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# Orthogonal Polar Spaces and Unitals

Markus Stroppel

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## Abstract

We use the fact that certain classical unitals contain affine parts of orthogonal polar spaces in order to determine their full groups of automorphisms.

We investigate a connection between suitable unitals and quadrics. This connection helps to determine the full group of automorphisms for hermitian unitals over commutative fields, or over quaternion fields if the form is trace-valued and the involution is the standard one, see Theorem 7.1 below. Thus our present approach extends the results in [14], where the characteristic two case and some of the non-commutative cases have been left open.

One of our tools will be the reconstruction of a hermitian form on a vector space  $V$  from its restriction to the diagonal  $\{(v, v) \mid v \in V\}$ , cf. Section 3; here we extend an old result by Jacobson [8], see 3.4.

## 1 Hermitian Forms and Unitals

Let  $K$  be a (not necessarily commutative) field, and let  $\sigma: K \rightarrow K: x \rightarrow \bar{x}$  be an involution: that is, an additive map with the properties  $\overline{\bar{y}} = y$  and  $\overline{\bar{x}} = x$ . In general, the set of fixed points of  $\sigma$  is not closed under multiplication. However, we will consider a special case:

**1.1 General Assumptions.** We will assume throughout that the *norm*  $N(x) := x\bar{x}$  and *trace*  $T(x) := x + \bar{x}$  are contained in the center  $Z$  of  $K$ , for each  $x \in K$ . Then the norms and traces lie in the subfield  $F := \{z \in Z \mid \bar{z} = z\}$  of  $Z$ , and the degree of the extension  $Z/F$  is at most 2. We also assume  $\sigma \neq \text{id}$ .

Let  $k$  be a non-degenerate  $\sigma$ -hermitian form of Witt index 1 on some left vector space  $W$  of dimension 3 over  $K$ . Moreover, assume that  $k$  is trace-valued (“tracique” in the sense of [4, I§ 10]), i.e.,  $k(v, w)$  is a trace (and thus lies in  $F$  by our assumption) for all  $v, w$ . By a suitable choice of coordinates we identify  $W$  with the space  $K^3$  of rows with 3 entries from  $K$  such that  $k$  is given by  $k((a, b, c), (x, y, z)) = a\bar{y} + b\bar{x} + c\bar{z}$ , see 1.5 below.

**1.2 Definition.** We put  $U := \{Kv \mid v \in K^3 \setminus \{0\}, k(v, v) = 0\}$ . For each secant  $\ell$  (i.e., each line  $\ell$  of the projective plane over  $K$  meeting  $U$  in more than one point) we call the trace  $b_\ell := \{X \in U \mid X \text{ lies on } \ell\}$  a *block* of  $U$  and denote by  $B$  the set of all these blocks. The incidence geometry  $(U, B)$  is called the (*hermitian*) *unital* corresponding to  $\sigma$ .

**1.3 Example.** The line  $S$  joining  $K(1, 0, 0)$  and  $K(0, 1, 0)$  induces on  $U$  the block

$$b_S = \{K(x, 1, 0) \mid \bar{x} = -x\} \cup \{K(1, 0, 0)\}.$$

**1.4 Remark.** The norm  $N(x) := x\bar{x}$  is a multiplicative quadratic form over  $F$ : in fact, using  $N(y) \in F \leq Z$  we compute  $N(xy) = (xy)(\overline{xy}) = xy\bar{y}\bar{x} = x\bar{x}y\bar{y} = N(x)N(y)$ . Thus  $(K, N)$  is an associative composition algebra over  $F$ , cf. [9, Thm. 7.5, p. 444]. By the generalized Hurwitz Theorem (see [9, p. 447, p. 450]), the field  $K$  is either commutative, or a quaternion field over  $F$ , and the involution  $\sigma$  is the standard involution of the composition algebra (fixing  $F$  pointwise, and inducing  $-\text{id}$  on  $\text{Pu}(K) := 1^\perp$ ). If  $K$  is commutative, we deal with a separable quadratic extension  $K/F$ , and  $\sigma$  generates the Galois group.

**1.5 Remarks.** The assumption on the explicit formula for  $k$  means no loss of generality: in fact, there exists a non-trivial vector  $u$  with  $k(u, u) = 0$  because  $k$  has Witt index 1, we find  $v$  such that  $k(u, v) \neq 0$  because the form is non-degenerate, and we may pick  $w \in \{u, v\}^\perp$  to obtain a basis  $u, v, w$  for the vector space. With respect to this basis, we have  $k((a, b, c), (x, y, z)) = a\bar{y} + b\bar{x} + cs\bar{z}$ , with  $s = k(w, w) \neq 0$ . Our assumption that  $k$  is trace-valued yields  $s \in F$ .

Since we are only interested in geometries described by the orthogonality relation defined by  $k$ , we may (and will) replace  $k$  by  $ks^{-1}$  (and  $v$  by  $s^{-1}v$ ) to obtain that  $k$  has the form that we assume in 1.1. Note that the form  $ks^{-1}$  is still  $\sigma$ -hermitian, because  $s$  is fixed by  $\sigma$  and belongs to the center of  $K$ .

Our assumption that norms and traces lie in the center is a weaker form of the condition that the fixed points of  $\sigma$  lie in the center of  $K$ . The latter condition would exclude non-commutative fields with characteristic 2.

If  $\text{char } K \neq 2$  or if  $K$  is commutative then every hermitian form is trace-valued. Over each quaternion field with characteristic 2 suitable choices of  $s$  in 1.5 yield hermitian forms (with respect to the standard involution) that are trace-valued and also forms that are not; see [5, p. 73]. The set  $U$  defined in 1.2 is contained in a line if the form is not trace-valued.

## 2 Semi-similitudes

Let  $D \leq E$  be (not necessarily commutative) fields, and assume that there exists an anti-automorphism  $\varphi$  of  $D$  such that  $\varphi^2 = \text{id}$  (we allow  $\varphi = \text{id}$  here). Let  $V$  be a left vector space over  $E$ ; we denote by  ${}_D V$  the vector space over  $D$  obtained by restricting the choice of scalars. Consider a semilinear map  $\alpha: V \rightarrow V$  with companion  $\mu \in \text{Aut}(E)$ , i.e., such that  $(xv + w)^\alpha = x^\mu v^\alpha + w^\alpha$  holds for all  $x \in E$  and all  $v, w \in V$ . Note that  $\alpha$  will be a semilinear endomorphism of  ${}_D V$  only if  $D$  is invariant under  $\mu$ .

Consider a non-degenerate  $\varphi$ -hermitian form  $d: {}_D V \times_D {}_D V \rightarrow D$ . The map  $\alpha: V \rightarrow V$  is called a semi-similitude of  $d$  if there exist  $\gamma \in \text{Aut}(D)$  and  $s \in D^\times$  such that  $d(v^\alpha, w^\alpha) = d(v, w)^\gamma s$  holds for all  $v, w \in V$ . Using non-degeneracy of  $d$  one easily sees that  $\alpha$  is additive.

**2.1 Lemma.** *Let  $\alpha: V \rightarrow V$  be semilinear with companion  $\mu \in \text{Aut}(E)$ . If there exist  $\gamma \in \text{Aut}(D)$  and  $s \in D^\times$  such that  $d(v^\alpha, w^\alpha) = d(v, w)^\gamma s$  holds for all  $v, w \in V$  (i.e., if  $\alpha$  is a semi-similitude of  $d: {}_D V \times_D {}_D V \rightarrow D$ ) then  $D^\mu = D$  and the restriction of  $\mu$  to  $D$  equals  $\gamma$ .*

*Proof.* Pick  $v \in V \setminus \{0\}$ . The semi-similitude  $\alpha$  maps  $Dv = v^{\perp\perp}$  onto  $(v^\alpha)^{\perp\perp} = Dv^\alpha$ . Now  $x^\mu v^\alpha = (xv)^\alpha \in Dv^\alpha$  yields  $x^\mu \in D$  for each  $x \in D$ . Comparing  $d((xv)^\alpha, w^\alpha) = d(xv, w)^\gamma s = x^\gamma d(v, w)^\gamma s$  and  $d((xv)^\alpha, w^\alpha) = d(x^\mu v^\alpha, w^\alpha) = x^\mu d(v^\alpha, w^\alpha) = x^\mu d(v, w)^\gamma s$  we obtain  $x^\gamma = x^\mu$ .  $\square$

### 3 Recovering Hermitian Forms

We will study affine polar spaces later on; these will be obtained by an interpretation of  $K^2$  as affine space over a subfield of  $K$ . For a discussion of the extendibility of automorphisms of the affine polar space to the unital, we need some basic information about hermitian forms over quaternion fields or separable quadratic extensions. The core of our method dates back to [8], see 3.4.

The results 3.2 and 3.3 of the present section will be used in 4.9 below in order to reconstruct the hermitian form  $k$ .

**3.1 Lemma.** *Let  $K$  be either a quaternion field or a separable quadratic extension over  $F$ . For each  $c \in K \setminus F$  with  $T(c) = 1$  we have  $c - \bar{c} \neq 0$ . Putting  $j_c := (c - \bar{c})^{-1}$  we obtain that  $\varphi: x \mapsto T(x) - j_c c T(x) + j_c T(c x)$  describes the orthogonal projection from  $K$  onto  $F \oplus Fc$ . If  $K$  is commutative this means  $\varphi = \text{id}$ . If  $K$  is not commutative we choose  $p \in \{1, c\}^\perp \setminus \{0\}$  and obtain  $x^\varphi + p^{-1}(p x)^\varphi = x$  for each  $x \in K$ .*

*Proof.* The polar form  $\beta_N$  is not degenerate on  $C := F + Fc$ , and  $K$  is the orthogonal direct sum of  $C$  and  $C^\perp$ . For any  $p \in C^\perp \setminus \{0\}$  we have  $C^\perp = pC = Cp$ . The relations  $\bar{c} = 1 - c$  and  $p \perp C$  imply  $pu = \bar{u}p$  for each  $u \in C$ . Now straightforward computations yield the formulae, as claimed.  $\square$

**3.2 Proposition.** *Let  $V$  be a vector space over a quaternion field  $H$ , and let  $h: V \times V \rightarrow H$  be a non-degenerate  $\sigma$ -hermitian form. Consider the quadratic map<sup>1</sup>  $v \mapsto h(v, v)$  and the corresponding polar form  $f(v, w) = h(v, w) + h(w, v) = h(v, w) + \overline{h(v, w)} = T(h(v, w))$ . As in 3.1 we choose  $c \in H$  with  $T(c) = 1$  and  $p \in \{1, c\}^\perp \setminus \{0\}$ , and put  $j_c := (c - \bar{c})^{-1}$ .*

a. *For all  $v, w \in V$  we have*

$$\begin{aligned} h(v, w) &= f(v, w) - j_c c f(v, w) + j_c f(c v, w) \\ &\quad + p^{-1} (f(p v, w) - j_c c f(p v, w) + j_c f(c p v, w)). \end{aligned}$$

b. *The form  $f$  is non-degenerate.*

c. *Every  $H$ -semilinear semi-similitude  $\alpha \in \text{GO}(f)$  is a semi-similitude of  $h$ .*

*Explicitly, assume that there exist  $\gamma \in \text{Aut}(F)$  and  $s \in F^\times$  such that  $f(v^\alpha, w^\alpha) = f(v, w)^\gamma s$  and let  $\mu$  be the companion automorphism of  $\alpha$ . Then  $\mu$  induces  $\gamma$  on  $F$  and  $h(v^\alpha, w^\alpha) = h(v, w)^\mu s$ .*

*Proof.* The first assertion follows from 3.1 and the observation

$$\begin{aligned} h(v, w) &= h(v, w)^\varphi + p^{-1}(p h(v, w))^\varphi \\ &= f(v, w) - j_c c f(v, w) + j_c f(c v, w) \\ &\quad + p^{-1} (f(p v, w) - j_c c f(p v, w) + j_c f(c p v, w)). \end{aligned}$$

If  $f$  were degenerate we would find  $w \in V \setminus \{0\}$  such that  $f(v, w) = 0$  holds for all  $v \in V$ . Then  $f(c v, w) = 0 = f(p v, w) = f(c p v, w)$  yields  $h(v, w) = 0$  for all  $v \in V$ , contradicting the fact that  $h$  is non-degenerate.

In order to prove assertion c let  $\alpha \in \text{GO}(f)$  be  $H$ -semilinear. The companion  $\mu$  leaves  $F$  invariant and restricts to  $\gamma$  on  $F$ , see 2.1. Using 3.1 together with the fact that the pair  $(c^\mu, p^\mu)$  satisfies the conditions imposed on  $(c, p)$  we see that  $h(v^\alpha, w^\alpha) = h(v, w)^\mu s$ , as claimed.  $\square$

<sup>1</sup> In general, the range of this quadratic map will consist of the set of fixed points of  $\sigma$ , and will not be contained in the center of  $H$ . However, the values of the polar form lie in  $F$  by 1.1.

**3.3 Lemma.** Let  $G$  be a commutative field, and let  $g: V \times V \rightarrow G$  be a  $\sigma$ -hermitian form. Consider the quadratic form  $v \mapsto g(v, v)$  and the corresponding polar form  $f(v, w) = g(v, w) + g(w, v)$ . As in 3.1 we choose  $c \in G \setminus F$  with  $\bar{c} = 1 + c$  and put  $j_c := (c - \bar{c})^{-1}$ .

- a. For all  $v, w \in V$ , we have  $g(v, w) = f(v, w) - j_c c f(v, w) + j_c f(c v, w)$ .
- b. Every  $G$ -semilinear semi-similitude  $\alpha$  of  $f$  is a semi-similitude of  $g$ : explicitly, we have  $g(v^\alpha, w^\alpha) = g(v, w)^\gamma s$  if  $f(v^\alpha, w^\alpha) = f(v, w)^\gamma s$ .

*Proof.* The first assertion follows from 3.1 and the observation that  $g(v, w)^\sigma = f(v, w) - j_c c f(v, w) + j_c f(c v, w)$ .

The companion automorphism  $\gamma$  of  $\alpha$  is an automorphism of  $G$  that leaves  $F$  invariant, thus it normalizes (and then centralizes) the Galois group  $\{\text{id}, \sigma\}$  of the extension  $G/F$ . Now another straightforward calculation yields the last assertion, in each one of the cases.  $\square$

**3.4 Remarks.** Let  $H$  be a quaternion field with  $\text{char } H \neq 2$ . Then [8, p.266] contains an alternative to 3.1 and 3.3, as follows.

For every nontrivial pure element  $p_1$  and every  $p_2 \in \{1, p_1\}^\perp \setminus \{0\}$ , we have  $p_1 p_2 = -p_2 p_1$  and  $p_1 p_2 \in \{1, p_1, p_2\}^\perp \setminus \{0\}$ . We call  $(p_1, p_2)$  a *Hamilton pair*.

If  $a \in \text{Pu}(H) \setminus \{0\}$  then the  $F$ -linear endomorphism  $\sigma_a: x \mapsto a^{-1} \bar{x} a$  of  $H$  is the reflection with axis  $a^\perp$ . For each Hamilton pair  $(p_1, p_2)$ , we obtain  $-2 \text{id}_H = \sigma_1 - \sigma_{p_1} - \sigma_{p_2} - \sigma_{p_1 p_2}$ . Finally, we have for all  $v, w \in V$ :

$$2h(v, w) = f(v, w) + p_1^{-1} f(p_1 v, w) + p_2^{-1} f(p_2 v, w) + (p_1 p_2)^{-1} f(p_1 p_2 v, w).$$

## 4 Affine Quadrics

We resume notation from 1.1 and consider the affine plane obtained by deleting the points on some line  $S$  from the projective plane over  $K$ ; we want to take a secant for  $S$ . By Witt's Theorem the unitary group  $\text{PU}(k)$  acts two-transitively on  $U$ . Thus it acts transitively on the set  $B$  of blocks and we may without loss of generality assume  $S = K(1, 0, 0) + K(0, 1, 0)$  as in 1.3.

Passing to inhomogeneous coordinates, we will identify the point  $K(x, y, 1)$  with  $(x, y) \in K^2$ . This affine plane also carries the richer structure of an affine space

$$\mathbb{A} := \left( K^2, \{a + Fv \mid a \in K^2, v \in K^2 \setminus \{0\}\} \right)$$

over the smaller (and commutative) field  $F$ , which will be utilized in the sequel. The dimension of this affine space is  $\dim_F(K^2) = 2 \dim_F K \in \{4, 8\}$ , see 1.4.

The projective hull  $\mathbb{P}$  of  $\mathbb{A}$  may be described by homogeneous coordinates  $F(x, y, s)$ , with  $x, y \in K$  and  $s \in F$ . We will identify the hyperplane  $\mathbb{A}_\infty := \mathbb{P} \setminus \mathbb{A}$  at infinity with the projective space consisting of all  $F$ -subspaces of  $S$ .

We interpret the affine part  $U \setminus b_S$  of the unital in terms of  $\mathbb{A}$ :

**4.1 Lemma.** a. The set  $A := U \setminus b_S = \{(x, y) \mid x\bar{y} + y\bar{x} + 1 = 0\}$  is the affine part of the quadric  $Q$  described by the quadratic form

$$\begin{aligned} q: K^2 \times F \rightarrow F: (x, y, s) &\mapsto k((x, y, s), (x, y, s)) \\ &= x\bar{y} + x\bar{\bar{y}} + s^2 = x\bar{y} + y\bar{x} + s^2, \end{aligned}$$

with polar form  $\beta_q((x, y, s), (u, v, t)) = x\bar{v} + v\bar{x} + y\bar{u} + u\bar{y} + 2st$ .



b. For all  $v, w \in K^2 \times F$ , we obtain  $q(v) = k(v, v)$  and

$$\beta_q(v, w) = k(v, w) + k(w, v) = k(v, w) + \overline{k(v, w)}.$$

The restriction  $q|_S$  is non-degenerate; its Witt index is  $\dim_F K$ . □

**4.2 Lemma.** Every block  $b$  of the unital that meets  $b_S$  induces an affine subspace  $X_b$  of  $\mathbb{A}$  such that the completion of  $X_b$  in  $\mathbb{P}$  is totally singular with respect to  $q$ . That subspace either is a line (if  $K$  is commutative) or has dimension 3 (if  $K$  is a quaternion field).

*Proof.* Let  $K(a, b, 1)$  and  $K(x, y, 1)$  be affine points on a block of the unital; then  $a\bar{b} + b\bar{a} + 1 = 0 = x\bar{y} + y\bar{x} + 1$ . The joining line  $K(a, b, 1) \oplus K(x, y, 1)$  meets  $S$  in the point  $K(a - x, b - y, 0)$ . This point belongs to the unital precisely if  $0 = (a - x)\overline{(b - y)} + (b - y)\overline{(a - x)} = a\bar{b} + b\bar{a} + x\bar{y} + y\bar{x} - (a\bar{y} + y\bar{a} + b\bar{x} + x\bar{b})$ . This yields  $a\bar{y} + y\bar{a} + b\bar{x} + x\bar{b} = a\bar{b} + b\bar{a} + x\bar{y} + y\bar{x} = -2$ , and  $\beta_q((a, x, 1), (b, y, 1)) = a\bar{y} + y\bar{a} + b\bar{x} + x\bar{b} + 2 = 0$ .

We have thus proved that any two affine points on the considered block are orthogonal with respect to  $\beta_q$ . Since points on the unital are singular, this means that the affine part of the block is contained in a totally singular subspace. Conversely, the affine part of this subspace is contained in the quadric (and thus in the unital), and coincides with the affine part of the block. □

In general, only a partial converse of 4.2 is true:

**4.3 Lemma.** Let  $T$  be a line of  $\mathbb{P}$  that is totally singular with respect to  $q$ . Then the intersection of  $T$  with  $\mathbb{A}$  is contained in a unique block of the unital that meets the block  $b_S$ .

*Proof.* Let  $(a, b)$  and  $(x, y)$  be two affine points that span  $T$ . Then we have  $a\bar{b} + b\bar{a} + 1 = q(a, b, 1) = 0 = q(x, y, 1) = x\bar{y} + y\bar{x} + 1$  and  $0 = q(a - x, b - y, 0)$  because the space  $T$  is totally singular. The line  $K(a, b, 1) \oplus K(x, y, 1)$  joining the two points in the projective plane over  $K$  meets the line  $S$  in the point  $K(a - x, b - y, 0)$ . This point belongs to the unital, and all affine points of  $T$  lie in the block induced by  $K(a, b, 1) + K(x, y, 0)$ , as claimed. □

**4.4 Definitions.** We use the polar space  $\mathcal{Q}$  (consisting of all totally singular subspaces of  $K^2 \times F$ , with respect to  $q$ ) in order to gain a deeper understanding of the unital. The affine part of  $\mathcal{Q}$  will be denoted by  $\mathcal{A}$ .

If (and only if)  $K$  is commutative, this affine part of the quadric forms an affine quadrangle with respect to a full weak subquadrangle: the maximal singular subspaces are lines. See [11] for an axiomatization of that class of geometries; cf. also [13] and [12]. We will denote the set of nonempty affine traces of maximal totally singular subspaces (with respect to  $q$ ) by  $\mathcal{M}$ .

**4.5 Remarks.** The lines in  $\mathcal{Q}$  are maximal totally singular subspaces if, and only if, the field  $K$  is commutative (i.e., if  $\dim_F K = 2$ ). In that case  $(A, B^S)$  is equal to  $(A, \mathcal{M})$ . If  $K$  is not commutative, pick two affine points  $a, b \in \mathcal{Q}$  that span a totally singular line  $\ell$ . Then the orthogonal space  $\ell^\perp$  has  $F$ -dimension greater than 3, and contains affine points of  $\mathcal{Q}$  that do not belong to the block of the unital that joins  $a$  and  $b$ . This shows that, in the non-commutative case, not every element of  $\mathcal{M}$  is induced by a block of the unital.

We collect our results so far:

- 4.6 Theorem.**
- a. The quadratic form  $q$  is non-degenerate; its Witt index equals  $\dim_F K \in \{2, 4\}$ .
  - b. The stabilizer  $\text{Aut}(U, B)_{b_S}$  acts faithfully by automorphisms of  $(A, \mathcal{M})$ .
  - c. If  $K$  is commutative then the affine part  $\mathcal{A}$  of  $\mathcal{Q}$  consists of the affine subspaces (in  $\mathbb{A}$ ) of the blocks of  $U$  that meet the block  $S$ .
  - d. If  $K$  is not commutative then there are totally singular subspaces whose affine part is not contained in any block of the unital.

*Proof.* The quadratic form  $q$  is non-degenerate because no point belongs to every totally singular subspace. From 4.2 and 4.3 we know that each element of the stabilizer induces a permutation of the point set  $A \cap U$  of the affine polar space  $\mathcal{A}$  such that collinearity in  $\mathcal{A}$  is preserved. The elements of  $\mathcal{M}$  are just the maximal sets of pairwise collinear points in  $\mathcal{A}$ . This proves b; the rest is clear from 4.1, 4.2, 4.3 and 4.5.  $\square$

**4.7 Remark.** The result 4.6 has been known in the finite (and necessarily commutative) case. In fact, Buekenhout [1] used the converse, observing that an affine quadric in a suitable finite projective space could be interpreted as a unital in a translation plane.

We want to apply 3.2 and 3.3 in order to reconstruct the hermitian form  $k$ . To this end, we have to extend the forms from  $K^2 \times F$  or  $K^2 \times \{0\}$  to  $K^3$ .

**4.8 Definition.** For  $v := (a, b, c)$  and  $w := (x, y, z) \in K^3$  we put  $\tilde{q}(v) := k(v, v) = a\bar{b} + b\bar{a} + c\bar{c}$  and obtain the polar form  $\beta_{\tilde{q}}: (v, w) \mapsto k(v, w) + k(w, v) = a\bar{y} + y\bar{a} + b\bar{x} + x\bar{b} + c\bar{z} + z\bar{c}$ .

Writing  $q'(a, b) := q(a, b, 0)$  and  $k'((a, b), (x, y)) := k((a, b, 0), (x, y, 0))$ , we find  $q(v) = q'(a, b) + c\bar{c}$  and  $k(v, w) = k'((a, b), (x, y)) + c\bar{z}$ .

**4.9 Lemma.** The forms  $q, q'$  and  $\beta_q, \beta_{q'}$  are restrictions of  $\tilde{q}$  and  $\beta_{\tilde{q}}$ , respectively.

- a. Each semi-similitude of  $q$  leaving  $S$  invariant induces a semi-similitude of  $q'$ .
- b. Let  $\alpha'$  be a semi-similitude of  $q'$ , satisfying  $q'(u^{\alpha'}) = q'(u)^{\gamma} s$ . The formula  $(u, z)^{\alpha} := (u^{\alpha'}, z^{\gamma} t)$  defines a semi-similitude  $\alpha$  of  $q$  if, and only if, we have  $t \in F$  and  $t^2 = s$ . In this case, the formula defining  $\alpha$  (for  $z \in F$ ) extends to the definition of a semi-similitude  $\tilde{\alpha}$  of  $\tilde{q}$ .
- c. In particular, the factor  $s$  is a square in  $F$  whenever  $\alpha'$  is obtained as the restriction of a semi-similitude of  $q$  or of  $\tilde{q}$ .
- d. If the restriction  $\alpha'$  of a semi-similitude  $\alpha$  is  $K$ -semilinear then  $\alpha$  allows a unique extension to a semi-similitude  $\tilde{\alpha}$  of  $k$ , and  $\tilde{\alpha}$  leaves  $K(0, 0, 1)$  and  $(0, 0, 1)^{\perp} = K^2 \times \{0\}$  invariant.

*Proof.* The first assertion is obvious, the second one is checked by comparing  $q(u, z)^{\gamma} s = q'(u)^{\gamma} s + (z\bar{z})^{\gamma} s$  and  $q(u^{\alpha'}, z^{\gamma} t) = q'(u)^{\gamma} s + z^{\gamma} t \bar{t} z^{\gamma} = q'(u)^{\gamma} s + z^{\gamma} z^{\gamma} t^2$ : we use that  $\sigma$  centralizes every automorphism of  $K$  that leaves  $F$  invariant. Now the third assertion is clear, and the last assertion follows by an application of 3.2 and 3.3.  $\square$

## 5 Automorphisms of Affine Polar Spaces

The following is shown in [2, 2.7]:

**5.1 Proposition.** *Every isomorphism between affine parts  $\mathcal{A}_j = \mathcal{Q}_j \setminus \mathcal{H}_j$  of polar spaces  $\mathcal{Q}_1$  and  $\mathcal{Q}_2$  of rank at least 2 (obtained by deleting geometric hyperplanes  $\mathcal{H}_1$  and  $\mathcal{H}_2$ , respectively) extends uniquely to an isomorphism between the polar spaces.  $\square$*

By 5.1 every automorphism of  $\mathcal{A}$  extends to a unique element of  $\text{Aut}(\mathcal{Q})$ . If  $K$  is not commutative then the polar space  $\mathcal{Q}$  can be recovered from the system  $\mathcal{M}$  of *all* affine maximal totally singular subspaces:

**5.2 Lemma.** *Every automorphism of  $(A, \mathcal{M})$  extends uniquely to an automorphism of the polar space  $\mathcal{Q}$  defined by  $q$ .*

*Proof.* We claim that for each totally singular subspace  $T$  of  $\mathcal{Q}$  there is a set  $M_T$  of maximal totally singular subspaces such that  $A \cap T = \bigcap_{X \in M_T} (A \cap X)$ .

In order to see this, choose first<sup>2</sup> a subspace  $V \leq K^2 \times F$  such that  $K^2 \times F = (K^2 \times F)^\perp \oplus V$  and  $T \leq V$ ; this is possible because the quadratic form is not degenerate. Now one may take  $M_T$  as the set of all maximal totally singular subspaces contained in  $T^\perp \cap V$ . The radical of  $T^\perp \cap V$  is just  $T$ , and  $M_T/T := \{X/T \mid X \in M_T\}$  is the set of all maximal totally singular subspaces of  $Z := (T^\perp \cap V)/T$ , with respect to the induced quadratic form  $q|_Z$ . As the polar form  $\beta_{q|_Z}$  is non-degenerate, we have  $\{0\} = \text{rad}(\beta_{q|_Z}) \geq \bigcap_{X \in M_T} X/T$ , and  $T = \bigcap_{X \in M_T} X$  follows. Consequently, every automorphism of  $(A, \mathcal{M})$  is an automorphism of the affine polar space  $\mathcal{A}$ . The extension to  $\mathcal{Q}$  exists by 5.1.  $\square$

Consider a non-degenerate quadratic or hermitian form of Witt index at least 2 on a vector space  $V$  with  $5 \leq \dim V < \infty$ . Then every automorphism of the corresponding polar space is induced by a semi-similitude, cf. [6, 8.1.5]. Therefore, we have:

**5.3 Corollary.** *Every automorphism of  $(A, \mathcal{M})$  is induced by a semi-similitude of  $q$  that leaves  $\mathbb{A}_\infty$  invariant.  $\square$*

## 6 Extending Automorphisms of the Quadric

The points at infinity for the affine plane over  $K$  are obtained by identifying those points that belong to the same orbit under multiplication (from the left) with elements of  $K^\times$ . Let  $\Sigma$  denote the group of automorphisms of the projective space  $\mathbb{P}$  induced by maps of the form  $(x, y, z) \mapsto (mx, my, z)$ , with  $m \in K^\times$ . Then the  $\Sigma$ -orbits are the classes  $[X]_\sim$  with respect to the equivalence relation  $\sim$  given by  $F(a, b, 0) \sim F(x, y, 0) \iff \exists m \in K^\times: F(a, b, 0) = F(mx, my, 0)$ .

**6.1 Theorem.** *An element  $\gamma \in \text{Aut}(\mathcal{A})$  induces an automorphism of the unital  $(U, B)$  if, and only if, it normalizes  $\Sigma$ . In particular, not every automorphism of  $(U \setminus b_S, B^S)$  extends to an automorphism of  $(U, B)$ .*

*Proof.* Recall from 5.1 and 5.3 that  $\gamma$  is induced by a semi-similitude of  $q$ , that is, by an element of  $\text{IO}(q)$ . Assume first that  $\gamma$  normalizes  $\Sigma$ . Then  $\gamma$  respects the relation  $\sim$  on  $\mathbb{A}_\infty$ . Thus the

<sup>2</sup> This precaution is necessary if  $\text{char } K = 2$  because then  $\beta_q$  is degenerate; cf. [4, §16].

action of  $\gamma$  extends to an action on the projective plane over  $K$ , induced by a  $K$ -semilinear map in  $\Gamma\text{O}(\widehat{q})$ . According to 3.2, this semilinear map is a semi-similitude of the hermitian form  $f$ . Thus it leaves  $U$  invariant, and respects the blocks because these (apart from  $b_S$ ) are obtained by joining a point of  $A$  with  $[X]_{\sim} \in \mathbb{A}_{\infty}$ , for some  $X \in Q \cap \mathbb{A}_{\infty}$ .

Now assume that  $\gamma$  induces an automorphism of the unital. Then  $\gamma$  preserves the relation  $\sim$ . This means that the group  $\gamma^{-1}\Sigma\gamma$  has the same orbits as  $\Sigma$  on  $K^2 \times \{0\}$ . We also know that  $\gamma^{-1}\Sigma\gamma$  is induced by a subgroup of  $\text{O}(q') \times \{\text{id}\}$  because the latter is a normal subgroup of  $\text{Aut}(\mathcal{A})$ , cf. 5.3. Looking at the images of  $F(1, 0, 0)$ ,  $F(0, 1, 0)$  and  $F(1, y, 0)$  under an arbitrary element  $\varphi \in \gamma^{-1}\Sigma\gamma$  we find that  $\varphi$  is induced by a map of the form  $(x, y, z) \mapsto (x^{\alpha}, y^{\beta}, z)$ , where  $\alpha$  and  $\beta$  are  $F$ -linear bijections of  $K$  such that

$$\forall s \in K^{\times} \forall y \in K \exists t \in K^{\times} : (s^{\alpha}, (sy)^{\beta}) = (t, ty).$$

Specializing  $y = 1$ , we find  $\alpha = \beta$ . Considering the general case again, we obtain  $s^{\alpha}t = (st)^{\alpha}$  for all  $s, t \in K$ . This means that  $\alpha$  centralizes the multiplications by elements of  $K$  from the right. Thus  $\alpha$  is contained in the group of multiplications by elements of  $K^{\times}$  from the left, see [10, 4.4, Lemma 2]. We have proved that  $\gamma$  normalizes  $\Sigma$ .  $\square$

**6.2 Examples.** There is no chance to reconstruct the relation  $\sim$  inside the affine part of the polar space: Let  $\alpha$  be an  $F$ -linear bijection of  $K$ . Since  $K \times \{0\} \times \{0\}$  and  $\{0\} \times K \times \{0\}$  are totally singular subspaces of  $S$ , there exists a unique  $F$ -linear bijection  $\widehat{\alpha}$  such that  $(x, y, 0) \mapsto (x^{\alpha}, y^{\widehat{\alpha}}, 0)$  gives an element  $\alpha' \in \text{O}(q')$ . For a suitable choice of  $\alpha$ , we find that  $\alpha'$  does not preserve the equivalence relation  $\sim$ . We give explicit examples:

Let  $G$  be commutative with  $\text{char } G \neq 2$ , and pick  $j \in G^{\times}$  with  $\bar{j} = -j$  (for instance, consider  $G = \mathbb{C}$  and  $F = \mathbb{R}$ , with  $j^2 = -1$ ). Then the map  $\alpha$  given by  $(u + vj)^{\alpha} = u + v + vj$  is a linear bijection, we have  $(u + vj)^{\widehat{\alpha}} = u + uj^{-1} + vj$ , and  $(1, 1, 0)^{\alpha'} = (1, 1 + j^{-1}, 0) \sim (j, j, 0)^{\alpha'} = (1 + j, j, 0)$  would imply  $j^{-1} = -2$ , contradicting the fact that  $j$  does not belong to  $F$ .

Consider a quaternion field  $H$  with  $\text{char } H \neq 2$ , and pick a Hamilton pair  $(p_1, p_2)$ , cf. 3.4. Put  $(x_0 + x_1p_1 + x_2p_2 + x_3p_1p_2)^{\alpha} := x_0 + x_1 + x_1p_1 + x_2p_2 + x_3p_1p_2$ , then  $(x_0 + x_1p_1 + x_2p_2 + x_3p_1p_2)^{\widehat{\alpha}} = x_0 + x_0p_1^{-1} + x_1p_1 + x_2p_2 + x_3p_1p_2$ , and  $\alpha'$  does not preserve the relation  $\sim$ , as before.

## 7 Automorphisms of the Unital

From 4.6.b we know that every automorphism  $\varphi$  of  $(U, B)$  that leaves  $b_S$  invariant induces an automorphism of  $(A, \mathcal{M})$ . Since the action of  $\varphi$  on  $\mathcal{A}$  extends to a unique action on  $(U, B)$  and to a unique action on the polar space  $Q$  (cf. 5.2), we have an injective group homomorphism  $\varphi \mapsto \widehat{\varphi}$  from the stabilizer  $\text{Aut}(U, B)_{b_S}$  into the stabilizer  $\text{Aut}(Q)_{Q_{\infty}}$ .

From 5.3 we know that  $\text{Aut}(Q)_{Q_{\infty}}$  is induced by the stabilizer of  $S$  in the group  $\Gamma\text{O}(q)$ , which in turn induces the group  $\Gamma\text{O}(q')$  on  $S$ . According to 6.1, the image of  $\text{Aut}(U, B)_{b_S}$  under the homomorphism  $\widehat{\phantom{\varphi}}$  induces the normalizer of  $\Sigma$  in  $\Gamma\text{O}(q')$ . This normalizer consists of the  $K$ -semilinear semi-similitudes of  $q'$ , and these extend to semi-similitudes of the hermitian form  $k$  by 3.3, cf. 4.9.

Since the group of (semi-)similitudes of the hermitian form  $k$  acts transitively on the set  $B$  of blocks of the unital, the full group of automorphisms of the unital is obtained as the product of  $\text{PU}(k)$  and the stabilizer of  $b_S$ . As we have just seen, this stabilizer is induced by semi-similitudes, and we have proved the following.

**7.1 Theorem.** Every automorphism of the unital is induced by a semi-similitude of the hermitian form; we have  $\text{Aut}(U, B) = \text{PFU}(k)$ .  $\square$

A translation with center  $p$  of a unital is an automorphism fixing every block through  $p$ . The group  $T$  generated by all translations is called the *little projective group* of the unital.

**7.2 Corollary.** The little projective group  $T$  is normal in  $\text{Aut}(U, B)$ , and the latter acts faithfully by conjugation on  $T$ . Thus  $\text{Aut}(U, B)$  is embedded in  $\text{Aut}(T)$ .  $\square$

**7.3 Remark.** Once we know  $\text{Aut}(U, B) = \text{PFU}(k)$  it is easy to compute the translations with center  $p$ ; they form a subgroup of the (abelian) group of elations with center  $p$  and axis  $p^\perp$  in the projective plane over  $K$ . Thus  $T$  is the group generated by all transvections in  $\text{PU}(k)$ . One knows (see [3, Thm. 1]) that  $T$  is a simple group. If  $\text{char } K \neq 2$  then a direct argument (cf. [14, 3.2, 1.5]) yields that  $T$  is perfect, and an application of Iwasawa's Lemma [7] shows that  $T$  is a simple group.

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