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Abstract

We prove that the little projective group of the unital is contained in the centralizer of the polarity defining the unital. This yields information about the full automorphism group of the unital. The action of a nilpotent regular normal subgroup is used to establish isomorphisms between unitals in spaces of different dimensions, and over different fields.

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1. Involutions, unitals and weak unitals

Let $\mathbf{R}$ be a commutative field, and let $\mathbf{K}$ be an alternative (not necessarily associative) algebra over $\mathbf{R}$ with involution σ; i.e., an $\mathbf{R}$-semilinear map $\sigma\colon \mathbf{K} \to \mathbf{K}$ such that $\sigma^2 = \mathrm{id}$ and $\sigma(xy) = \sigma(y)\sigma(x)$ holds for all $x, y \in \mathbf{K}$. Prominent examples are composition algebras with their standard involution $\kappa\colon x \mapsto \overline{x}$; cf. [11, 7.6], [24, Ch. 1]. We will also assume that the algebra has no zero divisors; then it can be used to construct a projective plane (see [4], [22], [3], [20, Ch. 1] for the non-associative case).

1.1 Remark. At some places we have to assume that the set of fixed elements of σ is contained in $\mathbf{R}$: this implies that $N(x) := \sigma(x)x$ becomes a multiplicative quadratic form, with polar form $f_N(x, y) = \sigma(x)y + \sigma(y)x$. The form N is anisotropic. If $\operatorname{char}\mathbf{R} \neq 2$ or $\sigma \neq \mathrm{id}$ we pick $x \in \mathbf{K} \smallsetminus \{x \in \mathbf{K} \mid \sigma(x) = -x\}$; then $f_N(1, x) = \sigma(x) + x \neq 0$ shows that f_N is not zero. As the left multiplications with elements of $\mathbf{K} \smallsetminus \{0\}$ generate a transitive group of similitudes, we infer that f_N is not degenerate.

In any case, we deal with a composition algebra (cf. [24]) and its standard involution: $\sigma = \kappa$. Involutions like $x \mapsto i\,\overline{x}\,i^{-1}$ on Hamilton's quaternions show that the restriction to standard involutions cannot be dropped completely if the fixed points of σ should form a subfield. The assumption that $\mathbf{R}$ contains the set of fixed elements of σ also rules out the non-commutative composition algebras of characteristic 2.

Indeed, involutions and polarities behave quite differently in characteristic two. We devote a separate paper to a detailed study of that case, see [13].

If $\mathbf{K}$ is associative, we are dealing either with a separable quadratic extension of commutative fields (and σ generates the Galois group $\mathrm{Gal}(\mathbf{K}/\mathbf{R})$), or with an involutory anti-automorphism of a skew field. In these cases we use homogeneous coordinates on the projective space $\mathrm{PG}(n-1, \mathbf{K})$: the set $\mathbf{K}^n$ will be regarded as the right vector space of columns, the points in the projective space will then be one-dimensional subspaces $v\mathbf{K}$, hyperplanes will be given as kernels of linear forms in matrix description (i.e., row vectors).

1.2 Definition. Let $h\colon \mathbf{K}^n \times \mathbf{K}^n \to \mathbf{K}$ be a non-degenerate σ-hermitian form of Witt index 1. The set $U := \{v\mathbf{K} \mid v \neq 0, h(v,v) = 0\}$ of points together with the set $\mathcal{B}$ of *blocks* (i.e., nontrivial traces of lines of the projective space $\mathrm{PG}(n-1,\mathbf{K})$ on U) will be called the *classical unital* defined by h.

The case of non-associative alternative algebras (i.e., octonion algebras) has to be treated differently, see Section 2.

In combinatorial geometry (finite) unitals are defined as incidence structures with q^3+1 points and $q+1$ points per block such that any two points are joined by a unique block. However, there appears to be no purely incidence-geometric definition (independent of a given embedding into a projective plane or even a connection with a polarity) for unitals with infinitely many points. One can use additional topological assumptions to replace the conditions on the order, cf. [8], [9], [10], [15], [16], [19], [28].

1.3 Definition. By a *weak unital* we mean an incidence structure $(U,\mathcal{B})$ consisting of a set U and a collection $\mathcal{B}$ of subsets of U called blocks such that the following hold:

- For any two points $P,Q \in U$ there exists a unique block $B \in \mathcal{B}$ with $\{P,Q\} \subseteq B$.
- Each block has at least 3 points.
- Each point belongs to at least 2 blocks.
- If B is a block and $P \in U \setminus B$ then there exists a block B' with $P \in B'$ and $B \cap B' = \emptyset$.

1.4 Examples. Being a weak unital is a weak condition, indeed:

a. Classical unitals defined by hermitian forms are weak unitals.

b. The unitals considered in finite geometry are precisely those weak unitals that satisfy the additional conditions that each block has the same (finite) size $q+1$, and that there are q^3+1 points in total.

c. Every affine plane of order at least 3 is a weak unital, but only the plane of order 3 is a unital in the sense of finite geometry; it is isomorphic to the classical unital corresponding to $\mathbb{F}_4/\mathbb{F}_2$.

d. If $(U,\mathcal{B})$ is a weak unital and $p \in U$ is a point such that each block through p has more than 3 points then $(U \setminus \{p\},\mathcal{B})$ is a weak unital, as well.

This last procedure may be iterated, leading to weak unitals with strange sizes of blocks.

2. Polarities and unitals in semifield planes

By an *octonion field* over $\mathbf{R}$ we mean a composition algebra $\mathbf{K}$ of dimension 8 over $\mathbf{R}$ with *anisotropic* norm form. If $\operatorname{char}\mathbf{R} = 2$ we also assume that the polar form of the norm is non-degenerate, cf. [24, Remark 1.2.2]. Such an algebra is never associative, and describing the projective plane over $\mathbf{K}$ is much more complicated than describing the affine plane. Therefore, we will describe polarities of octonion planes in suitable affine coordinates. We study a more general class of (non-associative) algebras.

2.1 Definitions. Let $\mathbf{K}$ be a semifield[1], cf. [20, 24.7, 25.8]: i.e., a (necessarily abelian) group $(\mathbf{K}, +)$ endowed with a bi-additive multiplication possessing a unit element 1 and such that each equation $ax = b$ or $xa = b$ with $a, b \in \mathbf{K}$ and $a \neq 0$ has a unique solution $x \in \mathbf{K}$. The *kernel* of $\mathbf{K}$ is the set $\ker_{\mathbf{K}} := \{k \in \mathbf{K} \mid \forall x, y \in \mathbf{K}\colon (xy)k = x(yk)\}$. The kernel is a skewfield, the *center* $Z(\mathbf{K}) := \{z \in \ker_{\mathbf{K}} \mid \forall x \in \mathbf{K}\colon zx = xz\}$ is a commutative field. Note that $Z(\mathbf{K})$ may be smaller than the set of all elements that commute with each member of $\mathbf{K}$.

Each semifield $\mathbf{K}$ can be used to define an affine plane $\mathcal{A}_{\mathbf{K}}$ with point set $\mathbf{K}^2$ and line set $\{[m, t] \mid m, t \in \mathbf{K}\} \cup \{[c] \mid c \in \mathbf{K}\}$ where $[m, t] := \{(x, mx + t) \mid x \in \mathbf{K}\}$ and $[c] := \{c\} \times \mathbf{K}$. The lines of the latter type are called *vertical*. The projective closure of $\mathcal{A}_{\mathbf{K}}$ will be denoted by $\mathcal{P}_{\mathbf{K}}$. We write L_∞ for the line at infinity and ∞ for the point on L_∞ that corresponds to the vertical lines.

2.2 Example. Any octonion field is a special semifield; the kernel and the center both coincide with the ground field $\mathbf{R}$. The special property of the octonion algebras is *alternativity* (cf. [21, Th. 3.1] or [11, §7.6, Th. 7.5]): any two elements of such an algebra are contained in an associative subalgebra. In particular, in any octonion algebra the equations $xa = 1 = ay$ imply $x = y$ (we write $a^{-1} := x$), and the *inverse property* holds: i.e., we have $a^{-1}(ab) = b = (ba)a^{-1}$ for all $a, b \in \mathbf{K}$ with $a \neq 0$.

In projective planes over alternative fields the line at infinity is not as special as it seems:

2.3 Proposition. *Let $\mathbf{K}$ be an alternative field.*

a. *Mapping $(x, y) \in \mathbf{K} \times (\mathbf{K} \smallsetminus \{0\})$ to (xy^{-1}, y^{-1}) and $[m, b]$ to $[-b^{-1}m, b^{-1}]$ extends to an involutory automorphism ρ of $\mathcal{P}_{\mathbf{K}}$ with axis $[0, 1]$ and center $(0, -1)$.*

b. *If $\operatorname{char} \mathbf{K} \neq 2$ then mapping $(x, y) \in \mathbf{K} \times (\mathbf{K} \smallsetminus \{-1/2\})$ to $(x(y + 1/2)^{-1}, (y + 1/2)^{-1} - 1/2)$ and $[m, b]$ to $[-(b + 1/2)^{-1}m, (b + 1/2)^{-1} - 1/2]$ extends to an involutory automorphism $\tilde{\rho}$ of $\mathcal{P}_{\mathbf{K}}$ with axis $[0, 1/2]$ and center $(0, -1/2)$.*

Proof. The distributive laws together with bi-associativity and the inverse property suffice to justify all the routine computations needed to prove the claims. □

2.4 Remarks. The axial involution ρ is taken from [18, 3.5 (22), p. 107]. The involution $\tilde{\rho}$ is obtained by conjugation of ρ with the translation mapping (x, y) to $(x, y + 1/2)$.

We recall that the following is true in any Moufang plane (cf. [18, 7.3.19, p. 198]):

a. If the plane has characteristic $\neq 2$ then for every anti-flag (p, L) there is precisely one involutory collineation $\rho_{p,L}$ with center p and axis L. There are no axial involutions with incident center and axis in that case.

b. If the plane has characteristic 2 then the axial involutions have their center on their axis; they are the elations. If the order of the plane is bigger than 2 then there is more than one involution for a given (incident) center and axis.

Here the characteristic of the plane is the characteristic of any coordinatizing alternative field. Note also that the existence of many axial involutions characterizes the Moufang planes, cf. [18, 8.4.13, p. 213].

Our aim is to achieve some understanding of the unitals defined by polarities of $\mathcal{P}_{\mathbf{K}}$ if $\mathbf{K}$ is a semifield. The unital in question has as points the *absolute* points of the polarity; i.e., those

[1] Semifields sometimes occur under the name "(non-associative) division algebra".

that lie on their image under the polarity. The blocks are the non-trivial traces of lines of $\mathcal{P}_{\mathbf{K}}$ on that point set.

Polarities of octonion planes are hard to classify in general because any classification of the elliptic polarities (those where no point lies on its image) involves a thorough knowledge of anisotropic quadratic forms over the ground field $\mathbf{R}$. For the octonions over the field $\mathbb{R}$ of real numbers, a classification has been achieved by J. Tits [33], [34]; see also [20, Section 18]. As we are interested in the unitals, we may exclude the elliptic cases from our considerations without any loss.

We will assume throughout that the flag (∞, L_∞) is absolute. This choice of a special absolute flag is not a crucial restriction of generality if we consider non-elliptic polarities of planes over semifields. Indeed the automorphism group of a plane over an alternative field acts transitively on the set of all flags. On the other hand, the plane over a proper (i.e., non-alternative) semifield has one special flag (describing the Lenz type, cf. [20, Sect. 24]) that is fixed by each automorphism and has to be absolute under each polarity, cf. [29, 1.3].

Polarities with this absolute flag can be treated as in [27], [29]. Many examples can be constructed using an involution of the semifield. Straightforward computations yield the following fundamental facts:

2.5 Lemma. *Let* $\mathbf{K}$ *be a semifield, and let* σ *be an involution of* $\mathbf{K}$.

a. *The map* $(u, v) \mapsto [\sigma(u), -\sigma(v)]$ *extends to a polarity* $\hat\sigma$*, with* $\hat\sigma([s, t]) = (\sigma(s), -\sigma(t))$.

b. *The set of affine absolute points is* $A_\sigma = \{(u, v) \mid u, v \in \mathbf{K}, \sigma(v) + v = \sigma(u)u\}$.

c. *The involutions* ρ *and* $\tilde\rho$ *in 2.3 centralize* $\hat\sigma$. □

2.6 Definition. The lines of $\mathcal{P}_{\mathbf{K}}$ meeting $U_\sigma := A_\sigma \cup \{\infty\}$ are called *secants*. Let $\mathcal{B}_\sigma$ be the set of all traces of secants on U_σ. Then $(U_\sigma, \mathcal{B}_\sigma)$ is called *the unital corresponding to* $(\mathbf{K}, \sigma)$.

In order to show that the construction 2.5 yields all non-elliptic polarities of a translation plane if the characteristic is different from two, we first need a second absolute point:

2.7 Theorem. *Let* $\mathcal{P}$ *be a translation plane with a polarity* π*. If* π *has at least one absolute point and* $\operatorname{char}\mathcal{P} \neq 2$ *then there exists at least one more absolute point.*

Proof. The projective plane $\mathcal{P}$ is self-dual. Therefore, it either has Lenz type V and then a distinguished flag (∞, L_∞) or it is a Moufang plane (by the Skornyakov–Sans Soucie Theorem, cf. [7, VI.6 and 7]). In the first case, we know that (∞, L_∞) is an absolute flag. In the Moufang case, we may choose any absolute flag for (∞, L_∞).

We choose an affine point o, put $u := \pi(o) \wedge L_\infty$, and pick an affine point e outside the lines $o \vee u$ and $o \vee \infty$. With respect to the quadrangle (o, u, ∞, e) the affine plane is then coordinatized by some semifield $\mathbf{K}$; we have $o = (0, 0)$, $o \vee \infty = [0]$, $o \vee u = [0, 0]$.

Our choice of u yields that there exists $b \in \mathbf{K}$ such that $\pi(o) = [0, b]$. For $t \in \mathbf{K}$ the map $\xi_{0,t} \colon \mathbf{K}^2 \to \mathbf{K}^2 \colon (x, y) \mapsto (x, y + t)$ is a translation with axis L_∞ and center ∞. Since $\pi(\infty) = L_\infty$ the conjugate $\pi \circ \xi_{0,t} \circ \pi^{-1}$ is again a translation of the form $\xi_{0,\tau(t)}$. We obtain a map $\tau \colon \mathbf{K} \to \mathbf{K}$ which is additive because conjugation induces a group homomorphism and $\xi_{0,s} \circ \xi_{0,t} = \xi_{0,s+t}$.

We compute $\pi(0, t) = \pi\big(\xi_{0,t}(0, 0)\big) = \big(\pi \circ \xi_{0,t} \circ \pi^{-1}\big)\big(\pi(0, 0)\big) = \xi_{0,\tau(t)}([0, b]) = [0, b + \tau(t)]$. Now $(0, b) \in [0, b] = \pi([0, 0])$ yields $(0, 0) \in \pi(0, b) = [0, b + \tau(b)]$; and $\tau(b) = -b$ follows.

Up to this point, we did not use the restriction on $\operatorname{char}\mathbf{K}$. A point $(0, s)$ is absolute if, and only if, the element $s \in \mathbf{K}$ is a solution of $s - \tau(s) = b$. If $\operatorname{char}\mathbf{K} \neq 2$ then there exists $w \in \mathbf{K}$ with $2w = b$, and $(0, w)$ is an absolute point. □

2.8 Remarks. The proof of 2.7 also gives a description of all absolute points on the vertical line $o \vee \infty$. We do not need to make this explicit here because we know from [14, 3.4] that in suitable coordinates the polarity is of the form $\hat\sigma$, where we know *all* absolute points from 2.5.

Our result 2.7 has been known to hold under additional assumptions such as finiteness ([1, Thm. 5], cf. [7, 12.1]) or a compact connected topology compatible with the geometric operations (cf. [29, 1.1]).

The assumption char $\mathbf{K} \neq 2$ is indispensable: even if $\mathbf{K}$ is in fact a commutative field (of characteristic 2) it may happen that a polarity has precisely one absolute point; cf. [12, 3.5] and [13, 9.1]. Polarities of Moufang planes of characteristic 2 show special features; among them the fact that the absolute points may form a proper subset of some line. See [13, 7.3].

The following has been established in [14, 3.4]:

2.9 Theorem. *Let $\mathcal{P}$ be a projective plane with a polarity π. If $\mathcal{P}$ is a translation plane and π has at least two absolute points then there is a semifield $\mathbf{K}$ with an anti-automorphism σ and an isomorphism $\eta\colon \mathcal{P} \to \mathcal{P}_{\mathbf{K}}$ such that $\eta \circ \pi \circ \eta^{-1} = \hat\sigma$, cf. 2.5.* □

2.10 Remark. Our affine point of view is especially useful if $\mathbf{K}$ is not associative (e.g., if $\mathbf{K}$ is an octonion field) because then the projective plane $\mathcal{P}_{\mathbf{K}}$ cannot be described by homogeneous coordinates. In order to fully understand the automorphism group of an octonion plane, some replacement for homogeneous coordinates is needed. A convenient generalization of homogeneous coordinates uses Jordan algebras, see [22], [23] (where algebras of characteristic 2 or 3 are excluded). A concise overview of this method is given in [3], treating also the case where char $\mathbf{K} \in \{2, 3\}$.

3. Automorphisms, translations and the little projective group of unitals

For any (weak) unital $(\mathcal{U}, \mathcal{B})$ one is interested in the group $\mathrm{Aut}(\mathcal{U}, \mathcal{B})$ of all automorphisms. If an embedding of $(\mathcal{U}, \mathcal{B})$ into a projective plane $\mathcal{P}$ is given then one may also ask whether this embedding is equivariant with respect to the actions of $\mathrm{Aut}(\mathcal{U}, \mathcal{B})$ and $\mathrm{Aut}(\mathcal{P})$, respectively. J. Tits [35] has given an affirmative answer to this question in the case of classical (polar) unitals over *commutative* fields in their natural embedding (M. O'Nan had treated the case of finite fields before in [17]). This affirmative answer has been extended to certain cases of non-commutative fields in [32, 4.2] and in [31]. We give a further extension in 7.1 below, including the classical unitals over octonion algebras.

It will be crucial to understand the little projective group that we introduce next.

3.1 Definition. Let $(\mathcal{U}, \mathcal{B})$ be a weak unital, and consider a point $p \in \mathcal{U}$. An automorphism $\tau \in \mathrm{Aut}(\mathcal{U}, \mathcal{B})$ is called a *translation with center p* if it fixes each block through p. The group of all translations of $(\mathcal{U}, \mathcal{B})$ with center p will be denoted by T_p. The group T generated by $\bigcup_{p \in \mathcal{U}} \mathrm{T}_p$ is called the *little projective group* of $(\mathcal{U}, \mathcal{B})$.

Clearly, the little projective group T is a normal subgroup of $\mathrm{Aut}(\mathcal{U}, \mathcal{B})$. We describe a big source of automorphisms of polar unitals next.

3.2 Definition. Let $\mathbf{K}$ be a semifield with an involution σ. By Ψ_σ we denote the centralizer of the polarity $\hat\sigma$ defined in 2.5, taken in the group of all automorphisms of the plane $\mathcal{P}_{\mathbf{K}}$. For $x, y \in \mathbf{K}$ we put

$$\xi_{x,y}\colon (u, v) \mapsto (u + x, v + \sigma(x)u + y) \ .$$

We write $\Xi_\sigma := \{\xi_{x,y} \mid x, y \in \mathbf{K}, \sigma(y) + y = \sigma(x)x\}$ and $\Xi'_\sigma := \{\xi_{0,p} \mid p \in \mathbf{K}, \sigma(p) = -p\}$.

3.3 Lemma. *The set Ξ_σ forms a subgroup of Ψ_σ. It acts sharply transitively on the affine part of the unital.*

The set Ξ'_σ is a normal subgroup of Ξ_σ. This subgroup fixes each vertical block and acts (sharply) transitively on the affine part of that block. In other words, we have $\Xi'_\sigma \leq \mathrm{T}_\infty$.

Proof. For arbitrary $x, y \in \mathbf{K}$ the map $\xi_{x,y}$ is the composition of a translation and a shear, and thus an automorphism of $\mathcal{A}_\mathbf{K}$ and of $\mathcal{P}_\mathbf{K}$.

A straightforward calculation (using only the distributive laws, the properties of σ, the relation $\sigma(y) + y = \sigma(x)x$ and associativity of addition) yields that Ξ centralizes $\hat{\sigma}$. Moreover, we compute $\xi_{x,y} \circ \xi_{v,w} = \xi_{x+v,y+w+\sigma(v)x}$ and $\xi_{x,y}^{-1} = \xi_{-x,-y+\sigma(x)x}$. Thus the first part of the first assertion is proved.

Clearly Ξ'_σ is a subgroup. Each vertical block is fixed; in fact, Ξ'_σ consists of those shears with axis $[0]$ that centralize the polarity $\hat{\sigma}$. The rest is obvious. □

3.4 Remark. If $\mathbf{K}$ is commutative then $\sigma = \mathrm{id}$ is an involution. The definitions of U_σ, Ψ_σ and Ξ_σ still make sense. However, the set U_σ becomes rather thin (it will form an oval or a line, in fact), and the group Ξ_σ becomes commutative.

3.5 Remark. If $\operatorname{char} \mathbf{K} \neq 2$ and $\sigma \neq \mathrm{id}$ then Ξ'_σ is the commutator subgroup of Ξ_σ. For $\operatorname{char} \mathbf{K} = 2$ there are examples where the commutator subgroup of Ξ_σ is a proper subgroup of Ξ'_σ; for instance, this happens if $\mathbf{K}$ is a quaternion field (of characteristic 2) and σ is its standard involution. It even occurs that Ξ_σ is elementary abelian, see [13, 7.5].

3.6 Definition. If the axis of an involution $\rho \in \Psi_\sigma$ is a secant (i.e., meets U_σ in more than one point) we call ρ an *exterior reflection*. If the axis does not meet the unital at all, we call ρ an *interior reflection*.

3.7 Theorem. *Let $\mathbf{K}$ be an alternative field with involution σ.*

a. *The centralizer Ψ_σ of the polarity $\hat{\sigma}$ acts two-transitively on U_σ.*
b. *The centralizer Ψ_σ acts transitively on the set of secants.*
c. *If $\operatorname{char} \mathbf{K} \neq 2$ then the centralizer Ψ_σ acts (via conjugation) transitively on the set of exterior involutions.*
d. *Every exterior reflection in Ψ_σ is contained in the little projective group T of U_σ.*

Proof. The stabilizer of the point ∞ in the centralizer contains Ξ_σ and thus acts transitively on the affine part. It suffices to exhibit a single element of the centralizer that moves ∞; we have seen in 2.5 that the reflection ρ constructed in 2.3 is such an element. Now the second assertion follows from two-transitivity and assertion c then follows from uniqueness 2.4.

In order to prove the last assertion it suffices to show $\rho_{[0]} \in \mathrm{T}$. We use inhomogeneous coordinates. Pick $p \in \mathbf{K} \setminus \{0\}$ such that $\sigma(p) = -p$ and abbreviate $q := -p^{-1}$. We consider $\psi_p := \xi_{0,p} \circ (\rho \circ \xi_{0,q} \circ \rho) \circ \xi_{0,p} \in \mathrm{T}$. For $y \in \mathbf{K} \setminus \{0, -p\}$ we use alternativity to verify $\left((y+p)^{-1} + q\right)^{-1} + p = -py^{-1}p$. This yields $\psi_p(0, y) = (0, -py^{-1}p)$ and $\psi_p^2(0, y) = (0, y)$ for each $y \in \mathbf{K} \setminus \{0\}$. Thus ψ_p^2 is a collineation with axis $[0]$ and $\psi_p^2 \in \mathrm{T} \leq \Psi_\sigma$ implies that the point $\hat{\sigma}([0])$ at infinity is the center of ψ_p^2.

There is a commutative (and associative) subfield C_p containing p. For $z \in C_p$ and any $x \in \mathbf{K}$ alternativity now yields $\psi_p(x,z) = (-xz^{-1}p, -zp^2)$. Using this we find that ψ_p^2 interchanges (x,z) with $(-x,z)$. Therefore ψ_p^2 equals the unique reflection with axis $[0]$ and center $\hat{\sigma}([0])$. □

3.8 Examples. If $\operatorname{char} \mathbf{K} \neq 2$ then mapping (x,y) to $(-x,y)$ defines an exterior reflection $\rho_{[0]}$ in Ψ_σ. In that case $\tilde{\rho}$ from 2.3 is an exterior reflection. The involution ρ in 2.3 is a reflection but need not be an exterior one in general; this depends on the existence of solutions for $\sigma(x)x = 2$. If $\operatorname{char} \mathbb{O} = 2$ then ρ in 2.3 is a translation that belongs to the little projective group.

4. Projections of blocks: the standard case

In this section, let $\mathbf{K}$ be a semifield of finite dimension over its center $\mathbf{R}$, and assume that there exists an involution $\kappa \neq \mathrm{id}$ of $\mathbf{K}$ such that $\{\kappa(x)x \mid x \in \mathbf{K}\}$ is contained in $\mathbf{R}$. Examples are given by composition algebras (of arbitrary characteristic) with standard involution κ, in particular by separable quadratic field extensions. There are also examples where $\mathbf{K}$ is a proper (non-alternative) semifield, cf. [29, 3.3]. We will write $\kappa(x) = \overline{x}$ also in the general (non-alternative) case. For $X \subseteq \mathbf{K}$ we put $\mathrm{Pu}(X) := \{p \in X \mid \overline{p} = -p\}$.

The *norm* $N(x) := \overline{x}\,x$ is an anisotropic quadratic form (it is multiplicative precisely if $\mathbf{K}$ is a composition algebra); the corresponding polar form will be denoted by $\beta(x,y) := \overline{x}\,y + \overline{y}\,x$. Orthogonality in $\mathbf{K}$ will be meant with respect to this form. The form β is not zero because $\kappa \neq \mathrm{id}$; we require, in addition, that β is not degenerate.

If $\mathbf{K}$ is a composition algebra then left multiplications generate a group of similitudes of N that acts transitively on $\mathbf{K} \smallsetminus \{0\}$. Thus our additional assumption of non-degeneracy comes for free in that case.

Now $A_\sigma = \{(x,y) \in \mathbf{K}^2 \mid \beta(y,1) = N(x)\}$, and for $m \in \mathbf{K}$ the set $\pi_m := \{x \in \mathbf{K} \mid (x,mx) \in A_\sigma\} = \{x \in \mathbf{K} \mid \beta(x,\overline{m}) = N(x)\}$ describes the projection of the block $[m,0] \cap U_\sigma$ into the pencil $\mathcal{B}_\infty$ because $\{[c] \mid [c] \cap [m,0] \subseteq A_\sigma\} = \{[c] \mid c \in \pi_m\}$.

4.1 Proposition. *Let $(U,\mathcal{B}) = (U_\kappa,\mathcal{B}_\kappa)$ be the unital corresponding to $(\mathbf{K},\kappa)$. For different blocks through $(0,0)$ the projections into the pencil $\mathcal{B}_\infty$ are never equal. In fact, we have $\pi_m \cap \pi_w = \{x \in \mathbf{K} \mid \beta(x,\overline{m}) = \overline{x}\,x = \beta(x,\overline{w})\} = \pi_m \cap \{\overline{w-m}\}^\perp \neq \pi_m$. More generally, for any subset $W \subseteq \mathbf{K}$ we obtain $\bigcap_{w \in W \cup \{m\}} \pi_w = \pi_m \cap (\overline{W-m})^\perp$.*

Proof. It suffices to consider non-vertical blocks. We consider an affine point of the unital and a non-vertical block through that point. From 3.7 we know that we may assume that the affine point is $(0,0)$; then the block is induced by some line $[m,0]$ with $m \neq 0$.

As $\beta(x,\overline{w}) = \beta(x,\overline{m})$ is equivalent to $\beta(x,\overline{w-m}) = 0$, we find $\pi_m \cap \pi_w = \pi_m \cap \{\overline{m-w}\}^\perp$. The set π_m is a non-degenerate affine quadric over the field $\mathbf{R}$, and it does not have any points at infinity because the norm form is anisotropic. The quadric π_m is not contained in any hyperplane because the vector space of homogeneous coordinates has a basis consisting of isotropic vectors, cf. [2, §11, 2)]. This means $\pi_m \cap \pi_w \neq \pi_m$ if $m \neq w$. □

5. Projections of blocks: the alternative case

In this section, we assume that the semifield $\mathbf{K}$ is alternative and consider an involutory anti-automorphism σ of $\mathbf{K}$. For $x \in \mathbf{K}$ we define the *norm* $\nu_\sigma(x) := \sigma(x)x$ and the *trace* $\tau_\sigma(x) = \sigma(x)+x$. Note that both the norm and the trace are maps from $\mathbf{K}$ to $\mathrm{Fix}(\sigma)$.

5.1 Proposition. *Assume that there exists $z \in Z(\mathbf{K})$ such that $\tau_\sigma(z) = 1$. Consider any two absolute points a and b of the polarity $\hat{\sigma}$. Then the projections of different blocks through a into the pencil $\mathcal{B}_b$ are never equal.*

Proof. It suffices to consider $a = \infty$ and $b = (0, 0)$ because the centralizer of the polarity acts two-transitively on the unital (cf. 3.7). As τ_σ is additive the set $P_\sigma := \{p \in \mathbf{K} \mid \tau_\sigma(p) = 0\}$ is a subgroup of $\mathbf{K}$. It is easy to see that $x_0 := z\,\nu_\sigma(x)$ satisfies $\tau_\sigma(x_0) = \nu_\sigma(x)$.

The set $\mu_x := \{m \in \mathbf{K} \mid \nu_\sigma(x) = \tau_\sigma(mx)\}$ consists of all slopes of lines through $(0, 0)$ that meet the vertical $[x]$ in an absolute point. Thus the projections of the blocks induced by $[x]$ and $[y]$ are equal if, and only if, the sets μ_x and μ_y coincide. This condition translates into the equality $x_0x^{-1} + P_\sigma x^{-1} = y_0y^{-1} + P_\sigma y^{-1}$. Cosets modulo subgroups can only be equal if the subgroups are the same, so we have $x_0x^{-1} - y_0y^{-1} \in P_\sigma x^{-1} = P_\sigma y^{-1}$.

Now $x_0x^{-1} - y_0y^{-1} = z\,\sigma(x - y)$ by our definition of x_0 and y_0, and we obtain that both $z\,(\sigma(x - y)x) = (z\,\sigma(x - y))x$ and $z\,(\sigma(x - y)y)$ lie in P_σ. This yields $0 = \tau_\sigma\,(z\,(\sigma(x - y)\,(x - y))) = \tau_\sigma\,(z\,\nu_\sigma(x - y)) = \nu_\sigma(x - y)\,\tau_\sigma(z) = \nu_\sigma(x - y)$. Thus $x - y = 0$, as claimed. □

5.2 Remarks. Our result 5.1 re-proves 4.1 for the special case where $\mathbf{K}$ is alternative with $\operatorname{char}\mathbf{K} \neq 2$ (and the involution is standard). Indeed, $z := \frac{1}{2}$ satisfies that requirement for any involutory anti-automorphism.

Note also that a suitable $z \in \mathbf{R}$ exists if the involution is not $\mathbf{R}$-linear. There is in fact only one case of involutions of composition algebras that is not covered by 4.1 or 5.1, namely, the case where $\operatorname{char}\mathbf{K} = 2$ and σ is $\mathbf{R}$-linear, cf. [13, 3.2].

5.3 Remark. For the case of a non-standard but $\mathbf{R}$-linear involution our result 5.1 could also be deduced from 4.1 because the unital defined by the non-standard involution is isomorphic to a standard unital in a projective space over the algebra $\mathbf{F}$ of fixed points of ι; see 6.7 below.

5.4 Open Problem. Can we avoid the use of the inverse property in 5.1 and thus generalize to non-alternative composition algebras, as in 4.1? Study the examples in [27], [29].

6. Generalized Heisenberg groups and an isomorphism between unitals

We consider a composition algebra $\mathbf{K}$ over a commutative field $\mathbf{R}$ with $\operatorname{char}\mathbf{R} \neq 2$, and an $\mathbf{R}$-linear involutory automorphism ι of $\mathbf{K}$. Then $\mathbf{F} := \operatorname{Fix}(\iota)$ is a subalgebra, and $\mathbf{K}$ may be recovered from $\mathbf{F}$ as $\mathbf{K} = \mathbf{F} \oplus \mathbf{F}w$ for any $w \in \mathbf{F}^\perp$ if we know $w^2 \in \mathbf{R}$. If $\mathbf{K}$ is a division algebra then $\mathbf{F}$ is a skewfield and the hermitian form $f_N\colon \mathbf{F}^2 \to \mathbf{F}\colon\ ((x_1, x_2)^t, (y_1, y_2)^t) \mapsto \overline{x_1}\,y_1 + \gamma\,\overline{x_2}\,y_2$ is anisotropic because $f_N(x, x) = N(x) := \bar{x}\,x$ is the norm form of $\mathbf{K}$.

Conversely, we may start with any *associative* composition algebra without zero divisors and any anisotropic κ-hermitian form $f\colon \mathbf{F}^2 \to \mathbf{F}$. Up to a similitude we may assume that $f\big((x_1, x_2)^t, (y_1, y_2)^t\big) = \overline{x_1}\,y_1 + \overline{x_2}\,\gamma y_2$ with some $\gamma \in \mathbf{R} \setminus \{0\}$. Now let $\mathbf{K}_\gamma$ be the γ-double of the associative composition algebra $\mathbf{F}$ over $\mathbf{R}$, cf. [24, 1.5.1] or [11, §7.6, Lemma 3]; thus $\mathbf{K}_\gamma = \mathbf{F} \oplus \mathbf{F}w$ with $w^2 = -\gamma$ and

$$\forall\, a, c \in \mathbf{F}\colon \quad a(cw) = (ca)w, \quad (aw)c = (a\,\bar{c})w, \quad (aw)(cw) = w^2\,\bar{c}\,a = -\gamma\,\bar{c}\,a\,.$$

The norm form of $\mathbf{K}_\gamma$ becomes $\overline{(x + yw)}(x + yw) = \bar{x}\,x + \gamma\,\bar{y}\,y = f\big((x, y), (x, y)\big)$ and is therefore anisotropic; the subalgebra $\mathbf{F}$ is the set of fixed points of the involutory automorphism mapping $x + yw$ to $x - yw$.

6.1 The unital in $\mathrm{PG}(3,\mathbf{F})$**.** The hermitian form

$$h\colon \mathbf{F}^4\times\mathbf{F}^4\to\mathbf{F}\colon \left(\begin{pmatrix}x_0\\x_1\\x_2\\x_3\end{pmatrix},\begin{pmatrix}y_0\\y_1\\y_2\\y_3\end{pmatrix}\right)\mapsto \overline{x_0}\,y_3+\overline{x_1}\,y_1+\gamma\,\overline{x_2}\,y_2+\overline{x_3}\,y_0$$

on $\mathbf{F}^4$ has Witt index 1, and defines a unital in the projective space $\mathrm{PG}(3,\mathbf{F})$. The stabilizer of $\infty := (0,0,0,1)^{\mathrm{t}}\mathbf{F}$ in the unitary group $\mathrm{U}(h)$ contains the nilpotent normal subgroup $\Lambda_{\mathbf{F}} := \{\lambda_{x_1,x_2,p} \mid x_1,x_2\in\mathbf{F}, p\in\mathrm{Pu}(\mathbf{F})\}$ where

$$\lambda_{x_1,x_2,p} := \begin{pmatrix}1&0&0&0\\x_1&1&0&0\\x_2&0&1&0\\p-\frac{\overline{x_1}\,x_1+\gamma\,\overline{x_2}\,x_2}{2}&-\overline{x_1}&-\gamma\,\overline{x_2}&1\end{pmatrix}.$$

Via $\lambda_{x_1,x_2,p}\mapsto(x_1,x_2,p)$ the group $\Lambda_{\mathbf{F}}$ is isomorphic to the *generalized Heisenberg group* $\mathrm{GH}(\mathbf{F}^2,\mathrm{Pu}(\mathbf{F}),\beta) := (\mathbf{F}^2\times\mathrm{Pu}(\mathbf{F}),*_\beta)$ with $(x,p)*_\beta(y,q) := (x+y,p+q+\frac{1}{2}\beta(x,y))$ where

$$\begin{aligned}\beta(x,y) &= f_N(y,x)-f_N(x,y)\\ &= \overline{y_1}\,x_1-\overline{x_1}\,y_1+\gamma\,(\overline{y_2}\,x_2-\overline{x_2}\,y_2)\,.\end{aligned}$$

For information about generalized Heisenberg groups see [5], [6], [25], [26], [30].

6.2 The unital in $\mathrm{PG}(2,\mathbf{K})$**.** We adopt an affine point of view because $\mathbf{K}$ need not be associative. The anti-automorphism $\alpha\colon\mathbf{K}\to\mathbf{K}\colon a+cw\mapsto\bar{a}+cw$ (for $a,c\in\mathbf{F}$) yields the polarity $\hat{\alpha}$, cf. 2.5. The subgroup Ξ_α of Ψ_α introduced in 3.2 is nilpotent and normal in the stabilizer of the (unique) non-affine absolute point in Ψ_α. Note that $\mathrm{Pu}(\mathbf{F})=\{y\in\mathbf{K}\mid \alpha(y)=-y\}$.

6.3 Lemma. *Via* $(x_1,x_2,p)\mapsto\xi_{x_1+x_2w,-p}$ *the group* $\mathrm{GH}(\mathbf{F}^2,\mathrm{Pu}(\mathbf{F}),\beta)$ *is isomorphic to* Ξ_α*.*

Proof. It suffices to note $\alpha(x_1+x_2w)\,(y_1+y_2w)-\alpha(y_1+y_2w)\,(x_1+x_2w) = -\beta\left((x_1,x_2)^{\mathrm{t}},(y_1,y_2)^{\mathrm{t}}\right)$. □

In both cases that we consider here, the nilpotent group acts sharply transitively on the set of affine absolute points. In general, we cannot identify the points of the plane $\mathrm{PG}(2,\mathbf{K})$ with the secants of the unital in $\mathrm{PG}(3,\mathbf{F})$. We concentrate on the absolute points (modeled by the generalized Heisenberg group $\mathrm{GH}(\mathbf{F}^2,\mathrm{Pu}(\mathbf{F}),\beta)$) via the isomorphisms onto the groups $\Lambda_{\mathbf{F}}$ and Ξ_α and their sharply transitive actions, respectively) and the blocks induced by secants on these sets of points.

6.4 Blocks of the unital in $\mathrm{PG}(3,\mathbf{F})$**.** We take an affine point of view, with $\ker(1,0,0,0)$ as the hyperplane at infinity. It suffices to concentrate on those blocks that pass through $o=(1,0,0,0)^{\mathrm{t}}\mathbf{F}$. Lines in $\ker(0,0,0,1)$ can be ignored because they are tangents. The affine points of the block induced by the vertical line through o form the set $\{(1,0,0,p)^{\mathrm{t}}\mathbf{F}\mid p\in\mathrm{Pu}(\mathbf{F})\}$; this set corresponds to the center of $\mathrm{GH}(\mathbf{F}^2,\mathrm{Pu}(\mathbf{F}),\beta)$. Each non-vertical block through o is of the form $B_{a,c} := \{(1,as,cs,s)^{\mathrm{t}}\mathbf{F}\mid s\in S_{a,c}\}$ where $S_{a,c} := \{s\in\mathbf{F}\mid N(a+cw)\,\bar{s}\,s+\bar{s}+s=0\}$.

6.5 Remark. Since $N|_{\mathbf{F}}\colon\mathbf{F}\to\mathbf{R}\colon s\mapsto\bar{s}\,s$ is an anisotropic quadratic form the sets $B_{a,c}$ and $S_{a,c}$ are elliptic quadrics in the affine spaces $\{(1,at,ct,t)^{\mathrm{t}}\mathbf{F}\mid t\in\mathbf{F}\}$ and $\mathbf{F}$, respectively, over $\mathbf{R}$.

6.6 Blocks of the unital in $\mathrm{PG}(2,\mathbf{K})$**.** The vertical block through $(0,0)^{\mathrm{t}}$ in the unital from 6.2 is $\{(0,p)^{\mathrm{t}} \mid p \in \mathrm{Pu}(\mathbf{F})\}$. Now consider $m := -\frac{\overline{a}+cw}{N(a+cw)} = -\frac{\alpha(a+cw)}{N(a+cw)}$ for $a+cw \in \mathbf{K}\setminus\{0\}$ and the block induced by the line $[m,0]$.

For $x = x_1+x_2w$ we compute $mx = (-\overline{a}\,x_1 + \gamma\,\overline{x_2}\,c - (x_2\,\overline{a} + c\,\overline{x_1})w)\,N(a+cw)^{-1}$. The condition $\alpha(mx) + mx = \alpha(x)x$ for $(x,mx)^{\mathrm{t}} \in A_\pi$ yields $-\overline{a}\,x_1 - \overline{x_1}\,a + \gamma(\overline{x_2}\,c + \overline{c}\,x_2) - 2(x_2\,\overline{a} + c\,\overline{x_1})w = N(a+cw)\left(\overline{x_1}\,x_1 - \gamma\,\overline{x_2}\,x_2 + 2(\overline{x_2}\,x_1)w\right)$, i.e.

$$-\overline{a}\,x_1 - \overline{x_1}\,a + \gamma(\overline{x_2}\,c + \overline{c}\,x_2) = N(a+cw)\big(N(x_1) - \gamma N(x_2)\big) \quad (1)$$

$$-x_2\,\overline{a} - c\,\overline{x_1} = N(a+cw)\left(\overline{x_2}\,x_1\right). \quad (2)$$

6.7 Theorem. *For $x,y \in \mathbf{F}$ and $p \in \mathrm{Pu}(\mathbf{F})$ we define*

$$\psi\left(\begin{pmatrix}1\\ x\\ y\\ p - \frac{\overline{x}\,y + \gamma\,\overline{y}\,x}{2}\end{pmatrix}\mathbf{F}\right) = \psi\left(\lambda_{x,y,p}\left(\begin{pmatrix}1\\0\\0\\0\end{pmatrix}\mathbf{F}\right)\right)$$

$$:= \xi_{x+yw,-p}\begin{pmatrix}0\\0\end{pmatrix} = \begin{pmatrix} x+yw \\ -p + \frac{\alpha(x+yw)(x+yw)}{2}\end{pmatrix}.$$

Then ψ is a bijection from the set of affine absolute points in $\mathrm{PG}(3,\mathbf{F})$ onto the set of affine absolute points in $\mathrm{PG}(2,\mathbf{K})$. This bijection maps each block onto a block, and thus extends to an isomorphism of unitals.

Proof. Since ψ translates the transitive action of the group $\Lambda_{\mathbf{F}}$ into that of Ξ_α it suffices to consider blocks through $o := (1,0,0,0)^{\mathrm{t}}\mathbf{F}$. The vertical block through o is mapped to the vertical block through $(0,0)^{\mathrm{t}}$ by our bijection ψ. For $(a,c)^{\mathrm{t}} \in \mathbf{F}^2 \setminus \{(0,0)^{\mathrm{t}}\}$ we claim that ψ maps $B_{a,c}$ onto the block induced by the line $[\frac{-\alpha(a+cw)}{N(a+cw)},0]$.

The point $(1,x,y,z)^{\mathrm{t}}\mathbf{F}$ lies on the unital precisely if $\overline{z} + z + N(x+yw) = 0$. Then $\mathrm{Pu}(z) = \frac{1}{2}(z - \overline{z}) = z + \frac{1}{2}N(x+yw)$ and

$$\psi\left((1,x,y,z)^{\mathrm{t}}\mathbf{F}\right) = \begin{pmatrix} x+yw \\ -\mathrm{Pu}(z) + \frac{1}{2}\alpha(x+yw)(x+yw)\end{pmatrix} = \begin{pmatrix} x+yw \\ -z - \gamma\,\overline{y}\,y + (y\,\overline{x})w\end{pmatrix}.$$

Applying this formula for ψ and using $s \in S_{a,c}$ we find that $\psi\left((1,as,cs,s)^{\mathrm{t}}\mathbf{F}\right)$ lies in the block B induced by $\left[-\frac{\overline{a}+cw}{N(a+cw)},0\right]$.

In order to show $\psi(B_{a,c}) = B$ we consider a point $(1,x,y,z)^{\mathrm{t}}\mathbf{F} \neq (1,0,0,0)^{\mathrm{t}}\mathbf{F}$ on the unital in $\mathrm{PG}(3,\mathbf{F})$ and assume that $\psi\left((1,x,y,z)^{\mathrm{t}}\mathbf{F}\right)$ lies in B. Then $(1,x,y,z)^{\mathrm{t}}\mathbf{F} \in B_{xz^{-1},yz^{-1}}$ implies $\psi(B_{xz^{-1},yz^{-1}}) \subseteq B$ by our reasoning above. This means $-\frac{\overline{a}+cw}{N(a+cw)} = -\frac{\overline{xz^{-1}} + (yz^{-1})w}{N\left(xz^{-1}+(yz^{-1})w\right)}$, thus $N(a+cw) = N\left(N(a+cw)(\overline{a}+cw)^{-1}\right) = N\left(\frac{N\left(xz^{-1}+(yz^{-1})w\right)}{\overline{xz^{-1}}+(yz^{-1})w}\right) = N(x+yw)$ and then

$$\begin{pmatrix}x\\y\end{pmatrix} = \begin{pmatrix}a\\c\end{pmatrix} z\,\frac{N\left(xz^{-1} + (yz^{-1})w\right)}{N(a+cw)} = \begin{pmatrix}a\\c\end{pmatrix} z\,.$$

Thus $(1,x,y,z)^{\mathrm{t}}\mathbf{F} = \left((1,0,0,0)^{\mathrm{t}} + (0,a,c,1)^{\mathrm{t}}z\right)\mathbf{F}$ lies in $B_{a,c}$, as required. □

7. The full automorphism group

In this final section, let σ be any involutory anti-automorphism of a composition algebra $\mathbf{K}$ over $\mathbf{R}$, and let $(\mathcal{U},\mathcal{B}) := (\mathcal{U}_\sigma,\mathcal{B}_\sigma)$ be the corresponding classical unital. Assume that there exists $z \in \mathrm{Z}(\mathbf{K})$ such that $\tau_\sigma(z) = 1$. (The only case of involutions of composition algebras that we exclude is the case where char $\mathbf{K} = 2$ and σ is $\mathbf{R}$-linear but not standard; then $\mathcal{U}_\sigma$ is not a unital but contained in a line, cf. [13, 3.2].)

In order to understand $\mathrm{Aut}(\mathcal{U},\mathcal{B})$ we will use the following.

7.1 Theorem. *For each point $p \in \mathcal{U}$ the group T_p is contained in the centralizer $\Psi_\sigma := \mathrm{C}_{\mathrm{Aut}(\mathcal{P}_\mathbf{K})}(\hat\sigma)$ of the polarity $\hat\sigma$. In particular, every translation of the unital is induced by an elation of the projective plane over $\mathbf{K}$. The group T_p is a normal subgroup of the point stabilizer $\mathrm{Aut}(\mathcal{U},\mathcal{B})_p$, and the group T is normal in $\mathrm{Aut}(\mathcal{U},\mathcal{B})$.*

Proof. It suffices to consider the point $p = \infty$ because the centralizer of the polarity forms a two-transitive subgroup of $\mathrm{Aut}(\mathcal{U},\mathcal{B})$, see 3.7. We claim that T_∞ acts semi-regularly on $B \setminus \{\infty\}$ for each block $B \in \mathcal{B}_\infty$ (and thus semi-regularly on $\mathcal{U} \setminus \{\infty\}$); then transitivity of Ξ'_σ (see 3.3) yields $\Xi'_\sigma = \mathrm{T}_\infty$.

Aiming at a contradiction, we assume that $\tau \in \mathrm{T}_\infty$ fixes a point $p \in \mathcal{U} \setminus \{\infty\}$ but $\tau \neq \mathrm{id}$. If τ fixes every block through p then it clearly fixes every point outside the block V joining p and ∞. The remaining points on V are then fixed because they lie on blocks with more than one point outside V. This contradicts our assumption $\tau \neq \mathrm{id}$.

So we are left with $\tau \in \mathrm{Aut}(\mathcal{U},\mathcal{B})_\infty$ such that τ fixes each block through ∞ and the point p but moves some block $B \in \mathcal{B}_p$. Up to conjugation by an element of Ξ_σ we may assume $p = (0,0)$. The blocks B and $\tau(B)$ have the same projection into the pencil $\mathcal{B}_\infty$. This contradicts 5.1 . □

7.2 Theorem. *The full group $\mathrm{Aut}(\mathcal{U}_\sigma,\mathcal{B}_\sigma)$ is a subgroup of $\mathrm{Aut}(\mathrm{T})$.*

Proof. After 7.1 we know that $\mathrm{Aut}(\mathcal{U}_\sigma,\mathcal{B}_\sigma)$ acts as a group of automorphisms on the little projective group T. The kernel of this action is the centralizer Φ of T. The group $\Xi'_\sigma = \mathrm{T}_\infty \le \mathrm{T}$ fixes precisely one point in $\mathcal{U}_\sigma \setminus \{\infty\}$. Therefore, this point is fixed by Φ. Since T acts transitively on $\mathcal{U}_\sigma$ we find that Φ fixes each point, and is thus trivial. □

7.3 Remark. In many cases one knows T and its full group $\mathrm{Aut}(\mathrm{T})$ of automorphisms quite well. For instance, consider the octonion field $\mathbb{O}$ over the field $\mathbb{R}$ of real numbers; the three[2] representatives of polarities and their centralizers (i.e., the corresponding *motion groups*) are discussed in detail in [20, Section 18]. The centralizers are simple groups, and thus coincide with the respective little projective groups on the unitals. Moreover, one knows that these groups do not allow outer automorphisms. Thus the full group of (abstract) automorphisms of the unital coincides with the little projective group in these cases.

[2] One of these polarities is elliptic, and not of interest here.

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