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HORO-TIGHTNESS AND TOTAL (ABSOLUTE) CURVATURES IN HYPERBOLIC SPACES

G. SOLANES AND E. TEUFEL

Abstract. We prove Gauß-Bonnet-type and Chern-Lashof-type formulas for immersions in hyperbolic space. Moreover we investigate the notion of tightness with respect to horospheres introduced by T.E. Cecil and P.J. Ryan. We introduce the notions of top-set and drop-set, and we prove fundamental properties of horo-tightness in hyperbolic spaces.

1. Introduction

For a smooth immersion $f : M \to \mathbb{R}^n$ of a closed manifold M into euclidean space, there are strong relations between the topology of M and the total (absolute) Lipschitz-Killing curvature of f. Firstly, the formula of Gauß-Bonnet relates the Euler characteristic $\chi(M)$ with the total curvature of the immersion. Secondly, the Chern-Lashof inequality states that the total absolute curvature is bigger or equal than $\beta(M)$, the sum of the Betti numbers of M. In case of equality the immersion is called *tight*. Tightness has been studied by N.H. Kuiper and many others till nowadays, cf. [Kui84], [Kui97], [CR85], [CC97]. Recent developments in this context deal with non-compact manifolds and manifolds with boundary (cf. [DK05],[KS11]).

For M smoothly immersed in hyperbolic space $\mathbb{H}^n$, the topology of M and total (absolute) curvatures of the immersion are not so closely related. Neither the total Lipschitz-Killing curvature of M is equal to the Euler characteristic of M, nor the total absolute Lipschitz-Killing curvature of M is in general bounded by the sum of the Betti numbers of M (cf. [LS03],[Sol07]). Despite of this, the following facts are known. For curves in hyperbolic spaces there are generalizations of the Fenchel inequality, cf. [Sze68], [BH74] [Tsu74]. For compact immersed submanifolds M lying inside a ball of radius R, there are lower bounds for the total absolute Lipschitz-Killing curvature in terms of $\beta(M)$ and the radius R, cf. [Teu82], [Teu88], [Oka98]. The Gauß-Bonnet theorem in hyperbolic spaces, especially for hypersurfaces, contains not only the Lipschitz-Killing curvature but also the other mean curvatures of M, cf. [San76], [Sol06]. In recent years there have been investigations on differential geometric quantities on M, other than the Lipschitz-Killing

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curvature, in order to obtain Gauß-Bonnet-type theorems and Chern-Lashof-type inequalities respectively, cf. [Koi03], [BISR10], [IRF06]. Concerning tightness there have been generalizations to hyperbolic spaces by T.E. Cecil and P.J. Ryan ([CR79, CR85]), and others (cf.[Bol82],[Sol07],[BIR10]).

In this paper we continue along these lines by using horospheres in hyperbolic space $\mathbb{H}^n$. Our basic construction is the *support map* (see Definition 2.1) assigning to each submanifold $M \subset \mathbb{H}^n$ the set of its enveloping horospheres. This set lies inside the space $\mathcal{H}$ of horospheres which has a conical structure. We show Gauß-Bonnet-type formulas and Chern-Lashof-type inequalities, and we investigate tightness with respect to horospheres.

In section 3 we use Morse theory applied to height functions defined by pencils of parallel horospheres. This leads to Chern-Lashof-type formulas (Propositions 3.1 and 3.2) and a Gauß-Bonnet-type formula (Proposition 3.3). By different methods these results were obtained in [Koi03], [IRF06] and [BISR10]. Moreover, we get other Gauß-Bonnet-type formulas involving integral geometric terms (Propositions 3.4 and 3.5).

In section 4, motivated by a question of T.E. Cecil and P.J. Ryan in [CR85], we study horo-tightness in hyperbolic spaces. This notion is again based on height functions defined by pencils of parallel horospheres. A smooth immersion $f : M \to \mathbb{H}^n$ of a closed manifold is called *horo-tight* if almost every such height function has $\beta(M)$ critical points along the immersion (see Definition 4.1 and Proposition 4.2). In parts, we follow along the euclidean line in [CR85]. Firstly, we look at the *horospherical two-piece property h*-TPP (see Definition 4.3). This is an analogue of the euclidean two-piece-property introduced by T.F. Banchoff in [Ban71]. Our main result here is Proposition 4.5: if a k-dimensional manifold M is immersed in $\mathbb{H}^n$ with the h-TPP and $k(k+3) < 2n$, then the image of M lies in a euclidean sphere in a horosphere. Then we introduce the notions of *top-sets* and *drop-sets*. For every height function, the associated top-set (resp. drop-set) is the set of points in M where the height function attains its maximum (resp. minimum). In contrast with the euclidean case, a drop-set must not coincide with the top-set of any height function. The reason is that at each point of the immersion there are two different tangent horospheres. One result here is Proposition 4.8: if an immersion is horo-tight, then top-sets and drop-sets are euclidean tight in the respective horospheres.

Next we focus on horo-tight surfaces in $\mathbb{H}^3$. Each point of such a surface is critical for exactly two height functions. For each non-degenerate critical point we have the following possibilities: either it is a relative maximum and a relative minimum (max/min-type), either it is a relative minimum and a saddle point(min/saddle-type), or it is a saddle point for both functions (saddle/saddle-type). In these terms, using the notions of top-cycle and drop-cycle (similar to the euclidean case) the description of horo-tight surfaces is rather complete (cf. Proposition 4.10). For instance, for an embedded horo-tight surface $M \subset \mathbb{H}^3$ the picture is the following. An open region $U_t \subseteq M$ is the boundary of the so-called *h-convex hull* of M (see Definition 4.2) with several disks removed. These disks are contained in horospheres and are euclidean convex there. The points of max/min-type are contained in U_t. A bigger region U_d with $U_t \subseteq U_d \subseteq M$ is contained in the boundary of the so called *h-concave hull* of M with several convex (in the sense above) disks removed. The min/saddle-type points are contained in $U_d \setminus U_t$. The points of saddle/saddle-type are contained in $M \setminus U_d$.

Finally, we complete the discussion of the relations between the different notions of tightness appearing in the literature ([CR79, CR85], [BIR10], [Sol07]). These notions come from several geometric height functions, namely those defined by pencils of parallel horospheres, or pencils of equidistants to a geodesic hyperplane, or pencils of hyperplanes orthogonal to a geodesic. We show that horo-tightness does not imply any of the other notions of tightness.

2. Preliminaries

We use the Lorentz space model for the Hyperbolic Geometry. The model lives in the Lorentz space $\mathbb{R}^{n+1}_1$ with its Lorentz product

$$\langle x, y\rangle = x_1y_1 + x_2y_2 + \cdots + x_ny_n - x_{n+1}y_{n+1}\,.$$

Therein the n-dimensional hyperbolic space $\mathbb{H}^n$ is realized as

$$\mathbb{H}^n = \{x \in \mathbb{R}^{n+1}_1 : \langle x, x\rangle = -1 \ \wedge \ x_{n+1} > 0\}\,,$$

which is the upper half of a two-sheeted hyperboloid with the light cone $\mathcal{C}^n = \{x \in \mathbb{R}^{n+1}_1 : \langle x, x\rangle = 0\}$ as asymptotic cone. The group G of hyperbolic motions of $\mathbb{H}^n$ is given by the subgroup of the Lorentz group leaving invariant $\mathbb{H}^n$.

The infinite or ideal boundary $\mathbb{H}^n_\infty$ of $\mathbb{H}^n$ is realized as the boundary of the projective closure of $\mathbb{H}^n$, or equivalently the boundary of the projective closure of $\mathcal{C}^n$ in the projective enlargement of $\mathbb{R}^{n+1}_1$. It is a $(n-1)$-dimensional sphere and it inherits a conformal structure invariant with respect to G.

Horospheres in $\mathbb{H}^n$ may be seen as limits of distance spheres through some given point the centers of which run on a geodesic towards infinity. In the model, distance spheres are realized by intersections of $\mathbb{H}^n$ with space-like affine hyperplanes. Therefore horospheres are realized by intersections of $\mathbb{H}^n$ with affine hyperplanes parallel to tangent hyperplanes of $\mathcal{C}^n$.

The space $\mathcal{H}$ of horospheres of $\mathbb{H}^n$ is represented by the upper half of the light cone, i.e.

$$\mathcal{H} = \mathcal{C}^n_+ = \{x \in \mathbb{R}^{n+1}_1 : \langle x, x\rangle = 0 \ \wedge \ x_{n+1} > 0\,.\}$$

Indeed, given $\theta \in \mathcal{C}^n_+$, the affine hyperplane $\Theta = \{x \in \mathbb{R}^{n+1}_1 : \langle x, \theta\rangle = -1\}$ is parallel to the tangent hyperplane $T_\theta\mathcal{C}^n_+ = \{x \in \mathbb{R}^{n+1}_1 : \langle x, \theta\rangle = 0\}$ of $\mathcal{C}^n_+$ at θ. Therefore Θ intersects $\mathbb{H}^n$ in a horosphere which we also denote by Θ. Vice versa, given a horosphere Θ as the intersection of $\mathbb{H}^n$ with an affine hyperplane Θ parallel to a hyperplane tangent to $\mathcal{C}^n_+$ along a half light-ray, there exists exactly one θ in this half light-ray such that $\Theta = \{x \in \mathbb{R}^{n+1}_1 : \langle x, \theta\rangle = -1\}$. (In the following we shall always denote horospheres in $\mathbb{H}^n$, or the underlying affine hyperplanes respectively, by capital Greek letters and the vectors in $\mathcal{C}^n_+$ representing them by the corresponding small Greek letters. Moreover, for a horosphere Θ the closed horoball bounded by Θ will be denoted B_Θ.) The correspondence between θ and the hyperplane Θ comes exactly from the polarity relation with respect to the quadric $\pm\mathbb{H}^n \subset \mathbb{R}^{n+1}_1$. The Lorentz product of $\mathbb{R}^{n+1}_1$ induces a degenerated product (isotropic metric) on $\mathcal{C}^n_+$. Despite the degeneracy of the product, there exists an invariant volume form ω on $\mathcal{C}^n_+$ (cf. [San67, San68]). This form is unique up to constant factors. We use the normalization such that the measure of horospheres containing a given point is equal to $O_{n-1}/(n-1)$ (here and in the following O_{n-1} denotes the $(n-1)-$dimensional volume of the unit sphere $\mathbb{S}^{n-1} \subset \mathbb{R}^n$).

The light-rays in the cone $\mathcal{C}^n_+$ represent the pencils of "parallel" horospheres. Two parallel horospheres Θ_1 and Θ_2 touch one another at a point at infinity, and they lie in constant hyperbolic distance to each other. A little computation in the model shows that this distance is equal to $|\ln \lambda|$, where $\lambda \in \mathbb{R}^+$ is given by $\theta_2 = \lambda\theta_1$. The signed distance from Θ_1 to Θ_2 is given by

$$\mathrm{d}(\Theta_1, \Theta_2) = -\ln \lambda\,. \tag{2.1}$$

For fixed Θ_1, as $\lambda \to +\infty$ the horospheres Θ_2 "shrink" to the common point at infinity whereas the signed distance $\mathrm{d}(\Theta_1, \Theta_2) \to -\infty$. On the other side, if $\lambda \to 0$, then Θ_2 expands over the whole $\mathbb{H}^n$ and $\mathrm{d}(\Theta_1, \Theta_2) \to +\infty$.

As geometric objects in $\mathbb{H}^n$, we take smooth (i.e. C^∞-differentiable) immersions $f : M \to \mathbb{H}^n$ of smooth closed (i.e. connected, compact and without boundary) manifolds M.

Our bridge between the point space $\mathbb{H}^n$ and the space of horospheres $\mathcal{C}^n_+$ is the following. Let $N^1 f$ denote the unit normal bundle of the immersion f (i.e. $(x, \xi) \in N^1 f$, iff $x \in M$ and $\xi \in T_{f(x)}\mathbb{H}^n$ with $\xi \perp df(T_x M)$ and $||\xi|| = 1$).

Definition 2.1. We call the map

$$\theta : N^1 f \longrightarrow \mathcal{C}^n_+ \quad , \quad (x, \xi) \mapsto f(x) + \xi \tag{2.2}$$

the *support map of f*.

Then $\theta(x, \xi)$ represents the horosphere $\Theta(x, \xi)$ which is tangent to $f(M)$ at $f(x)$ such that ξ is the inner unit normal of $\Theta(x, \xi)$, i.e. ξ points into its convex side.

Remark 2.1. Our definition of support maps corresponds to the notions in [Sch02] and in [IPS03], [IPRFT05] ("hyperbolic Gauss indicatrix").

The link to the topology of M comes up by Morse theory using height functions. Our height functions are based on pencils of parallel horospheres. In detail: For $u \in \mathbb{H}^n_\infty$, let $h_u : \mathbb{H}^n \to \mathbb{R}$ be the height function with level hypersurfaces given by the parallel horospheres of the pencil through u. As a measuring rod one may use any geodesic through u. In terms of $\mathcal{C}^n_+$, and if we fix a zero-level horosphere θ_0 in the pencil through u, the height of a level horosphere Θ is given by $h_u(\Theta) = -\ln \lambda$, where $\theta = \lambda\theta_0$ (cf. (2.1)). In the following, we use the height functions $h_u \circ f : M \to \mathbb{R}$, which generically are Morse functions, and we apply Morse theory to $h_u \circ f$.

3. Total (absolute) curvatures, Gauss-Bonnet type theorems and Chern-Lashof type inequalities

Let $f : M \to \mathbb{H}^n$ be a smoothly immersed closed hypersurface in $\mathbb{H}^n$. In this section, we consider relations between total (absolute) curvatures of f and the topology of M.

In order to see the connections between critical points of the height functions, their index and the curvature, let us first consider a general height function h on $\mathbb{H}^n$, i.e. a submersion $h : \mathbb{H}^n \to \mathbb{R}$ (defined at least locally). Let $p \in M$ be a critical point of the induced height function $h \circ f$, then some level hypersurface S of h is tangent to $f(M)$ at $f(p)$, and hence $\operatorname{grad} h(f(p)) = \lambda(p)\,\xi(p)$, where $\xi(p)$ is a unit normal vector of f at p. We have

Lemma 3.1.

$$\operatorname{hess}(h \circ f)(p) = \lambda(p)\, II_f(p) - |\lambda(p)|\, f^* II_S(p) \tag{3.1}$$

where $II_f(p)$ is the second fundamental form of f at p with respect to the unit normal $\xi(p)$, and $II_S(p)$ is the second fundamental form of S with respect to its unit normal $\operatorname{grad} h(f(p))/|\operatorname{grad} h(f(p))|$ at $f(p)$.

Proof. At least locally, we can write $df(\operatorname{grad}(h \circ f)) = \operatorname{grad} h - \lambda\,\xi$ with an appropriate function λ. Then, for $X, Y \in T_pM$,

$$\begin{aligned} & \operatorname{hess}(h \circ f)(p)(X,Y) = g(\nabla_X \operatorname{grad}(h \circ f), Y)|_p = \\ = \;& \langle D_{df(X)} \operatorname{grad} h, df(Y)\rangle|_{f(p)} - d\lambda(X)\,\langle \xi, df(Y)\rangle|_{f(p)} - \\ & \qquad -\lambda(p)\,\langle D_{df(X)}\xi, df(Y)\rangle|_{f(p)} = \\ = \;& -|\lambda(p)|\, II_S(df(X), df(Y)) + \lambda(p)\, II_f(X,Y) \end{aligned}$$

with $g = f^*\langle .,. \rangle =$ induced first fundamental form on M, $\nabla =$ Levi-Civita connection of g, and $D =$ usual derivative in $\mathbb{R}^{n+1}_1$. □

If $v_1, \ldots, v_{n-1}$ is a principal basis in T_xM with respect to $(x, \xi) \in N^1M$, we have $d\theta(v_i) = (1 - k_i)\, df(v_i)$ where $k_i = k_i(x, \xi)$ is the corresponding principal curvature. Then the area element of $\theta(N^1M)$ is

$$dA_\theta = |1 - k_1| \cdots |1 - k_{n-1}| dA_{(x,\xi)} \tag{3.2}$$

where $dA_{(x,\xi)}$ denotes the area element of N^1M at (x, ξ). This shows that the support map is an immersion if and only if f has no principal curvature (with respect to any ξ) which is equal to one. Moreover

$$\int_{N^1M} |1 - k_1| \cdots |1 - k_{n-1}| dA_{(x,\xi)} = \int_{\theta(N^1M)} dA_\theta\,. \tag{3.3}$$

Proposition 3.1. *Let $f : M \to \mathbb{H}^n$ be a smoothly immersed closed hypersurface. Assume that M is contained in some ball of radius r. Then*

$$\int_{N^1M} |1 - k_1| \cdots |1 - k_{n-1}| dA_{(x,\xi)} > e^{-(n-1)r} O_{n-1} \beta(M)\,. \tag{3.4}$$

Proof. Assume that M is contained in the ball $B_p(r)$ with radius $r > 0$ and center $p \in \mathbb{H}^n$. Each horosphere Θ tangent to M is interior to some parallel horosphere tangent to $B_p(r)$ leaving it to the convex side. Therefore we have $\langle \theta, -p \rangle \geq e^{-r}$; i.e. θ lies above the plane $\{\langle \theta, -p \rangle = e^{-r}\}$, which intersects $\mathcal{C}^n_+$ in a sphere $S(r)$ of radius e^{-r}. Hence the support image $\theta(N^1M)$ of N^1M lies above $S(r)$.

We take into account the following fact: Let S_1, S_2 be two hypersurfaces in the cone $\mathcal{C}^n_+$ with $\theta_2 = \lambda\theta_1$, $\theta_i \in S_i$. Then the projection $\pi : S_2 \to S_1$ (along the generators of $\mathcal{C}^n_+$) has

Jacobian λ^{1-n}.
In particular, $\pi : \theta(N^1M) \to S(r)$ locally reduces area.
Therefore, applying the co-area formula, cf. [How93], to π gives

$$\int_{\theta(N^1M)} dA_\theta \geq \int_{S(r)} \#(\pi^{-1}(\theta))\, dS(r)_\theta \tag{3.5}$$

Now, we use Differential Topology in particular Morse theory, cf. [Hir94]: For $\theta \in S(r)$, the associated pencil of parallel horospheres defines a height function h_θ. Because of the construction of the support map, the number of critical points of $h_\theta \circ f$ is just the number $\#(\pi^{-1}(\theta))$ of intersection points of $\mathbb{R}^+\theta$ with $\theta(N^1M)$. Generically $h_\theta \circ f$ is a Morse function. Therefore, by the Morse inequalities we have

$$\#(\pi^{-1}(\theta)) \geq \beta(M) \tag{3.6}$$

where $\beta(M)$ is the sum of the Betti numbers of M. This shows that π covers $S(r)$ at least $\beta(M)$ times. Finally, bringing together (3.3), (3.5) and (3.6), we arrive at (3.4). $\square$

Remark 3.1. Equality in (3.4) can never occur: In the proof we used two estimations, first the area-decreasing property of π and secondly the Morse inequalities. Although we may have equality in the second estimation (e.g. for horo-tight immersions, cf. Section 4), we never have equality in the first estimation. Note that for every $x \in M$ not both of the two tangent horospheres can lie in $S(r)$.

Remark 3.2. Proposition 3.1 was obtained with different methods in [Koi03].

To define a support function of f with respect to horospheres, we fix a point $p \in \mathbb{H}^n$. The signed distance $\rho(x,\xi)$ from p to the tangent horosphere $\Theta(x,\xi)$, represented by $\theta(x,\xi) = f(x) + \xi$, defines the *support function* $\rho : N^1f \to \mathbb{R}$ *of* f *based at* p. In terms of $\mathcal{C}^n_+$, the picture is as follows: $T_p\mathbb{H}^n \cap \mathcal{C}^n_+$ represents the pencil of horospheres through p. Some $\theta(x,\xi)_p \in T_p\mathbb{H}^n \cap \mathcal{C}^n_+$ represents the horosphere $\Theta(x,\xi)_p$ through p which is parallel to $\Theta(x,\xi)$. Then $\theta(x,\xi) = \lambda(x,\xi)\,\theta(x,\xi)_p$ and $\rho(x,\xi) = \mathrm{d}(\Theta(x,\xi)_p, \Theta(x,\xi)) = -\ln\lambda(x,\xi)$ (cf. (2.1)). Moreover $\rho(x,\xi) \geq 0$ iff p is in the convex side of $\Theta(x,\xi)$. (Cf. [San67], [San68], [Fil70].)

Proposition 3.2. *Let $f : M \to \mathbb{H}^n$ be a smoothly immersed closed hypersurface. Let ρ be the support function of f based at a fixed point p. Then*

$$\int_{N^1M} e^{(n-1)\rho}|1-k_1|\cdots|1-k_{n-1}|dA_{(x,\xi)} \geq O_{n-1}\beta(M). \tag{3.7}$$

Proof. Horospheres through p are represented by the section $S = T_p\mathbb{H}^n \cap \mathcal{C}^n_+$. The projection $\pi\colon \theta(N^1M) \to S$ has Jacobian $e^{(n-1)\rho}$, where $\rho = \rho(x,\xi)$ is the signed distance from p to the horosphere $\Theta(x,\xi)$ (positive when p is interior).
Therefore, the co-area formula applied to $\pi \circ \theta : N^1M \to S$ gives

$$\int_{N^1M} e^{(n-1)\rho}|1-k_1|\cdots|1-k_{n-1}|dA_{(x,\xi)} = \int_S \#(\pi^{-1}(\theta))\, dS_\theta.$$

Again, by Morse inequalities we have

$$\#(\pi^{-1}(\theta)) \geq \beta(M).$$

Altogether we get (3.7). □

Remark 3.3. For horo-tight hypersurfaces (cf. Section 4), we have equality in (3.7) with $\beta(M)$ being the Betti number with respect to $\mathbb{Z}_2$.

Remark 3.4. Proposition 3.2 was obtained with different methods in [Koi03] and [BISR10].

Remark 3.5. When M is oriented by a unit normal field $\nu(x), x \in M$, then its unit normal bundle N^1M splits into two copies of M, say M_+ with normals ν and M_- with normals $\hat{\nu} = -\nu$. Also its support map splits into two maps θ with $\theta(x) = f(x) + \nu(x)$ and $\hat{\theta}$ with $\hat{\theta}(x) = f(x) + \hat{\nu}(x) = f(x) - \nu(x)$ respectively. Then (3.7) writes

$$\int_M \left(e^{(n-1)\rho}|1-k_1|\cdots|1-k_{n-1}| + e^{(n-1)\hat{\rho}}|1+k_1|\cdots|1+k_{n-1}|\right) dA_x \geq O_{n-1}\beta(M) \quad (3.8)$$

where $k_1, \ldots, k_n$ are the principal curvatures of M with respect to ν, and $\rho, \hat{\rho}$ are the two support functions with base point p associated to $\theta, \hat{\theta}$.

Next we bring signs into game. First, we orient $T_x\mathbb{H}^n$ through x, and similarly we orient $\mathcal{C}^n_+$ through any vector $x \in \mathbb{H}^n$. Given a subspace $V \subset T_\theta C^n_+$ transverse to $\mathbb{R}\theta$ we orient it through θ. For $(x,\xi) \in N^1f$, we choose principal directions $v_1, \ldots, v_{n-1}$ in T_xM with respect to ξ such that $\{df(v_1), \ldots, df(v_{n-1}), \xi\}$ is a positive basis of $T_{f(x)}\mathbb{H}^n$. Then $d\theta(v_i) = (1-k_i)df(v_i)$, and $\{df(v_1), \ldots, df(v_{n-1}), \theta = f(x) + \xi\}$ is a positive basis of $T_\theta\mathcal{C}^n_+$. Thus, θ preserves orientations if and only if $(1-k_1)\ldots(1-k_{n-1}) > 0$. Hence, the signed area of $\theta(N^1M)$ is

$$A^+(\theta(N^1M)) = \int_{N^1M} (1-k_1)\cdots(1-k_{n-1})dA_{(x,\xi)}. \quad (3.9)$$

Proposition 3.3. *Let $f : M \to \mathbb{H}^n$ be a smoothly immersed closed hypersurface, oriented through a unit normal vector field ν. Let θ be the associated support map, i.e. $\theta(x) = f(x) + \nu(x), x \in M$. And let ρ be its support function based at a fixed point p. Then, if n is odd*

$$\int_M e^{(n-1)\rho}(1-k_1)\cdots(1-k_{n-1})dA_x = \frac{O_{n-1}}{2}\chi(M). \quad (3.10)$$

For general n, assume that f is an embedding, so that $f(M) = \partial Q$ for some compact domain Q. If ν points into Q, then

$$(-1)^{n-1}\int_M e^{(n-1)\rho}(1-k_1)\cdots(1-k_{n-1})dA_x = O_{n-1}\chi(Q). \quad (3.11)$$

Proof. The projection $\pi : \mathcal{C}^n_+ \to S,\ S = T_p\mathbb{H}^n \cap \mathcal{C}^n_+$, along the generators of $\mathcal{C}^n_+$ preserves orientations when restricted to hypersurfaces transverse to the light rays. In particular

$\pi : \theta(M) \to S$ preserves orientation, and it has Jacobian $e^{(n-1)\rho}$. Therefore, application of the co-area formula to $\pi \circ \theta : M \to S$ gives

$$\int_M e^{(n-1)\rho}(1-k_1)\cdots(1-k_{n-1})dA_x = \int_S \mu_f(\theta) dS_\theta, \tag{3.12}$$

where $\mu_f(\theta)$ is the algebraic intersection number of $\theta(M)$ with $\mathbb{R}^+\theta$.
For $\theta \in S$, let $h_\theta : \mathbb{H}^n \to \mathbb{R}$ be the height function with level hypersurfaces built by the pencil of horospheres parallel to Θ (i.e. represented by the light-ray $\mathbb{R}^+\theta$ in $\mathcal{C}^n_+$), and with heights given by the signed distance of these horospheres to the point p (positive when p lies in the convex side). Then, because of the construction of the support map θ,

$$\mu_f(\theta) = \sum_{\nabla h_\theta(f(x)) = -\nu(x)} (-1)^i \tag{3.13}$$

where $i = i(x, \theta)$ is the index of x as a critical point of $h_\theta \circ f$. Indeed, Lemma 3.1 gives

$$\begin{aligned}(-1)^i = \text{sign } \det \text{hess}(h_\theta \circ f)|_x = \text{sign } \det(Id - II_f)|_x = \\ = \text{sign}(1-k_1)\cdots(1-k_{n-1})|_x.\end{aligned} \tag{3.14}$$

Alternatively, we compute $\mu_f(\theta)$ by using a diffeomorphism $\Psi : \mathbb{H}^n \to \mathbb{R}^n$ such that $h_\theta = -x_n \circ \Psi$ (for instance, we can take the half-space model with Θ horizontal). Here, Lemma 3.1 gives

$$\begin{aligned}\text{sign} \det \text{hess}(h_\theta \circ f)|_x = \text{sign} \det \text{hess}(-x_n \circ \Psi \circ f)|_x = \\ = \text{sign} \det(-II^e_{\Psi\circ f})|_x = (-1)^{n-1}\text{sign}K_e(x),\end{aligned} \tag{3.15}$$

being $II^e_{\Psi\circ f}$ the euclidean second fundamental form (in the model), and K_e the euclidean Gauß curvature of $\Psi \circ f$ in $\mathbb{R}^n$ with respect to the normal $\Psi_*\nu(x)$. From (3.13), (3.14) and (3.15) we conclude that $(-1)^{n-1}\mu_f(\theta)$ is the degree of the euclidean Gauß map of $\Psi \circ f$.
In case n is odd we get

$$\mu_f(\theta) = \chi(M)/2.$$

This follows from

$$O_{n-1}\,\mu_f(\theta) = \int_M K_e\, dA' = \frac{O_{n-1}}{2}\,\chi(M),$$

where dA' is the area element of $\Psi \circ f$. Here the first equality comes by application of the co-area formula to the euclidean Gauß map, and the second one is just the formula of Gauß-Bonnet (cf. [CL57], [CL58]).

For general n, if $f(M) = \partial Q$ we have

$$\mu_f(\theta) = (-1)^{n-1}\chi(Q),$$

see e.g. [Mor29]. Altogether this proves the result. □

Remark 3.6. Proposition 3.3 was obtained with different methods in [Koi03],[IRF06] and [BISR10].

Remark 3.7. In case M is not orientable, one can apply the previous proposition to the unit normal bundle, which is oriented. If M is orientable, this is equivalent to taking two copies of M, each with a different orientation. In this case, we get for n odd

$$\int_M \big[e^{(n-1)\rho}(1-k_1)\cdots(1-k_{n-1}) \\ + e^{(n-1)\widehat{\rho}}(1+k_1)\cdots(1+k_{n-1})\big]\, dA_x = O_{n-1}\chi(M), \quad (3.16)$$

where $\widehat{\rho}$ is the support function induced by $\widehat{\theta}(x) = f(x) - \nu(x)$.

3.1. **Gauß-Bonnet-type theorems and Integral Geometry.** Let $f : M \to \mathbb{H}^n$ be a smoothly immersed closed hypersurface, oriented by a unit normal vector field ν. Let θ be the associated support map, i.e. $\theta(x) = f(x) + \nu(x), x \in M$.

Given any $\theta \in \mathcal{C}^n_+ \setminus \theta(M)$, we define $\mu^+_f(\theta)$ as the algebraic intersection number of the ray $(1,\infty)\theta = \{\lambda\theta | \lambda > 1\}$ with $\theta(M)$. Let $h_\theta : \mathbb{H}^n \to \mathbb{R}$ be the height function with level hypersurfaces built by the pencil of horospheres parallel to Θ, and normalized such that $h_\theta(\Theta) = 0$. According to (2.1), h_θ gives the signed distance of the points of $\mathbb{H}^n$ to Θ, negative in the convex side of Θ. Now, we consider the (signed) number of critical points of $h_\theta \circ f$ that occur inside $f^{-1}(B_\Theta)$ and such that $\nabla h_\theta = \nu$ (B_Θ = closed convex horoball bounded by Θ); i.e.

$$\mu^+_f(\theta) = \sum_{\substack{(\nabla h_\theta)(f(x)) = \nu(x) \\ h_\theta(f(x)) < 0}} \operatorname{sign}\det\operatorname{hess}(h_\theta \circ f)|_x.$$

Integrating with respect to θ over $\mathcal{C}^n_+$ we get

$$\int_{\mathcal{C}^n_+} \mu^+_f(\theta)\omega_\theta = \int_{\mathbb{S}^{n-1}} \int_0^\infty \mu^+_f(\theta) y_{n+1}^{n-2} dy_{n+1} d\mathbb{S}^{n-1}_v = \\ = \int_{\mathbb{S}^{n-1}} \sum_{y \in \mathbb{R}^+(v,1)\cap\theta(M)} (-1)^i \frac{(y_{n+1})^{n-1}}{n-1} d\mathbb{S}^{n-1}_v = \\ = \frac{(-1)^{n-1}}{n-1} \int_M (1-k_1)\cdots(1-k_{n-1}) dA_x. \quad (3.17)$$

Indeed, the first equality is just rewriting the density ω_θ of horospheres (cf. [San67], [San68]), $\theta = y_{n+1}(v,1), v \in \mathbb{S}^{n-1} = T^1_{(0,\ldots,0,1)}\mathbb{H}^n$). For the second equality we carry out the integration with respect to y_{n+1} for fixed $v \in \mathbb{S}^{n-1}$. This integration runs along the generator $\mathbb{R}^+(v,1)$ of $\mathcal{C}^n_+$. Note that along the generator the function $\mu^+_f(\theta)$ is locally constant with jumps exactly at the intersection points of $\mathbb{R}^+(v,1)$ and $\theta(M)$. The magnitude of the jump at a $\theta(x)$ is equal to $(-1)^i = \operatorname{sign}\det\operatorname{hess}(h_\theta \circ f)|_x$ by definition of μ^+_f. The third equality follows with (3.2), (3.3), taking into account (3.13) and $(y_{n+1})^{n-1}\, d\mathbb{S}^{n-1}_v = dA_\theta$ ($\theta = (y_{n+1})(v,1) = \theta(x)$).

The following question appears: is the number $\mu_f^+(\theta)$ determined by the topology of $M \cap\ f^{-1}(B_\Theta)$? We can answer this question in positive assuming M is embedded, or alternatively replacing M by its oriented cover, cf. (3.19).

Proposition 3.4. *Let $f : M \to \mathbb{H}^n$ be a smoothly embedded closed hypersurface, bounding a domain Q and oriented by its inner normals ν. Then,*

$$(-1)^{n-1}\int_M (1-k_1)\cdots(1-k_{n-1})dA_x = (n-1)\int_{\mathcal{C}_+^n} (\chi(Q\cap B_\Theta)-\chi(\Theta\cap Q))\,\omega_\theta. \quad (3.18)$$

Proof. Consider the domain $B_\Theta \cap Q$, which has piecewise smooth boundary $(M\cap B_\Theta)\cup(\Theta\cap Q)$. We can deform $S = \Theta\cap Q$ to a new hypersurface S' so that $\partial S = \partial S'$, and $(M\cap B_\Theta)\cup S'$ is a regular hypersurface bounding a domain R homotopic to $B_\Theta \cap Q$. Moreover, S' can be constructed so that the unit normal ν' on S' (obtained by transporting the orientation of S to S') fulfills $\langle \nu', \nabla h_\theta\rangle \leq 0$ everywhere on S. Let us consider the situation in the Poincaré half-space model. We can assume Θ is horizontal in the model, so that ∇h_θ is vertical and points downwards. Then, the degree of the Gauß map γ of ∂R (in the model) is $\chi(R)$. On the other hand, this degree can be computed as the signed number of preimages of the vector $(0, \ldots, 0, 1) \in \mathbb{S}^{n-1}$. Then we have

$$\chi(B_\Theta\cap Q) = \chi(R) = \deg\gamma = \chi(S') + \mu_f^+(\theta) = \chi(\Theta\cap Q) + \mu_f^+(\theta). \quad (3.19)$$

We finish by applying equation (3.17). □

Remark 3.8. For horo-convex Q (see Example 1 in 4.1), we have that $\chi(Q\cap B_\Theta)-\chi(\Theta\cap Q)$ is equal to 1 if $f(M) \subset B_\Theta$, otherwise it is equal to 0. Hence (3.18) gives

$$(-1)^{n-1}\int_M (1-k_1)\cdots(1-k_{n-1})dA_x = (n-1)\int_{f(M)\subset B_\Theta} \omega_\theta = (n-1)\,m_f \quad (3.20)$$

where m_f is the measure of horospheres having $f(M)$ entirely in their convex sides.

Proposition 3.5. *Let $f : M \to \mathbb{H}^n$ be a smoothly embedded closed hypersurface. Then*

$$\int_M (1+\sigma_2+\cdots+\sigma_{2k})dA_x = \frac{n-1}{2}\int_{\mathcal{C}_+^n} \chi(f(M)\cap B_\Theta)\omega_\theta \quad (3.21)$$

where $\sigma_i = \sum_{1\leq j_1\leq\cdots\leq j_i\leq n-1} k_{j_1}\cdots k_{j_i}$, and $2k \leq n-1 \leq 2k+1$.

Proof. To prove the formula, we consider $M' = N^1 f$. Then

$$\mu_{f'}^+(\theta) = \sum_i (-1)^i c_i^+(h_\theta\circ f)$$

where $c_i^+(h_\theta\circ f)$ is the number of critical points of index i of $h_\theta\circ f$ restricted to $M\cap\ f^{-1}(B_\Theta)$. By Morse theory we know

$$\mu_{f'}^+(\theta) = \chi(f(M)\cap B_\Theta),$$

and we get formula (3.21) by using equation (3.17). □

Remark 3.9. If n is even, then $\chi(f(M)\cap B_\Theta) = \chi(f(M)\cap\Theta)/2$ and formula (3.21) coincides with a result of [GNS04].
If n is even and Q is horo-convex, then formula (3.21) writes

$$\int_M (1+\sigma_2+\cdots+\sigma_{n-2})dA_x = \frac{n-1}{2}\int_{\mathcal{C}^n_+} \chi(f(M)\cap B_\Theta)\,\omega_\theta = \frac{n-1}{2}\,(m_2+m_1)$$

where m_2 is the measure of horospheres containing Q in the interior, and m_1 denotes the measure of the horospheres intersecting Q. This together with (3.20) gives

$$\int_{f(M)\cap\Theta\neq\emptyset} \omega_\theta = \frac{2}{n-1}\int_M (\sigma_1+\sigma_3+\cdots+\sigma_{n-2})\,dA_x,$$

which coincides with one of the results in [GNS04].

Remark 3.10. For an immersion f, Proposition 3.5 remains true with the integrand on the right-hand side of (3.21) replaced by $\chi(f^{-1}(f(M)\cap B_\Theta))$.

4. Horo-tightness

Let $f : M \to \mathbb{H}^n$ be a smooth immersion of a closed manifold M. In this section, we investigate the horo-tightness of f by applying Morse theory (e.g. [Hir94]) to the height functions $h_u \circ f$, $u \in \mathbb{H}^n_\infty$.

Proposition 4.1. *For almost all $u \in \mathbb{H}^n_\infty$ the height function $h_u \circ f$ is a Morse function.*

Proof. We take the *hyperbolic Gauß map* $\nu : N^1 f \to \mathbb{H}^n_\infty$ which assigns to $(x,\xi) \in N^1 f$ the endpoint of the geodesic half ray emanating form $f(x)$ in direction ξ. Then $x \in M$ is a critical point of $h_u \circ f$ if and only if $u = \nu(x,\xi)$ for some $(x,\xi) \in N^1 f$. Moreover in this case, x is non-degenerate if and only if ν is regular at (x,ξ). Then the assertion follows by application of the Theorem of Morse-Sard to ν. □

Remark 4.1. For the notion *hyperbolic Gauß map* $\nu : N^1 f \to \mathbb{H}^n_\infty$ and its applications, cf. [Eps86], [Bry87], [Kob89], [IPS03].

We begin with some nomenclature and data from Morse theory. Let φ be a Morse function on M, let $\mu_k(\varphi)$ denote the number of critical points of φ of index k and $\mu(\varphi)$ the number of critical points of φ. If $\beta_k(M;F)$ is the k-th Betti number of M with respect to a field F, then there are the Morse inequalities ([MC69])

$$\beta_k(M;F) \leq \mu_k(\varphi). \tag{4.1}$$

Let $\gamma(M)$ be the Morse number of M, i.e. $\gamma(M) = \min\{\mu(\varphi) : \varphi \text{ is a Morse function on } M\}$, then

$$\sum_{k=0}^{\dim M} \beta_k(M;F) =: \beta(M;F) \leq \gamma(M).$$

If $\mu(\varphi) = \gamma(M)$, then φ is polar, i.e. φ has exactly one local maximum and one local minimum ([Mor60]).
In particular, if M is a closed 2-dimensional surface, then $\gamma(M) = \beta(M;\mathbb{Z}_2) = 4 - \chi(M)$

([Kui62]). Moreover, if M is a closed 2-dimensional surface and φ a Morse function on M, then $\mu(\varphi) = \gamma(M)$ iff φ is polar ([CR85] Prop. 5.6).
Let φ be a Morse function on a closed manifold M. For a given field F, $\mu_k(\varphi, r) = \beta_k(\varphi, r; F)$ for all $r \in \mathbb{R}$ and all integers k if and only if the map on homology $H_*(M_{\leq r}(\varphi); F) \to H_*(M; F)$ induced by inclusion is injective for all $r \in \mathbb{R}$ ([MC69]), where $\mu_k(\varphi, r)$ is the number of critical points of φ of index k which lie in $M_{\leq r}(\varphi) = \{x \in M : \varphi(x) \leq r\}$ and $\beta_k(r, \varphi; F)$ is the kth Betti number of $M_{\leq r}(\varphi)$ over F.

Definition 4.1. The smooth immersion $f : M \to \mathbb{H}^n$ is called *horo-tight* if for every closed horoball B_Θ, the induced homomorphism

$$H_*(f^{-1}(B_\Theta)) \longrightarrow H_*(M)$$

in Cech homology with $\mathbb{Z}_2$ coefficients is injective.

Remark 4.2. Equivalent to this definition of horo-tightness is the property that

$$H_*(f^{-1}(\mathbb{H}^n \setminus \text{int } B_\Theta)) \longrightarrow H_*(M)$$

in Cech homology with $\mathbb{Z}_2$ coefficients is injective for every closed horoball B_Θ.

Remark 4.3. The use of Cech homology is motivated by its continuity property which is necessary to eliminate the requirement that the horoball be determined by a non-degenerate height function. For smooth manifolds, Cech homology agrees with singular homology, [CR85].

Through Morse theory, Definition 4.1 is equivalent to

Proposition 4.2. *A smooth immersion $f : M \to \mathbb{H}^n$ of a closed manifold M is horo-tight if and only if $\mu(h_u \circ f) = \beta(M; \mathbb{Z}_2)$ for almost all $u \in \mathbb{H}^n_\infty$.*

Proof. Analogous the euclidean case, [CR85] Theorem 5.4. □

4.1. **Examples. Ex. 1.** A subset $Q \subset \mathbb{H}^n$ is called *horo-convex* if every point $p \in \partial Q$ belongs to a horosphere bounding a closed horoball containing Q. Boundaries of compact horo-convex bodies in $\mathbb{H}^n$ are horo-tight. Every smooth horo-tight hypersurface homeomorphic to a sphere is of this type (cf. [BIR10]).

Ex. 2. Let M be homeomorphic to $\mathbb{S}^k \times \mathbb{S}^{n-1-k}$, embedded in $\mathbb{H}^n$ as rotation-symmetric hypersurface as follows: We fix a $(n-1-k)$-dimensional hyperbolic plane L as axis of rotation. In a $(n-k)$-dimensional hyperbolic plane through L we choose the boundary of an horo-convex body not intersecting L as $(n-1-k)$-dimensional meridian surface. Rotating around L, with k-dimensional distance spheres in $(k+1)$-dimensional hyperbolic planes orthogonal and complementary to L as orbits, we get the embedded M. Then M is horo-tight in $\mathbb{H}^n$.
To this: Let $u \in \mathbb{H}^n_\infty$, then $x \in M$ is a critical point of $h_u|_M$ if and only if there exists $\xi \in N^1_x M$ such that $u \in \nu(x, \xi)$. M is rotation-symmetric, therefore any normal geodesic of M intersects the axis of rotation L or is parallel to it in the common $(n-k)$-dimensional hyperbolic plane. This shows that the critical points of $h_u|_M$ are exactly the critical points of $h_u|_{M \cap E}$, where E is the $(n-k)$-dimensional hyperbolic plane determined by L and

u. Now $M \cap E$ consists of two copies of the horo-convex meridian surface. Therefore $h_u|_{M\cap E}$ generically has exactly 4 critical points, and hence h_u generically has exactly 4 critical points. The non-vanishing Betti numbers of M with respect to the field $\mathbb{R}$ are $\beta_0 = 1, \beta_k = 1, \beta_{n-1-k} = 1$ and $\beta_{n-1} = 1$. Therefore $h_u|_M$ generically has exactly $\gamma(M) = 4$ critical points, namely one minimum point, one maximum point and two saddle points, hence M is horo-tight.
Specially, a rotation-symmetric embedded torus in $\mathbb{H}^3$ with horo-convex meridian curve is horo-tight.

Ex. 3. Orientable closed surfaces M_g of genus $g \geq 1$, horo-tightly immersed in $\mathbb{H}^3$.
First we construct a horo-tightly embedded torus by starting with a distance sphere, removing two opposite disk caps and connecting their boundary circles by a cylinder type "wormhole". More precisely, we start with a geodesic line ℓ and fix a point $O \in \ell$. Then we choose a hyperbolic plane E through ℓ. Inside E we draw the distance circle of hyperbolic radius r and center O. Moreover inside E we draw a horocycle with endpoint on a geodesic half-ray emanating from O orthogonal to ℓ, and with distance ϵ to ℓ , $0 < \epsilon < r$. These two curves intersect in two points, and rounding near them produces a closed h-convex curve in E, which we use as meridian curve for a rotational surface in $\mathbb{H}^3$ with rotational axis ℓ. This way we get a horo-tightly embedded torus depending on the parameters r and ϵ. The entrances to the wormhole, i.e. essentially the removed disk caps, behave in terms of r and ϵ as follows: If r and ϵ decrease, then the apex angle at O of the cone of geodesic half-rays from O through the disk caps becomes smaller and smaller. If r increases, then the apex angle becomes bigger and bigger.
Second, using the above "wormhole" construction for g properly chosen geodesics through O as axis of rotation, and keeping r and ϵ sufficiently small, we get an orientable closed surface M_g horo-tightly immersed in $\mathbb{H}^3$.

Ex. 4. Veronese manifolds. The embedding of the n-dimensional real projective space into the unit sphere of $\mathbb{R}^N$ ($N = \frac{n(n+3)}{2}$) as Veronese manifold is substantial, tight and taut, cf. [CR85] Chpt. 1.7. Example 7.3 and Chpt. 1.9.. Therefore, in view of Proposition 4.9, embedding the euclidean unit sphere or spheres of arbitrary radii isometrically into $\mathbb{H}^N$ as distance spheres (or as horospheres in the limit), yields horo-tight and substantial Veronese manifolds in hyperbolic spaces.

Definition 4.2. (1) The *h-convex hull* of $f(M)$ in $\mathbb{H}^n$ is defined as

$$H_+f(M) = \bigcap_{f(M)\subset B_\Theta} B_\Theta.$$

(2) The *h-concave hull* of $f(M)$ in $\mathbb{H}^n$ is defined as

$$H_-f(M) = \bigcap_{f(M)\subset \mathbb{H}^n\setminus \mathrm{int}(B_\Theta)} (\mathbb{H}^n \setminus \mathrm{int}(B_\Theta)).$$

Direct consequences are

- $f(M_{max}) \subseteq \partial H_+f(M)$
- $f(M_{min}) \subseteq \partial H_-f(M)$

- $M_{max} \subseteq M_{min}$
- $f(M_{min}) \cap \partial H_+ f(M) = f(M_{max})$,

where $M_{max}, and M_{min} \subset M$ are the points which appear as a maximum point resp. a minimum point for some $h_u \circ f$.

4.2. **The horospherical two-piece property.**

Definition 4.3. A smooth immersion $f : M \to \mathbb{H}^n$ of a closed manifold M is said to have the *horospherical two-piece property* (h-TPP) if $f^{-1}(B_\Theta)$ is connected for every closed horoball B_Θ.

Remark 4.4. Taking into account Proposition 4.3 and Morse theory, equivalent to this definition of the h-TPP is the property that $f^{-1}(\mathbb{H}^n \setminus \mathrm{int} B_\Theta)$ is connected for every closed horoball B_Θ.

Proposition 4.3. *(1) Let $f : M \to \mathbb{H}^n$ be a smooth immersion of a closed manifold M. If f is horo-tight, then f has the h-TPP.*
(2) Let $f : M \to \mathbb{H}^n$ be a smooth immersion of a closed manifold M. Then f has the h-TPP if and only if every non-degenerate height function $h_u \circ f$ is polar.
(3) A h-TPP immersed 2-dimensional closed surface M in $\mathbb{H}^3$ is horo-tight.
(4) Let $f : M \to \mathbb{H}^n$ be a smooth immersion of a closed manifold M. Then the h-TPP is equivalent to ($f(M_{relmax}) \subseteq \partial H_+ f(M) \wedge f(M_{relmin}) \subseteq \partial H_- f(M)$).

Proof. The proofs run similar to the the euclidean situation. In particular, cf. [CR85] Thm. 5.9, Thm. 5.11 and Cor. 5.12 for (1), (2) resp. (3). The proofs of the euclidean case use Prop. 5.13 and Lemma 5.14, which hold in our hyperbolic situation just as in the euclidean one. Finally, (4) is just a reformulation of (2). □

Proposition 4.4. *Let $f : M \to \mathbb{H}^n$ be a smooth immersion of a closed manifold M. Then the following are equivalent.*

(1) f has the h-TPP.
(2) every local support horosphere of $f(M)$ is a global support horosphere of $f(M)$.
(3) every local extremum of a non-degenerate height function $h_u \circ f$ is an absolute extremum.

Proof. The proofs run similar to the euclidean case, cf. [CR85] Thm. 5.17. □

Proposition 4.5. *Let $f : M \to \mathbb{H}^n$ be a smooth immersion of a closed k-dimensional manifold M in $\mathbb{H}^n$ with the h-TPP. If $\frac{k(k+3)}{2} < n$, then $f(M)$ lies in a euclidean sphere in a horosphere, or equivalent, $f(M)$ lies in a distance sphere in a hyperbolic hyperplane. In particular, $f(M)$ is not substantial in $\mathbb{H}^n$ with respect to horospheres or hyperbolic hyperplanes respectively.*

Proof. Let us fix a $p \in M$ with $f(p) \in \partial H_+ f(M)$. Then p is an absolute maximum point of some, w.l.o.g non-degenerate, $h_v \circ f$. Hence there exists a $\xi \in N_p^1 f$ with $v = \nu(p, -\xi)$.

Moreover p is an absolute minimum point of $h_u \circ f$, w.l.o.g. non-degenerate, with $u = \nu(p, \xi)$. Therefore,

$$\mathrm{hess}(h_u \circ f)(p) = -II_f(p,\xi) + f^* II_h(p,\xi) = -II_f(p,\xi) + I_f(p)$$

is positive definite, cf. Lemma 3.1 (3.1) (note: $\lambda(p) = -1$), where $II_f(p,\xi)$ is the second fundamental form of f at p with respect to the unit normal ξ, and $II_h(f(p),\xi)$ is the second fundamental form of the critical level horosphere at $f(p)$ with respect to its inner unit normal ξ, i.e. $II_h(f(p),\xi) = f_* I_f(p)$.
Let V be the vector space of symmetric bilinear forms on T_pM. Consider the linear map $\mathrm{osc} : N_pf \to V$, defined by

$$\mathrm{osc}(\zeta)(X,Y) = II_f(p,\zeta)(X,Y) = I_f(p)(A_\zeta X, Y) \quad , X, Y \in T_pM,$$

where A_ζ is the shape operator of f at p with respect to the normal ζ. The dimension of N_pf is $n-k$, while the dimension of V is $\frac{k(k+1)}{2}$. Because

$$\frac{k(k+3)}{2} < n \quad , \text{i.e.} \quad \frac{k(k+1)}{2} < n-k\,,$$

the kernel of osc is non-trivial. Hence there exists $\eta \in N_p^1 f$ with $A_\eta = 0$. (W.l.o.g. we can assume $A_\xi \neq 0$.) (W.l.o.g we can assume $\xi \perp \eta$, otherwise a positive normalization factor comes into game which however does not disturb the relevant positive definiteness.) Then, for $\zeta(t) = \cos t \cdot \xi + \sin t \cdot \eta \in N_p^1 M$, ($t \in [-\pi, \pi]$), we get

$$A_{\zeta(t)} = \cos t \cdot A_\xi + \sin t \cdot A_\eta = \cos t \cdot A_\xi\,.$$

Because p is a non-degenerate absolute minimum point for $h_u \circ f$, the eigenvalues of $\mathrm{hess}(h_u \circ f)(p)$ are positive, hence the principal curvatures of A_ξ are less than one. Therefore, $h_{u(t)} \circ f$ with $u(t) = \nu(f(p), \zeta(t))$ has a non-degenerate relative minimum at p for all $t \in (-\pi/2, \pi/2)$. Because f has the h-TPP, p is the absolute minimum point of $h_{u(t)} \circ f$ for all $t \in (-\pi/2, \pi/2)$. Therefore, $f(M)$ does not intersect the open horoballs $\mathrm{int} B_{\Theta(t)}$ for $t \in (-\pi/2, \pi/2)$, where $\Theta(t)$ is the horosphere through p with inner normal $\zeta(t)$. On the other hand, p is the absolute maximum point of $h_v \circ f = h_{u(-\pi)} \circ f$. Hence $f(M)$ is contained in the horoball $B_{\Theta(-\pi)}$. All in all, we have

$$f(M) \subset D := B_{\Theta(-\pi)} \setminus \bigcup_{-\pi/2 < t < \pi/2} \mathrm{int}\ B_{\Theta(t)}.$$

Now, the families of horospheres $\{\Theta(t)\}$ with $-\pi \le t \le -\pi/2$ resp. $\pi/2 \le t \le \pi$ induce two foliations of D by parts of horospheres. We claim, that $f(M)$ must be contained in one of the leaves of each of these two foliations. Otherwise there exists some t_0, say $-\pi < t_0 < -\pi/2$, such that $f(M)$ lies on both sides of $\Theta(t_0)$. At the point p, we have the shape operators $A_\eta = 0$, and $A_{-\xi}$ with principal curvatures bigger than one (maximum point). Moreover $-\xi$ points into $B_{\Theta(t_0)}$. Therefore, through a small geodesic parallel translation of $\Theta(t_0)$ along the geodesic ray from $f(p)$ in direction $-\xi$ we get a horosphere Θ' such that $f^{-1}(\mathbb{H}^n \setminus B_{\Theta'})$ is not connected. But this is contrary to the h-TPP, and hence proves the claim. Altogether, $f(M)$ lies in the intersection of two horospheres, hence $f(M)$ lies in a euclidean sphere in a horosphere. $\square$

4.3. **Horo-tight immersions.** Let $f : M \to \mathbb{H}^n$ be a smooth immersion of a closed manifold M.

Definition 4.4. (1) For $u \in \mathbb{H}^n_\infty$ the *top-set* $\Omega(u)$ is defined as

$$\Omega(u) = \{x \in M : (h_u \circ f)(x) = \max\{(h_u \circ f)(y) : y \in M\}\}\,.$$

(2) For $u \in \mathbb{H}^n_\infty$ the *drop-set* $\omega(u)$ is defined as

$$\omega(u) = \{x \in M : (h_u \circ f)(x) = \min\{(h_u \circ f)(y) : y \in M\}\}\,.$$

Proposition 4.6. *If $f : M \to \mathbb{H}^n$ is a horo-tight immersion, then*

$$M_{relmax} = M_{max}\ ,\ f(M_{relmax}) = f(M_{max}) \subseteq \partial H_+ f(M)\ ,\ \textit{and}$$
$$M_{relmin} = M_{min}\ ,\ f(M_{relmin}) = f(M_{min}) \subseteq \partial H_- f(M)\,.$$

Proof. f is supposed to be horo-tight, therefore $h_u \circ f$ is polar for almost all $u \in \mathbb{H}^n_\infty$. Hence $M_{relmax} = M_{max}$ and $M_{relmin} = M_{min}$. □

Lemma 4.1. *Let $f : M \to \mathbb{H}^n$ be a smooth immersion of a closed manifold M.*

(1) Suppose $\Omega(u)$ is a top-set of f and Σ is a closed euclidean half-space in the maximum-level horosphere $\Theta_{max}(u)$ of $h_u \circ f$, such that $\Omega(u) \cap f^{-1}(\Sigma)$ is contained in an open set U in M. Then there exists a $v \in \mathbb{H}^n_\infty$ and a real number r such that

$$\Omega(u) \cap f^{-1}(\Sigma) \subset M_{>r}(v) \subset M_{\geq r}(v) \subset U.$$

Moreover, the same holds for all v' sufficiently near v.

(2) Suppose $\omega(u)$ is a drop-set of f and Σ is a closed euclidean half-space in the minimum-level horosphere $\Theta_{min}(u)$ of $h_u \circ f$, such that $\omega(u) \cap f^{-1}(\Sigma)$ is contained in an open set U in M. Then there exists a $v \in \mathbb{H}^n_\infty$ and a real number r such that

$$\omega(u) \cap f^{-1}(\Sigma) \subset M_{<r}(v) \subset M_{\leq r}(v) \subset U.$$

Moreover, the same holds for all v' sufficiently near v.

Proof. The top-set case. In the Poincaré half-space model we see the following euclidean picture: $\Theta_{max}(u)$ is a euclidean sphere tangent at a point u to the euclidean hyperplane representing $\mathbb{H}^n_\infty$. In the model, the $(n-2)$-dimensional euclidean hyperplane $\partial\Sigma$ in $\Theta_{max}(u)$ is represented by a $(n-2)$-dimensional sphere in $\Theta_{max}(u)$ through u. Therefore the tangent cone of $\Theta_{max}(u)$ along $\partial\Sigma$ is a euclidean rotational-symmetric cone which lies tangent to the hyperplane representing $\mathbb{H}^n_\infty$. Thus, picking an enveloping sphere of this cone nearby $\Theta_{max}(u)$ on the proper side, we get a horosphere Θ_v through some $v \in \mathbb{H}^n_\infty$ such that Θ_v separates the compact sets $f(\Omega(u) \cap f^{-1}(\Sigma))$ and $f(M \setminus U)$. Moreover $(f(\Omega(u) \cap f^{-1}(\Sigma))) \cap B_{\Theta_v} = \emptyset$ and $f(M \setminus U) \subset \operatorname{int} B_{\Theta_v}$. Then, $h_v \circ f$ with $r = h_v(\Theta_v)$ leads to the assertion in the top-set case.
The proof in the drop-set case runs analogously. □

Proposition 4.7. *Let $f : M \to \mathbb{H}^n$ be a smooth immersion of a closed manifold M. If f has the h-TPP, then every top-set and drop-set of f has the euclidean TPP with respect to its immersion f into its respective level horosphere.*

Proof. The top-set case. Suppose that a top-set $\Omega(u)$ of f has not the euclidean TPP with respect to the immersion f into its respective level horosphere $\Theta_{max}(u)$. Then there exists a closed euclidean half-space Σ in $\Theta_{max}(u)$ such that $\Omega(u)\cap f^{-1}(\Sigma)$ is not connected. Moreover, there are disjoint open sets U_1 and U_2 in M with $U_1 \cup U_2 \supseteq \Omega(u) \cap f^{-1}(\Sigma)$, each having non-empty intersection with $\Omega(u) \cap f^{-1}(\Sigma)$. According to Lemma 4.1 with $U = U_1 \cup U_2$, there exists a $v \in \mathbb{H}^n_\infty$ and a real number r such that

$$\Omega(u) \cap f^{-1}(\Sigma) \subset M_{\geq r}(v) \subset U\,.$$

Thus $M_{\geq r}(v) \cap U_i$ is non-empty for $i = 1, 2$, hence $M_{\geq r}(v)$ is not connected. This implies that $h_v \circ f$ has at least two local maxima. Hence it is not polar, contradicting according to Proposition 4.3 the assumption that f has the h-TPP.
The proof in the drop-set case runs analogously. □

Proposition 4.8. *Let $f : M \to \mathbb{H}^n$ be a smooth immersion of a closed manifold M. If f is horo-tight, then every top-set resp. drop-set of f is euclidean tight with respect to its immersion f into the respective level horosphere.*

Proof. Similar to the euclidean case, [CR85] Theorem 7.11. □

Let U^{n-1} denote a complete totally-umbilical $(n-1)$-dimensional submanifold in $\mathbb{H}^n$, i.e. either a distance sphere, a horosphere, or an equidistant to a hyperbolic hyperplane (which is a hyperbolic space with curvature between 0 and -1), or an $\mathbb{H}^{n-1}$.

Proposition 4.9. *Let $f : M \to U^{n-1} \subset \mathbb{H}^n$ be a smooth immersion of a closed manifold M. Then f is horo-tight in $\mathbb{H}^n$ if and only if f is taut in U^{n-1} (w.r.t. distance functions to points).*

Proof. We use the Poincaré model where the involved U^{n-1} etc. are euclidean spheres in the model space. Then we see: For $p \in U^{n-1}$ the geodesic γ in $\mathbb{H}^n$ through p orthogonal to U^{n-1} intersects $\mathbb{H}^n_\infty$ in $u, \bar{u}$. Then because of the rotation symmetry around γ, $h_u|_{U^{n-1}}$ and $h_{\bar{u}}|_{U^{n-1}}$ are the distance function in U^{n-1} to p. On the other hand, for $u \in \mathbb{H}^n_\infty$ there exists a geodesic γ starting at u and intersecting U^{n-1} orthogonally in q. Then $h_u|_{U^{n-1}}$ is the distance function in U^{n-1} to q. From this the assertion follows directly by the definitions of horo-tight and taut. □

4.4. **Horo-tight surfaces.** Let $f : M \to \mathbb{H}^3$ be a smooth immersion of a closed 2-dimensional manifold M. For every $p \in M$ there are two opposite horospheres tangent to f in $f(p)$. Generically, we see the following types of non-degenerate critical points, or equivalently the contact-types with the two tangent horospheres there, and their relations to the principal curvatures k_i, k_j $(i, j \in \{1, 2\}, i \neq j)$ of the surface there.

max/min-contact	$(\ k_i, k_j < -1\ ) \vee (1 < k_i, k_j)$
min/min-contact	$-1 < k_i, k_j < 1$
min/saddle-contact	$(k_i < -1 < k_j < 1) \vee (-1 < k_i < 1 < k_j)$
saddle/saddle-contact	$k_i < -1 < 1 < k_j$

Remark 4.5. If f has the h-TPP, then there are no points of type min/min-contact. To this, let $x \in M$ be a relative minimum point of $h_u \circ f$ (wlog. non-degenerate) with associated minimum level horosphere $\Theta_{min}(u)$. The geodesic g in $\mathbb{H}^3$ through $f(x)$ orthogonal to $\Theta_{min}(u)$ has the endpoints u and say v in $\mathbb{H}^3_\infty$. Suppose now that x is of type min/min-contact, then x is a relative minimum point also of $h_v \circ f$. Because of the h-TPP, x is the absolute minimum point for both, $h_u \circ f$ and $h_v \circ f$. Hence $f(x)$ is the only intersection point of g and $f(M)$. But M is closed, hence a generic geodesic has an even number of intersection points with $f(M)$. Therefore there exists an $x' \in M$ with $f(x) = f(x')$ and such that the immersed surface touches itself at $f(x)$ orthogonal to g. But then $h_u \circ f$ has at least two minimum points, namely x and x', violating the h-TPP of f.

Lemma 4.2. *Let $f : M \to \mathbb{H}^3$ be a horo-tight smooth immersion of a closed surface M.*

(1) For any top-set $\Omega(u)$,
  - *(a) if the euclidean convex hull $H_e f(\Omega(u))$ of $f(\Omega(u))$ in its respective maximum level horosphere $\Theta_{max}(u)$ is 0-dimensional or 1-dimensional, then f is an embedding of $\Omega(u)$ onto $H_e f(\Omega(u))$.*
  - *(b) if $H_e f(\Omega(u))$ is 2-dimensional, then $\partial H_e f(\Omega(u)) \subset f(\Omega(u))$, and f is an embedding on $\Gamma = f^{-1}(\partial H_e f(\Omega(u)))$. Further, if Γ separates M, then $\Omega(u)$ is the closure of one of the components of $M \setminus \Gamma$, and f embeds $\Omega(u)$ onto the disk $H_e f(\Omega(u))$.*

(2) For any drop-set $\omega(u)$,
  - *(a) if the euclidean convex hull $H_e f(\omega(u))$ of $f(\omega(u))$ in its respective minimum level horosphere $\Theta_{min}(u)$ is 0-dimensional or 1-dimensional, then f is an embedding of $\omega(u)$ onto $H_e f(\omega(u))$.*
  - *(b) if $H_e f(\omega(u))$ is 2-dimensional, then $\partial H_e f(\omega(u)) \subset f(\omega(u))$, and f is an embedding on $\gamma = f^{-1}(\partial H_e f(\omega(u)))$. Further, if γ separates M, then $\omega(u)$ is the closure of one of the components of $M \setminus \gamma$, and f embeds $\omega(u)$ onto the disk $H_e f(\omega(u))$.*

Proof. The top-set case *(1)*. *Ad (a).* If $H_e f(\Omega(u))$ is 0-dimensional, then $f(\Omega(u))$ is a point. If $H_e f(\Omega(u))$ is 1-dimensional, then it is a segment $[Q_1, Q_2] \subset \Theta_{max}(u)$. Now f is horo-tight, hence it has the h-TPP, so $[Q_1, Q_2] = f(\Omega(u))$ (cf. proof of Proposition 4.7). In both cases, f is a continuous map of $\Omega(u)$ into $\Theta_{max}(u)$ with the euclidean TPP (cf. Proposition 4.7), and such that $f(\Omega(u))$ is euclidean convex. Hence, by [CR85] Lemma 7.13, f is injective on $\Omega(u)$.

Ad (b). Every support line ℓ of $H_e f(\Omega(u))$ in the euclidean plane $\Theta_{max}(u)$ is also a support line of $f(\Omega(u))$. For such a support line ℓ, the set $f(\Omega(u)) \cap \ell$ has dimension 0 or 1. If it has dimension 0, then $f(\Omega(u)) \cap \ell \in f(\Omega(u))$. If it has dimension 1, then the boundary points Q_1, Q_2 of the compact interval $f(\Omega(u)) \cap \ell$ are in $f(\Omega(u))$. Moreover, because $f(\Omega(u))$ has the euclidean TPP in $\Theta_{max}(u)$ (cf. Proposition 4.7), we claim that all of $[Q_1, Q_2]$ is in $f(\Omega(u))$. To see this, assume there is a point $P \in (Q_1, Q_2)$ but $P \notin f(\Omega(u))$. By the compactness of $f(\Omega(u))$ there exists an open neighborhood of P disjoint to $f(\Omega(u))$. Therefore, an appropriate line parallel to ℓ cuts $f(\Omega(u))$ in at least three pieces, but this contradicts the TPP of $f(\Omega(u))$; hence the claim is proved. All in all $\partial H_e f(\Omega(u)) \subset$

$f(\Omega(u))$.
By the proof of the first part of [CR85] Lemma 7.15, we see that f is an embedding on Γ. Finally, suppose Γ separates M. Then, because f has the h-TPP, one of the components of $M \setminus \Gamma$, call it V, is mapped into the convex disk $H_e f(\Omega(u))$. For otherwise, $M_{<r}(u)$, $r = \max(h_u \circ f)$, is disconnected in violation of the h-TPP. Since f is an immersion, $f(V)$ is an open subset of $\mathrm{int} H_e f(\Omega(u))$. We claim that $f(V)$ is also closed in $\mathrm{int} H_e f(\Omega(u))$. To prove the claim, let y be a limit point of $f(V)$ in $\mathrm{int} H_e f(\Omega(u))$, and let $y_1, y_2, \ldots$ be a sequence in $f(V)$ converging to y. Choose $x_i \in V$ such that $f(x_i) = y_i$. The sequence x_i has a limit point x in the compact set $\bar{V} = V \cup \Gamma$. By continuity, $f(x) = y$, and x cannot be in Γ, since $f(\Gamma)$ is disjoint from $\mathrm{int} H_e f(\Omega(u))$. Thus $x \in V$, and so $y \in f(V)$ and $f(V)$ is closed in $\mathrm{int} H_e f(\Omega(u))$. Therefore $f(V) = \mathrm{int} H_e f(\Omega(u))$, and $f(\Omega(u))$ is a convex set in $\Theta_{max}(u)$. Again by [CR85] Lemma 7.13, f embeds $\bar{V}$ onto $H_e f(\Omega(u))$.

The proof in the drop-set case *(2)* runs analogously. □

Remark 4.6. At horo-tight surfaces in $\mathbb{H}^3$ we see the following picture: We can close a "wormhole" by an euclidean convex *horospherical cap* tangent to $\partial H_+ f(M)$ along the respective boundary component of $f(M_{max})$. Moreover, in the entrance area of a "wormhole" it is possible to put in a *horospherical drop* tangent to the wormhole along the respective boundary component of $f(M_{min})$.

We see from Lemma 4.2 that if $H_e f(\Omega(u))$ is 2-dimensional and $H_e f(\Omega(u)) \neq f(\Omega(u))$, then $\Gamma = f^{-1}(\partial H_e f(\Omega(u)))$ does not separate M. Such a curve Γ is called a *top-cycle* of M. Accordingly, if $H_e f(\omega(u))$ is 2-dimensional and $H_e f(\omega(u)) \neq f(\omega(u))$, then $\gamma = f^{-1}(\partial H_e f(\omega(u)))$ does not separate M. Such a curve γ is called a *drop-cycle* of M.

Lemma 4.3. *Let M be a closed smooth surface horo-tightly immersed in $\mathbb{H}^3$. Then the numbers of top-cycles and drop-cycles are finite.*

Proof. Analogous to the euclidean case, [CR85] Lemma 7.17. □

Lemma 4.4. *Let M be a closed smooth surface, horo-tightly immersed in $\mathbb{H}^3$. Then, for any drop-set $\omega(u)$, $f^{-1}(\partial H_e f(\omega(u)))$ does not separate M.*

Proof. Let's assume that $H_e f(\omega(u))$ is 2-dimensional and $f^{-1}(\partial H_e f(\omega(u)))$ separates M. Then, by Lemma 4.2 we have $f(\omega(u)) = H_e f(\omega(u))$. Pick a point $x \in \mathrm{int}\, \omega(u)$. The geodesic in $\mathbb{H}^3$ through $f(x)$ orthogonal to $\Theta_{min}(u)$ has the endpoints u and say v in $\mathbb{H}^3_\infty$. Then, the height function $h_v \circ f$ (wlog. non-degenerated) has a relative minimum at x. M is closed, hence the geodesic half-ray from $f(x)$ to v intersects $f(M)$ in a further point different from $f(x)$ (cf. Remark 4.5). Therefore, x is not the absolute minimum point for $h_v \circ f$, hence $h_v \circ f$ is not polar. But, by Proposition 4.3 this contradicts the horo-tightness of f. □

Lemma 4.5. *Let M be a closed smooth surface, horo-tightly immersed in $\mathbb{H}^3$. Then*

$$\alpha_t(M) \leq \alpha_d(M), \tag{4.2}$$

where $\alpha_t(M)$ and $\alpha_d(M)$ are the numbers of top-cycles and drop-cycles of M respectively.

Proof. Let Γ be a top-cycle of M. Then $f(\Gamma)$ bounds a euclidean convex disk D in the respective maximum level horosphere $\Theta_{max}(u)$. For $y \in D$ let $v(y)$ denote the endpoint of the geodesic through y orthogonal to $\Theta_{max}(u)$ different from u. Then $v(.)$ maps D bijective onto a disk D' in $\mathbb{H}^3_\infty$. We consider the set-valued map w from D' into $f(M)$, defined by $w(u') = H_e f(\omega(u'))$, $u' \in D'$. Suppose now that there are no drop-cycles for $u' \in D'$. Then, taking into account Lemma 4.2, $w(D')$ is a disk in $f(M)$ bounded by $f(\Gamma)$. Hence Γ separates M, contradicting the property of being a top-cycle. $\square$

Proposition 4.10. *Let $f : M \to \mathbb{H}^3$ be a smooth horo-tight immersion of a closed surface M not homeomorphic to the sphere $\mathbb{S}^2$. Then*

(1) M is the union of two non-empty disjoint open sets U_t and V_t and a finite number of top-cycles $\Gamma_1, ..., \Gamma_k$ such that: The set U_t is embedded onto the complement in $\partial H_+ f(M)$ of a finite number of horospherical-plane closed euclidean convex disks $D_1, ..., D_k$, where $\Gamma_i = f^{-1}(\partial D_i)$ for $1 \leq i \leq k$. Moreover, the points of type max/min-contact are contained in U_t, and the points of type min/saddle- or saddle/saddle-contact respectively are contained in V_t.

(2) M is the union of two non-empty disjoint open sets U_d and V_d and a finite number of drop-cycles $\gamma_1, ..., \gamma_\ell$ such that: The set U_d is embedded onto the complement in $\partial H_- f(M)$ of the intersection of $\partial H_- f(M)$ with a finite number of horospherical-plane closed euclidean convex disks $D_1, ..., D_\ell$, where $\gamma_i = f^{-1}(\partial D_i)$ for $1 \leq i \leq \ell$. Moreover, the points of type max/min- or min/saddle-contact are contained in U_d, and the points of type saddle/saddle-contact are contained in V_d.

Proof. Ad (1). There must exist some top-cycles, for otherwise f embeds M onto $\partial H_+ f(M)$, and M is a sphere. Let $\Gamma_1, ..., \Gamma_k$ be the finite number of top-cycles, and let $D_1, ..., D_k$ be the associated horospherical-plane closed euclidean convex disks such that $\Gamma_i = f^{-1}(\partial D_i)$ for $1 \leq i \leq k$. We know that $\partial H_+ f(M)$ is the union of the $H_e f(\Omega(u)$ as $\Omega(u)$ ranges over the top-sets of M. If a point y is in the set

$$W_t = \partial H_+ f(M) \setminus (D_1 \cup ... \cup D_k) \,,$$

then by Lemma 4.2, $y \in f(\Omega(u)) = H_e f(\Omega(u))$ for an appropriate $u \in \mathbb{H}^3_\infty$, such that $H_e f(\Omega(u))$ has dimension 0 or 1, or such that $H_e f(\Omega(u))$ is 2-dimensional and f embeds $\Omega(u)$ onto $H_e f(\Omega(u))$. Thus f is an embedding on the open set $U_t = f^{-1}(W_t)$. With

$$V_t = M \setminus (U_t \cup \Gamma_1 \cup ... \cup \Gamma_k) \,,$$

we get the first part of the assertion.

Now, let $x \in U_t$. Then x is the absolute maximum point of some $h_u \circ f$. Let v denote the endpoint of the geodesic in $\mathbb{H}^3$ through $f(x)$ and orthogonal to $\Theta_{max}(u)$ different from u. Then x is an absolute minimum point of $h_v \circ f$. By Proposition 4.3 any height function is polar, hence all relative maximum points lie in U_t. By Remark 4.5, there are no points of type min/min-contact. This shows the second part of the assertion.

Ad (2). There must exist some drop-cycles, for otherwise M is a sphere. Let $\gamma_1, ..., \gamma_\ell$ be the finite number of drop-cycles, and let $D_1, ..., D_\ell$ be the associated horospherical-plane closed euclidean convex disks such that $\gamma_i = f^{-1}(\partial D_i)$ for $1 \leq i \leq \ell$. We know that

$\partial H_- f(M)$ is contained in the union of the $H_e f(\omega(u))$ as $\omega(u)$ ranges over the drop-sets of M. If a point y is in the set

$$W_d = \partial H_- f(M) \setminus ((D_1 \cap \partial H_- f(M)) \cup ... \cup (D_\ell \cap \partial H_- f(M))),$$

then by Lemma 4.2, $y \in f(\omega(u)) = H_e f(\omega(u))$ for an appropriate $u \in \mathbb{H}^3_\infty$, such that $H_e f(\omega(u))$ has dimension 0 or 1, or such that $H_e f(\omega(u))$ is 2-dimensional and f embeds $\omega(u)$ onto $H_e f(\omega(u))$. Thus f is an embedding on the open set $U_d = f^{-1}(W_d)$. With

$$V_d = M \setminus (U_d \cup \gamma_1 \cup ... \cup \gamma_\ell),$$

we get the first part of the assertion.
Now, let $x \in U_d$. Then x is the absolute minimum point of some some $h_u \circ f$, hence $f(x) \in \partial H_- f(M)$. By Proposition 4.3 any height function is polar. Therefore all relative minimum points are absolute minimum points. Hence all relative minimum points lie in U_d. By Remark 4.5, there are no points of type min/min-contact. This shows the second part of the assertion. □

Lemma 4.6. *Let $f : M \to \mathbb{H}^3$ be a smooth horo-tight immersion of a closed surface M.*

(1) If for a top-set $\Omega(u)$, the euclidean convex hull $H_e f(\Omega(u))$ is 2-dimensional, then f embeds $\Omega(u)$ onto a horospherical-plane closed euclidean convex disk with k disjoint open euclidean convex disks removed, where $0 \le k \le \beta_1(M; \mathbb{Z}_2)$.

(2) If for drop-set $\omega(u)$, the euclidean convex hull $H_e f(\omega(u))$ is 2-dimensional, then f embeds $\omega(u)$ onto a horospherical-plane closed euclidean convex disk with k disjoint open euclidean convex disks removed, where $0 \le k \le \beta_1(M; \mathbb{Z}_2)$.

Proof. Ad (1). By Proposition 4.8 the top-set $\Omega(u)$ of f is euclidean tight with respect to its immersion f into the respective level horosphere. Hence, $f(\Omega(u))$ is a closed euclidean convex disk in Θ_{max} with a possibly infinite number of disjoint open euclidean convex disks removed. The boundary of each disk removed carries a generator for $H_1(\Omega(u))$. By horo-tightness, the homomorphism $H_1(\Omega(u)) \to H_1(M)$ is injective, and so the number of disks removed satisfies $0 \le k \le \beta_1(M; \mathbb{Z}_2)$.

Ad (2). The proof runs as in the top-set case. □

Proposition 4.11. *Let $f : M \to \mathbb{H}^3$ be a smooth horo-tight immersion of a closed surface M not homeomorphic to the sphere $\mathbb{S}^2$.*

(1) Let $\alpha_t(M)$ be the number of top-cycles of M. Then

$$2 \le \alpha_t(M) \le 2 - \chi(M). \tag{4.3}$$

Moreover, if $\alpha_t(M) = 2 - \chi(M)$, then the top-cycles come in pairs, each joined by a topological cylinder.

(2) Let $\alpha_d(M)$ be the number of drop-cycles of M. Then

$$2 \le \alpha_d(M) \le 2 - \chi(M). \tag{4.4}$$

Moreover, if $\alpha_d(M) = 2 - \chi(M)$, then the drop-cycles come in pairs, each joined by a topological cylinder.

Proof. Ad (1). Similar to the euclidean case, [CR85] Theorem 7.20.
Ad (2). The proof runs as in the top-cycle case. □

4.5. **Height functions in hyperbolic spaces.** We compare some species of height functions in hyperbolic spaces relating to tightness. Firstly, we take our h_u, i.e. height functions defined by pencils of parallel horospheres. Secondly, we look at height functions defined by the signed distance to a hyperbolic hyperplane (i.e. with equdistants as level hypersurfaces). And thirdly, we look at height functions defined by pencils of hyperplanes orthogonal to a geodesic. These sets of height function are homogeneous spaces with respect to G, and their dimensions are $n-1$, n and $2(n-1)$ respectively. The associated concepts of tightness, according to Definition 4.1 or Proposition 4.2 we call horo-tightness, e-tightness and g-tightness respectively. (In [CR79], [CR85], [BIR10], e-tightness is called H-tightness.) The relations between these concepts are as follows.
By an approximation argument, namely through equidistants to horospheres, [CR79], [CW72] showed

$$e - \text{tightness} \Rightarrow \text{horo-tightness.} \tag{4.5}$$

Similar, by approximating hyperbolic hyperplanes through equidistants, one gets

$$e - \text{tightness} \Rightarrow g - \text{tightness.} \tag{4.6}$$

Now, geodesically convex bodies are not necessarily horo-convex. Also, geodesically convex bodies in $\mathbb{H}^2$ having some geodesic segment contained in their boundaries are not e-tight. Hence

$$g - \text{tightness} \quad \not\Rightarrow \quad \text{horo} - \text{tightness} \tag{4.7}$$

$$g - \text{tightness} \quad \not\Rightarrow \quad e - \text{tightness.} \tag{4.8}$$

Moreover

$$\text{horo} - \text{tightness} \quad \not\Rightarrow \quad g - \text{tightness,} \tag{4.9}$$

which can be seen by the following counterexample: We start with a geodesic ℓ in a hyperbolic plane E in $\mathbb{H}^3$. We choose a distance circle c in E with $c \cap \ell = \emptyset$, and a point $p \in c$ such that the tangent geodesic t of c at p separates c and the reflection image c' of c at ℓ, and moreover such that c and c' both lie in the concave side of the horocycle σ which is tangent to c at p. Then, rotating c around ℓ in $\mathbb{H}^3$, we obtain a horo-tightly embedded torus M in $\mathbb{H}^3$ (cf. Ex. 2). By the construction, $p \in \partial H_- M$ and $p \notin \partial H_+ M$. By an appropriate choice of the distance of c to ℓ, we can arrange the principal curvature of M along the rotation orbit at p with respect to the outer unit normal ξ to be greater than 0 but near 0. Now, we dent M around p in direction ξ to an embedded torus $\tilde{M}$ such that in the dented part, the two principal curvatures with respect to the outer unit normals remain less than -1 and between 1 and -1 respectively, and such that at some $\tilde{p}$, the second principal curvature becomes negative. Then, in the dented part, the points of $\tilde{M}$ have contact with tangent horospheres of type min/saddle. Hence $\tilde{M}$ is horo-tight. At $\tilde{p}$, there is elliptic contact of $\tilde{M}$ and its hyperbolic tangent plane $T_{\tilde{p}}\tilde{M}$. By the construction, $\tilde{M}$ lies

on both sides of $T_{\tilde{p}}\tilde{M}$, hence $\tilde{M}$ cannot be g-tight.
At last

$$\text{horo} - \text{tightness} \quad \not\Rightarrow \quad e - \text{tightness}, \tag{4.10}$$

because otherwise (4.6) implies g-tightness, which contradicts (4.9).

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