 & -\left(A^{d_i}_{\phi_\nu}\right)^T & 0 & \left(C^{d_i}_{\phi_\nu}\right)^T \\ \hline C^{d_i}_{\psi_\nu} & 0 & D^{d_i}_{\psi_\nu} & 0 \\ 0 & -\left(B^{d_i}_{\phi_\nu}\right)^T & 0 & \left(D^{d_i}_{\phi_\nu}\right)^T \\ \hline 0 & 0 & I & 0 \\ 0 & 0 & 0 & I \end{array}\right) \succ 0 \tag{7}$$

for $i = 1 : m$ *and*

$$\left(\begin{array}{ccc|ccc} X_{11} - R_{11} & X_{12} & X_{13} & -R_{12} & 0 & 0 \\ X_{21} & X_{22} + R_{11} & X_{23} & 0 & -R_{12} & 0 \\ X_{31} & X_{32} & X_{33} & 0 & 0 & I \\ \hline -R_{21} & 0 & 0 & Y_{11} - R_{22} & Y_{12} & Y_{13} \\ 0 & -R_{21} & 0 & Y_{21} & Y_{22} + R_{22} & Y_{23} \\ 0 & 0 & I & Y_{31} & Y_{32} & Y_{33} \end{array}\right) \succ 0. \tag{8}$$

Note that, for reasons of computational complexity, one can impose different lengths of the basis vectors (4) for different sub-blocks of the uncertainty Δ. Actually, the "if" statement in Theorem 1 remains true for arbitrary vectors of transfer functions replacing b_ν for each uncertainty block, as long as they are all proper and stable.

Remark 2: Based on solutions of the synthesis LMIs one can construct a controller that has dynamic order $\dim(A) + 2\nu d$, which is reduced in case that the left-hand side of (8) loses rank. Similarly, the dimension d_{K_i} of the scheduling block $\delta_i I_{d_{K_i}}$ is determined by the rank of the left-hand side of (7); if the rank is full the dimension equals $d_i(1 + 2\nu)$ which leads to the overall dimension $d(1 + 2\nu)$ of Δ_K.

Remark 3: When external disturbances (w) and controlled outputs (z) are present in the system, the problem of designing robustly stabilizing controllers that achieve a closed-loop

$\mathcal{H}_\infty$-gain less than γ can be solved if replacing the plant by

$$\left[\begin{array}{c|ccc} A & B_p & B_w & B_u \\ \hline C_q & D_{qp} & D_{qw} & D_{qu} \\ C_z & D_{zp} & D_{zw} & D_{zu} \\ C_y & D_{yp} & D_{yw} & 0 \end{array}\right],$$

the multiplier $\mathbf{diag}\left(\psi_i^* M_i \psi_i, -\psi_i^* M_i \psi_i\right)$ by

$$\mathbf{diag}\left(\psi_i^* M_i \psi_i, \gamma^{-1} I, -\psi_i^* M_i \psi_i, -\gamma I\right)$$

and $\mathbf{diag}\left(\phi_i N_i \phi_i^*, -\phi_i N_i \phi_i^*\right)$ by

$$\mathbf{diag}\left(\phi_i N_i \phi_i^*, \gamma I, -\phi_i N_i \phi_i^*, -\gamma^{-1} I\right).$$

In this formulation, γ can be treated as a variable which, after taking the Schur-complement, enters the LMIs linearly. The minimization of the upper bound of the robust $\mathcal{H}_\infty$-norm is then cast as an SDP.

IV. Proof of Theorem 1

A. Necessity

Suppose that there exists a $\mathbf{K}$ that nominally stabilizes $\mathcal{G}_{cl}$ and satisfies (3) for some structured $\psi_{cl} \in R\mathcal{H}_\infty$. Our goal is to apply Part (ii) of Theorem 3 in [18]. For this purpose we parameterize $\psi_{cl}^* \psi_{cl}$ as follows. The sub-blocks of ψ_{cl} are described with free coefficient matrices $U_{i_{G_\nu}}$ and $U_{i_{K_\nu}}$ as

$$U_{i_{G_\nu}} \left[\begin{array}{c|c} A_{\psi_\nu}^{d_i} & B_{\psi_\nu}^{d_i} \\ \hline C_{\psi_\nu}^{d_i} & D_{\psi_\nu}^{d_i} \end{array}\right] \quad \text{and} \quad U_{i_{K_\nu}} \left[\begin{array}{c|c} A_{\psi_\nu}^{d_{K_i}} & B_{\psi_\nu}^{d_{K_i}} \\ \hline C_{\psi_\nu}^{d_{K_i}} & D_{\psi_\nu}^{d_{K_i}} \end{array}\right]$$

respectively. If

$$\left(U_{G_\nu} \;\; U_{K_\nu} \right) := \left(\mathbf{diag}_{i=1}^m \left(U_{i_{G_\nu}}\right) \;\; \mathbf{diag}_{i=1}^m \left(U_{i_{K_\nu}}\right) \right)$$

then ψ_{cl} is parameterized as

$$\left(U_{G_\nu} \;\; U_{K_\nu} \right) \left[\begin{array}{c|c} A_{\psi_\nu}^{d+d_K} & B_{\psi_\nu}^{d+d_K} \\ \hline C_{\psi_\nu}^{d+d_K} & D_{\psi_\nu}^{d+d_K} \end{array}\right],$$

which leads to the description of $\psi_{cl}^*\psi_{cl}$ as

$$\left[\begin{array}{c|c} A_{\psi_\nu}^{d+d_K} & B_{\psi_\nu}^{d+d_K} \\ \hline C_{\psi_\nu}^{d+d_K} & D_{\psi_\nu}^{d+d_K} \end{array}\right]^* \hat{M} \left[\begin{array}{c|c} A_{\psi_\nu}^{d+d_K} & B_{\psi_\nu}^{d+d_K} \\ \hline C_{\psi_\nu}^{d+d_K} & D_{\psi_\nu}^{d+d_K} \end{array}\right]$$

in which

$$\hat{M} := \begin{pmatrix} U_{G_\nu} & U_{K_\nu} \end{pmatrix}^T \begin{pmatrix} U_{G_\nu} & U_{K_\nu} \end{pmatrix}.$$

Similarly, the inverse of $\psi_{cl}^*\psi_{cl}$ is written as $\phi_{cl}\phi_{cl}^*$ with stable $\phi_{cl} = \begin{pmatrix} \phi_G \\ \phi_K \end{pmatrix}$ and

$$\phi_G = \operatorname*{\mathbf{diag}}_{i=1}^{m}(\phi_{i_G}), \quad \phi_K = \operatorname*{\mathbf{diag}}_{i=1}^{m}(\phi_{i_K}).$$

The diagonal sub-blocks of ϕ_{cl} are approximated with $V_{i_{G_\nu}}$ and $V_{i_{K_\nu}}$ as

$$\left[\begin{array}{c|c} A_{\phi_\nu}^{d_i} & B_{\phi_\nu}^{d_i} \\ \hline C_{\phi_\nu}^{d_i} & D_{\phi_\nu}^{d_i} \end{array}\right] V_{i_{G_\nu}} \quad \text{and} \quad \left[\begin{array}{c|c} A_{\phi_\nu}^{d_{K_i}} & B_{\phi_\nu}^{d_{K_i}} \\ \hline C_{\phi_\nu}^{d_{K_i}} & D_{\phi_\nu}^{d_{K_i}} \end{array}\right] V_{i_{K_\nu}}.$$

Then the approximation of $\phi_{cl}\phi_{cl}^*$ is described as

$$\left[\begin{array}{c|c} A_{\phi_\nu}^{d+d_K} & B_{\phi_\nu}^{d+d_K} \\ \hline C_{\phi_\nu}^{d+d_K} & D_{\phi_\nu}^{d+d_K} \end{array}\right] \hat{N} \left[\begin{array}{c|c} A_{\phi_\nu}^{d+d_K} & B_{\phi_\nu}^{d+d_K} \\ \hline C_{\phi_\nu}^{d+d_K} & D_{\phi_\nu}^{d+d_K} \end{array}\right]^*$$

where

$$\begin{pmatrix} V_{G_\nu} \\ V_{K_\nu} \end{pmatrix} := \begin{pmatrix} \mathbf{diag}_{i=1}^m \left(V_{i_{G_\nu}}\right) \\ \mathbf{diag}_{i=1}^m \left(V_{i_{K_\nu}}\right) \end{pmatrix}$$

and

$$\hat{N} := \begin{pmatrix} V_{G_\nu} \\ V_{K_\nu} \end{pmatrix} \begin{pmatrix} V_{G_\nu} \\ V_{K_\nu} \end{pmatrix}^T.$$

Recall that $\mathcal{G}_{cl}$ in Figure 1 can be viewed as the interconnection of G^e realized as

$$\left[\begin{array}{c|cc} A & B_p^e & B_u^e \\ \hline C_q^e & D_{qp}^e & D_{qu}^e \\ C_y^e & D_{yp}^e & 0 \end{array}\right] := \left[\begin{array}{c|cc|cc} A & B_p & 0 & B_u & 0 \\ \hline C_q & D_{qp} & 0 & D_{qu} & 0 \\ 0 & 0 & 0 & 0 & I_{\rho_K} \\ \hline C_y & D_{yp} & 0 & 0 & 0 \\ 0 & 0 & I_{\mu_K} & 0 & 0 \end{array}\right]$$

with the controller (2). Then LPV-synthesis boils down to robust controller synthesis for this extended system against $\mathbf{diag}(\Delta, \Delta_K)$ as shown in Figure 2.

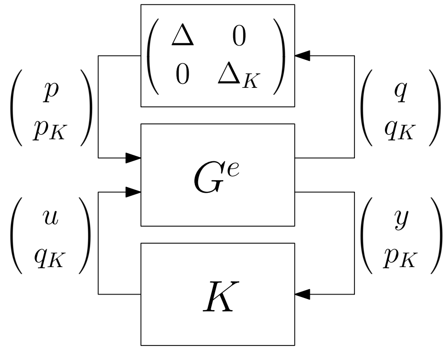

Fig. 2. Equivalent robust control synthesis problem.

Despite the fact that the multiplier structure is somewhat different, we can hence apply Theorem 3, Part (ii) in [18] and conclude that, for some sufficiently large ν and with the annihilators

$$\mathcal{U}^e := \begin{pmatrix} 0 \\ 0 \\ \left(C_y^e\right)^T \\ \left(D_{yp}^e\right)^T \end{pmatrix}_\perp \quad \text{and} \quad \mathcal{V}^e := \begin{pmatrix} 0 \\ 0 \\ B_u^e \\ D_{qu}^e \end{pmatrix}_\perp$$

for the extended system, the LMIs

$$\star^T \star^T \mathcal{M}\left(\hat{X}, J(\hat{M})\right) \left(\begin{array}{ccc:c} I & 0 & 0 & 0 \\ 0 & I & 0 & 0 \\ 0 & 0 & I & 0 \\ \hdashline A_{\psi_\nu}^{d+d_K} & 0 & B_{\psi_\nu}^{d+d_K} C_q^e & B_{\psi_\nu}^{d+d_K} D_{qp}^e \\ 0 & A_{\psi_\nu}^{d+d_K} & 0 & B_{\psi_\nu}^{d+d_K} \\ 0 & 0 & A & B_p^e \\ \hdashline C_{\psi_\nu}^{d+d_K} & 0 & D_{\psi_\nu}^{d+d_K} C_q^e & D_{\psi_\nu}^{d+d_K} D_{qp}^e \\ 0 & C_{\psi_\nu}^{d+d_K} & 0 & D_{\psi_\nu}^{d+d_K} \end{array}\right) \mathcal{U}^e \prec 0 \tag{9}$$

$$\star^T \star^T \mathcal{M}\left(\hat{Y}, J(\hat{N})\right)$$

$$
\left(\begin{array}{ccc:c}
-\left(A_{\phi_\nu}^{d+d_K}\right)^T & 0 & 0 & \left(C_{\phi_\nu}^{d+d_K}\right)^T \\
0 & -\left(A_{\phi_\nu}^{d+d_K}\right)^T & -\left(C_{\phi_\nu}^{d+d_K}\right)^T \left(B_p^e\right)^T & -\left(C_{\phi_\nu}^{d+d_K}\right)^T \left(D_{qp}^e\right)^T \\
0 & 0 & -A^T & -\left(C_q^e\right)^T \\
\hdashline
I & 0 & 0 & 0 \\
0 & I & 0 & 0 \\
0 & 0 & I & 0 \\
\hdashline
-\left(B_{\phi_\nu}^{d+d_K}\right)^T & 0 & 0 & \left(D_{\phi_\nu}^{d+d_K}\right)^T \\
0 & -\left(B_{\phi_\nu}^{d+d_K}\right)^T & -\left(D_{\phi_\nu}^{d+d_K}\right)^T \left(B_p^e\right)^T & -\left(D_{\phi_\nu}^{d+d_K}\right)^T \left(D_{qp}^e\right)^T
\end{array}\right) \mathcal{V}^e \succ 0 \quad (10)
$$

$$
\star^T \mathcal{M}\left(\hat{R}, \left(\begin{array}{cc:cc}
\hat{M} & 0 & 0 & 0 \\
0 & \hat{N} & 0 & 0 \\
\hdashline
0 & 0 & 0 & I \\
0 & 0 & I & 0
\end{array}\right)\right)
\left(\begin{array}{cc:cc}
I & 0 & 0 & 0 \\
0 & I & 0 & 0 \\
\hdashline
A_{\psi_\nu}^{d+d_K} & 0 & B_{\psi_\nu}^{d+d_K} & 0 \\
0 & -\left(A_{\phi_\nu}^{d+d_K}\right)^T & 0 & \left(C_{\phi_\nu}^{d+d_K}\right)^T \\
\hdashline
C_{\psi_\nu}^{d+d_K} & 0 & D_{\psi_\nu}^{d+d_K} & 0 \\
0 & -\left(B_{\phi_\nu}^{d+d_K}\right)^T & 0 & \left(D_{\phi_\nu}^{d+d_K}\right)^T \\
\hdashline
0 & 0 & I & 0 \\
0 & 0 & 0 & I
\end{array}\right) \succ 0, \qquad (11)
$$

$$
\left(\begin{array}{ccc:ccc}
\hat{X}_{11} - \hat{R}_{11} & \hat{X}_{12} & \hat{X}_{13} & -\hat{R}_{12} & 0 & 0 \\
\hat{X}_{21} & \hat{X}_{22} + \hat{R}_{11} & \hat{X}_{23} & 0 & -\hat{R}_{12} & 0 \\
\hat{X}_{31} & \hat{X}_{32} & \hat{X}_{33} & 0 & 0 & I \\
\hat{X}_{c1} & \hat{X}_{c2} & \hat{X}_{c3} & 0 & 0 & 0 \\
\hdashline
-\hat{R}_{21} & 0 & 0 & \hat{Y}_{11} - \hat{R}_{22} & \hat{Y}_{12} & \hat{Y}_{13} \\
0 & -\hat{R}_{21} & 0 & \hat{Y}_{21} & \hat{Y}_{22} + \hat{R}_{22} & \hat{Y}_{23} \\
0 & 0 & I & \hat{Y}_{31} & \hat{Y}_{32} & \hat{Y}_{33}
\end{array}\right) \succ 0 \qquad (12)
$$

are feasible. Exploiting the structure of the realization of G^e and that induced for $\mathcal{U}^e$, $\mathcal{V}^e$, the inequalities (9) and (10) simplify to

$$
\star^T \star^T \mathcal{M}\left(\hat{X}, J(\hat{M})\right)
$$

$$\begin{pmatrix} I & 0 & 0 & 0 \\ 0 & I & 0 & 0 \\ 0 & 0 & I & 0 \\ \hline \begin{pmatrix} A^d_{\psi_\nu} & 0 \\ 0 & A^{d_K}_{\psi_\nu} \end{pmatrix} & 0 & \begin{pmatrix} B^d_{\psi_\nu} \\ 0 \end{pmatrix} C_q & \begin{pmatrix} B^d_{\psi_\nu} \\ 0 \end{pmatrix} D_{qp} \\ 0 & \begin{pmatrix} A^d_{\psi_\nu} & 0 \\ 0 & A^{d_K}_{\psi_\nu} \end{pmatrix} & 0 & \begin{pmatrix} B^d_{\psi_\nu} \\ 0 \end{pmatrix} \\ 0 & 0 & A & B_p \\ \hline \begin{pmatrix} C^d_{\psi_\nu} & 0 \\ 0 & C^{d_K}_{\psi_\nu} \end{pmatrix} & 0 & \begin{pmatrix} D^d_{\psi_\nu} \\ 0 \end{pmatrix} C_q & \begin{pmatrix} D^d_{\psi_\nu} \\ 0 \end{pmatrix} D_{qp} \\ 0 & \begin{pmatrix} C^d_{\psi_\nu} & 0 \\ 0 & C^{d_K}_{\psi_\nu} \end{pmatrix} & 0 & \begin{pmatrix} D^d_{\psi_\nu} \\ 0 \end{pmatrix} \end{pmatrix} \mathcal{U} \prec 0 \quad (13)$$

and

$$\star^T \star^T \mathcal{M}\left(\hat{Y}, J(\hat{N})\right)$$

$$\begin{pmatrix} -\begin{pmatrix} A^d_{\phi_\nu} & 0 \\ 0 & A^{d_K}_{\phi_\nu} \end{pmatrix}^T & 0 & 0 & \begin{pmatrix} \left(C^d_{\phi_\nu}\right)^T \\ 0 \end{pmatrix} \\ 0 & -\begin{pmatrix} A^d_{\phi_\nu} & 0 \\ 0 & A^{d_K}_{\phi_\nu} \end{pmatrix}^T & -\begin{pmatrix} \left(C^d_{\phi_\nu}\right)^T \\ 0 \end{pmatrix} B_p^T & -\begin{pmatrix} \left(C^d_{\phi_\nu}\right)^T \\ 0 \end{pmatrix} D_{qp}^T \\ 0 & 0 & -A^T & -C_q^T \\ \hline I & 0 & 0 & 0 \\ 0 & I & 0 & 0 \\ 0 & 0 & I & 0 \\ \hline -\begin{pmatrix} B^d_{\phi_\nu} & 0 \\ 0 & B^{d_K}_{\phi_\nu} \end{pmatrix}^T & 0 & 0 & \begin{pmatrix} \left(D^d_{\phi_\nu}\right)^T \\ 0 \end{pmatrix} \\ 0 & -\begin{pmatrix} B^d_{\phi_\nu} & 0 \\ 0 & B^{d_K}_{\phi_\nu} \end{pmatrix}^T & -\begin{pmatrix} \left(D^d_{\phi_\nu}\right)^T \\ 0 \end{pmatrix} B_p^T & -\begin{pmatrix} \left(D^d_{\phi_\nu}\right)^T \\ 0 \end{pmatrix} D_{qp}^T \end{pmatrix} \mathcal{V} \succ 0. \quad (14)$$

By defining $E := \begin{pmatrix} I \\ 0 \end{pmatrix}$, $F := \begin{pmatrix} I \\ 0 \end{pmatrix}$ of appropriate dimensions, these imply, by canceling the columns of the outer terms related to uncontrollable modes of the multiplier dynamics, the inequalities (5) and (6) where

$$
\begin{aligned}
X &:= \mathbf{diag}\,(E, E, I)^T\, \hat{X}\, \mathbf{diag}\,(E, E, I)\,, \\
Y &:= \mathbf{diag}\,(F, F, I)^T\, \hat{Y}\, \mathbf{diag}\,(F, F, I)\,, \\
M &:= E^T \hat{M} E \quad \text{and} \quad N := F^T \hat{N} F.
\end{aligned}
$$

The congruence transformation $T^T(\star)T$ with $T := \mathbf{diag}\,(E, E, I, F, F, I)$ on (12) yields (8) with the definition $R := \mathbf{diag}\,(E, F)^T\, \hat{R}\, \mathbf{diag}\,(E, F)$. Now delete columns from (11) to obtain

$$
\star^T \mathcal{M}\left(\hat{R}, \left(\begin{array}{cc|cc} \hat{M} & 0 & 0 & 0 \\ 0 & \hat{N} & 0 & 0 \\ \hline 0 & 0 & 0 & I \\ 0 & 0 & I & 0 \end{array}\right)\right)
\left(\begin{array}{cc|cc}
I & 0 & 0 & 0 \\
0 & I & 0 & 0 \\ \hline
\begin{pmatrix} A^d_{\psi_\nu} & 0 \\ 0 & A^{d_K}_{\psi_\nu} \end{pmatrix} & 0 & \begin{pmatrix} B^d_{\psi_\nu} \\ 0 \end{pmatrix} & 0 \\
0 & -\begin{pmatrix} A^d_{\phi_\nu} & 0 \\ 0 & A^{d_K}_{\phi_\nu} \end{pmatrix}^T & 0 & \begin{pmatrix} C^d_{\phi_\nu} \\ 0 \end{pmatrix}^T \\ \hline
\begin{pmatrix} C^d_{\psi_\nu} & 0 \\ 0 & C^{d_K}_{\psi_\nu} \end{pmatrix} & 0 & \begin{pmatrix} D^d_{\psi_\nu} \\ 0 \end{pmatrix} & 0 \\
0 & -\begin{pmatrix} B^d_{\phi_\nu} & 0 \\ 0 & B^{d_K}_{\phi_\nu} \end{pmatrix}^T & 0 & \begin{pmatrix} D^d_{\phi_\nu} \\ 0 \end{pmatrix}^T \\ \hline
0 & 0 & \begin{pmatrix} I \\ 0 \end{pmatrix} & 0 \\
0 & 0 & 0 & \begin{pmatrix} I \\ 0 \end{pmatrix}
\end{array}\right) \succ 0.
$$

By uncontrollability again, we have

$$\star^T \mathcal{M}\left(R, \left(\begin{array}{cc|cc} \hat{M} & 0 & 0 & 0 \\ 0 & \hat{N} & 0 & 0 \\ \hline 0 & 0 & 0 & I \\ 0 & 0 & I & 0 \end{array}\right)\right) \left(\begin{array}{cc|cc} I & 0 & 0 & 0 \\ 0 & I & 0 & 0 \\ \hline A^d_{\psi_\nu} & 0 & B^d_{\psi_\nu} & 0 \\ 0 & -\left(A^d_{\phi_\nu}\right)^T & 0 & \left(C^d_{\phi_\nu}\right)^T \\ \hline \begin{pmatrix} C^d_{\psi_\nu} \\ 0 \end{pmatrix} & 0 & \begin{pmatrix} D^d_{\psi_\nu} \\ 0 \end{pmatrix} & 0 \\ 0 & -\begin{pmatrix} B^d_{\phi_\nu} \\ 0 \end{pmatrix}^T & 0 & \begin{pmatrix} D^d_{\phi_\nu} \\ 0 \end{pmatrix}^T \\ \hline 0 & 0 & \begin{pmatrix} I \\ 0 \end{pmatrix} & 0 \\ 0 & 0 & 0 & \begin{pmatrix} I \\ 0 \end{pmatrix} \end{array}\right) \succ 0.$$

Simplification yields (7). This proves necessity.

B. Sufficiency

Step 1: Squaring of the outer factors. Note that the inequalities (5) and (7) involve the multiplier $\left[\begin{array}{c|c} A^d_{\psi_\nu} & B^d_{\psi_\nu} \\ \hline C^d_{\psi_\nu} & D^d_{\psi_\nu} \end{array}\right]^* M \left[\begin{array}{c|c} A^d_{\psi_\nu} & B^d_{\psi_\nu} \\ \hline C^d_{\psi_\nu} & D^d_{\psi_\nu} \end{array}\right]$ with an outer factor that is typically tall; a similar observation can be made for (6) and (7) and the dual multiplier. For technical reasons, we need to work with square outer factors in inequalities (5), (6) and (7). This can be achieved by *shifting* X and Y using solutions of AREs related to the spectral factorizations of the multiplier sub-blocks $\psi_i^* M_i \psi_i$ and $\phi_i N_i \phi_i^*$.

Primal Inequality. Suppose inequalities (5)-(8) are satisfied. Let $\hat{Z}_i$ represent the stabilizing solution of

$$\left(A^{n_i}_{\psi_\nu}\right)^T \hat{Z}_i + \hat{Z}_i A^{n_i}_{\psi_\nu} + \left(C^{n_i}_{\psi_\nu}\right)^T M_i C^{n_i}_{\psi_\nu} - \left(\star\right)^T \hat{M}_i^{-1} \left(\left(B^{n_i}_{\psi_\nu}\right)^T \hat{Z}_i + \left(D^{n_i}_{\psi_\nu}\right)^T M_i C^{n_i}_{\psi_\nu}\right) = 0,$$

where $\hat{M}_i := \left(D^{n_i}_{\psi_\nu}\right)^T M_i D^{n_i}_{\psi_\nu}$. In that case, we have $\psi_i^* M_i \psi_i = \uppsi_i^* \uppsi_i$, where

$$\uppsi_i = \left[\begin{array}{c|c} A_{\uppsi_i} & B_{\uppsi_i} \\ \hline C_{\uppsi_i} & D_{\uppsi_i} \end{array}\right] =: \left[\begin{array}{c|c} A^{n_i}_{\psi_\nu} & B^{n_i}_{\psi_\nu} \\ \hline C_{\uppsi_i} & \hat{M}_i^{1/2} \end{array}\right]$$

with $C_{\psi_i} := \hat{M}_i^{-1/2}\left(\left(B_{\psi_\nu}^{n_i}\right)^T \hat{Z}_i + \left(D_{\psi_\nu}^{n_i}\right)^T M_i C_{\psi_\nu}^{n_i}\right)$. Let $\hat{Z} := \mathbf{diag}_{i=1}^m(\hat{Z}_i)$ and

$$\left[\begin{array}{c|c} A_\psi & B_\psi \\ \hline C_\psi & D_\psi \end{array}\right] := \left[\begin{array}{c|c} \mathbf{diag}_{i=1}^m(A_{\psi_i}) & \mathbf{diag}_{i=1}^m(B_{\psi_i}) \\ \hline \mathbf{diag}_{i=1}^m(C_{\psi_i}) & \mathbf{diag}_{i=1}^m(D_{\psi_i}) \end{array}\right].$$

As in the proof of Lemma 1 in [18], inequality (5) can now be rewritten as

$$\star^T \star^T \mathcal{M}(\mathcal{X}, J(I)) \left(\begin{array}{ccc:c} I & 0 & 0 & 0 \\ 0 & I & 0 & 0 \\ 0 & 0 & I & 0 \\ \hdashline A_\psi & 0 & B_\psi C_q & B_\psi D_{qp} \\ 0 & A_\psi & 0 & B_\psi \\ 0 & 0 & A & B_p \\ \hdashline C_\psi & 0 & D_\psi C_q & D_\psi D_{qp} \\ 0 & C_\psi & 0 & D_\psi \end{array}\right) \mathcal{U} \prec 0, \tag{15}$$

where $\mathcal{X} := X + \mathbf{diag}\left(-\hat{Z}, \hat{Z}, 0\right)$.

Dual Inequality. Similarly, let $\hat{W}_i$ represent the stabilizing (smallest) solution of

$$A_{\phi_\nu}^{n_i}\hat{W}_i + \hat{W}_i\left(A_{\phi_\nu}^{n_i}\right)^T - B_{\phi_\nu}^{n_i} N_i \left(B_{\phi_\nu}^{n_i}\right)^T + \left(\hat{W}_i\left(C_{\phi_\nu}^{n_i}\right)^T - B_{\phi_\nu} N_i \left(D_{\phi_\nu}^{n_i}\right)^T\right)\hat{N}_i^{-1}\left(\star\right) = 0,$$

where $\hat{N}_i := D_{\phi_\nu}^{n_i} N_i \left(D_{\phi_\nu}^{n_i}\right)^T$. Then, $\phi_i N_i \phi_i^* = \phi_i \phi_i^*$, where

$$\phi_i = \left[\begin{array}{c|c} A_{\phi_i} & B_{\phi_i} \\ \hline C_{\phi_i} & D_{\phi_i} \end{array}\right] =: \left[\begin{array}{c|c} A_{\phi_\nu}^{n_i} & B_{\phi_i} \\ \hline C_{\phi_\nu}^{n_i} & \hat{N}_i^{1/2} \end{array}\right]$$

with $B_{\phi_i} := -\left(\hat{W}_i\left(C_{\phi_\nu}^{n_i}\right)^T - B_{\phi_\nu}^{n_i} N_i \left(D_{\phi_\nu}^{n_i}\right)^T\right)\hat{N}_i^{-1/2}$. Let $\hat{W} := \mathbf{diag}_{i=1}^m\left(\hat{W}_i\right)$ and

$$\left[\begin{array}{c|c} A_\phi & B_\phi \\ \hline C_\phi & D_\phi \end{array}\right] := \left[\begin{array}{c|c} \mathbf{diag}_{i=1}^m(A_{\phi_i}) & \mathbf{diag}_{i=1}^m(B_{\phi_i}) \\ \hline \mathbf{diag}_{i=1}^m(C_{\phi_i}) & \mathbf{diag}_{i=1}^m(D_{\phi_i}) \end{array}\right].$$

Hence, inequality (6) can be rewritten as

$$\star^T \star^T \mathcal{M}\left(\mathcal{Y}, J(I)\right) \left(\begin{array}{ccc|c} -A_\phi^T & 0 & 0 & C_\phi^T \\ 0 & -A_\phi^T & -C_\phi^T B_p^T & -C_\phi^T D_{qp}^T \\ 0 & 0 & -A^T & -C_q^T \\ \hline I & 0 & 0 & 0 \\ 0 & I & 0 & 0 \\ 0 & 0 & I & 0 \\ \hline -B_\phi^T & 0 & 0 & D_\phi^T \\ 0 & -B_\phi^T & -D_\phi^T B_p^T & -D_\phi^T D_{qp}^T \end{array}\right) \mathcal{V} \succ 0, \tag{16}$$

where $\mathcal{Y} := Y + \mathbf{diag}\left(-\hat{W}, \hat{W}, 0\right)$.

Multiplier Coupling. Also inequality (7) can now be rewritten as

$$\star^T \mathcal{M}\left(\left(\mathcal{R}_i, \left(\begin{array}{cc|cc} I & 0 & 0 & 0 \\ 0 & I & 0 & 0 \\ \hline 0 & 0 & 0 & I \\ 0 & 0 & I & 0 \end{array}\right)\right)\right) \left(\begin{array}{cc|cc} I & 0 & 0 & 0 \\ 0 & I & 0 & 0 \\ \hline A_{\psi_i} & 0 & B_{\psi_i} & 0 \\ 0 & -A_{\phi_i}^T & 0 & C_{\phi_i}^T \\ \hline C_{\psi_i} & 0 & D_{\psi_i} & 0 \\ 0 & -B_{\phi_i}^T & 0 & D_{\phi_i}^T \\ \hline 0 & 0 & I & 0 \\ 0 & 0 & 0 & I \end{array}\right) \succ 0 \tag{17}$$

for $i = 1 : m$, where $\mathcal{R}_i := R_i + \mathbf{diag}\left(-\hat{Z}_i, -\hat{W}_i\right)$.

X-Y **Coupling:** Trivially the coupling condition can be expressed as

$$\left(\begin{array}{c|c} \mathcal{X} + \mathbf{diag}\left(-\mathcal{R}_{11}, \mathcal{R}_{11}, 0\right) & \mathbf{diag}\left(-\mathcal{R}_{12}, -\mathcal{R}_{12}, I\right) \\ \hline \mathbf{diag}\left(-\mathcal{R}_{12}, -\mathcal{R}_{12}, I\right) & \mathcal{Y} + \mathbf{diag}\left(-\mathcal{R}_{22}, \mathcal{R}_{22}, 0\right) \end{array}\right) \succ 0.$$

Step 2: Coordinate transformation on ψ_i and ϕ_i. Observe that $\mathcal{R}_{i_{12}}$ are square for $i = 1 : m$ and that they can be rendered non-singular by perturbation. With an appropriate coordinate transformation on the states of either ψ_i or ϕ_i, we obtain $\mathcal{R}_{i_{12}} = \mathcal{R}_{i_{21}}^T = -I$ for each

$i = 1 : m$. Let the transformed realizations for ψ_i and ϕ_i be denoted by

$$\left[\begin{array}{c|c} \hat{A}_{\psi_i} & \hat{B}_{\psi_i} \\ \hline \hat{C}_{\psi_i} & \hat{D}_{\psi_i} \end{array}\right] \quad \text{and} \quad \left[\begin{array}{c|c} \hat{A}_{\phi_i} & \hat{B}_{\phi_i} \\ \hline \hat{C}_{\phi_i} & \hat{D}_{\phi_i} \end{array}\right]$$

and define

$$\left[\begin{array}{c|c} \hat{A}_{\psi} & \hat{B}_{\psi} \\ \hline \hat{C}_{\psi} & \hat{D}_{\psi} \end{array}\right] := \left[\begin{array}{c|c} \mathbf{diag}_{i=1}^{m}\left(\hat{A}_{\psi_i}\right) & \mathbf{diag}_{i=1}^{m}\left(\hat{B}_{\psi_i}\right) \\ \hline \mathbf{diag}_{i=1}^{m}\left(\hat{C}_{\psi_i}\right) & \mathbf{diag}_{i=1}^{m}\left(\hat{D}_{\psi_i}\right) \end{array}\right],$$

$$\left[\begin{array}{c|c} \hat{A}_{\phi} & \hat{B}_{\phi} \\ \hline \hat{C}_{\phi} & \hat{D}_{\phi} \end{array}\right] := \left[\begin{array}{c|c} \mathbf{diag}_{i=1}^{m}\left(\hat{A}_{\phi_i}\right) & \mathbf{diag}_{i=1}^{m}\left(\hat{B}_{\phi_i}\right) \\ \hline \mathbf{diag}_{i=1}^{m}\left(\hat{C}_{\phi_i}\right) & \mathbf{diag}_{i=1}^{m}\left(\hat{D}_{\phi_i}\right) \end{array}\right].$$

Conditions (5)-(8) can now be rewritten as

$$\star^T \star^T \mathcal{M}\left(\hat{\mathcal{X}}, J(I)\right) \left(\begin{array}{ccc:c} I & 0 & 0 & 0 \\ 0 & I & 0 & 0 \\ 0 & 0 & I & 0 \\ \hdashline \hat{A}_{\psi} & 0 & \hat{B}_{\psi} C_q & \hat{B}_{\psi} D_{qp} \\ 0 & \hat{A}_{\psi} & 0 & \hat{B}_{\psi} \\ 0 & 0 & A & B_p \\ \hdashline \hat{C}_{\psi} & 0 & \hat{D}_{\psi} C_q & \hat{D}_{\psi} D_{qp} \\ 0 & \hat{C}_{\psi} & 0 & \hat{D}_{\psi} \end{array}\right) \mathcal{U} \prec 0 \tag{18}$$

$$\star^T \star^T \mathcal{M}\left(\hat{\mathcal{Y}}, J(I)\right) \left(\begin{array}{ccc:c} -\hat{A}_{\phi}^T & 0 & 0 & \hat{C}_{\phi}^T \\ 0 & -\hat{A}_{\phi}^T & -\hat{C}_{\phi}^T B_p^T & -\hat{C}_{\phi}^T D_{qp}^T \\ 0 & 0 & -A^T & -C_q^T \\ \hdashline I & 0 & 0 & 0 \\ 0 & I & 0 & 0 \\ 0 & 0 & I & 0 \\ \hdashline -\hat{B}_{\phi}^T & 0 & 0 & \hat{D}_{\phi}^T \\ 0 & -\hat{B}_{\phi}^T & -\hat{D}_{\phi}^T B_p^T & -\hat{D}_{\phi}^T D_{qp}^T \end{array}\right) \mathcal{V} \succ 0. \tag{19}$$

$$\star^T \mathcal{M}\left(\begin{pmatrix} \hat{\mathcal{R}}_{11} & -I \\ -I & \hat{\mathcal{R}}_{22} \end{pmatrix}, \begin{pmatrix} I & 0 & 0 & 0 \\ 0 & I & 0 & 0 \\ \hline 0 & 0 & 0 & I \\ 0 & 0 & I & 0 \end{pmatrix}\right) \begin{pmatrix} I & 0 & 0 & 0 \\ 0 & I & 0 & 0 \\ \hline \hat{A}_\psi & 0 & \hat{B}_\psi & 0 \\ 0 & -\hat{A}_\phi^T & 0 & \hat{C}_\phi^T \\ \hline \hat{C}_\psi & 0 & \hat{D}_\psi & 0 \\ 0 & -\hat{B}_\phi^T & 0 & \hat{D}_\phi^T \\ \hline 0 & 0 & I & 0 \\ 0 & 0 & 0 & I \end{pmatrix} \succ 0 \tag{20}$$

$$\begin{pmatrix} \hat{\mathcal{X}}_{11} - \hat{\mathcal{R}}_{11} & \hat{\mathcal{X}}_{12} & \hat{\mathcal{X}}_{13} & I & 0 & 0 \\ \hat{\mathcal{X}}_{21} & \hat{\mathcal{X}}_{22} + \hat{\mathcal{R}}_{11} & \hat{\mathcal{X}}_{23} & 0 & I & 0 \\ \hat{\mathcal{X}}_{31} & \hat{\mathcal{X}}_{32} & \hat{\mathcal{X}}_{33} & 0 & 0 & I \\ \hline I & 0 & 0 & \hat{\mathcal{Y}}_{11} - \hat{\mathcal{R}}_{22} & \hat{\mathcal{Y}}_{12} & \hat{\mathcal{Y}}_{13} \\ 0 & I & 0 & \hat{\mathcal{Y}}_{21} & \hat{\mathcal{Y}}_{22} + \hat{\mathcal{R}}_{22} & \hat{\mathcal{Y}}_{13} \\ 0 & 0 & I & \hat{\mathcal{Y}}_{31} & \hat{\mathcal{Y}}_{32} & \hat{\mathcal{Y}}_{33} \end{pmatrix} \succ 0, \tag{21}$$

where $\hat{\mathcal{X}}$, $\hat{\mathcal{Y}}$, $\hat{\mathcal{R}}_{11}$ and $\hat{\mathcal{R}}_{22}$ are obtained by congruence transformations in accordance with the performed coordinate changes.

Step 3. Construction of the extended multiplier. In this key step which gracefully exploits (20) (as a diagonal combination of inequalities similar to (17)), we now extend each $\left(\left[\begin{array}{c|c} \hat{A}_{\psi_i} & \hat{B}_{\psi_i} \\ \hline \hat{C}_{\psi_i} & \hat{D}_{\psi_i} \end{array}\right], \left[\begin{array}{c|c} \hat{A}_{\phi_i} & \hat{B}_{\phi_i} \\ \hline \hat{C}_{\phi_i} & \hat{D}_{\phi_i} \end{array}\right]\right)$ pair as in Lemma 7. When the extended multipliers are placed block-diagonally, we obtain A_ψ, B_{ψ_G}, B_{ψ_K}, C_{ψ_G}, C_{ψ_K}, $D_{\psi_{GG}}$, $D_{\psi_{GK}}$ and A_ϕ, B_{ϕ_G}, B_{ϕ_K}, C_{ϕ_G}, C_{ϕ_K}, $D_{\phi_{GG}}$, $D_{\phi_{GK}}$ such that

(i) $$\left[\begin{array}{c|c} \hat{A}_\psi & \hat{B}_\psi \\ \hline \hat{C}_\psi & \hat{D}_\psi \end{array}\right]^* \left[\begin{array}{c|c} \hat{A}_\psi & \hat{B}_\psi \\ \hline \hat{C}_\psi & \hat{D}_\psi \end{array}\right] = \left[\begin{array}{c|c} A_\psi & B_{\psi_G} \\ \hline C_{\psi_G} & D_{\psi_{GG}} \\ C_{\psi_K} & 0 \end{array}\right]^* \left[\begin{array}{c|c} A_\psi & B_{\psi_G} \\ \hline C_{\psi_G} & D_{\psi_{GG}} \\ C_{\psi_K} & 0 \end{array}\right],$$

(ii) $$\left[\begin{array}{c|c} \hat{A}_\phi & \hat{B}_\phi \\ \hline \hat{C}_\phi & \hat{D}_\phi \end{array}\right] \left[\begin{array}{c|c} \hat{A}_\phi & \hat{B}_\phi \\ \hline \hat{C}_\phi & \hat{D}_\phi \end{array}\right]^* = \left[\begin{array}{c|cc} A_\phi & B_{\phi_G} & B_{\phi_K} \\ \hline C_{\phi_G} & D_{\phi_{GG}} & D_{\phi_{GK}} \end{array}\right] \left[\begin{array}{c|cc} A_\phi & B_{\phi_G} & B_{\phi_K} \\ \hline C_{\phi_G} & D_{\phi_{GG}} & D_{\phi_{GK}} \end{array}\right]^*,$$

$$\text{(iii)} \quad \left[\begin{array}{c|cc} A_\psi & B_{\psi_G} & B_{\psi_K} \\ \hline C_{\psi_G} & D_{\psi_{GG}} & D_{\psi_{GK}} \\ C_{\psi_K} & 0 & I \end{array}\right]^{-1} = \left[\begin{array}{c|cc} A_\phi & B_{\phi_G} & B_{\phi_K} \\ \hline C_{\phi_G} & D_{\phi_{GG}} & D_{\phi_{GK}} \\ C_{\phi_K} & 0 & I \end{array}\right].$$

Step 4. Construction of the controller. Recall from Lemma 7 that (i) and (ii) are certified by $\hat{\mathcal{R}}_{11}$ and $\hat{\mathcal{R}}_{22}$ respectively. We can then apply the Gluing Lemma ([18, Lemma 6]) to infer that conditions (18) and (19) become

$$\star^T \star^T \mathcal{M}\left(\hat{\mathcal{X}} + \mathbf{diag}\left(-\hat{\mathcal{R}}_{11}, \hat{\mathcal{R}}_{11}, 0\right), J\begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix}\right)$$

$$\left(\begin{array}{ccc:c} I & 0 & 0 & 0 \\ 0 & I & 0 & 0 \\ 0 & 0 & I & 0 \\ \hdashline A_\psi & 0 & B_{\psi_G} C_q & B_{\psi_G} D_{qp} \\ 0 & A_\psi & 0 & B_{\psi_G} \\ 0 & 0 & A & B_p \\ \hdashline \begin{pmatrix} C_{\psi_G} \\ C_{\psi_K} \end{pmatrix} & \begin{pmatrix} 0 \\ 0 \end{pmatrix} & \begin{pmatrix} D_{\psi_{GG}} \\ 0 \end{pmatrix} C_q & \begin{pmatrix} D_{\psi_{GG}} \\ 0 \end{pmatrix} D_{qp} \\ \begin{pmatrix} 0 \\ 0 \end{pmatrix} & \begin{pmatrix} C_{\psi_G} \\ C_{\psi_K} \end{pmatrix} & \begin{pmatrix} 0 \\ 0 \end{pmatrix} & \begin{pmatrix} D_{\psi_{GG}} \\ 0 \end{pmatrix} \end{array}\right) \mathcal{U} \prec 0 \quad (22)$$

and

$$\star^T \star^T \mathcal{M}\left(\hat{\mathcal{Y}} + \mathbf{diag}\left(-\hat{\mathcal{R}}_{22}, \hat{\mathcal{R}}_{22}, 0\right), J\begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix}\right)$$

$$\begin{pmatrix} -A_\phi^T & 0 & 0 & C_{\phi_G}^T \\ 0 & -A_\phi^T & -C_{\phi_G}^T B_p^T & -C_{\phi_G}^T D_{qp}^T \\ 0 & 0 & -A^T & -C_q^T \\ \hline I & 0 & 0 & 0 \\ 0 & I & 0 & 0 \\ 0 & 0 & I & 0 \\ \hline -\begin{pmatrix} B_{\phi_G}^T \\ B_{\phi_K}^T \end{pmatrix} & \begin{pmatrix} 0 \\ 0 \end{pmatrix} & \begin{pmatrix} 0 \\ 0 \end{pmatrix} & \begin{pmatrix} D_{\phi_{GG}}^T \\ D_{\phi_{GK}}^T \end{pmatrix} \\ \begin{pmatrix} 0 \\ 0 \end{pmatrix} & -\begin{pmatrix} B_{\phi_G}^T \\ B_{\phi_K}^T \end{pmatrix} & -\begin{pmatrix} D_{\phi_{GG}}^T \\ D_{\phi_{GK}}^T \end{pmatrix} B_p^T & -\begin{pmatrix} D_{\phi_{GG}}^T \\ D_{\phi_{GK}}^T \end{pmatrix} D_{qp}^T \end{pmatrix} \mathcal{V} \succ 0 \quad (23)$$

respectively. Due to (21), we can expand $\hat{X}$ as $\hat{\mathcal{X}}^a := \begin{pmatrix} \hat{\mathcal{X}}_{11} - \hat{\mathcal{R}}_{11} & \hat{\mathcal{X}}_{21} & \hat{\mathcal{X}}_{13} & \hat{\mathcal{X}}_{1c} \\ \hat{\mathcal{X}}_{21} & \hat{\mathcal{X}}_{22} + \hat{\mathcal{R}}_{11} & \hat{\mathcal{X}}_{23} & \hat{\mathcal{X}}_{2c} \\ \hat{\mathcal{X}}_{31} & \hat{\mathcal{X}}_{32} & \hat{\mathcal{X}}_{33} & \hat{\mathcal{X}}_{3c} \\ \hat{\mathcal{X}}_{c1} & \hat{\mathcal{X}}_{c2} & \hat{\mathcal{X}}_{c3} & \hat{\mathcal{X}}_{cc} \end{pmatrix} \succ 0$, so that $\left(\hat{\mathcal{X}}^a\right)^{-1} =: \hat{\mathcal{Y}}^a$ has the form

$$\hat{\mathcal{Y}}^a = \begin{pmatrix} \hat{\mathcal{Y}}_{11} - \hat{\mathcal{R}}_{22} & \hat{\mathcal{Y}}_{12} & \hat{\mathcal{Y}}_{13} & \hat{\mathcal{Y}}_{1c} \\ \hat{\mathcal{Y}}_{21} & \hat{\mathcal{Y}}_{22} + \hat{\mathcal{R}}_{22} & \hat{\mathcal{Y}}_{23} & \hat{\mathcal{Y}}_{2c} \\ \hat{\mathcal{Y}}_{31} & \hat{\mathcal{Y}}_{33} & \hat{\mathcal{Y}}_{33} & \hat{\mathcal{Y}}_{3c} \\ \hat{\mathcal{Y}}_{c1} & \hat{\mathcal{Y}}_{c2} & \hat{\mathcal{Y}}_{c3} & \hat{\mathcal{Y}}_{cc} \end{pmatrix} \succ 0.$$

We can now expand inequalities (22) and (23) as

$$\star^T \star^T \mathcal{M}\left(\hat{\mathcal{X}}^a, J(I)\right) \begin{pmatrix} I & 0 & 0 & 0 \\ 0 & I & 0 & 0 \\ 0 & 0 & I & 0 \\ \hline A_\psi & 0 & B_{\psi_{cl}} C_q^a & B_{\psi_{cl}} D_{qp}^a \\ 0 & A_\psi & 0 & B_{\psi_{cl}} \\ 0 & 0 & A^a & B_p^a \\ \hline C_{\psi_{cl}} & 0 & D_{\psi_{cl}} C_q^a & D_{\psi_{cl}} D_{qp}^a \\ 0 & C_{\psi_{cl}} & 0 & D_{\psi_{cl}} \end{pmatrix} \mathcal{U}^a \prec 0,$$

and

$$\star^T \star^T \mathcal{M}\left(\hat{\mathcal{Y}}^a, J(I)\right) \left(\begin{array}{ccc:c} -A_\phi^T & 0 & 0 & C_{\phi_{cl}}^T \\ 0 & -A_\phi^T & -C_{\phi_{cl}}^T B_p^T & -C_{\phi_{cl}}^T D_{qp}^{a\ T} \\ 0 & 0 & -A^{aT} & -C_q^{aT} \\ \hdashline I & 0 & 0 & 0 \\ 0 & I & 0 & 0 \\ 0 & 0 & I & 0 \\ \hdashline -B_{\phi_{cl}}^T & 0 & 0 & D_{\phi_{cl}}^T \\ 0 & -B_{\phi_{cl}}^T & -D_{\phi_{cl}}^T B_p^{aT} & -D_{\phi_{cl}}^T D_{qp}^{a\ T} \end{array}\right) \mathcal{V}^a \succ 0,$$

where $\mathcal{U}^a := \begin{pmatrix} 0 \\ 0 \\ \left(C_y^a\right)^T \\ \left(D_{yp}^a\right)^T \end{pmatrix}_\perp$, $\mathcal{V}^a := \begin{pmatrix} 0 \\ 0 \\ B_u^a \\ D_{qu}^a \end{pmatrix}_\perp$ and

$$\left[\begin{array}{c|c} A_\psi & B_{\psi_{cl}} \\ \hline C_{\psi_{cl}} & D_{\psi_{cl}} \end{array}\right] = \left[\begin{array}{c|cc} A_\psi & B_{\psi_G} & B_{\psi_K} \\ \hline C_{\psi_G} & D_{\psi_{GG}} & D_{\psi_{GK}} \\ C_{\psi_K} & 0 & I \end{array}\right], \quad \text{and} \quad \left[\begin{array}{c|c} A_\phi & B_{\phi_{cl}} \\ \hline C_{\phi_{cl}} & D_{\phi_{cl}} \end{array}\right] = \left[\begin{array}{c|cc} A_\phi & B_{\phi_G} & B_{\phi_K} \\ \hline C_{\phi_G} & D_{\phi_{GG}} & D_{\phi_{GK}} \\ C_{\phi_K} & 0 & I \end{array}\right],$$

just because the left-hand sides of the respective inequalities turn out to be identical.

Now the controller construction is relatively routine. In fact, by Lemma 4, we arrived at the conditions for the existence of $\mathbf{K}$ such that

$$(\mathcal{U}_A + \mathcal{U}_B \mathbf{K} \mathcal{U}_C)^T \Pi (\mathcal{U}_A + \mathcal{U}_B \mathbf{K} \mathcal{U}_C) \prec 0,$$

where $\Pi := \mathcal{M}\left(\hat{\mathcal{X}}^a, J(I)\right)$ and

$$\mathcal{U}_A := \left(\begin{array}{ccc|c} I & 0 & 0 & 0 \\ 0 & I & 0 & 0 \\ 0 & 0 & I & 0 \\ \hline A_\psi & 0 & B_{\psi_{cl}} C_q^a & B_{\psi_{cl}} D_{qp}^a \\ 0 & A_\psi & 0 & B_{\psi_{cl}} \\ 0 & 0 & A^a & B_p^a \\ \hline C_{\psi_{cl}} & 0 & D_{\psi_{cl}} C_q^a & D_{\psi_{cl}} D_{qp}^a \\ 0 & C_{\psi_{cl}} & 0 & D_{\psi_{cl}} \end{array}\right), \quad \mathcal{U}_B := \left(\begin{array}{c} 0 \\ 0 \\ 0 \\ \hline B_{\psi_{cl}} D_{qu}^a \\ 0 \\ B_u^a \\ \hline D_{\psi_{cl}} D_{qu}^a \\ 0 \end{array}\right), \quad \mathcal{U}_C := \left(\begin{array}{ccc|c} 0 & 0 & C_y^a & D_{yp}^a \end{array}\right).$$

For clarity, let $\alpha \times \beta$ and $\kappa \times \lambda$ be the dimensions of $\mathcal{U}_A$ and $\mathbf{K}$, respectively. Since

$$\mathbf{in}(\Pi) = \mathbf{in}\begin{pmatrix} 0 & \hat{\mathcal{X}}^a \\ \hat{\mathcal{X}}^a & 0 \end{pmatrix} + \mathbf{in}\begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix},$$

it is easily verified that $\mathbf{in}(\Pi) = (\alpha - \beta, \beta, 0)$. The desired $\mathbf{K}$ can now be obtained as follows. Defining $\Theta := \begin{pmatrix} \mathcal{U}_A & \mathcal{U}_B \end{pmatrix}^T \Pi \begin{pmatrix} \mathcal{U}_A & \mathcal{U}_B \end{pmatrix}$ of dimension $\beta + \kappa$, we can rewrite

$$\begin{pmatrix} I_\beta \\ \mathbf{K}\mathcal{U}_C \end{pmatrix}^T \Theta \begin{pmatrix} I_\beta \\ \mathbf{K}\mathcal{U}_C \end{pmatrix} \prec 0.$$

Hence, $n_-(\Theta) \geq \beta$. However, since Θ is obtained by restricting Π to a certain subspace, we also have $n_-(\Theta) \leq n_-(\Pi) = \beta$. The conclusion is that $\mathbf{in}(\Theta) = (\kappa, \beta, 0)$. Then, by [15], the inequality above can be written as

$$\begin{pmatrix} -\mathcal{U}_C^T \mathbf{K}^T \\ I_\kappa \end{pmatrix}^T \Theta^{-1} \begin{pmatrix} -\mathcal{U}_C^T \mathbf{K}^T \\ I_\kappa \end{pmatrix} = \star^T \underbrace{\begin{pmatrix} -\mathcal{U}_C^T & 0 \\ 0 & I_\kappa \end{pmatrix}^T \Theta^{-1} \begin{pmatrix} -\mathcal{U}_C^T & 0 \\ 0 & I_\kappa \end{pmatrix}}_{\Omega} \begin{pmatrix} \mathbf{K}^T \\ I_\kappa \end{pmatrix} \succ 0.$$

Similarly, we can also conclude that $\mathbf{in}(\Omega) = (\kappa, \lambda, 0)$. Now choose a matrix $\begin{pmatrix} \mathcal{S}_1 \\ \mathcal{S}_2 \end{pmatrix} \in \mathbb{R}^{(\kappa+\lambda)\times\kappa}$ with $\mathcal{S}_2$ invertible such that

$$\begin{pmatrix} \mathcal{S}_1 \\ \mathcal{S}_2 \end{pmatrix}^T \Omega \begin{pmatrix} \mathcal{S}_1 \\ \mathcal{S}_2 \end{pmatrix} \succ 0 \iff \begin{pmatrix} \mathcal{S}_1 \mathcal{S}_2^{-1} \\ I_\kappa \end{pmatrix}^T \Omega \begin{pmatrix} \mathcal{S}_1 \mathcal{S}_2^{-1} \\ I_\kappa \end{pmatrix} \succ 0.$$

Then, can take $\mathbf{K} = \left(\mathcal{S}_1 \mathcal{S}_2^{-1}\right)^T$. Finally, nominal stability of the closed-loop system is guaranteed by the fact that $\hat{X}^a \succ 0$.

Generically $\hat{\mathcal{X}}^a$ has a dimension equal to that of the left-hand side of (21), which is $2(\dim(A) + 2\dim(\hat{A}_\psi)) = 2\dim(A) + 4\nu N$. Hence the dynamic order of the controller is $\dim(A) + 2\nu N$. The size of the scheduling block is determined by the numbers of added rows/columns in the extended primal/dual multipliers, i.e., the row/colum dimension of C_{ϕ_K}/B_{ϕ_K} respectively. According to Lemma 7 this equals $n_i + 2n_i\nu$ for each individual block, which sums up to the dimension $N(1+2\nu)$ for Δ_K.

This completes the proof.

V. NUMERICAL EXAMPLE

Consider the mechanical system shown in Figure 3.

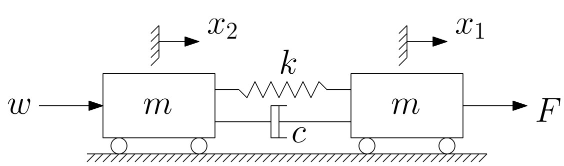

Fig. 3. Mechanical system with uncertain spring and damper.

We assume that the values of k and c are constant, but they vary around their nominal values, k_0 and c_0, as $k = k_0(1 + k^*\delta_k)$ and $c = c_0(1 + c^*\delta_c)$, where $|\delta_k| \leq 1$ and $|\delta_c| \leq 1$. Take x_1 as the measured output and x_2 as the controlled output. We can now express the system as

$$\begin{pmatrix} q \\ z \\ y \end{pmatrix} = \left[\begin{array}{cccc|cc:c:c}
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
-\frac{k_0}{m} & \frac{k_0}{m} & -\frac{c_0}{m} & \frac{c_0}{m} & -\sqrt{\frac{k_0 k^*}{m}} & -\sqrt{\frac{c_0 c^*}{m}} & 0 & \frac{1}{m} \\
\frac{k_0}{m} & -\frac{k_0}{m} & \frac{c_0}{m} & -\frac{c_0}{m} & \sqrt{\frac{k_0 k^*}{m}} & \sqrt{\frac{c_0 c^*}{m}} & \frac{1}{m} & 0 \\
\hline
\sqrt{\frac{k_0 k^*}{m}} & -\sqrt{\frac{k_0 k^*}{m}} & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & \sqrt{\frac{c_0 c^*}{m}} & -\sqrt{\frac{c_0 c^*}{m}} & 0 & 0 & 0 & 0 \\
\hdashline
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
\hdashline
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{array}\right] \begin{pmatrix} p \\ w \\ u \end{pmatrix}$$

and $p = \begin{pmatrix} \delta_k & 0 \\ 0 & \delta_c \end{pmatrix} q$. For the numerical values $m_0 = 10$ kg, $k_0 = 10$ N/m, $c_0 = 10$ Ns/m and $k^* = c^* = 0.75$, we calculate the minimum achievable $\mathcal{L}_2$-gains for different dynamic orders (ν), and we obtain the figures shown in the following table:

ν	0	1	2	3	4
γ	4.2	0.45	0.44	0.44	0.44
n_{A_K}	4	8	12	16	20
$n_{K_{\delta_k}}$	1	3	5	7	9
$n_{K_{\delta_c}}$	1	3	5	7	9

The rows below the γ values indicate the dynamic order of the resulting controller (*i.e.*, n_{A_K}) and the sizes of the δ_k and δ_c blocks in Δ_K (*i.e.*, $n_{K_{\delta_k}}$ and $n_{K_{\delta_c}}$). Simulation results in response to a step disturbance of magnitude 10 are shown for different ν values in Figure 4. The results are given for $\delta_k = \delta_c = 0.75$.

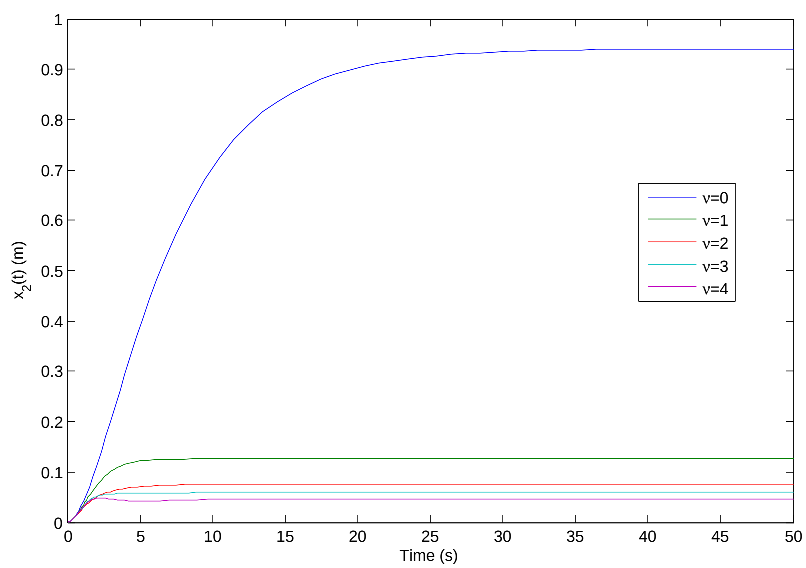

Fig. 4. Responses to a step disturbance for different ν values.

VI. Summary and Discussion

We have given necessary and sufficient conditions for the existence of robustly stabilizing gain-scheduled controllers for uncertain LFT systems using dynamic D-scales. The existence conditions consist of finite-dimensional LMIs where the specific structure of the D-scales allows us to search for suitable multipliers with arbitrary accuracy. The application of the main result to a numerical example shows significant reduction in conservatism as the dynamic order of the D-scales is increased.

The extension of the proposed method to the general setting of IQCs with dynamic multipliers is still an open problem. The range of applications of such techniques is large. On the one hand, one can systematically reduce conservatism for the synthesis of controllers that are scheduled with non-linearities [13], delays [5], or any other uncertainty blocks for which IQC-results are available. On the other hand, since the design of distributed controllers in [4] is based on static IQC techniques, our results are expected to have impact for the reduction of conservatism in structured controller synthesis.

Acknowledgement. The author Carsten W. Scherer would like to thank the German Research Foundation (DFG) for financial support of the project within the Cluster of Excellence in Simulation Technology (EXC 310/1) at the University of Stuttgart.

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APPENDIX

A. Quadratic Elimination

Lemma 4: [8] Let $\mathcal{A} \in \mathbb{R}^{(k+n)\times n}$, $\mathcal{B} \in \mathbb{R}^{(k+n)\times m}$, $\mathcal{C} \in \mathbb{R}^{p\times n}$ and $\Pi = \Pi^T \in \mathbb{R}^{(k+n)\times(k+n)}$ be given. Assume $\mathbf{in}(\Pi) = (k, n, 0)$. Then, there exists a $K \in \mathbb{R}^{m\times p}$ such that

$$(\mathcal{A} + \mathcal{B}K\mathcal{C})^T \Pi (\mathcal{A} + \mathcal{B}K\mathcal{C}) \prec 0 \tag{24}$$

if and only if

$$\left(\mathcal{C}^T\right)_\perp^T \mathcal{A}^T \Pi \mathcal{A} \left(\mathcal{C}^T\right)_\perp \prec 0 \tag{25a}$$

$$\begin{pmatrix} \mathcal{A} & \mathcal{B} \end{pmatrix}_\perp^T \Pi^{-1} \begin{pmatrix} \mathcal{A} & \mathcal{B} \end{pmatrix}_\perp \succ 0. \tag{25b}$$

B. Multiplier Extension

Before proceeding to the main result of this section, Lemma 7, let us first formulate two elementary auxiliary facts.

Lemma 5: If $D = \tilde{D}^{-1}$ then $\left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right]^{-1} = \left[\begin{array}{c|c} \tilde{A} & \tilde{B} \\ \hline \tilde{C} & \tilde{D} \end{array}\right]$ iff

$$\begin{pmatrix} C^TC & (\tilde{A}-A)^T & C^TD & -\tilde{C}^T \\ \star & \tilde{B}\tilde{B}^T & -B & -\tilde{B}\tilde{D}^T \\ \star & \star & D^TD & I \\ \star & \star & \star & \tilde{D}\tilde{D}^T \end{pmatrix} \succeq 0.$$

Proof: After elimination of the blocks C^TD and D^TC by congruence, the inequality is equivalent to

$$\begin{pmatrix} 0 & (\tilde{A}-A+BD^{-1}C)^T & 0 & -(\tilde{C}+D^{-1}C)^T \\ \star & \tilde{B}\tilde{B}^T & -B & -\tilde{B}\tilde{D}^T \\ \star & \star & D^TD & I \\ \star & \star & \star & \tilde{D}\tilde{D}^T \end{pmatrix} \succeq 0.$$

This holds iff $\tilde{A} = A - BD^{-1}C$, $\tilde{C} = -D^{-1}C$ and (if exploiting $D = \tilde{D}^{-1}$ and taking the Schur complement)

$$\begin{pmatrix} \tilde{B}\tilde{B}^T & -B \\ -B^T & D^TD \end{pmatrix} - \begin{pmatrix} -\tilde{B} \\ D^T \end{pmatrix}\begin{pmatrix} -\tilde{B} \\ D^T \end{pmatrix}^T \succeq 0.$$

The latter is, in turn, equivalent to $B = \tilde{B}D = \tilde{B}\tilde{D}^{-1}$. ∎

Lemma 6: Let B and C have full column and row rank and suppose that

$$\begin{pmatrix} C^TC & A^T \\ A & BB^T \end{pmatrix} \succeq 0. \tag{26}$$

Then $A = BLC$ with some (unique) L satisfying $\|L\| \leq 1$. If (26) is strict then L is a strict contraction.

Proof: By (26), $x^TB = 0$ and $Cy = 0$ imply $x^TA = 0$ and $Ay = 0$. Hence there exists a solution L of the equation $A = BLC$. With the left- and right-inverses B^+ and C^+, it is actually given by $L = B^+AC^+$. Right-multiplying (26) with $\mathbf{diag}(C^+, B^+)$ and left-multiplying the transpose implies $\begin{pmatrix} I & (B^+AC^+)^T \\ B^+AC^+ & I \end{pmatrix} \succeq 0$ which reveals that L is a contraction. A strict inequality leads to a strict contraction. ∎

Lemma 7: Suppose that $\left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right]$ and $\left[\begin{array}{c|c} \tilde{A} & \tilde{B} \\ \hline \tilde{C} & \tilde{D} \end{array}\right]$ with square and non-singular D and $\tilde{D}$ satisfy

$$\begin{pmatrix} A^TR+RA+C^TC & (\tilde{A}-A)^T & RB+C^TD & -\tilde{C}^T \\ \tilde{A}-A & -\tilde{A}\tilde{R}-\tilde{R}\tilde{A}^T+\tilde{B}\tilde{B}^T & -B & \tilde{R}\tilde{C}^T-\tilde{B}\tilde{D}^T \\ B^TR+D^TC & -B^T & D^TD & I \\ -\tilde{C} & \tilde{C}\tilde{R}-\tilde{D}\tilde{B}^T & I & \tilde{D}\tilde{D}^T \end{pmatrix} \succeq 0. \tag{27}$$

for some $R=R^T$ and $\tilde{R}=\tilde{R}^T$. Then, there exist extensions such that

(i) $$\left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right]^* \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right] = \left[\begin{array}{c|c} A & B \\ \hline \mathcal{C}_1 & D \\ \mathcal{C}_2 & 0 \end{array}\right]^* \left[\begin{array}{c|c} A & B \\ \hline \mathcal{C}_1 & D \\ \mathcal{C}_2 & 0 \end{array}\right]$$

(ii) $$\left[\begin{array}{c|c} \tilde{A} & \tilde{B} \\ \hline \tilde{C} & \tilde{D} \end{array}\right] \left[\begin{array}{c|c} \tilde{A} & \tilde{B} \\ \hline \tilde{C} & \tilde{D} \end{array}\right]^* = \left[\begin{array}{c|cc} \tilde{A} & \tilde{\mathcal{B}}_1 & \tilde{\mathcal{B}}_2 \\ \hline \tilde{C} & \tilde{\mathcal{D}}_{11} & \tilde{\mathcal{D}}_{12} \end{array}\right] \left[\begin{array}{c|cc} \tilde{A} & \tilde{\mathcal{B}}_1 & \tilde{\mathcal{B}}_2 \\ \hline \tilde{C} & \tilde{\mathcal{D}}_{11} & \tilde{\mathcal{D}}_{12} \end{array}\right]^*$$

(iii) $$\left[\begin{array}{c|cc} A & B & \mathcal{B}_2 \\ \hline \mathcal{C}_1 & D & \mathcal{D}_{12} \\ \mathcal{C}_2 & 0 & I \end{array}\right]^{-1} = \left[\begin{array}{c|cc} \tilde{A} & \tilde{\mathcal{B}}_1 & \tilde{\mathcal{B}}_2 \\ \hline \tilde{C} & \tilde{\mathcal{D}}_{11} & \tilde{\mathcal{D}}_{12} \\ \tilde{\mathcal{C}}_2 & 0 & I \end{array}\right].$$

The factorizations (i) and (ii) are certified by R and $\tilde{R}$ respectively. If (27) is strict, the dimension of the extended outer factors in (iii) is $\dim(A)+\text{rowdim}(C)$ plus $\dim(\tilde{A})+\text{rowdim}(\tilde{C})$.

Proof: Motivated by (iii) and the inversion formula for realizations we choose

$$\tilde{\mathcal{D}}_{11}=D^{-1} \quad \text{and} \quad \tilde{\mathcal{B}}_1 := \mathcal{B}_1\tilde{D}_{11}^{-1}=BD^{-1}. \tag{28}$$

By congruence let us eliminate all off-diagonal blocks in the third row/column of (27). For this purpose define $\mathcal{C}_1$ uniquely by solving the equation

$$RB+C^TD=\mathcal{C}_1^TD \tag{29}$$

and note that $B(D^TD)^{-1} = \tilde{\mathcal{B}}_1\tilde{\mathcal{D}}_{11}^T$, $B(D^TD)^{-1}B^T = \tilde{\mathcal{B}}_1\tilde{\mathcal{B}}_1^T$ and $(D^TD)^{-1} = \tilde{\mathcal{D}}_{11}\tilde{\mathcal{D}}_{11}^T$. We then arrive at

$$\begin{pmatrix} A^TR+RA+C^TC-\mathcal{C}_1^T\mathcal{C}_1 & (\tilde{A}-A)^T+\mathcal{C}_1^TD^{-T}B^T & 0 & -\tilde{C}^T-\mathcal{C}_1^TD^{-T} \\ \tilde{A}-A+BD^{-1}\mathcal{C}_1 & -\tilde{A}\tilde{R}-\tilde{R}\tilde{A}^T+\tilde{B}\tilde{B}^T-\tilde{\mathcal{B}}_1\tilde{\mathcal{B}}_1^T & 0 & \tilde{R}\tilde{C}^T-\tilde{B}\tilde{D}^T+\tilde{\mathcal{B}}_1\tilde{\mathcal{D}}_{11}^T \\ 0 & 0 & D^TD & 0 \\ -\tilde{C}-D^{-1}\mathcal{C}_1 & \tilde{C}\tilde{R}-\tilde{D}\tilde{B}^T+\tilde{\mathcal{D}}_{11}\tilde{\mathcal{B}}_1^T & 0 & \tilde{D}\tilde{D}^T-\tilde{\mathcal{D}}_{11}\tilde{\mathcal{D}}_{11}^T \end{pmatrix} \succeq 0. \tag{30}$$

Since the left-upper block is positive semi-definite, we can solve for a full-row-rank matrix $\mathcal{C}_{2_0}$ such that

$$\mathcal{C}_{2_0}^T\mathcal{C}_{2_0} = A^TR+RA+C^TC-\mathcal{C}_1^T\mathcal{C}_1.$$

We then obtain

$$A^TR+RA+C^TC = \mathcal{C}_1^T\mathcal{C}_1+\mathcal{C}_2^T\mathcal{C}_2, \tag{31}$$

for any $\mathcal{C}_2$ that is given by

$$\mathcal{C}_2 = \mathbb{U}\mathcal{C}_{2_0}, \quad \text{where} \quad \mathbb{U}^T\mathbb{U} = I. \tag{32}$$

Note that $\mathbb{U}$ can be tall. Clearly (29) and (31) certify (i).

Canceling the first and third column in (30) reveals that the left-hand side of (33) is positive semi-definite. We can thus determine a full-column-rank matrix $\begin{pmatrix} -\tilde{\mathcal{B}}_{2_0}^T & \tilde{\mathcal{D}}_{12_0}^T \end{pmatrix}^T$ such that

$$\begin{pmatrix} -\tilde{A}\tilde{R}-\tilde{R}\tilde{A}^T & \tilde{R}\tilde{C}^T \\ \tilde{C}\tilde{R} & 0 \end{pmatrix} + \begin{pmatrix} -\tilde{B} \\ \tilde{D} \end{pmatrix}\begin{pmatrix} -\tilde{B} \\ \tilde{D} \end{pmatrix}^T - \begin{pmatrix} -\tilde{\mathcal{B}}_1 \\ \tilde{\mathcal{D}}_{11} \end{pmatrix}\begin{pmatrix} -\tilde{\mathcal{B}}_1 \\ \tilde{\mathcal{D}}_{11} \end{pmatrix}^T = \begin{pmatrix} -\tilde{\mathcal{B}}_2 \\ \tilde{\mathcal{D}}_{12} \end{pmatrix}\begin{pmatrix} -\tilde{\mathcal{B}}_2 \\ \tilde{\mathcal{D}}_{12} \end{pmatrix}^T \tag{33}$$

for all $\begin{pmatrix} -\tilde{\mathcal{B}}_2^T & \tilde{\mathcal{D}}_{12}^T \end{pmatrix}^T$ given by

$$\begin{pmatrix} -\tilde{\mathcal{B}}_2 \\ \tilde{\mathcal{D}}_{12} \end{pmatrix} = \begin{pmatrix} -\tilde{\mathcal{B}}_{2_0} \\ \tilde{\mathcal{D}}_{12_0} \end{pmatrix}\mathbb{V}^T, \quad \text{where} \quad \mathbb{V}^T\mathbb{V} = I. \tag{34}$$

Observe that (33) certifies (ii).

For the subsequent step we note that we can cancel the third block row/column in (30) and exploit (31), (32) and (33), (34) to arrive at

$$\begin{pmatrix} \mathcal{C}_{2_0}^T\mathcal{C}_{2_0} & (\tilde{A}-A)^T+\mathcal{C}_1^TB^TD^{-T} & -\tilde{C}^T-\mathcal{C}_1^TD^{-T} \\ (\tilde{A}-A)+BD^{-1}\mathcal{C}_1 & \tilde{\mathcal{B}}_{2_0}\tilde{\mathcal{B}}_{2_0}^T & -\tilde{\mathcal{B}}_{2_0}\tilde{\mathcal{D}}_{12_0}^T \\ -\tilde{C}-D^{-1}\mathcal{C}_1 & -\tilde{\mathcal{D}}_{12_0}\tilde{\mathcal{B}}_{2_0}^T & \tilde{\mathcal{D}}_{12_0}\tilde{\mathcal{D}}_{12_0}^T \end{pmatrix} \succeq 0. \tag{35}$$

Let us finally consider (iii). By Lemma 5 and using (28), this relation is enforced by

$$\left(\begin{array}{cc|cc|cc}
\mathcal{C}_1^T\mathcal{C}_1+\mathcal{C}_2^T\mathcal{C}_2 & (\tilde{A}-A)^T & \mathcal{C}_1^T D & \mathcal{C}_1^T\mathcal{D}_{12}+\mathcal{C}_2^T & -\tilde{C}^T & -\tilde{\mathcal{C}}_2^T \\
\tilde{A}-A & B(D^TD)^{-1}B^T+\tilde{\mathcal{B}}_2\tilde{\mathcal{B}}_2^T & -B & -\mathcal{B}_2 & -B(D^TD)^{-1}-\tilde{\mathcal{B}}_2\tilde{\mathcal{D}}_{12}^T & -\tilde{\mathcal{B}}_2 \\
\hline
D^T\mathcal{C}_1 & -B^T & D^TD & D^T\mathcal{D}_{12} & I & 0 \\
\mathcal{D}_{12}^T\mathcal{C}_1+\mathcal{C}_2 & -\mathcal{B}_2^T & \mathcal{D}_{12}^T D & I+\mathcal{D}_{12}^T\mathcal{D}_{12} & 0 & I \\
\hline
-\tilde{C} & \star & I & 0 & (D^TD)^{-1}+\tilde{\mathcal{D}}_{12}\tilde{\mathcal{D}}_{12}^T & \tilde{\mathcal{D}}_{12} \\
-\tilde{\mathcal{C}}_2 & -\tilde{\mathcal{B}}_2^T & 0 & I & \tilde{\mathcal{D}}_{12}^T & I
\end{array}\right) \succeq 0.$$

Again, we eliminate all off-diagonal blocks in the third row/column by congruence to arrive at

$$\left(\begin{array}{cc|cc|cc}
\mathcal{C}_2^T\mathcal{C}_2 & \star & 0 & \mathcal{C}_2^T & -\tilde{C}^T-\mathcal{C}_1^T D^{-T} & -\tilde{\mathcal{C}}_2^T \\
\tilde{A}-A+BD^{-1}\mathcal{C}_1 & \tilde{\mathcal{B}}_2\tilde{\mathcal{B}}_2^T & 0 & -\mathcal{B}_2+BD^{-1}\mathcal{D}_{12} & -\tilde{\mathcal{B}}_2\tilde{\mathcal{D}}_{12}^T & -\tilde{\mathcal{B}}_2 \\
\hline
0 & 0 & D^TD & 0 & 0 & 0 \\
\mathcal{C}_2 & \star & 0 & I & -\mathcal{D}_{12}^T D^{-T} & I \\
\hline
-\tilde{C}-D\mathcal{C}_1 & -\tilde{\mathcal{D}}_{12}\tilde{\mathcal{B}}_2^T & 0 & -D^{-1}\mathcal{D}_{12} & \tilde{\mathcal{D}}_{12}\tilde{\mathcal{D}}_{12}^T & \tilde{\mathcal{D}}_{12} \\
-\tilde{\mathcal{C}}_2 & -\tilde{\mathcal{B}}_2^T & 0 & I & \tilde{\mathcal{D}}_{12}^T & I
\end{array}\right) \succeq 0.$$

Subtracting the last row/column from the fourth and dropping the trivial third row/column leads to the equivalent inequality

$$\left(\begin{array}{cc|c|cc}
\mathcal{C}_2^T\mathcal{C}_2 & \star & \mathcal{C}_2^T+\tilde{\mathcal{C}}_2^T & -\tilde{C}^T-\mathcal{C}_1^T D^{-T} & -\tilde{\mathcal{C}}_2^T \\
\tilde{A}-A+BD^{-1}\mathcal{C}_1 & \tilde{\mathcal{B}}_2\tilde{\mathcal{B}}_2^T & -\mathcal{B}_2+BD^{-1}\mathcal{D}_{12}+\tilde{\mathcal{B}}_2 & -\tilde{\mathcal{B}}_2\tilde{\mathcal{D}}_{12}^T & -\tilde{\mathcal{B}}_2 \\
\hline
\mathcal{C}_2+\tilde{\mathcal{C}}_2 & \star & 0 & -\mathcal{D}_{12}^T D^{-T}-\tilde{\mathcal{D}}_{12}^T & 0 \\
\hline
-\tilde{C}-D\mathcal{C}_1 & -\tilde{\mathcal{D}}_{12}\tilde{\mathcal{B}}_2^T & -D^{-1}\mathcal{D}_{12}-\tilde{\mathcal{D}}_{12} & \tilde{\mathcal{D}}_{12}\tilde{\mathcal{D}}_{12}^T & \tilde{\mathcal{D}}_{12} \\
-\tilde{\mathcal{C}}_2 & -\tilde{\mathcal{B}}_2^T & 0 & \tilde{\mathcal{D}}_{12}^T & I
\end{array}\right) \succeq 0.$$

This inequality is guaranteed to hold if we choose

$$\tilde{\mathcal{C}}_2=-\mathcal{C}_2,\quad \mathcal{D}_{12}=-D\tilde{\mathcal{D}}_{12},\quad \mathcal{B}_2=BD^{-1}\mathcal{D}_{12}+\tilde{\mathcal{B}}_2,$$

and if

$$\left(\begin{array}{c|cc}
\mathcal{C}_2^T\mathcal{C}_2 & \star & \star \\
\hline
\tilde{A}-A+BD^{-1}\mathcal{C}_1 & \tilde{\mathcal{B}}_2\tilde{\mathcal{B}}_2^T & -\tilde{\mathcal{B}}_2\tilde{\mathcal{D}}_{12}^T \\
-\tilde{C}-D^{-1}\mathcal{C}_1 & -\tilde{\mathcal{D}}_{12}\tilde{\mathcal{B}}_2^T & \tilde{\mathcal{D}}_{12}\tilde{\mathcal{D}}_{12}^T
\end{array}\right) - \left(\begin{array}{c}
\mathcal{C}_2^T \\ \hline -\tilde{\mathcal{B}}_2 \\ \tilde{\mathcal{D}}_{12}
\end{array}\right)\left(\begin{array}{c}
\mathcal{C}_2^T \\ \hline -\tilde{\mathcal{B}}_2 \\ \tilde{\mathcal{D}}_{12}
\end{array}\right)^T \succeq 0. \qquad (36)$$

Since the diagonal blocks of (36) in the given partition vanish, the inequality is enforced through

$$\begin{pmatrix} \tilde{A} - A + BD^{-1}\mathcal{C}_1 \\ -\tilde{C} - D^{-1}\mathcal{C}_1 \end{pmatrix} - \begin{pmatrix} -\tilde{\mathcal{B}}_2 \\ \tilde{\mathcal{D}}_{12} \end{pmatrix} \mathcal{C}_2 = 0. \tag{37}$$

As the very last step it remains to show that we can indeed adjust $\mathbb{U}$ and $\mathbb{V}$ to render (37) valid. By Lemma 6 and (35) we infer the existence of some L with $\|L\| \leq 1$ such that

$$\begin{pmatrix} \tilde{A} - A + BD^{-1}\mathcal{C}_1 \\ -\tilde{C} - D^{-1}\mathcal{C}_1 \end{pmatrix} - \begin{pmatrix} -\tilde{\mathcal{B}}_{2_0} \\ \tilde{\mathcal{D}}_{12_0} \end{pmatrix} L\mathcal{C}_{2_0} = 0.$$

It then suffices to choose the partial isometries such that $\mathbb{V}^T\mathbb{U} = L$ (whose existence is guaranteed since L is a contraction) and to recall (32), (34) in order to conclude that (37) holds.

Suppose that (27) is strict. Then the left-hand side of (33) is positive definite which implies that the row dimension r of L is equal to that of $\begin{pmatrix} \tilde{A} \\ \tilde{C} \end{pmatrix}$. Similarly, since the left-upper block of (30) is positive definite, the column dimension c of L is $\dim(A)$. Moreover L is a strict contraction. We can then take

$$\mathbb{U} = \begin{pmatrix} L \\ (I - L^T L)^{\frac{1}{2}} \end{pmatrix} \quad \text{and} \quad \mathbb{V} = \begin{pmatrix} I \\ 0 \end{pmatrix}$$

of dimension $(r + c) \times c$ and $(r + c) \times r$ respectively. ■

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