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# Baer involutions and polarities in Moufang planes of characteristic two

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**Abstract**

We classify those polarities of Moufang planes in characteristic two that have at least one absolute point. In many cases the absolute points form a reasonable unital but it turns out that the set of absolute points may also form a rather small collinear set. Along the way, we determine the Baer involutions of the planes in question and show that the corresponding Baer subplanes will be pappian or non-desarguesian Moufang planes; there do exist desarguesian Baer subplanes that are not pappian but these are not fixed pointwise by any involution.

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**Keywords.** Moufang plane, translation plane, Baer involution, polarity, conjugacy, semifield, division algebra, alternative algebra, composition algebra, octonion field, automorphism, autotopism.

## Introduction

We consider *proper* (i.e., non-desarguesian) Moufang planes. Such a plane is coordinatized by an octonion field $\mathbb{O}$ (i.e., a non-associative, alternative division ring). The first two chapters of [8] form a convenient source for the basic theory of such algebras; see also Chapters 9 and 20 in [11]. Throughout the present note, we will concentrate on the case where char $\mathbb{O} = 2$.

For our present purposes it will be convenient to consider the projective plane $\mathrm{P}_2(\mathbb{O})$ as the projective hull of the affine plane. We describe the affine points by pairs $(x, y) \in \mathbb{O}^2$, lines are either of the form $[m, b] := \{(x, mx + b) \mid x \in \mathbb{O}\}$ or $[c] := \{c\} \times \mathbb{O}$ with $m, b, c \in \mathbb{O}$, cf. [3, p. 13]. The line at infinity will be denoted by $L_\infty$, it contains the parallel class $\infty$ of all "vertical" lines (of the form $[c]$) and the class $(m) := \{[m, b] \mid b \in \mathbb{O}\}$ for each $m \in \mathbb{O}$. See [2] for a more projective viewpoint of Moufang planes in characteristic 2.

We are interested in polarities and their absolute points. It comes as no surprise that involutory (anti-)automorphisms of octonion fields play an important role in this context.

According to a fundamental observation by R. Baer [1] involutory collineations may be elations (here our assumption char $\mathbb{O} = 2$ is used for the first time) or Baer involutions (i.e., involutions fixing a non-degenerate quadrangle). The fixed elements of a Baer involution form a subplane with the property that every point of the ambient plane is incident with at least one line of the subplane and, dually, every line is incident with at least one point of the subplane (such a subplane is called a *Baer subplane*). Since the collineation group of our Moufang plane acts transitively ([7, 7.3.14], see also [5, 2.7]) on the set of quadrangles, each one of the Baer involutions is conjugate to one of the form $(x, y) \mapsto (\iota(x), \iota(y))$ where $\iota$ is an automorphism of $\mathbb{O}$.

# 1 Octonion fields

**1.1 The norm and the standard involution.** We will consider $\mathbb{O}$ as a composition algebra over its center $Z := Z(\mathbb{O})$. There is a (unique, cf. [8, 1.2.4]) multiplicative quadratic form $N\colon \mathbb{O} \to Z$. This form is anisotropic because $\mathbb{O}$ contains no divisors of zero. The corresponding polar form

$$f_N\colon (x,y) \mapsto \langle x|y\rangle := N(x+y) - N(x) - N(y)$$

is non-degenerate (and alternating because $\operatorname{char} Z = 2$).

The *standard involution* of $\mathbb{O}$ is the anti-automorphism given by

$$\kappa\colon \mathbb{O} \to \mathbb{O}\colon a \mapsto \bar{a} := f_N(a,1) - a\,.$$

From $a^2 = (f_N(a,1) - \bar{a})\,a$ we see that $\mathbb{O}$ is a quadratic algebra over $Z$: for each $a \in \mathbb{O}$ there are unique elements $n, t \in Z$ such that $a^2 = -ta - n$, namely the norm $n = N(a) = \bar{a}a$ and trace $t = f_N(a,1) = \bar{a} + a$ of $a$; cf. [8, 1.2.3].

The involution is indeed a standard one (cf. [5, 2.2]):

**1.2 Lemma.** *Every ring automorphism of $\mathbb{O}$ centralizes the standard involution. Consequently, every automorphism and every anti-automorphism of $\mathbb{O}$ is a semi-similitude of the norm form.* □

# 2 Involutory automorphisms of octonion fields

Of course, every automorphism of $\mathbb{O}$ leaves $Z$ invariant. Automorphisms thus come in two flavors; they may be $Z$-linear or fail to do so. From experience outside the realm of characteristic two one might be tempted to expect that quaternion fields occur as sets of fixed points of involutory $Z$-linear automorphisms. This is not the case, indeed we prove in this section that the set $\operatorname{Fix}(\iota)$ of fixed points of an involutory automorphism $\iota$ either forms an octonion field over some subfield of $Z$ (this happens if $\iota$ is not $Z$-linear) or is a four-dimensional commutative $Z$-subalgebra, see 2.1.

The reader should be warned that the following results heavily depend on the fact that our algebra has no divisors of zero; indeed there are subalgebras of dimensions 5 and 6 in *split* octonion algebras, and some of these occur as fixed point sets of involutory automorphisms (see [2, 4.11]).

**2.1 Theorem.** *If $\iota$ is a $Z$-linear involutory automorphism of $\mathbb{O}$ then $\operatorname{Fix}(\iota)$ is a commutative field, and thus a totally inseparable extension of degree 4 over $Z$.*

*Proof.* The trace map $\operatorname{Tr}_\iota\colon \mathbb{O} \to \mathbb{O}\colon x \mapsto \iota(x) + x$ is a $Z$-linear map; its kernel is $F := \operatorname{Fix}(\iota)$ and its range is contained in $F$. In particular, we have $\dim F \geq 4$. Choosing any $a \in \mathbb{O} \setminus F$ we obtain $F \cap aF = \{0\}$, and $\dim F \leq 4$ follows. Thus $\operatorname{Tr}_\iota$ may be regarded as a surjection onto $F$.

From 1.2 we know that $\iota$ commutes with $\kappa$. Thus the trace maps $\operatorname{Tr}_\iota$ and $\operatorname{Tr}_\kappa$ commute, and $F$ lies in the kernel $1^\perp = \operatorname{Fix}(\kappa)$ of $\operatorname{Tr}_\kappa$. This means that $F$ is totally isotropic with respect to the polar form $f_N$. Now we use $0 = f_N(x,y) = xy + yx$ and $\operatorname{char}\mathbb{O} = 2$ to conclude $xy = yx$ for $x, y \in F$. □

**2.2 Theorem.** *Let $\iota$ be an involutory automorphism of $\mathbb{O}$ that acts non-trivially on $Z$ and put $S := \operatorname{Fix}_Z(\iota)$. Then $Z/S$ is a separable quadratic field extension, and $\operatorname{Fix}(\iota)$ is an octonion field over $S$.*

*Proof.* We abbreviate $F := \mathrm{Fix}(\iota)$. Since $\iota$ commutes with the standard involution it is easy to see that the norm $N$ induces a (multiplicative) quadratic form $N^\iota\colon F \to S$.

In order to show $\dim_S F = 8$ we consider the $S$-linear endomorphism $\varphi := \iota + \mathrm{id}$ of $\mathbb{O}$ and pick $c \in Z$ with $\iota(c) = c + 1$. Then $\varphi(cF) = F \subseteq \varphi(\mathbb{O}) \subseteq F = \ker\varphi$ yields $\dim_S F = \dim_S(\mathbb{O}/F) = 16 - \dim_S F$.

Since the values of $f_N$ on $\mathbb{O} = F \oplus cF$ can be computed from values of the restriction $f_{N^\iota}$ of $f_N$ to $F$ we see that this restriction is non-degenerate, and $F$ is indeed an octonion algebra (over $S$). Of course, there are still no divisors of zero, and we have an octonion field. □

**2.3 Lemma.** *For each involution $\iota \in \mathrm{Aut}(\mathbb{O})$ the trace map $\mathrm{Tr}_\iota\colon \mathbb{O} \to \mathrm{Fix}(\iota)\colon a \mapsto \iota(a) + a$ is surjective.*

*Proof.* We abbreviate $S := \mathrm{Fix}_Z(\iota)$. Clearly $\mathrm{Tr}_\iota$ is $S$-linear with kernel $\mathrm{Fix}(\iota)$ and $\mathrm{Tr}_\iota(\mathbb{O}) \subseteq \mathrm{Fix}(\iota)$. Comparing dimensions over $S$ we obtain the assertion. □

**2.4 Theorem.** *If $\beta \in \mathrm{Aut}(\mathrm{P}(\mathbb{O}))$ is a Baer involution then the Baer subplane consisting of fixed points and fixed lines of $\beta$ is either pappian or not desarguesian.*

*Proof.* As $\mathrm{Aut}(\mathrm{P}(\mathbb{O}))$ acts transitively on quadrangles we may assume that $\beta$ is of the form $(x, y) \mapsto (\iota(x), \iota(y))$ with some involution $\iota \in \mathrm{Aut}(\mathbb{O})$. Now the assertion follows from 2.1 and 2.2. □

## 3 Involutory anti-automorphisms

Apart from the standard involution $\kappa$ our octonion field $\mathbb{O}$ will also admit other involutions that reverse the multiplication.

**3.1 Theorem.** *If $\sigma$ is an involutory anti-automorphism of $\mathbb{O}$ then either $\sigma = \kappa$ or $\sigma = \iota \circ \kappa$ where $\iota$ is one of the involutions discussed in 2.1 and 2.2.*

*Proof.* The automorphism $\iota := \sigma \circ \kappa$ commutes with the standard involution $\kappa$, see 1.2. Thus $\iota^2 = \mathrm{id}$ and either $\iota = \mathrm{id}$ or $\iota$ is known from 2.1 and 2.2. □

**3.2 Proposition.** *Consider $\iota \in \mathrm{Aut}(\mathbb{O})$ with $\iota^2 = \mathrm{id}$ and put $\sigma := \iota \circ \kappa$. The trace map $\mathrm{Tr}_\sigma\colon \mathbb{O} \to \mathrm{Fix}(\sigma)\colon a \mapsto \sigma(a) + a$ is surjective if, and only if, the restriction $\sigma|_Z = \iota|_Z$ of $\iota$ to the center $Z$ is not trivial. More precisely:*

- **a.** *If $\iota = \mathrm{id}$ then $\mathrm{Tr}_\sigma(\mathbb{O}) = Z$ and $\mathrm{Fix}(\sigma) = 1^\perp$. Thus $\dim_Z \mathrm{Tr}_\sigma(\mathbb{O}) = 1 < 7 = \dim_Z \mathrm{Fix}(\sigma)$ and $\mathrm{Tr}_\sigma(\mathbb{O})$ is the radical of the restriction $f_N|_{\mathrm{Fix}(\sigma)\times\mathrm{Fix}(\sigma)}$ of the polar form.*
- **b.** *If $\iota \neq \mathrm{id}$ but $\iota|_Z = \mathrm{id}$ then $\dim_Z \mathrm{Tr}_\sigma(\mathbb{O}) = 3 < 5 = \dim_Z \mathrm{Fix}(\sigma)$.*
  *More explicitly, we have $\mathrm{Fix}(\sigma) = \mathrm{Fix}(\iota) \oplus Zb$ for any $b \in \mathbb{O}$ satisfying $\mathrm{Tr}_\iota(b) \in Z \smallsetminus \{0\}$. Again, $\mathrm{Tr}_\sigma(\mathbb{O})$ is the radical of the restriction $f_N|_{\mathrm{Fix}(\sigma)\times\mathrm{Fix}(\sigma)}$ of the polar form.*
- **c.** *If $\iota|_Z \neq \mathrm{id}$ then $\mathrm{Fix}(\sigma) = \mathrm{Tr}_\sigma(\mathbb{O})$ and $\dim_S \mathrm{Fix}(\sigma) = 8$.*

*Proof.* As in 2.2 we abbreviate $S := \mathrm{Fix}_Z(\iota)$; then $\mathrm{Tr}_\sigma$ is $S$-linear and the kernel of $\mathrm{Tr}_\sigma$ is $\mathrm{Fix}(\sigma)$. We treat the three cases separately:

If $\iota = \mathrm{id}$ then $S = Z$ and $\dim_Z \mathrm{Tr}_\sigma(\mathbb{O}) = \dim_Z(\mathbb{O}/\mathrm{Fix}(\sigma)) = 1 < 7 = \dim_Z \mathrm{Fix}(\sigma)$. The observation $Z \leq \mathrm{Tr}_\sigma(\mathbb{O})$ now yields $\mathrm{Tr}_\sigma(\mathbb{O}) = Z = 1^\perp = \mathrm{Fix}(\sigma)^\perp$.

If $\iota \neq \mathrm{id}$ but $S = Z$ then $\mathrm{Fix}(\iota) \leq \mathrm{Fix}(\kappa)$ by 2.1. We pick $b \in \mathbb{O}$ such that $\mathrm{Tr}_\iota(b) \in Z \smallsetminus \{0\}$; such a $b$ exists by 2.3. Then the subalgebra $Z(b) = Z + Zb$ is invariant under $\iota$ and also invariant under $\kappa$ by 1.1. Both $\iota$ and $\kappa$ induce the generator of the Galois group $\mathrm{Gal}(Z(b)/Z)$ on $Z(b)$. Thus $\sigma(b) = b$, and $\dim_Z \mathbb{O} = \dim_Z \mathrm{Fix}(\sigma) + \dim_Z \mathrm{Tr}_\sigma(\mathbb{O})$ yields $\dim_Z \mathrm{Tr}_\sigma(\mathbb{O}) \le 3$. From $\mathrm{Fix}(\kappa) \cap \mathrm{Fix}(\sigma) = \mathrm{Fix}(\iota)$ we infer that $\mathrm{Tr}_\sigma(\mathrm{Fix}(\kappa))$ has dimension 3. Thus $\dim_Z \mathrm{Tr}_\sigma(\mathbb{O}) = 3$. Now $\mathrm{Fix}(\sigma)$ contains $\mathrm{Fix}(\iota)$ and $b \notin \mathrm{Fix}(\iota)$. Comparing dimensions we find $\mathrm{Fix}(\sigma) = \mathrm{Fix}(\iota) \oplus Zb$.

For any $x \in \mathrm{Fix}(\sigma)$ and $y \in \mathbb{O}$ we use 1.2 to compute $\langle x|\mathrm{Tr}_\sigma(y)\rangle = \langle x|y\rangle + \langle x|\sigma(y)\rangle = \langle x|y\rangle + \langle \sigma(x)|\sigma(y)\rangle = \langle x|y\rangle + \langle x|y\rangle = 0$. Thus $\mathrm{Tr}_\sigma(\mathbb{O}) \le \mathrm{Fix}(\sigma)^\perp$, and equality follows because the dimension is the right one.

Finally, consider the case where $S \ne Z$. Pick $z \in Z$ such that $\iota(z) = z + 1$. Then $\mathbb{O} = \mathrm{Fix}(\iota) \oplus z\,\mathrm{Fix}(\iota)$ and for any $x, y \in \mathrm{Fix}(\iota)$ we compute $\sigma(x + zy) = \bar{x} + \bar{y}(z + 1) = \overline{x + y} + z\bar{y}$. Thus $x + zy$ is fixed by $\sigma$ precisely if $y = x + \bar{x}$, and $\dim_S \mathrm{Fix}(\sigma) = 8 = \dim_S \mathrm{Tr}_\sigma(\mathbb{O})$. This means that $\mathrm{Fix}(\sigma)$ and $\mathrm{Tr}_\sigma(\mathbb{O})$ coincide. □

In 7.4 and 8.1 below we will locate the norms and traces of non-standard involutions more explicitly.

## 4 Existence of involutions

**4.1 Lemma** ([11, 20.16]). *Let $B$ and $C$ be subalgebras of $A$ with $B \le C$. If the restriction of the polar form $f_N$ of the norm $N$ to $B$ is non-zero then the restriction of $f_N$ to $C$ is non-degenerate.* □

**4.2 Lemma.** *Let $A$ be a composition algebra over $R$, with standard involution $\kappa\colon x \mapsto \bar{x}$ and multiplicative norm $N\colon x \mapsto N(x) := \bar{x}x$. Consider a subalgebra $B$ such that $\dim_R A = 2\dim_R B$ and the restriction of the polar form $f_N$ to $B$ is non-zero, and pick any $a \in B^\perp$ with $\gamma := N(a) \ne 0$. Then $A$ is the $\gamma$-double of $B$, i.e., we have $A = B \oplus Ba$ and the multiplication is given by*

$$(u + xa)(v + ya) = (uv + \gamma\bar{y}x) + (yu + x\bar{v})a$$

*for all $u, x, v, y \in B$. If $\alpha$ is a $Z$-linear automorphism of $\mathbb{O}$ with $\alpha(B) = B$ then there exist $c, p \in B$ with $N(c) \ne 0$ and $N(p) = 1$ such that $\alpha(v + ya) = cvc^{-1} + (pcyc^{-1})a$ holds for all $v, y \in B$.*

*Proof.* See [8, 1.5.1] and [8, Sec. 2.1]. □

**4.3 Proposition.** *For each quaternion subfield $\mathbb{H}$ of $\mathbb{O}$ there exists an involutory $Z$-linear automorphism of $\mathbb{O}$ leaving $\mathbb{H}$ invariant (but acting non-trivially on $\mathbb{H}$).*

*Proof.* Pick $a \in \mathbb{H}^\perp \setminus \{0\}$, then $\mathbb{O} = \mathbb{H} \oplus \mathbb{H}a$ is the $N(a)$-double of $\mathbb{H}$, see 4.2. The automorphism $\alpha\colon \mathbb{O} = \mathbb{H} \oplus \mathbb{H}a\colon x + ya \mapsto cxc^{-1} + (pcyc^{-1})a$ is an involution whenever we take $c \in \mathbb{H} \setminus \{0\}$ such that $c$ is orthogonal to both 1 and $p$ because then $\bar{c} = c$ yields $c^2 \in Z$ and $pcpc^{-1} = 1$. □

Our next aim is to construct involutory $Z$-linear automorphisms with given sets of fixed points. We will also see that such an involution is not determined by its set of fixed points. This is in marked contrast to the case of non-linear automorphisms, or the situation if the characteristic is different from 2; see [5, 3.4].

We start with an observation by Faulkner [2, 4.11]:

**4.4 Lemma.** *Let $D = D^\perp$ be a (four-dimensional commutative) subalgebra of $\mathbb{O}$. For each $s \in \mathbb{O} \setminus D$ there exists a unique $Z$-linear automorphism $\alpha_s$ of $\mathbb{O}$ with $D \le \mathrm{Fix}(\alpha_s)$ such that $\alpha_s^2 = \mathrm{id}$ and $\alpha_s(s) = \bar{s}$.* □

**4.5 Theorem.** *Let $D = D^\perp$ be a (four-dimensional commutative) subalgebra of $\mathbb{O}$, and pick $c \in \mathbb{O} \smallsetminus D$. Then $\mathbb{O} = D \oplus Dc$, and a map $\alpha\colon \mathbb{O} \to \mathbb{O}$ belongs to the global stabilizer of $D$ in* $\mathrm{Aut}(\mathbb{O})$ *precisely if there exists $u \in D$ with $u^2 = \langle c|u\rangle$ such that $\alpha(x + yc) = x + yu + yc$ for all $x, y \in D$. In particular, we have $\alpha^2 = \mathrm{id}$.*

*Proof.* If a $Z$-linear automorphism $\gamma$ of $\mathbb{O}$ fixes $D$ globally then it induces an element of the Galois group $\mathrm{Gal}(D/Z)$ on $D$. As the extension $D/Z$ is purely inseparable, this group is trivial, and $\gamma$ fixes $D$ pointwise. We write $\gamma(c) = u + vc$ with $u, v \in D$. The intersection $D \cap c^\perp$ is invariant under $\gamma$ because the latter acts trivially on $D$. Thus $(D \cap c^\perp)^\perp = D^\perp + c^{\perp\perp} = D + Zc$ is also invariant, and $v \in Z$ follows. If $\bar{c} \neq c$ then $\bar{c} + c = \gamma(\bar{c} + c) = v(\bar{c} + c)$ yields $v = 1$, and $\gamma^2 = \mathrm{id}$ follows. If $\bar{c} = c$ we use temporarily replace $c$ by an element with non-vanishing trace to see $\gamma^2 = \mathrm{id}$ and then compute $c = \gamma^2(c) = u + v(u + vc) = (u + vu) + v^2c \in D \oplus Dc$ to see $v = 1$, again.

From $N(c) = \gamma(N(c)) = N(\gamma(c)) = (\overline{u + c})(u + c) = (u + \bar{c})(u + c) = u^2 + \langle c|u\rangle + N(c)$ we now infer $u^2 = \langle c|u\rangle$. Thus every $Z$-linear automorphism stabilizing $D$ is an involution of the form given in the statement of the theorem.

Conversely, consider $u \in D$ with $u^2 = \langle c|u\rangle$. In order to show that $\alpha(x + yc) := x + yu + yc$ defines an automorphism of $\mathbb{O}$ we put $s := uc$, then $\bar{s} = \bar{c}\bar{u} = \bar{c}u = uc + u^2$. For any $y \in D$ we obtain $yc = (u^{-1}yu)c = N(u)^{-1}((uy)u)c = N(u)^{-1}u(y(uc))$ by one of the Moufang identities. Using the automorphism $\alpha_s$ from 4.4 we compute $\alpha_s(x + yc) = \alpha_s\left(x + N(u)^{-1}u(y(uc))\right) = \alpha_s(x) + \alpha_s\left(N(u)^{-1}u(ys)\right) = x + N(u)^{-1}u(y\bar{s}) = x + N(u)^{-1}u(y(uc + u^2)) = x + yc + uy = x + yc + yu$. Thus $\alpha = \alpha_s$ is an automorphism. □

**4.6 Corollary.** *The (global) stabilizer of a four-dimensional commutative subalgebra of $\mathbb{O}$ in the group of $Z$-linear automorphisms is an elementary abelian group.* □

Every involutory $Z$-linear anti-automorphism arises from an involution as in 4.3:

**4.7 Lemma.** *Let $\sigma$ be a $Z$-linear involutory anti-automorphism of $\mathbb{O}$. Then every element of $\mathbb{O}$ lies in a quaternion subalgebra which is invariant both under $\sigma$ and under $\iota = \sigma \circ \kappa$. Moreover, if $\sigma \neq \kappa$ then the restrictions of $\sigma$ and $\kappa$ to that subalgebra are distinct.*

*Proof.* Every $Z$-subalgebra $A$ of $\mathbb{O}$ is invariant under $\kappa$ because $a \in A$ implies $\bar{a} \in Z + Za \subseteq A$. Thus the case $\sigma = \kappa$ is clear; we concentrate on the case $\sigma \neq \kappa$.

Consider $x \in \mathbb{O}$. We will construct a subalgebra $\mathbb{H}$ containing $x$ such that $\dim_Z \mathbb{H} = 4$ and $\kappa|_{\mathbb{H}} \neq \mathrm{id}$; then $\kappa_{\mathbb{H}}$ has non-degenerate polar form (by [11, 20.16], see 4.1), and $\mathbb{H}$ is indeed a quaternion algebra.

If $x \in Z$ then we choose $y \in \mathbb{O} \smallsetminus (\mathrm{Fix}(\sigma) \cup \mathrm{Fix}(\kappa))$. Then either $y$ and $\sigma(y)$ generate a four-dimensional subalgebra $\mathbb{H}$ or $\sigma(y) \in Z(y)$. In the latter case we adjoin $w \in \mathrm{Fix}(\sigma) \smallsetminus Z(y)$ to $Z(y)$ and put $\mathbb{H} := Z(y) + Z(y)w$.

If $x = \sigma(x) \notin Z$ we choose $w \in \mathrm{Fix}(\sigma) \smallsetminus (Z(x) \cup \mathrm{Fix}(\kappa))$ and put $\mathbb{H} := Z(x) + Z(x)w$.

There remains the case where $x \neq \sigma(x)$. The $Z$-algebra $A$ generated by $x$ and $\sigma(x)$ cannot be purely inseparable because $\sigma|_A \neq \mathrm{id}$ is an involution. From 4.1 we infer that either $A$ is a quaternion algebra (and we put $\mathbb{H} := A$) or $\dim_Z A = 2$ and $A = Z(x)$. In the latter case, we pick $w \in \mathrm{Fix}(\sigma) \smallsetminus Z(x)$ and put $\mathbb{H} := Z(x) + Z(x)w$.

In any case, we have defined a non-commutative four-dimensional subalgebra $\mathbb{H}$ which is invariant under $\sigma$. If the restrictions $\sigma|_{\mathbb{H}}$ and $\kappa|_{\mathbb{H}}$ were the same then $\mathbb{H} \leq \mathrm{Fix}(\iota)$ would imply that $\mathbb{H}$ is commutative, see 2.1. This would contradict the fact that $\kappa|_{\mathbb{H}} \neq \mathrm{id}$ by construction. □

## 5 Polarities

A *polarity* of a projective plane is a map $\pi$ interchanging points with lines such that incidence is preserved and $\pi^2 = \mathrm{id}$. We consider the set $\mathrm{Abs}(\pi)$ of *absolute points*; i.e. points incident with their image under $\pi$. The pair $(p, \pi(p))$ is called an *absolute flag* if $p$ lies on $\pi(p)$.

If there exists an absolute point we may use flag-transitivity of the automorphism group of $\mathrm{P}_2(\mathbb{O})$ to achieve that the point $\infty$ is absolute, with $\pi(\infty) = L_\infty$. Since $\mathbb{O}$ has characteristic 2 there do exist polarities with precisely one absolute flag; cf. 9.1 below.

Polarities with this absolute flag and at least one more absolute point can be treated as in [9], [10]. In particular, we have the following source of examples:

**5.1 Construction** ([6, 4.5])**.** Let $\sigma$ be an involutory anti-automorphism of $\mathbb{O}$.

**a.** The map $(u, v) \mapsto [\sigma(u), -\sigma(v)]$ extends to a polarity $\hat\sigma$ with $\hat\sigma([c]) = (\sigma(c))$ and $\hat\sigma(\infty) = L_\infty$.

**b.** The set of affine absolute points is $A_\sigma = \{(u, v) \mid u, v \in \mathbb{O}, \sigma(v) + v = \sigma(u)u\}$, and $U_\sigma := A_\sigma \cup \{\infty\}$ is the set of all absolute points.

**c.** For $x, y \in K$ we put

$$\xi_{x,y} \colon (u, v) \mapsto (u + x, v + \sigma(x)u + y)\ .$$

Then $\Xi_\sigma := \{\xi_{x,y} \mid x, y \in K, \sigma(y) + y = \sigma(x)x\}$ is a subgroup of the centralizer of the polarity $\hat\sigma$.

**d.** The subgroup $\mathrm{T}_\sigma := \{\xi_{0,p} \mid p \in \mathrm{Fix}(\sigma)\}$ of $\Xi_\sigma$ fixes each line through $\infty$ and acts regularly on the set of affine absolute points on any vertical line. □

**5.2 Proposition.** *The group $\mathrm{T}_\sigma$ contains the commutator group of $\Xi_\sigma$. Equality holds precisely if $\sigma|_Z \neq \mathrm{id}$.*

*Proof.* A straightforward computation yields $\xi_{a,b}^{-1} \circ \xi_{x,y}^{-1} \circ \xi_{a,b} \circ \xi_{x,y} = \xi_{0,\sigma(a)x+\sigma(x)a}$. Thus the set of commutators equals $\{\xi_{0,t} \mid \exists y \in \mathbb{O} \colon t = \mathrm{Tr}_\sigma(y)\}$. Now 3.2 gives the assertion. □

**5.3 Remark.** If $\sigma$ is a non-standard $Z$-linear involutory anti-automorphism then it turns out (see 7.5 below) that $\Xi_\sigma = \mathrm{T}_\sigma$ is an elementary abelian group.

In fact, the construction in 5.1 yields (up to conjugation) *each* polarity with more than one absolute point.

**5.4 Theorem** ([5, 3.4])**.** *Let $\mathcal{P}$ be a projective plane with a polarity $\pi$. If $\mathcal{P}$ is a translation plane and $\pi$ has at least two absolute points then there is a semifield $K$ with an anti-automorphism $\sigma$ and an isomorphism $\eta \colon \mathcal{P} \to \mathcal{P}_K$ such that $\eta \circ \pi \circ \eta^{-1} = \hat\sigma$, as defined in 5.1.* □

Since $\mathrm{Aut}(\mathrm{P}_2(\mathbb{O}))$ acts transitively on quadrangles, we obtain:

**5.5 Corollary.** *Every polarity of $\mathrm{P}_2(\mathbb{O})$ with at least two absolute points is conjugate to a polarity $\hat\sigma$ defined by an involutory anti-automorphism $\sigma$ of $\mathbb{O}$.* □

Combining 5.5 with 2.2 and 2.1 we thus know all polarities of $\mathrm{P}_2(\mathbb{O})$ that have at least two absolute points. See 9.1 for examples of polarities with only one absolute point and Section 10 for a very brief discussion of the problem of existence of polarities with no absolute points at all.

## 6 The centralizer, and conjugacy

**6.1 Example** (see [7, 3.5 (22), p. 107]). Mapping $(x, y) \in \mathbb{O} \times (\mathbb{O} \smallsetminus \{0\})$ to $(xy^{-1}, y^{-1})$ and $[m, b]$ to $[-b^{-1}m, b^{-1}]$ extends to a collineation that centralizes each one of the polarities $\hat{\sigma}$ where $\sigma$ is an involutory anti-automorphism of $\mathbb{O}$.

Together with the group $\Xi_\sigma$ from 5.1 we obtain:

**6.2 Theorem.** *The centralizer $\Psi_\sigma$ of $\hat{\sigma}$ acts two-transitively on $U_\sigma$.*

Since $\Xi_\sigma$ acts sharply transitively on $U_\sigma \smallsetminus \{\infty\}$ it remains to determine the stabilizer $\nabla_\sigma$ of the two absolute points $o = (0, 0)$ and $\infty$ in order to find the stabilizer of $\infty$ in the centralizer $\Psi_\sigma$. The group $\nabla_\sigma$ also fixes the pole $\hat{\sigma}([0]) = (0)$. We study $\nabla_\sigma$ in [5].

**6.3 Proposition.** *Let $\sigma$ and $\vartheta$ be involutory anti-automorphisms of $\mathbb{O}$. Then $\nabla_\sigma$ acts transitively on $[0, 0] \smallsetminus \{(0, 0)\}$, and $\hat{\sigma}$ and $\hat{\vartheta}$ are conjugates under* $\mathrm{Aut}(\mathrm{P}_2(\mathbb{O}))$ *if, and only if, the involutions $\sigma$ and $\vartheta$ are conjugates under* $\mathrm{Aut}(\mathbb{O})$.

*Proof.* The assertion about transitivity is proved in [5, 5.3]. Clearly $\hat{\sigma}$ and $\hat{\vartheta}$ are conjugates under $\mathrm{Aut}(\mathrm{P}_2(\mathbb{O}))$ if $\sigma$ and $\vartheta$ are conjugates under $\mathrm{Aut}(\mathbb{O})$. The converse has been proved in [5, 4.3]. □

## 7 Polarities induced by linear involutions

From 5.2 we infer for the standard involution $\kappa$ :

**7.1 Theorem.** $\mathrm{T}_\kappa = \mathrm{Z}(\Xi_\kappa) = \{\xi_{0,p} \mid p \in \mathrm{Fix}(\kappa)\}$ *and* $\Xi'_\kappa = \{\xi_{0,t} \mid t \in \mathrm{Z}(\mathbb{O})\}$. □

The rest of this section is concerned with the fact that the set of all absolute points is collinear (in fact, contained in the vertical [0]) if $\sigma$ is $Z$-linear but $\sigma \neq \kappa$. We study the quaternion case first.

**7.2 Theorem.** *Let $\mathbb{H}$ be a quaternion field of characteristic 2 and let $\sigma$ be an involutory anti-automorphism that is linear over the center $Z$ of $\mathbb{H}$. If $\sigma$ is not the standard involution then the absolute points of the polarity $\hat{\sigma}$ are collinear. Explicitly, the set of absolute points is* $\{(0, y) \mid y \in \mathrm{Fix}(\sigma)\} \cup \{\infty\}$.

*Proof.* Every $Z$-linear automorphism of $\mathbb{H}$ is inner (by the Skolem-Noether Theorem). Thus each $Z$-linear anti-automorphism $\sigma$ is of the form $\sigma(w) = a\overline{w}a^{-1}$ with some $a \in \mathbb{H}$, and $\sigma^2 = \mathrm{id} \iff \bar{a} = \pm a \iff a^2 \in Z$. If this is the case then there exists a subfield $B \leq \mathbb{H}$ which is a separable extension of $Z$ and contained in $a^\perp$. Thus $\mathbb{H}$ is the $\gamma$-double of $B$, see 4.2.

For any $w = v + ya \in \mathbb{H}$ with $v, y \in B$ we compute $\sigma(w) = (a(\overline{v + ya}))\, a^{-1} = (a\overline{v})a^{-1} + ay = v + \overline{y}a$ and then $\sigma(w)\, w = (v + \overline{y}a)(v + ya) = (v^2 + \overline{y}^2\gamma) + (vy + \overline{vy})a$ and $\sigma(w) + w = (y + \overline{y})a$. Thus the set of traces of $\sigma$ is $Za$.

Now $\gamma = a^2$ cannot be the square of an element of $B$ because the extension $B/Z$ is separable while $Z(a)/Z$ is inseparable (it is here that we use $\mathrm{char}\, \mathbb{H} = 2$). Thus $\sigma(w)\, w$ is a trace of $\sigma$ only if $w = 0$. This means that the set $U_\sigma$ of absolute points of the polarity $\hat{\sigma}$ is contained in a single line, cf. 5.1.b. □

**7.3 Theorem.** *Let $\mathbb{O}$ be an octonion field of characteristic 2 and let $\sigma$ be an involutory anti-automorphism that is linear over the center $Z$ of $\mathbb{O}$. If $\sigma$ is not the standard involution then the absolute points of the polarity $\hat{\sigma}$ are collinear. Explicitly, the set of absolute points is* $\{(0, y) \mid y \in \mathrm{Fix}(\sigma)\} \cup \{\infty\}$.

*Proof.* It is clear that the vertical $[0]$ contains infinitely many absolute points, namely those in $\{(0, y) \mid y \in \mathrm{Fix}(\sigma)\}$.

Aiming at a contradiction, we assume that there exists an absolute point $(u, v)$ with $u \neq 0$. Now 4.7 yields that $u$ is contained in some $\sigma$-invariant quaternion field $\mathbb{H}$. According to 7.2 there is no $y \in \mathbb{H}$ with $\sigma(u)u = \sigma(y) + y$. Thus the affine subspaces $v + \mathrm{Fix}(\sigma)$ and $\mathbb{H}$ have empty intersection. This means $\mathbb{H} \subseteq \mathrm{Fix}(\sigma)$, contradicting the fact that $\sigma$ induces an anti-automorphism on the non-commutative quaternion field $\mathbb{H}$. □

**7.4 Proposition.** *Let $\sigma$ be a non-standard $Z$-linear involutory anti-automorphism of $\mathbb{O}$, and consider the "norm" $\mathrm{N}_\sigma(x) := \sigma(x)x$ and the "trace" $\mathrm{Tr}_\sigma(x) := \sigma(x) + x$.*

**a.** *The intersection of $\{\sigma(x)x \mid x \in \mathbb{O}\}$ and $\{\sigma(y) + y \mid y \in \mathbb{O}\}$ is $\mathrm{N}_\sigma(\mathbb{O}) \cap \mathrm{Tr}_\sigma(\mathbb{O}) = \{0\}$.*

**b.** *The map $\mathrm{N}_\sigma \colon \mathbb{O} \to \mathbb{O}$ is injective, and it induces an injective* frob*-semilinear map*

$$\hat{\mathrm{N}}_\sigma \colon \mathbb{O} \to \mathrm{Fix}(\sigma)/\mathrm{Tr}_\sigma(\mathbb{O}) \colon x \mapsto \mathrm{N}_\sigma(x) + \mathrm{Tr}_\sigma(\mathbb{O})$$

*where* $\mathrm{frob} \colon s \to s^2$ *is the Frobenius endomorphism.*

**c.** *If $\mathrm{N}_\sigma(x) + \mathrm{Tr}_\sigma(u) = \mathrm{N}_\sigma(y) + \mathrm{Tr}_\sigma(v)$ then $x = y$ and $\mathrm{Tr}_\sigma(u) = \mathrm{Tr}_\sigma(v)$.*

**d.** *We have $\mathrm{Fix}(\iota) = Z \oplus \mathrm{Tr}_\sigma(\mathbb{O})$ and $\mathrm{Fix}(\sigma) = \mathrm{Fix}(\iota) \oplus Zb$ for any $b \in \mathbb{O}$ with $\mathrm{Tr}_\iota(b) \in Z \smallsetminus \{0\}$.*

*Proof.* The first assertion a is clear from 7.3. For the second one we assume $\mathrm{N}_\sigma(x) = \mathrm{N}_\sigma(y)$ and compute $\mathrm{N}_\sigma(x - y) = \mathrm{N}_\sigma(x) + \mathrm{N}_\sigma(y) + \sigma(x)y + \sigma(y)x$. Using our assumption we find $\mathrm{N}_\sigma(z) = \sigma(\sigma(y)x) + \sigma(y)x = \mathrm{Tr}_\sigma(\sigma(y)x)$. Then assertion a yields $x = y$ as claimed.

From $\mathrm{N}_\sigma(x) + \mathrm{Tr}_\sigma(u) = \mathrm{N}_\sigma(y) + \mathrm{Tr}_\sigma(v)$ we infer $\mathrm{Tr}_\sigma(u) + \mathrm{Tr}_\sigma(v) = \mathrm{N}_\sigma(x) + \mathrm{N}_\sigma(y) = \mathrm{N}_\sigma(x + y) + \mathrm{Tr}_\sigma(\sigma(y)x)$. Thus $\mathrm{N}_\sigma(x + y)$ is a trace, assertion a yields $x = y$, and assertion c is established.

Finally, choose any $b \in \mathbb{O}$ such that $\iota(b) + b \in Z \smallsetminus \{0\}$ and recall from 3.2 the decomposition $\mathrm{Fix}(\sigma) = \mathrm{Fix}(\iota) \oplus Zb$. The set of traces forms a hyperplane in $\mathrm{Fix}(\iota)$, cf. 3.2. Since each element of $Z \smallsetminus \{0\}$ yields a non-trivial element in $Z \cap \mathrm{N}_\sigma(\mathbb{O})$ assertion a yields $\mathrm{Fix}(\iota) = Z \oplus \mathrm{Tr}_\sigma(\mathbb{O})$. It remains to note $\mathrm{N}_\sigma(b) = b^2 = \bar{b}b + b \in Z(b) \smallsetminus Z$ to establish $\mathrm{Fix}(\sigma) = \mathrm{Fix}(\iota) \oplus Zb$. □

**7.5 Corollary.** *If $\sigma$ is a non-standard $Z$-linear involutory anti-automorphism of $\mathbb{O}$ then $\Xi_\sigma = \mathrm{T}_\sigma = \{\xi_{0,p} \mid p \in \mathrm{Fix}(\sigma)\}$ is elementary abelian.* □

**7.6 Proposition.** *Let $\sigma$ be a non-standard involutory $Z$-linear anti-automorphism of $\mathbb{O}$. Put $D := \mathrm{Fix}(\kappa \circ \sigma)$. Then $\mathrm{Fix}(\sigma) = \mathrm{N}_\sigma(\mathbb{O}) + \mathrm{Tr}_\sigma(\mathbb{O})$ holds precisely if $Z = \{u^2 \mid u \in D\}$.*

*Proof.* The set $D$ is a four-dimensional purely inseparable extension of $Z$, see 2.1. From 4.4 we know that there exists $s \in \mathbb{O} \smallsetminus D$ such that $(\kappa \circ \sigma)(x + ys) = x + y\bar{s}$ for $x, y \in D$. This gives $\sigma(x + ys) = x + sy$. We have $\bar{s} \neq s$ because $\sigma \neq \kappa$; in particular, we have $(Z + Zs) \cap D = Z$.

Clearly $D + Zs$ is contained in $\mathrm{Fix}(\sigma)$, and $\dim \mathrm{Fix}(\sigma) = 5$ yields $D + Zs = \mathrm{Fix}(\sigma) = \ker \mathrm{Tr}_\sigma$, see 3.2.b. For $x, y \in D$ we compute $\mathrm{Tr}_\sigma(x + ys) = x + sy + x + ys = sy + ys = \langle s|y\rangle + (s + \bar{s})y$. These elements lie in $s^\perp$. The intersection of $Z$ and $\mathrm{Tr}_\sigma(\mathbb{O})$ is trivial because $\langle s|1\rangle = s + \bar{s} \neq 0$. Comparing dimensions we get $\mathrm{Tr}_\sigma(\mathbb{O}) = \{\langle s|y\rangle + (s + \bar{s})y \mid y \in D\} = D \cap s^\perp$.

We abbreviate $Q := \{u^2 \mid u \in D\} = \mathrm{N}_\sigma(D) = \mathrm{N}_\kappa(D) \subseteq Z$ and obtain

$$\begin{array}{rcl} \mathrm{N}_\sigma(x + ys) & = & (x + sy)(x + ys) \\ & = & x^2 + (sy)(ys) + x(ys) + (sy)x \\ & = & x^2 + s^2y^2 + \langle x|ys\rangle + (\bar{s} + s)yx \\ & = & x^2 + \bar{s}sy^2 + \langle x|ys\rangle + (\bar{s} + s)yx + (s + \bar{s})sy^2 \\ & \in & Z + Z + Z + D + Qs = D + Qs. \end{array}$$

Now $Zs \le \mathrm{N}_\sigma(\mathbb{O}) + \mathrm{Tr}_\sigma(\mathbb{O}) \subseteq \mathrm{N}_\sigma(\mathbb{O}) + D \subseteq D \oplus Qs$ implies $Z = (\bar{s}+s)Z \le Q$ and then $Z = Q$.

Conversely, assume $Z = Q$. Then $Z$ and $Zs$ are both contained in $\mathrm{N}_\sigma(\mathbb{O}) + \mathrm{Tr}_\sigma(\mathbb{O})$. We already know $(Z \oplus Zs) \cap \mathrm{Tr}_\sigma(\mathbb{O}) \le (Z \oplus Zs) \cap D = \{0\}$. Thus $Z \oplus Zs \oplus \mathrm{Tr}_\sigma(\mathbb{O})$ has dimension 5, coincides with $\mathrm{Fix}(\sigma)$ and forces $\mathrm{N}_\sigma(\mathbb{O}) + \mathrm{Tr}_\sigma(\mathbb{O}) = \mathrm{Fix}(\sigma)$, as claimed. □

**7.7 Remark.** From 7.4.d we also see that $\mathrm{N}_\sigma(\mathbb{O})$ is far from being additively closed because $Z(b)$ and $Z(c)$ may be quite different if $\mathrm{Tr}_\iota(b)$ and $\mathrm{Tr}_\iota(c)$ both lie in $Z$.

## 8 Polarities induced by involutions that are not linear

**8.1 Lemma.** *Let $\sigma$ be an involutory anti-automorphism with $\sigma|_Z \ne \mathrm{id}$ and put $\iota = \sigma \circ \kappa$.*

- **a.** *If $\mathrm{char}\,\mathbb{O} = 2$ then $\mathrm{Fix}(\sigma) \cap \mathrm{Fix}(\iota) = \{x \in \mathrm{Fix}(\iota) \mid \bar{x} = x\}$ is a hyperplane in $\mathrm{Fix}(\iota)$. Picking $z \in Z$ with $\iota(z) = z + 1$ $( = \sigma(z))$ and $w \in \mathrm{Fix}(\sigma)$ with $\overline{w} = w + 1$ $( = \sigma(w))$ we obtain $w + z \in \mathrm{Fix}(\sigma)$ and thus $\mathrm{Fix}(\sigma) = (\mathrm{Fix}(\sigma) \cap \mathrm{Fix}(\iota)) \oplus S(w + z)$.*
- **b.** *If $\mathrm{char}\,\mathbb{O} \ne 2$ then the intersection $\mathrm{Fix}(\sigma) \cap \mathrm{Fix}(\iota) = S$ has dimension 1. We pick any $z \in Z \setminus \{0\}$ with $\iota(z) = -z$. For each $w \in \mathrm{Fix}(\sigma)$ with $\overline{w} = -w$ we then have $wz \in \mathrm{Fix}(\sigma)$, and $\mathrm{Fix}(\sigma) = S \oplus \{w \in \mathrm{Fix}(\iota) \mid \overline{w} = -w\}z$.*

*Proof.* From 3.2 we know $\dim_S \mathrm{Fix}(\sigma) = 8$. Thus it suffices to check that the given sets are contained in $\mathrm{Fix}(\sigma)$; this is easy. □

## 9 Polarities with precisely one absolute point

One knows (see [6, 2.7]) that a polarity of a translation plane cannot have precisely one absolute point if the characteristic of that plane is different from 2. For the case that we study here, the situation is different:

**9.1 Proposition.** *Let $A$ be a non-commutative composition algebra with $\mathrm{char}\,A = 2$ and without divisors of zero; and let $\sigma$ be an involutory anti-automorphism of $A$. If $\mathrm{Fix}(\sigma) \ne \mathrm{N}_\sigma(A) + \mathrm{Tr}_\sigma(A)$ then the projective plane over $A$ admits polarities with precisely one absolute point. More explicitly:*

- **a.** *The product of $\hat{\sigma} \circ \xi_{0,z}$ has more than one absolute flag whenever $z \in \mathrm{N}_\sigma(A) + \mathrm{Tr}_\sigma(A)$.*
- **b.** *The polarity $\hat{\sigma} \circ \xi_{0,x}$ has precisely one absolute flag whenever $y \in \mathrm{Fix}(\sigma) \setminus (\mathrm{N}_\sigma(A) + \mathrm{Tr}_\sigma(A))$.*

*Proof.* The (involutory) translation $\xi_{0,y}$ mapping $(u, v)$ to $(u, v + y)$ centralizes the polarity $\hat{\sigma}$ precisely if $\sigma(y) = y$, see 5.1. In that case the product $\pi_y := \hat{\sigma} \circ \xi_{0,y}$ is a polarity, and $(\infty, L_\infty)$ is an absolute flag of $\pi_y$. There are no other absolute points on $L_\infty$, and the affine absolute points are of the form $(u, v)$ with $\mathrm{N}_\sigma(u) + y = \mathrm{Tr}_\sigma(v)$. Choosing $y \in \mathrm{Fix}(\sigma) \setminus (\mathrm{N}_\sigma(A) + \mathrm{Tr}_\sigma(A))$ we obtain that $\pi_y$ has no affine absolute point. □

**9.2 Examples.** **a.** The standard involution $\kappa$ of $A$ fixes each point of the hyperplane $\mathrm{Fix}(\kappa) = 1^\perp$. We have $\mathrm{N}_\kappa(A) \subseteq \mathrm{Tr}_\kappa(A) = Z \subsetneq \mathrm{Fix}(\kappa)$ where $Z$ is the center of $A$. Thus $\kappa$ satisfies the requirement on $\sigma$ in 9.1.

- **b.** If $\sigma$ is a non-standard $Z$-linear involutory anti-automorphism then it depends on the finer structure of squares in $A$ whether $\mathrm{N}_\sigma(A) + \mathrm{Tr}_\sigma(A)$ fills all of $\mathrm{Fix}(\sigma)$, see 7.6.
- **c.** For involutions that are not linear over $Z$ we always have that every fixed point is a trace, see 3.2. Thus we cannot construct polarities with only one absolute point from the corresponding polarities.

**9.3 Remark.** There do exist (orthogonal) polarities of pappian planes over suitable commutative fields of characteristic 2 with precisely one absolute point (cf. [4, 3.5]): if there are sufficiently many square classes one may use a non-degenerate diagonal bilinear form which, considered as a semilinear map with respect to the Frobenius endomorphism $\mathrm{frob}\colon s \mapsto s^2$, has a kernel of dimension one. These polarities may be interpreted as analogues of those constructed in 9.1; the involution is the identity, the norm is the square, and the traces are zero.

The following general result shows that every polarity with precisely one absolute point arises from a polarity with more absolute points, as in 9.1.

**9.4 Theorem.** *If $\pi$ is a polarity of a translation plane with precisely one absolute point then the translation plane has characteristic* 2 *and there exists a translation $\tau$ in the centralizer of $\pi$ such that $\pi \circ \tau$ is a polarity with at least two absolute points.*

*Proof.* Let $\pi$ be such a polarity. From [6, 2.7] we know that the plane has characteristic two.

If the translation plane is not a Moufang plane then it has Lenz type V and the absolute flag is the (uniquely determined) flag $(\infty, L_\infty)$ such that $L_\infty$ is the translation axis and $\infty$ forms the translation axis of the dual plane. If we have a Moufang plane and $(\infty, L_\infty)$ is already used to denote a flag then flag-transitivity allows to choose the absolute flag as $(\infty, L_\infty)$.

We start introducing coordinates from a semifield $S$ by choosing any line $[0] \neq L_\infty$ through $\infty$, any point $(0,0) \neq \infty$ on $[0]$ and putting $(0) := \pi([0])$. The intersection point of $[0]$ and $\pi((0,0))$ will be denoted by $a$; then $\pi(a)$ is the line $[0,0]$ joining $\pi([0]) = (0)$ and $\pi^2((0,0)) = (0,0)$. Let $\tau$ denote the translation mapping $(0,0)$ to $a$.

We claim that the duality $\pi \circ \tau$ is a polarity, i.e. that $\lambda := (\pi \circ \tau)^2$ is the identity. The conjugate $\tau' := \pi \circ \tau \circ \pi$ is a translation because $(\infty, L_\infty)$ is an absolute flag. As the plane has characteristic two it suffices to show $\tau'((0,0)) = \tau((0,0))$; this equality follows from $\pi(\tau(\pi((0,0)))) = \pi(\tau((0) \vee a)) = \pi((0) \vee (0,0)) = \pi([0,0]) = a = \tau((0,0))$. □

We return to the study of the octonion plane again. The translation $\tau$ in 9.4 centralizes $\pi$ and thus of course also centralizes $\pi \circ \tau$ which is a conjugate of $\hat\sigma$ for some involutory anti-automorphism $\sigma$ of $\mathbb{O}$, cf. 5.5. For any involutory anti-automorphism $\sigma$ of $\mathbb{O}$ and any translation $\tau$ we have $\hat\sigma \circ \tau = \tau \circ \hat\sigma$ precisely if $\tau \in \mathrm{T}_\sigma$, i.e., if $\tau(0,0) = (0,t)$ with $t \in \mathrm{Fix}_{\mathbb{O}}(\sigma)$. We know from 9.1 that $\hat\sigma \circ \tau$ has more than one absolute flag precisely if $t \in \mathrm{N}_\sigma(\mathbb{O}) + \mathrm{Tr}_\sigma(\mathbb{O})$. All the polarities obtained in this way are conjugates of $\hat\sigma$:

**9.5 Lemma.** *Let $\sigma$ be an involutory anti-automorphism of $\mathbb{O}$. For $a, c \in \mathbb{O}$ define*

$$\mu_{a,b}\colon \mathbb{O}^2 \to \mathbb{O}^2\colon (x,y) \mapsto (x+a, y+\sigma(a)x+c)\,.$$

*Then $\mu_{a,c}$ describes an element of* $\mathrm{Aut}(\mathrm{P}_2(\mathbb{O}))$ *and $\mu_{a,c} \circ \hat\sigma \circ \mu_{a,c}^{-1} = \hat\sigma \circ \xi_{0,z}$ with $z := \mathrm{N}_\sigma(a) + \mathrm{Tr}_\sigma(c)$.*

*Proof.* The map $\mu_{a,c}$ is the product of a shear and a translation, and thus a collineation; the line map is given by $\mu_{a,c}([m,b]) = [m+\sigma(a), b+c-ma-\mathrm{Tr}_\sigma(a)]$. A straightforward computation yields $\mu_{a,c} \circ \hat\sigma = \hat\sigma \circ \xi_{0,z} \circ \mu_{a,c}$. This means $\mu_{a,c} \circ \hat\sigma \circ \mu_{a,c}^{-1} = \hat\sigma \circ \xi_{0,z}$, as claimed. □

**9.6 Theorem.** *Let $\sigma$ be an involutory anti-automorphism of $\mathbb{O}$, and consider $\tau, \eta \in \mathrm{T}_\sigma$. Then the polarities $\hat\sigma \circ \tau$ and $\hat\sigma \circ \eta$ are conjugates under the flag stabilizer $\mathrm{Aut}(\mathrm{P}_2(\mathbb{O}))_{\infty, L_\infty}$ if, and only if, there exist $a, c \in \mathbb{O}$ and $\psi \in \nabla_\sigma$ such that $\tau \circ \psi \circ \eta^{-1} \circ \psi^{-1} = \xi_{0,z}$ for $z := \mathrm{N}_\sigma(a) + \mathrm{Tr}_\sigma(c)$.*

*Proof.* Assume first that there exists $\varphi \in \mathrm{Aut}(\mathrm{P}_2(\mathbb{O}))_{\infty,L_\infty}$ such that $\varphi \circ \hat\sigma \circ \eta \circ \varphi^{-1} = \hat\sigma \circ \tau$. Then $\varphi \circ \hat\sigma \circ \varphi^{-1} = \hat\sigma \circ \tau \circ \varphi \circ \eta^{-1} \circ \varphi^{-1}$ is a polarity with more than one absolute flag. From 9.1 we know that $\tau \circ \varphi \circ \eta^{-1} \circ \varphi^{-1} = \xi_{0,z}$ where $z = \mathrm{N}_\sigma(a) + \mathrm{Tr}_\sigma(c)$ for some $a, c \in \mathbb{O}$.

Now 9.5 says $\mu_{a,c} \circ \hat\sigma \circ \mu_{a,c}^{-1} = \hat\sigma \circ \xi_{0,z} = \varphi \circ \hat\sigma \circ \varphi^{-1}$, and $\mu_{a,c}^{-1} \circ \varphi$ lies in the centralizer $\Psi_\sigma = \nabla_\sigma \Xi_\sigma$. Thus there exist $\psi \in \nabla_\sigma$ and $\xi \in \Xi_\sigma$ such that $\varphi = \mu_{a,c} \circ \psi \circ \xi$. As $\mathrm{T}_\sigma$ is centralized by $\xi, \mu_{a,c}$ and normalized by $\psi$, we obtain $\xi_{0,z} = \tau \circ \varphi \circ \eta^{-1} \circ \varphi^{-1} = \tau \circ \mu_{a,c} \circ \psi \circ \xi \circ \eta^{-1} \circ \xi^{-1} \circ \psi^{-1} \circ \mu_{a,c}^{-1} = \tau \circ \mu_{a,c} \circ \psi \circ \eta^{-1} \circ \psi^{-1} \circ \mu_{a,c}^{-1} = \tau \circ \psi \circ \eta^{-1} \circ \psi^{-1}$.

In order to prove the converse, we assume the existence of $\psi$ and $a, c$ with $\tau \circ \psi \circ \eta^{-1} \circ \psi^{-1} = \xi_{0,z}$ for $z = \mathrm{N}_\sigma(a) + \mathrm{Tr}_\sigma(c)$. Using 9.5 and the fact that $\mu_{a,c}$ centralizes $\mathrm{T}_\sigma = \psi \circ \mathrm{T}_\sigma \circ \psi^{-1}$ we verify $\hat\sigma \circ \tau = \left(\hat\sigma \circ \xi_{0,z}\right) \circ \left(\xi_{0,z}^{-1} \circ \tau\right) = \left(\mu_{a,c} \circ \hat\sigma \circ \mu_{a,c}^{-1}\right) \circ \left(\psi \circ \eta \circ \psi^{-1}\right) = \left(\mu_{a,c} \circ (\psi \circ \hat\sigma \circ \psi^{-1}) \circ \mu_{a,c}^{-1}\right) \circ \left(\mu_{a,c} \circ (\psi \circ \eta \circ \psi^{-1}) \circ \mu_{a,c}^{-1}\right) = (\mu_{a,c} \circ \psi) \circ (\hat\sigma \circ \eta) \circ (\mu_{a,c} \circ \psi)^{-1}$. □

**9.7 Remark.** If one of the polarities $\hat\sigma \circ \tau$ and $\hat\sigma \circ \eta$ in 9.6 has only one absolute flag then the extra restriction is superfluous because any conjugating element has to fix the unique absolute flag.

**9.8 Theorem.** *Let $\vartheta$ and $\sigma$ be involutory anti-automorphisms of $\mathbb{O}$ and consider $\xi \in \mathrm{T}_\vartheta$ and $\tau \in \mathrm{T}_\sigma$.*

- **a.** *If the polarities $\hat\vartheta \circ \xi$ and $\hat\sigma \circ \tau$ are conjugates under the stabilizer of* (0) *in* $\mathrm{Aut}(\mathrm{P}_2(\mathbb{O}))$ *then $\vartheta$ and $\sigma$ are conjugates under* $\mathrm{Aut}(\mathbb{O})$.
- **b.** *If $\vartheta = \kappa$ then $\sigma = \kappa$ holds whenever $\hat\vartheta \circ \xi$ and $\hat\sigma \circ \tau$ are conjugates under* $\mathrm{Aut}(\mathrm{P}_2(\mathbb{O}))$.
- **c.** *In general, there may exist translations $\zeta, \tau \in \mathrm{T}_\sigma$ such that $\hat\sigma \circ \tau$ and $\hat\sigma \circ \zeta$ are polarities with precisely one absolute flag but are not conjugates.*

*Proof.* After 6.3 it suffices to consider the case where $(\infty, L_\infty)$ is the unique absolute flag of both $\hat\vartheta \circ \xi$ and $\hat\sigma \circ \tau$. Assume that $\varphi$ is an automorphism of $\mathrm{P}_2(\mathbb{O})$ such that $\varphi \circ (\hat\vartheta \circ \xi) \circ \varphi^{-1} = \hat\sigma \circ \tau$. Then $\varphi$ fixes the absolute flag, and is thus of the form $\varphi = \zeta \circ \psi$ with a translation $\zeta\colon (u, v) \mapsto (u + z_1, v + z_2)$ and some $\mathrm{Z}(\mathbb{O})$-semilinear bijection $\psi\colon \mathbb{O}^2 \to \mathbb{O}^2$. The latter is given by $\psi(u, v) = (A(u), B(u) + C(v))$ with semilinear bijections $A$, $B$, and $C$; the line map is of the form $\psi([m, b]) = [D(m) + p, C(b)]$ with a semilinear bijection $D$ and a constant $p \in \mathbb{O}$.

If $\varphi$ fixes (0) then $B = 0$ and $p = 0$. We compute the values of $\varphi \circ (\hat\vartheta \circ \xi) = (\hat\sigma \circ \tau) \circ \varphi$ at an arbitrary line $[u, v]$ and evaluate the results in the special cases $u = 0$ and $v = 0$, respectively. Returning to the general case gives the conditions $z_1 = 0$, $A \circ \vartheta = \sigma \circ D$, and $C \circ \vartheta = \sigma \circ C$. This yields $\psi \circ \hat\vartheta = \hat\sigma \circ \psi$, and assertion a is established.

If $\vartheta = \kappa$ we recall from 5.1c that the subgroup $\Xi_\kappa$ of the centralizer of $\hat\kappa$ centralizes $\xi \in \mathrm{T}_\kappa$ and then also $\hat\kappa \circ \xi$. The group $\Xi_\kappa$ acts transitively[1] on $L_\infty \smallsetminus \{\infty\}$, and we can thus replace any conjugating element by one that fixes (0). Now a yields that $\sigma$ is a conjugate of $\kappa$ in $\mathrm{Aut}(\mathbb{O})$, and assertion b follows from the fact that $\kappa$ is central $\mathrm{Aut}(\mathbb{O})$, cf. 1.2.

The last (admittedly vague) claim is justified by 9.6. □

## 10 Elliptic polarities

A polarity is called *elliptic* if it has no absolute points at all. It appears that elliptic polarities are hard to understand even if we consider projective planes over commutative fields; there the existence of elliptic polarities depends on the existence of anisotropic bilinear or hermitian forms. Studying the product of a (hypothetical) elliptic polarity with a suitable conjugate

[1] This argument breaks down if $\vartheta$ is linear but not standard because then $\Xi_\vartheta = \mathrm{T}_\vartheta$ acts trivially on $L_\infty$.

of $\hat{\kappa}$ and the involution from 6.1 one can at least derive a necessary (but in general by no means sufficient) condition for the existence of elliptic polarities: one needs an anisotropic quadratic map from $\mathbb{O} \times \mathrm{Z}(\mathbb{O})$ to $\mathbb{O}$.

## 11 Unitals

The *polar unital* $(\mathrm{Abs}(\pi), \mathcal{B})$ of a polarity $\pi$ is the set $\mathrm{Abs}(\pi)$ of absolute points together with the system $\mathcal{B}$ of secants (lines that meet $\mathrm{Abs}(\pi)$ in more than one point). Often one replaces each secant with the subset of points of $\mathrm{Abs}(\pi)$ that are collinear with it; these subsets are called the *blocks* of the unital. Note that it may happen that the unital degenerates to a collinear set of points (possibly consisting of less than two points). In our context, unitals with just one block occur if $\pi$ is a conjugate of $\hat{\sigma}$ for a linear involution $\sigma \neq \kappa$, see 7.3. Unitals with just one point do occur, as well, see 9.1. We have left open whether elliptic polarities (with empty unitals) are possible.

**11.1 Definition.** Let $\pi$ be a polarity, and let $a$ be an absolute point of $\pi$. By a *tangent* to $\mathrm{Abs}(\pi)$ in $a$ we mean a line $L$ such that $L \cap \mathrm{Abs}(\pi) = \{a\}$.

**11.2 Proposition.** *Let $\pi$ be any polarity of* $\mathrm{P}_2(\mathbb{O})$ *such that* $\mathrm{Abs}(\pi)$ *is not contained in a line. Then there is a unique tangent to* $\mathrm{Abs}(\pi)$ *in any absolute point.*

*Proof.* On any absolute line there is exactly one absolute point (namely, the pole of that line). Thus $\pi(a)$ is a tangent to $\mathrm{Abs}(\pi)$ in $a \in \mathrm{Abs}(\pi)$. If $\mathrm{Abs}(\pi)$ is empty, we have nothing to prove. If $\mathrm{Abs}(\pi)$ is non-empty but contained in a line $L$ then any other line meeting $L$ in an absolute point is a tangent.

In order to see uniqueness in the remaining cases, we infer from 5.4, 3.1 and 3.2 that it suffices to consider $\pi = \hat{\sigma}$ for an involutory anti-automorphism $\sigma$ that is not $Z$-linear and for $\sigma = \kappa$. Transitivity of $\Psi_\sigma$ on $\mathrm{Abs}(\pi)$ allows to concentrate on the non-absolute (i.e., affine) lines through $\infty$. Now $[c]$ meets $\mathrm{Abs}(\pi)$ in an affine point because the norm $\mathrm{N}_\sigma(c)$ is a trace of $\sigma$ by 3.2 and 5.1. Thus $[c]$ is not a tangent. □

**11.3 Remark.** In a translation plane with characteristic different from two the uniqueness of tangents to the set of absolute points of a polarity of the projective plane over $S$ is not a problem. Indeed by [5, 5.3]. such a polarity is either elliptic or of the form $\hat{\sigma}$ for a suitable involutory anti-automorphism of some semifield $S$ with $\mathrm{char}\, S \neq 2$, and $\mathrm{N}_\sigma(c) = \mathrm{Tr}_\sigma\left(\frac{1}{2}\,\mathrm{N}_\sigma(c)\right)$ holds for any $c \in S$. This shows that no vertical line is a tangent; thus the tangent through $\infty$ is unique.

Since $\Xi_\sigma$ acts transitively on $\mathrm{Abs}(\pi) \setminus \{\infty\}$ it remains to consider tangents through $(0,0)$; here it suffices to show that $[m,0]$ is a secant if $m \neq 0$. Now that line contains the absolute point $(2\sigma(m), 2m\,\sigma(m)) \neq (0,0)$.

For any unital $(\mathrm{Abs}(\pi), \mathcal{B})$ we study the group $\mathrm{Aut}(\mathrm{Abs}(\pi), \mathcal{B})$ of *abstract* automorphisms, i.e. permutations of $\mathrm{Abs}(\pi)$ that preserve the system of blocks (but need not necessarily be induced by collineations of $\mathcal{P}$ that leave $\mathrm{Abs}(\pi)$ invariant). Experience has shown that the groups of translations of the unital are important: For any point $x$ of the unital the group of translations of the unital with center $x$ consists of all elements of $\mathrm{Aut}(\mathrm{Abs}(\pi), \mathcal{B})$ that fix each block through $x$. We will denote by $\mathrm{T}$ the group generated by all translations of the unital.

**11.4 Theorem** ([6, 7.2]). *If $\sigma = \kappa$ or $\sigma|_Z \neq \mathrm{id}$ then $\mathrm{T}_\sigma$ is the full group of translations at $\infty$, and* $\mathrm{Aut}(U_\sigma, \mathcal{B}_\sigma)$ *is a subgroup of* $\mathrm{Aut}(\mathrm{T})$. □

## 12 Open questions

**12.1 Problem.** Determine the centralizer in $\mathrm{Aut}(\mathrm{P}_2(\mathbb{O}))$ for each one of the polarities considered in the present paper, and clarify its structure.

**12.2 Problem.** Let $\sigma$ be an involutory anti-automorphism of $\mathbb{O}$, and assume $\sigma = \kappa$ or $\sigma|_Z \neq \mathrm{id}$. Is it then true that the centralizer of $\hat{\sigma}$ coincides with the group of all collineations of $\mathcal{P}$ that leave the unital invariant?

The restrictions in 12.2 are due to the following observation:

**12.3 Example.** It is *not true* that the centralizer of $\hat{\sigma}$ coincides with the group of all collineations of $\mathcal{P}$ that leave the unital invariant, if $\sigma$ is a $Z$-linear involutory anti-automorphism different from the standard one.

In order to be explicit, we consider for $r \in Z \smallsetminus \{0\}$ the collineations

$$\delta_r\colon \quad (x,y) \mapsto (x,ry), \quad [m,b] \mapsto [rm,rb]\,, \qquad \delta'_r\colon \quad (x,y) \mapsto (rx,y), \quad [m,b] \mapsto [r^{-1}m,b]\,.$$

If $\sigma \neq \kappa$ is a $Z$-linear anti-involution then $\delta_r$ and $\delta'_r$ both leave the (collinear) set $\mathrm{Abs}(\hat{\sigma})$ invariant but none of them centralizes $\hat{\sigma}$ if $r \neq 1$.

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