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Abstract

We show that conjugacy classes of Baer involutions and non-elliptic polarities, respectively, of proper (i.e., non-desarguesian) Moufang planes are interrelated. Restriction of the conjugating group to the stabilizer of a triangle or a quadrangle does not refine the classes. These results are applied to prove transitivity properties for the centralizers of these polarities. Along the way, a new proof is obtained for the fact that the automorphism group of a Moufang plane acts transitively on quadrangles.

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Introduction

In the present paper we study collineations and correlations (i.e., dualities) of Moufang planes. We concentrate on the non-desarguesian case. Specifically, we study involutions: among the collineations, we are interested in those involutions (traditionally, named to honor R. Baer [1]) that fix a subplane pointwise. We establish an intimate interrelation between Baer involutions and polarities (i.e., involutory correlations). Since we exclude the desarguesian case, we have an exposed class of polarities, corresponding to the standard involution of the coordinate structure.

We will show (Theorem 2.3) that conjugacy classes of Baer involutions are not refined if we restrict the conjugating element to the stabilizer of a quadrangle; this means that conjugacy of Baer involutions precisely corresponds to conjugacy of involutory automorphisms of the coordinate structure. In Theorem 4.3 we then establish a precise interrelation between (conjugacy classes of) Baer involutions and polarities with at least two absolute flags. Finally, we use our results to prove a transitivity result for centralizers of polarities, see 5.3.

Every Moufang plane is coordinatized by an alternative field, see 1.2. Whenever feasible, we will consider the more general situation of semifield, i.e, a not necessarily associative division ring.

The characteristic two case presents special features (such as polarities with precisely one absolute flag, and Baer subplanes that are pappian planes). In the present paper we do not exclude this case. However, a detailed study of specific details in characteristic two is contained in a separate paper [7].

1 Autotopisms

1.1 Definition. Let $\mathbf{S}$ be a semifield. An *autotopism* (A, B, C) of $\mathbf{S}$ is a triplet of additive bijections of $\mathbf{S}$ such that $B(sx) = C(s)A(x)$ holds for all $s, x \in \mathbf{S}$.

Semifields may be used to coordinatize projective planes of Lenz type V, i.e., translation planes that are also dual translation planes. The affine plane $\mathcal{A}_\mathbf{S}$ over $\mathbf{S}$ has point set $\mathbf{S}^2$ and line set $\mathcal{L} := \{[m,b] \mid m,b \in \mathbf{S}\} \cup \{[c] \mid c \in \mathbf{S}\}$ where $[m,b] := \{(x, mx+b) \mid x \in \mathbf{S}\}$ and $[c] := \{c\} \times \mathbf{S}$. The projective hull $\mathcal{P}_\mathbf{S}$ of $\mathcal{A}_\mathbf{S}$ is obtained by adding a line L_∞ containing points ∞ and (m) for each $m \in \mathbf{S}$ that correspond to the parallel classes $\{[c] \mid c \in \mathbf{S}\}$ and $\{[m,b] \mid b \in \mathbf{S}\}$, respectively.

In the group $\mathrm{Aut}(\mathcal{P}_\mathbf{S})$ the autotopisms describe each element in the stabilizer Δ of the triangle consisting of the three points $o = (0,0)$, ∞ and (0) as maps $(x,y) \mapsto (A(x), B(y))$; cf. [5, VIII 4]. The map C gives the action on the line at infinity via $(s) \mapsto (C(s))$. We will be interested in those elements of the triangle stabilizer Δ that centralize a certain polarity.

Note that any autotopism is already determined by any one of the maps A, B, or C together with a single non-zero value of any one of the remaining two maps. For instance, we have $C(1)A(x) = B(x) = C(x)A(1)$ for each $x \in \mathbf{S}$.

1.2 Octonion fields. In general, it may be quite hard to find the autotopisms for a given semifield. This task becomes easier if we consider an *alternative field*, i.e., a semifield $\mathbf{A}$ with the *inverse property*: for $a \in \mathbf{A} \setminus \{0\}$ the unique element $a^{-1} \in \mathbf{A}$ with $aa^{-1} = 1$ also satisfies $a^{-1}(ax) = x = (xa)a^{-1}$ for all $x \in \mathbf{A}$. The inverse property yields (cf. [10, 6.1.2 (18)–(21), p. 160]) the *Moufang identities*:

$$\forall\, x, y, a \in \mathbf{A}\colon \quad a(x(ay)) = ((ax)a)y\,, \quad x(a(ya)) = ((xa)y)a\,, \quad (ax)(ya) = (a(xy))a\,.$$

The center[1] $Z := Z(\mathbf{A}) := \{z \in \mathbf{A} \mid \forall x \in \mathbf{A}\colon zx = xz\}$ is closed under addition, multiplication, and passage to the inverse. Moreover, any two elements $a, b \in \mathbf{A}$ are contained in an associative subalgebra, namely $Z(a,b) := Z + Za + Zb + Zab$. We refer to this fact as *bi-associativity*.

It is known (see [2], [6], cf. [10, Ch. 6] or [16, Ch. 10]) that every non-associative alternating field is an *octonion field*, i.e., a composition algebra $\mathbb{O}$ of dimension 8 with anisotropic norm form N and non-degenerate polar form $f_N\colon (x,y) \mapsto \langle x|y\rangle := N(x+y) - N(x) - N(y)$. Such a composition algebra has the *standard involution* $\kappa\colon x \mapsto \bar{x} := \langle 1|x\rangle - x$. This involution also allows to recover the norm as $N(x) := \bar{x}x = x\bar{x}$ and the polar form as $f_N(x,y) = \bar{x}y + \bar{y}x$. The space $\mathrm{Pu}(\mathbb{O}) := 1^\perp = \{x \in \mathbb{O} \mid \langle 1|x\rangle = 0\} = \{x \in \mathbb{O} \mid \bar{x} = -x\}$ is called the space of *pure* elements.

See [13] for a general discussion of composition algebras.

1.3 Lemma. *Every ring automorphism of $\mathbb{O}$ centralizes the standard involution. Consequently, every automorphism and every anti-automorphism of $\mathbb{O}$ is a semi-similitude of the norm form, i.e., a Z-semilinear bijection α such that $N(\alpha(x)) = \beta_\alpha(N(x))s$ holds for all $x \in \mathbb{O}$, with some fixed $\beta_\alpha \in \mathrm{Aut}(Z)$ and $s \in Z \setminus \{0\}$.*

Proof. Let α be an arbitrary ring automorphism of $\mathbb{O}$. Since α leaves the center $Z \subseteq \mathrm{Fix}(\kappa)$ invariant it suffices to consider the effect of κ and α on $a \in \mathbb{O} \setminus Z$. Now α induces an isomorphism from $Z(a) := Z + Za$ onto $Z(\alpha(a))$. Thus it transports the Galois group $\mathrm{Gal}(Z(a)/Z)$ onto $\mathrm{Gal}(Z(\alpha(a))/Z)$. It remains to note that κ induces the (possibly trivial) generators of these Galois groups. We may take $\beta_\alpha = \alpha|_Z$ and $s = 1$. □

[1] For a general (non-alternative) semifield one has to be more careful when defining the center.

1.4 Examples. If $\mathbb{O}$ is an octonion field then straightforward calculations (using bi-associativity and the Moufang identities) yield that for each $a \in \mathbb{O} \smallsetminus \{0\}$ the following maps (inspired by [11], see [4, 3.1]) are collineations:

$$\begin{array}{llll} \gamma_a\colon & (x,y) \mapsto (a^{-1}x, ay), & (s) \mapsto (asa)\,; \\ \gamma'_a\colon & (x,y) \mapsto (axa, ya), & (s) \mapsto (sa^{-1})\,; \\ \gamma''_a\colon & (x,y) \mapsto (xa, aya), & (s) \mapsto (as)\,. \end{array}$$

For each $p \in \mathrm{Pu}(\mathbb{O}) \smallsetminus \{0\}$ we also have the collineation (cf. [12, 11.22])

$$\mu_p\colon \quad (x,y) \mapsto (xp, -p^{-1}yp), \quad (s) \mapsto (-p^{-1}s)\,;$$

this follows from $-p^{-1} = \frac{1}{N(p)}p \in Zp$ and one of the Moufang identities.
Finally, for $r, t \in \mathrm{Z}(\mathbb{O}) \smallsetminus \{0\}$ we have the collineation

$$\tau_{r,t}\colon \quad (x,y) \mapsto (xr, yt), \qquad (s) \mapsto (str^{-1})\,.$$

1.5 Lemma. *Elements φ and ψ of the triangle stabilizer Δ induce the same bijection on* [0] *if, and only if, the quotient $\varphi \circ \psi^{-1}$ is a homothety of the form $\tau_{r,1}\colon (x,y) \mapsto (rx,y)$ with $r \in \mathrm{Z}(\mathbb{O}) \smallsetminus \{0\}$.*

Proof. As $\varphi \circ \psi^{-1}$ has axis [0] and fixes (0) it is a homothety with axis [0] and center (0). Thus it is of the form $\tau_{r,1}\colon (x,y) \mapsto (rx,y)$ with a nonzero element r of the middle nucleus $N_m := \{a \in \mathbb{O} \mid \forall x,y \in \mathbb{O}\colon (xa)y = (x(ay)\}$, cf. [5, 8.2]. This nucleus equals $\mathrm{Z}(\mathbb{O})$, (see [2, Thm. 3.1], [6], cf. [10, 6.2, 6.4]) and the assertion follows. □

The full group of automorphisms of a Moufang plane is transitive on quadrangles. Usually, this fact is proved algebraically via a discussion of isotopies between alternative fields. We offer another proof (generalizing arguments from [12, 12.17, 12.18, 17.11 ff]) which provides more information about the triangle stabilizer Δ and also about centralizers of planar collineations (see 2.1 below). We need a lemma first.

1.6 Lemma. *Let S be a subfield of $\mathrm{Z}(\mathbb{O})$ such that $\dim_S \mathrm{Z}(\mathbb{O})$ is finite. If $W \subset \mathbb{O}$ is a subspace of $\mathbb{O}$ such that $2\dim_S W > \dim_S \mathbb{O}$ and $\{w^{-1} \mid w \in W \smallsetminus \{0\}\} \subset W$ then every element of $\mathbb{O}$ is the product of two elements of W.*

In particular, every element of $\mathbb{O}$ is the product of two elements that are perpendicular to 1.

Proof. For $a \in \mathbb{O} \smallsetminus \{0\}$ we use that $\dim_S aW = \dim_S W$ implies that there exists $w \in W \cap aW$ with $w \neq 0$. Then there exists $u \in W$ such that $w = au$, and $wu^{-1} = a$ as required. □

1.7 Theorem. *The group Λ generated by $\{\gamma_a \mid a \in \mathbb{O} \smallsetminus \{0\}\} \cup \{\mu_p \mid p \in \mathrm{Pu}(\mathbb{O}) \smallsetminus \{0\}\}$ acts transitively on the set $\{(x,y) \mid x,y \in \mathbb{O} \smallsetminus \{0\}\}$. The full triangle stabilizer Δ is the product of Λ with the stabilizer of $(1,1)$. The latter is in fact the stabilizer of a quadrangle, and thus isomorphic to the full automorphism group of the alternative division ring $\mathbb{O}$.*

Proof. Consider $(x,y) \in \mathbb{O}^2$ with $x \neq 0 \neq y$. Applying γ_y^{-1} we map (x,y) to $(w,1)$ with $w := yx$. According to 1.6 we find p,q with $\bar{p} = -p$ and $\bar{q} = -q$ such that $w^{-1} = pq$. Now $\mu_q \circ \mu_p$ maps $(w,1)$ to $(1,1)$, as required. □

1.8 Corollary ([10, 7.3.14])**.** *The full group of collineations of the projective plane over $\mathbb{O}$ acts transitively on the set of non-degenerate quadrangles. The stabilizer of any triangle (a,b,c) acts transitively on $(a \vee b) \smallsetminus \{a,b\}$, and for $d \in (a \vee b) \smallsetminus \{a,b\}$ the stabilizer of the degenerate quadrangle (a,b,c,d) acts transitively on $(c \vee d) \smallsetminus \{c,d\}$.* □

1.9 Corollary. *For every autotopism (A, B, C) of $\mathbb{O}$ the maps A, B and C are semi-similitudes of the norm form.* □

2 Conjugacy of planar collineations

In general, not every automorphism of a subplane Q of the projective plane $\mathcal{P}_{\mathbb{O}}$ will extend to an automorphism of $\mathcal{P}_{\mathbb{O}}$. For instance, a subfield of the center Z of $\mathbb{O}$ may admit automorphisms that do not extend to automorphisms of Z, let alone $\mathbb{O}$.

2.1 Lemma. *Assume that $\mathbb{K} \subseteq \mathbb{O}$ contains 1 and is closed under addition, multiplication and both additive and multiplicative inverses. Then the stabilizer of $\mathcal{P}_{\mathbb{K}}$ in $\mathrm{Aut}(\mathcal{P}_{\mathbb{O}})$ acts transitively on the set of non-degenerate quadrangles in $\mathcal{P}_{\mathbb{K}}$.*

Proof. Let $\Gamma_{\mathbb{K}}$ be the subgroup consisting of those automorphisms of $\mathcal{P}_{\mathbb{O}}$ that leave $\mathcal{P}_{\mathbb{K}}$ invariant. We note that the subplane $\mathcal{P}_{\mathbb{K}}$ is a Moufang plane, again; the elations required to see that $\Gamma_{\mathbb{K}}$ acts transitive on triangles in $\mathcal{P}_{\mathbb{K}}$ are those translations and shears that may be described in terms of $\mathbb{K}$. For $a \in \mathbb{K} \smallsetminus \{0\}$ we have $\gamma_a, \gamma'_a \in \Gamma_{\mathbb{K}}$. Thus the proof of 1.7 also applies to the action of $\Gamma_{\mathbb{K}}$ on $\mathcal{P}_{\mathbb{K}}$. □

Examples of subsets $\mathbb{K}$ as in 2.1 occur as sets of fixed points of elements of $\mathrm{Aut}(\mathbb{O})$.

2.2 Corollary. *For each element $\tilde{\alpha}$ of the stabilizer of a quadrangle in $\mathrm{Aut}(\mathcal{P}_{\mathbb{O}})$ the centralizer of $\tilde{\alpha}$ acts transitively on the set of non-degenerate quadrangles in $\mathrm{Fix}(\tilde{\alpha})$.* □

2.3 Theorem. *Let α_1 and α_2 be automorphisms of $\mathbb{O}$, and consider the corresponding automorphisms given by $\tilde{\alpha}_1(x, y) := (\alpha_1(x), \alpha_1(y))$ and $\tilde{\alpha}_2(x, y) := (\alpha_2(x), \alpha_2(y))$, respectively. Then the following are equivalent:*

- **a.** *$\tilde{\alpha}_1$ and $\tilde{\alpha}_2$ are conjugates in $\mathrm{Aut}(\mathcal{P}_{\mathbb{O}})$.*
- **b.** *$\tilde{\alpha}_1$ and $\tilde{\alpha}_2$ are conjugates in the stabilizer of the standard quadrangle.*
- **c.** *α_1 and α_2 are conjugates in $\mathrm{Aut}(\mathbb{O})$.*

Proof. If $\delta \in \mathrm{Aut}(\mathcal{P}_{\mathbb{O}})$ satisfies $\delta \circ \tilde{\alpha}_1 \circ \delta^{-1} = \tilde{\alpha}_2$ then the image of the standard quadrangle under δ is a quadrangle in $\mathrm{Fix}(\tilde{\alpha}_2)$. From 2.2 we know that there exists φ in the centralizer of $\tilde{\alpha}_2$ such that $\varphi \circ \delta$ fixes the standard quadrangle. Thus there exists $\psi \in \mathrm{Aut}(\mathbb{O})$ such that $\varphi \circ \delta = \tilde{\psi}$, and the first assertion implies each one of the other two. The rest is clear. □

In general, planar automorphisms need not be conjugates if their respective planes of fixed elements are in the same orbit under $\mathrm{Aut}(\mathcal{P}_{\mathbb{O}})$. This is rather obvious if we deal with automorphisms that have order greater than 2. For Baer involutions, the situation depends on the characteristic of $\mathbb{O}$:

2.4 Theorem. *Consider the automorphisms $\tilde{\alpha}_j$ induced by involutions $\alpha_1, \alpha_2 \in \mathrm{Aut}(\mathbb{O})$.*

- **a.** *If $\alpha_1|_{Z(\mathbb{O})} \neq \mathrm{id}$ then $\mathrm{Fix}(\tilde{\alpha}_1) = \mathrm{Fix}(\tilde{\alpha}_2) \iff \alpha_1 = \alpha_2$.*
- **b.** *If $\mathrm{char}\,\mathbb{O} \neq 2$ then $\mathrm{Fix}(\tilde{\alpha}_1) = \mathrm{Fix}(\tilde{\alpha}_2) \iff \alpha_1 = \alpha_2$.*
- **c.** *If $\mathrm{char}\,\mathbb{O} = 2$ then there exist different Baer involutions with the same set of fixed points.*

Proof. We abbreviate $Z := \mathrm{Z}(\mathbb{O})$, again. If $\mathrm{Fix}(\tilde{\alpha}_1) = \mathrm{Fix}(\tilde{\alpha}_2)$ then $F := \mathrm{Fix}_{\mathbb{O}}(\alpha_1) = \mathrm{Fix}_{\mathbb{O}}(\alpha_2)$. Now F is a vector space over $S := \mathrm{Fix}_Z(\alpha_1) = \mathrm{Fix}_Z(\alpha_2)$, and $\dim_S \mathbb{O} = 2 \dim_S F$.

If $\alpha_1|_{Z(\mathbb{O})} \neq \mathrm{id}$ then both involutions induce the generator of $\mathrm{Gal}(Z/S)$. Thus $\alpha_1 \circ \alpha_2$ is an automorphism of $\mathbb{O}$ with $\dim_S \mathrm{Fix}_{\mathbb{O}}(\alpha_1 \circ \alpha_2) \geq \dim_S(Z + F) > \frac{1}{2} \dim_S \mathbb{O}$. This means $\alpha_1 \circ \alpha_2 = \mathrm{id}$, and $\alpha_1 = \alpha_2$, as claimed.

Now assume char $\mathbb{O} \neq 2$. It suffices to consider the case where α_j is linear over Z. Then both involutions are orthogonal maps, and $\mathrm{Fix}(-\alpha_1) = F^\perp = \mathrm{Fix}(-\alpha_2)$. Again, we obtain $\alpha_1 = \alpha_2$.

If char $\mathbb{O} = 2$ there are indeed examples of Z-linear involutory automorphisms sharing the same set of fixed points; see [7, 4.5]. □

3 Some polarities of semifield planes

Let $\mathbf{S}$ be a semifield. Easy computations show that every involutory anti-automorphism σ of $\mathbf{S}$ yields a polarity $\hat{\sigma}$ of $\mathcal{P}_\mathbf{S}$ interchanging (x, y) with $[\sigma(x), -\sigma(y)]$. The set of affine absolute points is $A_\sigma := \{(x, y) \in \mathbf{S}^2 \mid \sigma(y) + y = \sigma(x)x\}$, and the point ∞ is the unique absolute point at infinity.

3.1 Lemma. *Let σ and τ be involutory anti-automorphisms, and let $\delta := (A, B, C)$ be an autotopism. Then the following are equivalent:*

a. $\delta \circ \hat{\sigma} \circ \delta^{-1} = \hat{\tau}$.
b. $C \circ \sigma = \tau \circ A$.
c. $B \circ \sigma = \tau \circ B$ *and* $C(1) = \tau(A(1))$.

Proof. The autotopism (A, B, C) maps $[m, t]$ to $[C(m), B(t)]$. Thus $\delta \circ \hat{\sigma} = \hat{\tau} \circ \delta$ implies that $[C(\sigma(x), -B(\sigma(y))] = [\tau(A(x)), -\tau(B(y))]$ holds for all $x, y \in \mathbb{O}$. This shows that assertion a implies both b and c.

Now assume $C \circ \sigma = \tau \circ A$, then $\tau \circ C = A \circ \sigma$ because $\sigma^2 = \mathrm{id} = \tau^2$. For $x \in \mathbb{O}$ we compute $B(\sigma(x)) = B(\sigma(x)\sigma(1)) = C(\sigma(x))\, A(\sigma(1)) = \tau(A(x))\, \tau(C(1)) = \tau(C(1)A(x)) = \tau(B(x))$ and $B \circ \sigma = \tau \circ B$ follows. Thus b implies c.

Finally, assume that $B \circ \sigma = \tau \circ B$ and $C(1) = \tau(A(1))$. Then $\tau(C(1)) = A(1)$ because τ is an involution. Now $C(\sigma(x))A(1) = C(\sigma(x))A(\sigma(1)) = B(\sigma(x)\sigma(1) = B(\sigma(x)) = \tau(B(x)) = \tau(C(1)A(x)) = \tau(A(x))\tau(C(1)) = \tau(A(x))\tau^2(A(1)) = \tau(A(x))A(1)$ yields $C \circ \sigma = \tau \circ A$. A straightforward computation yields $\delta \circ \hat{\sigma} = \hat{\tau} \circ \delta$. Thus we have shown that c implies a. □

3.2 Proposition. *The stabilizer ∇_σ also fixes the pole $\hat{\sigma}([0]) = (0)$. Its elements correspond to those autotopisms $\delta = (A, B, C)$ of $\mathbb{O}$ that have one of the following equivalent properties:*

a. $\delta \circ \hat{\sigma} = \hat{\sigma} \circ \delta$.
b. $C \circ \sigma = \sigma \circ A$.
c. $B \circ \sigma = \sigma \circ B$ *and* $C(1) = \sigma(A(1))$.

Proof. The pole of $[0]$ is fixed by each element that centralizes $\hat{\sigma}$ and fixes $[0]$. The rest is a specialization of 3.1 for the case where $\sigma = \tau$. □

3.3 Corollary. *For each $\alpha \in \mathrm{Aut}(\mathbb{O})$ we have that $\tilde{\alpha} := (\alpha, \alpha, \alpha)$ is an autotopism which centralizes the polarity $\hat{\kappa}$.*

Proof. This follows from 1.3 and 3.2 because (α, α, α) is an autotopism with $\alpha(1) = 1$. □

Polarities of translation planes can be described quite explicitly if they have more than one absolute flag:

3.4 Theorem. *Let $\mathcal{P}$ be a projective plane with a polarity π. If $\mathcal{P}$ is a translation plane and π has at least two absolute flags then there is a semifield $\mathbf{S}$ with an anti-automorphism σ and an isomorphism $\eta \colon \mathcal{P} \to \mathcal{P}_\mathbf{S}$ such that $\eta \circ \pi \circ \eta^{-1} = \hat{\sigma}$.*

Proof. The projective plane $\mathcal{P}$ is self-dual. Therefore, it either has Lenz type V and then a distinguished flag (∞, L_∞) or it is a Moufang plane (by the Skornyakov–Sans Soucie Theorem, cf. [5, VI.6 and 7]). In the first case, we know that (∞, L_∞) is an absolute flag. In the Moufang case, we may choose any absolute flag for (∞, L_∞).

We pick a second absolute point o. Then $o \notin L_\infty$ and $\pi(o)$ meets L_∞ in a point $u \neq \infty$. We pick another point $w \in L_\infty \smallsetminus \{\infty, u\}$ and put $e := (o \vee w) \cap \pi(w)$. Then o, u, ∞, e form a quadrangle Q, the line L_∞ is a translation axis, and the point ∞ is a translation axis for the dual plane. Therefore, the ternary field defined by the quadrangle Q is a semifield $\mathbf{S}$.

In coordinates with respect to Q we have $o = (0,0)$, $e = (1,1)$, $\pi(o) = [0,0]$, and $\pi(w) = [1]$. The lines through w apart from L_∞ are those of the form $[1,t]$ with $t \in \mathbf{S}$.

There are bijections σ and τ of $\mathbf{S}$ such that $\pi(x,y) = [\sigma(x), \tau(y)]$ holds for each $(x,y) \in \mathbf{S}^2$. In particular, we know $\sigma(1) = 1 = -\tau(1)$ from

$$\pi(1,1) = \pi\Big((o \vee w) \wedge \pi(w)\Big) = \Big(\pi(o) \wedge \pi(w)\Big) \vee w = ([0,0] \wedge [1]) \vee w = [1,-1]\,.$$

Using the assumption $\pi^2 = \mathrm{id}$ we find

$$\forall\, x,y,z \in \mathbf{S}\colon \sigma(z)x + \tau\Big(\sigma(x)z + \tau(y)\Big) = y\,.$$

We specialize $z = 0$ and obtain $\sigma(0)x + \tau^2(y) = y$ for all $x, y \in \mathbf{S}$. This yields $\sigma(0) = 0$ and then $\tau^2 = \mathrm{id}$. Now we specialize $y = 0$ and $x = 1$ and find $\sigma(z) + \tau(z) = 0$ for each $z \in \mathbf{S}$. Using this for $x = 1$ and $y = \tau(c)$ we see from $\sigma(z) + \tau(z + c) = \tau(c)$ that τ is additive. Then $\sigma^2(z) = \sigma\Big(-\tau(z)\Big) = -\tau(-\tau(z)\Big) = \tau^2(z) = z$ yields $\sigma^2 = \mathrm{id}$. Finally, we specialize $y = 0$ and $x = \sigma(c)$ to see from $\sigma(z)\sigma(c) - \sigma(cz) = \sigma(z)\sigma(c) + \tau(cz) = 0$ that σ is in fact an anti-automorphism of $\mathbf{S}$. The rest is clear. □

3.5 Remark. If we start with $\mathcal{P} = \mathcal{P}_{\mathbf{K}}$ for some semifield $\mathbf{K}$ in 3.4 then it may well happen that the semifield $\mathbf{S}$ constructed in the proof is not isomorphic to $\mathbf{K}$; in general, the two semifields will only be isotopic. In a Moufang plane (where the automorphism group is transitive on quadrangles, see 1.8) we have that $\mathbf{S}$ and $\mathbf{K}$ are isomorphic.

3.6 Theorem. *Let π be a polarity of $\mathcal{P}_{\mathbb{O}}$.*

- **a.** *If π has at least two absolute flags then a conjugate of π has the absolute flags $((0,0),[0,0])$ and (∞, L_∞).*
- **b.** *If π has the absolute flags $((0,0),[0,0])$ and (∞, L_∞) then there exists an involutory anti-automorphism σ of $\mathbb{O}$ and an element δ of the stabilizer of the (degenerate) quadrangle $(\infty,(0),(0,0),(1,0))$ such that $\delta \circ \pi \circ \delta^{-1} = \hat{\sigma}$.*

Proof. Choose absolute flags (a, A) and (b, B) of π. Then $(a, b, A \cap B)$ is a non-degenerate triangle. Since the full automorphism group of $\mathcal{P}_{\mathbb{O}}$ acts transitively on quadrangles (see 1.8) we find $\varphi \in \mathrm{Aut}(\mathcal{P}_{\mathbb{O}})$ mapping (a, A) and (b, B) to $((0,0),[0,0])$ and (∞, L_∞), respectively. Then $((0,0),[0,0])$ and (∞, L_∞) are absolute flags of $\varphi \circ \pi \circ \varphi^{-1}$, as required.

Now assume that π has these absolute flags. Then $\pi(1,0)$ is a line through $(0,0)$ and $\pi(1,0) \neq [0]$. Let $(1,c) := [1] \wedge \pi(1,0)$; then $\pi(1,0) = (0,0) \vee (1,c) = [c,0]$ and $\pi[1] = \pi(\infty \vee (1,0)) = L_\infty \wedge \pi(1,0) = (c)$ yield $\pi(1,c) = \pi([1] \wedge [c,0]) = (c) \vee (1,0) = [c,-c]$. From 1.8 we know that there exists δ in the stabilizer of $(\infty,(0),(0,0),(1,0))$ such that $\delta(1,c) = (1,1)$; we obtain $\delta[c,-c] = [1,-1]$. Now $\delta \circ \pi \circ \delta^{-1}$ still has the absolute flags $((0,0),[0,0])$ and (∞, L_∞), and maps $(1,1)$ to $\delta(\pi(1,c)) = \delta[c,-c] = [1,-1]$. As in the proof of 3.4 we conclude that there exists an involutory anti-automorphism σ of $\mathbb{O}$ such that $\delta \circ \pi \circ \delta^{-1} = \hat{\sigma}$. □

4 Conjugacy of polarities of Moufang planes

In this section, we consider the projective plane $\mathcal{P}_\mathbb{O}$ over an octonion field $\mathbb{O}$, again.

4.1 Proposition. *An autotopism* (A,B,C) *centralizes* $\hat\kappa$ *precisely if* $\overline{A(1)}\,A(1) = B(1)$.

Proof. If (A,B,C) centralizes $\hat\kappa$ then $C(1) = \overline{A(1)}$ by 3.1, and $B(1) = C(1)\,A(1) = \overline{A(1)}\,A(1)$.

Assume now that $\overline{A(1)}\,A(1) = B(1)$. Then $B(1) \in Z$, and we get $B(\mathrm{Pu}(\mathbb{O})) = B(1^\perp) = B(1)^\perp = 1^\perp = \mathrm{Pu}(\mathbb{O})$ because B is a semi-similitude of the norm form, cf. 1.9. For $x \in \mathrm{Pu}(\mathbb{O})$ we find $B(x) + \overline{B(x)} = 0$ and hence $B(x) = \overline{B(\bar x)}$.

Define $S,T\colon \mathbb{O} \to \mathbb{O}$ by $S(x) := \overline{B(\bar x)}$ and $T := S^{-1} \circ B$. Then $T(x) = x$ for all $x \in \mathrm{Pu}(\mathbb{O})$. Thus the semi-similitude T is in fact an orthogonal (in particular, a linear) map.

For $x \in \mathrm{Pu}(\mathbb{O})$ and $y \in \mathbb{O}$ we obtain $\langle x|y\rangle = \langle T(x)|T(y)\rangle = \langle x|T(y)\rangle$ and therefore $\langle x|T(y)-y\rangle = 0$. It follows that $\varphi := T - \mathrm{id}$ maps $\mathbb{O}$ to $\mathrm{Pu}(\mathbb{O})^\perp = 1^{\perp\perp} = Z$.

Since T preserves the norm we get for any $x \in \mathbb{O}$:

$$\bar x x = N(T(x)) = (\bar x + \varphi(x))(x + \varphi(x)) = \bar x x + (x + \bar x + \varphi(x))\,\varphi(x)\,.$$

For each $x \in \mathbb{O}$ we either have $\varphi(x) = 0$ or $\varphi(x) = -x - \bar x$. In any case, we find $\mathrm{Pu}(\mathbb{O}) \le \ker\varphi$. So consider $x \in \mathbb{O} \smallsetminus \mathrm{Pu}(\mathbb{O})$. If $\varphi(x) \neq 0$ then $T(x) = x + \varphi(x) = -\bar x$ yields $B(x) = S(-\bar x) = -\overline{B(x)}$. Thus $x \in B^{-1}(\mathrm{Pu}(\mathbb{O})) \cap (\mathbb{O} \smallsetminus \mathrm{Pu}(\mathbb{O}))$, contradicting the fact that B is a semi-linear bijection leaving $\mathrm{Pu}(\mathbb{O})$ invariant. Thus $\varphi = 0$ and $T = \mathrm{id}$. This means that $B = S$, and B centralizes κ. The result now follows from 3.1 because $B(1) = C(1)\,A(1) = \overline{A(1)}\,A(1)$ yields $C(1) = \overline{A(1)}$. □

4.2 Lemma. *Every autotopism* (A,B,C) *can be uniquely written as a product* $\psi_c \circ \gamma_a \circ \delta$, *where* $\delta = (A',B',C')$ *centralizes* $\hat\kappa$ *and* $cB'(1) = 1$. *The elements a and c are determined by* $a = B(1)$ *and* $c^{-1} = N(B(1)\,A(1))$.

Proof. Assume that such a decomposition exists, then $A(x) = a^{-1}A'(x)$, $B(x) = acB'(x)$ and $C(x) = ca\,C'(x)\,a$. Since $c\,B'(1) = 1$ we get $B(1) = a$. Using 4.1 we obtain $c^{-1} = B'(1) = \overline{A'(1)}\,A'(1) = \overline{aA(1)}\,a\,A(1) = N(aA(1)) = N(B(1)\,A(1))$. This proves uniqueness of a and c, and then also of δ.

To prove existence we define $a := B(1)$ and $c := N(B(1)\,A(1))$ and use 4.1 to show that $\delta := \tau_{1,c}^{-1} \circ \gamma_a^{-1} \circ (A,B,C)$ centralizes $\hat\kappa$. □

Recall that Δ is the stabilizer of the standard triangle $(\infty, (0), (0,0))$; the group $\Delta_{(1,1)}$ is the stabilizer of the standard quadrangle.

4.3 Theorem. *Let* α *and* β *be automorphisms of* $\mathbb{O}$ *with* $\alpha^2 = \mathrm{id} = \beta^2$, *and consider the corresponding elements* $\tilde\alpha, \tilde\beta \in \Delta_{(1,1)}$ *given by* $\tilde\alpha(x,y) := (\alpha(x),\alpha(y))$ *and* $\tilde\beta(x,y) := (\beta(x),\beta(y))$, *respectively. We put* $\sigma = \kappa \circ \alpha$ *and* $\tau = \kappa \circ \beta$. *Then the following are equivalent:*

- **a.** *The polarities* $\hat\sigma$ *and* $\hat\tau$ *are conjugate under an element of* $\mathrm{Aut}(\mathcal{P}_\mathbb{O})$.
- **b.** *The polarities* $\hat\sigma$ *and* $\hat\tau$ *are conjugate under an element of the triangle stabilizer* Δ.
- **c.** *The polarities* $\hat\sigma$ *and* $\hat\tau$ *are conjugate under an element of* $\Delta_{(1,1)}$.
- **d.** *The anti-automorphisms* σ *and* τ *are conjugates under* $\mathrm{Aut}(\mathbb{O})$.
- **e.** *The collineations* $\tilde\alpha$ *and* $\tilde\beta$ *are conjugates in* $\mathrm{Aut}(\mathcal{P}_\mathbb{O})$.
- **f.** *The collineations* $\tilde\alpha$ *and* $\tilde\beta$ *are conjugates in* Δ.
- **g.** *The collineations* $\tilde\alpha$ *and* $\tilde\beta$ *are conjugates in* $\Delta_{(1,1)}$.
- **h.** *The automorphisms* α *and* β *are conjugates in* $\mathrm{Aut}(\mathbb{O})$.

Proof. Clearly assertion d implies c, that assertion implies b, and that implies a. Transitivity of $\mathrm{Aut}(\mathcal{P}_{\mathbb{O}})$ on triangles (see 1.8) yields that the first two assertions are equivalent.

We will show next that assertion b implies f; this will be the major task in the proof of the theorem. We will use the autotopism (cf. 1.4)

$$\psi_c := \tau_{1,c} \colon (x,y) \mapsto (x,cy), \quad (s) \mapsto (cs)\,.$$

Assume that there exists $\delta \in \Delta$ such that $\hat{\tau} = \delta \circ \hat{\sigma} \circ \delta^{-1}$. Write $\delta = \psi_c \circ \gamma_a \circ \delta'$ where δ' centralizes $\hat{\kappa}$, cf. 4.2. Then $\hat{\kappa} \circ \tilde{\beta} = \hat{\tau} = \psi_c \circ \gamma_a \circ \hat{\kappa} \circ \gamma_a^{-1} \circ \psi_c^{-1} \circ \delta \circ \tilde{\alpha} \circ \delta^{-1}$, and hence $\psi_c \circ \gamma_a \circ \hat{\kappa} \circ \gamma_a^{-1} \circ \psi_c^{-1} \circ \hat{\kappa} \circ \tilde{\beta} = \delta \circ \tilde{\alpha} \circ \delta^{-1}$.

Let $\delta = (A,B,C)$. We know from 3.1 that $\tau(a) = \tau(B(1)) = B(1) = a$ and hence $\beta(a) = \bar{a}$. Using $c \in Z$ and bi-associativity we compute that the composition $\psi_c \circ \gamma_a \circ \hat{\kappa} \circ \gamma_a^{-1} \circ \psi_c^{-1} \circ \hat{\kappa} \circ \tilde{\beta}$ is given by

$$(x,y) \mapsto \left(\beta(x)\big(N(a)\bar{a}c\big)^{-1},\, a\beta(y)\overline{a^{-1}}\right).$$

This mapping coincides with an involution (namely, $\delta \circ \tilde{\alpha} \circ \delta^{-1}$). Evaluating the square at the first component we find the condition $c\beta(c)N(a)^3 = 1$. Since $\beta(a) = \bar{a}$ this condition is equivalent to

$$\left(ca\beta(a)^2\right)\,\beta\left(ca\beta(a)^2\right) = 1.$$

Assume first that β is not the identity on $Z + Za$. Then Hilbert's Theorem 90 (cf. [8, VI 6.1]) implies that there exists an element $b \in Z + Za$ with $ca\beta(a)^2 = \beta(b)b^{-1}$. For the conjugate of $\gamma_a \circ \psi_c \circ \hat{\kappa} \circ \psi_c^{-1} \circ \gamma_a^{-1} \circ \hat{\kappa} \circ \tilde{\beta} = \delta \circ \tilde{\alpha} \circ \delta^{-1}$ under $(\gamma''_b \circ \psi_c)^{-1}$ we get

$$(x,y) \mapsto \left(\beta(x)\big(N(a)\bar{a}c\big)^{-1}\beta(b)b^{-1},\, b^{-1}c^{-1}a\beta(cbyb)(b\bar{a})^{-1}\right) = (\beta(x),\beta(y))\,.$$

Hence $\tilde{\beta}$ is conjugate to $\tilde{\alpha}$.

If β is the identity on $Z + Za$, then $a = \bar{a}$, and $(ca^3)^2 = 1$ yields $\varepsilon := ca^3 \in \{1,-1\}$. So the composition $\psi_c \circ \gamma_a \circ \hat{\kappa} \circ \gamma_a^{-1} \circ \psi_c^{-1} \circ \hat{\kappa} \circ \tilde{\beta} = \delta \circ \tilde{\alpha} \circ \delta^{-1}$ is given by

$$(x,y) \mapsto (\varepsilon\beta(x), a\beta(y)a^{-1})\,.$$

If $\beta = \mathrm{id}$ then the involution $\delta \circ \tilde{\alpha} \circ \delta^{-1}$ either has axis $[0,0]$ (if $\varepsilon = 1$) or it fixes precisely one affine point on that line (if $\varepsilon \neq 1$). In any case, our involution is not a Baer involution, and $\alpha = \mathrm{id} = \beta$ follows.

If $\beta \neq \mathrm{id}$ we choose $p \in \mathbb{O} \smallsetminus \{0\}$ such that $\beta(p) = -p$. Then the conjugate $\omega := \gamma''_p \circ (\delta \circ \tilde{\alpha} \circ \delta^{-1}) \circ (\gamma''_p)^{-1}$ is obtained as

$$(x,y) \mapsto \left(\beta(x),\, p\big(a(p^{-1}\beta(y)p^{-1}\big)a^{-1})p\right).$$

Now the product $\omega \circ \tilde{\beta}^{-1}$ fixes each point on the line $[0,0]$ and both the points (∞) and $(0,p^2)$ outside that line. This means $\omega \circ \tilde{\beta}^{-1} = \mathrm{id}$, and $\delta \circ \tilde{\alpha} \circ \delta^{-1} = \tilde{\beta}$ follows.

The last two assertions are clearly equivalent to each other; they are equivalent to assertion e by 2.3. Finally, assertion h implies d because $\mathrm{Aut}(\mathbb{O})$ centralizes the standard involution, see 1.3, 3.3. □

5 Centralizers of polarities

In this section, let σ be an involutory anti-automorphism of $\mathbb{O}$. The centralizer Ψ_σ of $\hat\sigma$ in $\mathrm{Aut}(\mathcal{P}_\mathbb{O})$ contains the group $\Xi_\sigma := \{\xi_{x,y} \mid (x,y) \in A_\sigma\}$ where $\xi_{x,y}$ maps (u,v) to $(u+x, v+\sigma(x)u+y)$.

Since Ξ_σ acts sharply transitively on A_σ it remains to understand the stabilizer $\nabla_\sigma := (\Psi_\sigma)_{o,\infty}$ of the two absolute points $o = (0,0)$ and ∞; then $\Psi_\sigma = \Xi_\sigma \circ \nabla_\sigma$.

5.1 Lemma. *The group ∇_σ acts faithfully on the points rows of $[0,0]$ and of L_∞.*

Proof. The polarity translates the action on L_∞ into the action on the line pencil in the pole $\hat\sigma(L_\infty) = \infty$. Thus if $\psi \in \nabla$ acts trivially on L_∞ then ψ has axis L_∞ and center ∞. Since ψ also fixes $(0,0)$ it is trivial. The argument for the kernel of the restriction to $[0,0]$ is analogous. □

5.2 Example (see [10, 3.5 (22), p. 107])**.** Mapping $(x,y) \in \mathbb{O} \times (\mathbb{O} \smallsetminus \{0\})$ to (xy^{-1}, y^{-1}) and $[m,b]$ to $[-b^{-1}m, b^{-1}]$ extends to a collineation that centralizes each one of the polarities $\hat\sigma$ where σ is an involutory anti-automorphism of $\mathbb{O}$.

5.3 Theorem. *Let π be a polarity of $\mathcal{P}_\mathbb{O} = (P,\mathcal{L})$ and let Ψ denote the centralizer of π in* $\mathrm{Aut}(\mathcal{P}_\mathbb{O})$*. Then Ψ acts two-transitively on the set U of absolute points.*

- **a.** *If $|U| \geq 2$ then π is a conjugate of $\hat\sigma$, for some involutory anti-automorphism σ of $\mathbb{O}$.*
- **b.** *The stabilizer $\Psi_{x,y}$ of two absolute points $x,y \in U$ acts transitely on $\pi(x) \smallsetminus \{x, \pi(x) \cap \pi(y)\}$.*
- **c.** *If U is not contained in a line then the stabilizer Ψ_x of $x \in U$ acts two-transitively on $\mathcal{L}_x \smallsetminus \{\pi(x)\}$.*

Proof. It suffices to consider the case where π has at least two absolute points. According to 3.6 we may then assume $\pi = \hat\sigma$ for some involutory anti-automorphism σ of $\mathbb{O}$.

Since Ξ_σ acts (sharply) transitively on A_σ the existence of a single element moving ∞ yields that $\Psi = \Psi_\sigma$ acts 2-transitively. Such an element was exhibited in 5.2.

It remains to show that ∇_σ acts transitively on $[0,0] \smallsetminus \{(0,0)\}$; then joining with ∞ gives transitivity of ∇_σ on $\mathcal{L}_\infty \smallsetminus \{[0], L_\infty\}$. If U is not contained in a line then two-transitivity of Ψ yields that $[0]$ can be moved by the stabilizer of ∞, and the last assertion follows.

Consider $x \in \mathbb{O} \smallsetminus \{0\}$. Transitivity of the triangle stabilizer (cf. 1.8) yields that there exists $\delta \in \Delta$ with $\delta(x,0) = (1,0)$. The conjugate $\delta \circ \hat\sigma \circ \delta^{-1}$ is a polarity having $((0,0),[0,0])$ and (∞, L_∞) as absolute flags. From 3.6 we know that there exists $\varphi \in \Delta_{(1,0)}$ and some involutory anti-automorphism τ of $\mathbb{O}$ such that $\varphi \circ (\delta \circ \hat\sigma \circ \delta^{-1}) \circ \varphi^{-1} = \hat\tau$. By 4.3 there exists $\gamma \in \Delta_{(1,1)}$ with $\gamma \circ \hat\tau \circ \gamma^{-1} = \hat\sigma$. We have thus found $\psi := \gamma \circ \varphi \circ \delta \in \nabla_\sigma$ with $\psi(x,0) = (1,0)$, as required. □

5.4 Remark. The extra assumption in 5.3.c may look strange to a reader who is not familiar with characteristic two. In fact, if we take for σ a Z-linear but not standard involutory automorphism of an octonion field of characteristic 2 then the absolute points of $\hat\sigma$ form a infinite proper subset of $[0] \cup \{\infty\}$, see [7, 7.2]. Note also that polarities with precisely one absolute point do exist in the characteristic two case, cf. [7, 9.1].

5.5 Remarks. Consider $a \in \mathbb{O} \smallsetminus \{0\}$. Straightforward computation yields:

- **a.** The map γ''_a belongs to the centralizer of $\hat\sigma$ precisely if $a \in \mathrm{Fix}(\sigma)$.
- **b.** The maps γ'_a or γ_a belong to that centralizer only if $a \in Z \cap \mathrm{Fix}(\sigma)$ and $a^3 = 1$. If this is the case then $\gamma'_a = \gamma_a = \gamma''_{a^{-1}}$.
- **c.** The map $\tilde\gamma_a := \gamma'_{-1} \circ \gamma''_a$ centralizes $\hat\sigma$ precisely if $a \in \mathrm{Fix}(-\sigma)$; i.e., if $\sigma(a) = -a$.
- **d.** For $r,t \in Z(\mathbb{O}) \smallsetminus \{0\}$ the collineation $\tau_{r,t}$ centralizes $\hat\sigma$ precisely if $\sigma(r)r = t$.

If σ is Z-linear one can use this to exhibit transitive subsets in ∇_σ quite explicitly. To this end, consider the vector spaces $W_\sigma^s := \{x \in \mathbb{O} \mid \sigma(x) = sx\}$ for $s \in \{+,-\}$.

If $\sigma = \kappa$ then 1.6 yields that $\{\gamma_a'' \mid a \in W_\kappa^- \smallsetminus \{0\}\}$ generates a subgroup of ∇_κ that is transitive on $[0,0] \smallsetminus \{(0,0)\}$.

If $\sigma \neq \kappa$ is $Z(\mathbb{O})$-linear then 1.6 yields that $\{\gamma_{-1}' \circ \gamma_a'' \mid a \in W_\sigma^+ \smallsetminus \{0\}\}$ generates a subgroup of ∇_σ that is transitive on $[0,0] \smallsetminus \{(0,0)\}$.

If σ is not Z-linear then $\{\gamma_a'' \mid a \in W_\kappa^- \smallsetminus \{0\}\} \cup \{\gamma_{-1}' \circ \gamma_a'' \mid a \in W_\sigma^+ \smallsetminus \{0\}\} \subseteq \nabla_\sigma$ generates a subgroup Γ of ∇_σ which is *not* transitive on $[0,0] \smallsetminus \{(0,0)\}$. In fact, each $a \in W_\sigma^s$ satisfies $N(a) \in S := \mathrm{Fix}_Z(\sigma)$. Thus the orbit of $(1,0)$ under the group Γ is contained in $\{x \in \mathbb{O} \mid N(x) \in S\} \times \{0\} \neq [0,0]$.

5.6 Remark. If we derive the transitivity properties 5.3 of ∇_σ without using the solution 4.3 of the conjugacy problem (as we did in 5.5 for linear σ), we can deduce that solution in a more direct fashion, as follows.

Assume that there exists $\varphi \in \mathrm{Aut}(\mathcal{P}_\mathbb{O})$ such that $\varphi \circ \hat{\vartheta} \circ \varphi^{-1} = \hat{\sigma}$. Using the transitivity properties of the centralizer Ψ_σ (see 5.3) of $\hat{\sigma}$ together with the transitivity of ∇_σ we may assume that φ fixes the points ∞, $(0,0)$, and $(1,0)$. Now φ fixes the line $[1]$ that joins $(0,0)$ with ∞, and $[1,0] = \hat{\sigma}(1,0) = \hat{\sigma}(\varphi(1,0)) = \varphi(\hat{\vartheta}(1,0)) = \varphi([1,0])$ yields that $[1,0]$ is also fixed by φ. Thus φ fixes the standard quadrangle, and there exists an automorphism α of $\mathbb{O}$ such that $\varphi(x,y) = (\alpha(x), \alpha(y))$ holds for all $(x,y) \in \mathbb{O}^2$. Evaluating $\varphi \circ \hat{\vartheta} \circ \varphi^{-1} = \hat{\sigma}$ at $(x,0)$ we obtain $\alpha \circ \vartheta \circ \alpha^{-1} = \sigma$.

5.7 Remark. The group Ξ_σ is nilpotent of class at most 2; in fact we have $\Xi_\sigma' \leq \mathrm{T}_\sigma := \{\xi_{0,y} \mid y \in \mathbb{O}, \sigma(y) = -y\}$. If $\operatorname{char} \mathbb{O} \neq 2$ then $\Xi_\sigma' = \mathrm{T}_\sigma$ and Ξ_σ is a generalized Heisenberg group. In general, such groups are good candidates for groups with many automorphisms, see [9], [14], [15], [3].

The group ∇_σ normalizes Ξ_σ; by 5.3 it induces a transitive group on $\Xi_\sigma/\mathrm{T}_\sigma \smallsetminus \{\mathrm{T}_\sigma\}$. However, transitivity of the action on T_σ is a strong condition, see [17].

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