Universität
Stuttgart

Fachbereich
Mathematik

Embedded eigenvalues for the elastic strip with cracks

André Hänel, Christiane Schulz, Jens Wirth

Preprint 2011/018

Universität Stuttgart

Fachbereich Mathematik

Embedded eigenvalues for the elastic strip with cracks

André Hänel, Christiane Schulz, Jens Wirth

Preprint 2011/018

Fachbereich Mathematik
Fakultät Mathematik und Physik
Universität Stuttgart
Pfaffenwaldring 57
D-70 569 Stuttgart

E-Mail: preprints@mathematik.uni-stuttgart.de
WWW: http://www.mathematik.uni-stuttgart.de/preprints

ISSN **1613-8309**

EMBEDDED EIGENVALUES FOR THE ELASTIC STRIP WITH CRACKS

ANDRÉ HÄNEL, CHRISTIANE SCHULZ, AND JENS WIRTH

Abstract. The elasticity operator with zero Poisson coefficient is considered in a strip or a plate with an interior crack. It is shown that there exist embedded eigenvalues in the continuous spectrum due to the presence of the crack and asymptotic bounds in terms of the size of the crack are provided.

1. Introduction

In this paper we consider a two-dimensional strip $\mathbb{R} \times (-\pi/2, \pi/2)$ or a three-dimensional plate $\mathbb{R}^2 \times (-\pi/2, \pi/2)$ of a homogenous, linear elastic and isotropic material with zero Poisson coefficient having a crack. In the two-dimensional case we consider cracks of the form $\Gamma := \{0\} \times [-l, l]$ (first case, $l < \pi/2$) or $\Gamma := [-l, l] \times \{0\}$ (second case), whereas in the three-dimensional case cracks of the form $\Gamma := U \times \{0\}$ will be considered for an open subset U of $\mathbb{R}^2$.

Mathematically, the problem is described by a self-adjoint operator A_Γ and its associated quadratic form a_Γ. It is known, that the absolute continuous spectrum of the operator $A_\varnothing$ is $[0, \infty)$, if there is no crack in the material. Roitberg, Vassiliev and Weidl showed in [7] that there is an embedded eigenvalue, if we consider the half-strip, which is the limit situation of the first case with $l = \pi/2$. So it seems likely that there is also an eigenvalue for a smaller crack. Indeed, in this work we will show that there exists such an eigenvalue for arbitrarily small cracks.

1.1. **The model.** We will start by introducing the mathematical model. The elasticity operator A_Γ will be defined in terms of its sesquilinear form.

<u>The two-dimensional case:</u> Let $\Omega := \mathbb{R} \times (-\pi/2, \pi/2)$ and let Γ be the empty set or chosen as above. We denote

$$a_\Gamma[u, v] := \int_\Omega \Big(2\partial_1 u_1 \overline{\partial_1 v_1} + 2\partial_2 u_2 \overline{\partial_2 v_2} + (\partial_1 u_2 + \partial_2 u_1)\,\overline{(\partial_1 v_2 + \partial_2 v_1)}\Big)\ \mathrm{d}x, \tag{1.1}$$

for $u, v \in d[a_\Gamma] := H^1(\Omega \backslash \Gamma; \mathbb{C}^2)$. Hence the quadratic forms for different Γ differ only in their form domain. From the definition of a_Γ it is obvious that

$$a_\Gamma[u, v] \le 2\|u\|_{H^1}\|v\|_{H^1}.$$

It is important to point out that a converse is also valid. The inequality

$$c\|u\|^2_{H^1(\Omega\backslash\Gamma;\mathbb{C}^2)} \le a_\Gamma[u, u] + \|u\|^2_{L^2(\Omega;\mathbb{C}^2)} \tag{1.2}$$

is known as Korn's inequality, see, e.g., [6, Chapter 10] and [7]. It implies that the form a_Γ is closed on the domain $d[a_\Gamma]$ and as the form is lower semi-bounded there is a unique self-adjoint operator A_Γ associated to it. On smooth functions the operator A_Γ acts as

$$A_\Gamma = -\Delta - \operatorname{grad} \operatorname{div} \tag{1.3}$$

and is endowed with stress-free boundary conditions on $\partial\Omega \cup \Gamma$. If the strip has no crack, i.e., $\Gamma = \varnothing$, the domain of the operator is

$$D(A_\varnothing) = \left\{ u \in H^2(\Omega;\mathbb{C}^2) : \partial_2 u_2|_{x_2=\pm\frac{\pi}{2}} = \partial_2 u_1 + \partial_1 u_2|_{x_2=\pm\frac{\pi}{2}} = 0 \right\}, \tag{1.4}$$

with additional boundary conditions on Γ in the general case. Our approach is based on the symmetry of the domain $\Omega \setminus \Gamma$. It allows to decompose the form domain and therefore also the operator into symmetric pieces. We point out that the situation differs depending on wether we consider horizontal or vertical cracks. Using the ideas from [4, 7] we denote $H := L^2(\Omega;\mathbb{C}^2)$ and define H_j to be the following subspaces

$$H_j := \left\{ u \in H : u_k(x_1,x_2) = (-1)^{j+k} u_k(x_1,-x_2),\ k=1,2 \right\}, \quad j = 1,2,$$
$$H_3 := \{ u \in H : \partial_2 u_1(x_1,x_2) = 0,\ u_2(x_1,x_2) = 0 \}.$$

The set H_3 is a subspace of H_1. Let H_4 be the orthogonal complement of H_3 in H_1, thus

$$H_4 = \left\{ v \in H_1 : \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} v_1(x_1,x_2)\,\mathrm{d}x_2 = 0 \text{ for a.e. } x_1 \in \mathbb{R} \right\},$$

then we get the following decomposition

$$H = H_3 \oplus H_4 \oplus H_2$$

Denote $P^{(j)}$ the orthognal projection on H_j, $j = 1,2,3,4$. For horizontal cracks the operator A_Γ and the quadratic form a_Γ decompose into orthogonal sums

$$A_\Gamma = A_\Gamma|_{D(A_\Gamma)\cap H_2} \oplus A_\Gamma|_{D(A_\Gamma)\cap H_3} \oplus A_\Gamma|_{D(A_\Gamma)\cap H_4}$$
$$a_\Gamma = a_\Gamma|_{d[a_\Gamma]\cap H_2} \oplus a_\Gamma|_{d[a_\Gamma]\cap H_3} \oplus a_\Gamma|_{d[a_\Gamma]\cap H_4}$$

due to the fact, that the subspaces H_3, H_4 and H_2 form invariant subspaces for the operator A_Γ respectively for the quadratic form [7]. We use the short notation $A_\Gamma^{(j)}$ for $A_\Gamma|_{D(A_\Gamma)\cap H_j}$. The absolute continuous spectrum of $A_\varnothing$ is the set $[0,\infty)$. Förster showed in [3] that there is a constant $\Lambda \neq 0$, such that the essential spectrum of $A_\varnothing^{(4)}$ is $[\Lambda,\infty)$ and that for some perturbations of that operator eigenvalues below the threshold Λ can be found.

The decomposition does no longer hold true for vertical cracks as $H^1(\Omega\backslash\Gamma;\mathbb{C}^2)$ is not invariant with respect to the projection $P^{(3)}$. In order to show the existence of embedded eigenvalues we introduce the subspace

$$H_{4'} := \{ u \in H_4 : u_k(x_1,x_2) = (-1)^{k-1} u_k(-x_1,x_2),\ k=1,2 \}$$

which reduces the form a_Γ and and hence the associated operator A_Γ. Let $A_\Gamma^{(4')} := A_\Gamma|_{D(A_\Gamma)\cap H_{4'}}$. If there is no crack in the material then the operator $A_\varnothing^{(4')}$ is the restriction of the operator $A_\varnothing^{(4)}$ to $D(A_\varnothing) \cap H_{4'}$ and its essential spectrum is also $[\Lambda,\infty)$.

<u>The three-dimensional case:</u> Let $\Omega := \mathbb{R}^2 \times (-\pi/2,\pi/2)$ and $\Gamma := U \times \{0\}$ for $U \subseteq \mathbb{R}^2$, open. We denote

$$a_\Gamma[u,v] := \int_\Omega \left(\sum_{1\le i\le j\le 3} (\partial_i u_j + \partial_j u_i)\overline{(\partial_i v_j + \partial_j v_i)} \right) \mathrm{d}x \tag{1.5}$$

for $u, v \in d[a_\Gamma] := H^1(\Omega\backslash\Gamma; \mathbb{C}^3)$. Korn's equality (1.2) applies analogously for the three-dimensional case and the operator can be decomposed similarly, denoting:

$$\begin{aligned} H_j &:= \{u \in H : u_k(x_1, x_2, x_3) = (-1)^{j-1} u_k(x_1, x_2, -x_3),\ k = 1, 2 \text{ and} \\ &\qquad u_3(x_1, x_2, x_3) = (-1)^j u_3(x_1, x_2, -x_3)\}, \quad j = 1, 2, \\ H_3 &:= \{u \in H : \partial_3 u_1(x_1, x_2, x_3) = \partial_3 u_2(x_1, x_2, x_3) = 0,\ u_3(x_1, x_2, x_3) = 0\}. \end{aligned}$$

Let $H_4 := H_1 \ominus H_3$, thus

$$H_4 = \left\{ v \in H_1 : \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} v_k(x_1, x_2, x_3)\, \mathrm{d}x_3 = 0 \text{ for a.e. } (x_1, x_2) \in \mathbb{R}^2,\ k = 1, 2 \right\}$$

and let $P^{(j)}$ be the orthogonal projection on H_j, $j = 1, 2, 3, 4$. Similarly the absolute continuous spectrum of $A_\varnothing$ is the set $[0, \infty)$ and the essential spectrum of $A^{(4)}_\varnothing$ is $[\Lambda, \infty)$ with the same constant Λ as in the two-dimensional case (cf. [3]).

1.2. **Results.** In the two-dimensional case we get the following results:

Result (Theorem 3.5)**.** *The two-dimensional elasticity operator A_Γ has an embedded eigenvalue in the interval $[0, \Lambda)$ for every crack Γ of the form $\{0\} \times [-l, l]$ or $[-l, l] \times \{0\}$.*

To obtain this result we will construct a test function $u \in H_4 \cap d[a_\Gamma]$ resp. $u \in H_{4'} \cap d[a_\Gamma]$ and show that the variational coefficient of this test function u will be below the threshold Λ. In addition we will show that the distance between the value of the variational coefficient and Λ for certain test functions and small size l of the crack is about l^4 for horizontal cracks and l^8 for vertical ones. This gives a lower bound for the distance of the eigenvalue to the threshold.

Result (Theorem 3.6, 3.7)**.** *For $l > 0$ let $\Gamma^{(l)}$ denote the crack of size l and let $k := 4$ for horizontal cracks and $k := 8$ for vertical cracks. Then there exists a constant $C > 0$ and an eigenvalue $\lambda(l) \in \sigma_p(A_{\Gamma^{(l)}})$ with*

$$|\lambda(l) - \Lambda| \geq C l^k. \tag{1.6}$$

In the three-dimensional case we obtain:

Result (Theorem 4.2, 4.3)**.** *The three-dimensional elasticity operator A_Γ has an embedded eigenvalue in the interval $[0, \Lambda)$ for every crack Γ of the form $U \times \{0\}$ with U open subset of $\mathbb{R}^2$.*

If U denotes the unit disc we can take advantage of further spacial symmetries to decompose the operator $A^{(4)}_\Gamma$. Due to this decomposition we obtain:

Result (Corollary 4.6)**.** *The elasticity operator $A^{(4)}_\Gamma$ has infinitely many eigenvalues (counted by multiplicity) in the interval $[0, \Lambda)$.*

In the last section we will also obtain a lower bound for the distance of the eigenvalue to the threshold which depends on the considered subspace.

2. Cracks and essential spectra

Lemma 2.1. *The essential spectra of A_Γ and $A_\Gamma^{(j)}$, $j = 2, 3, 4$ (horizontal cracks) resp. $A_\Gamma^{(4')}$ (vertical cracks) in the two-dimensional and in the three-dimensial case, are independent of the crack Γ.*

Proof. We will sketch the proof for the two-dimensional case with horizontal cracks, the other cases are proven analogously. It is sufficent to show that $(A_\Gamma - \lambda)^{-1} - (A_\varnothing - \lambda)^{-1}$ is a compact operator for some $\lambda < 0$ since we have

$$(A_\Gamma^{(j)} - \lambda)^{-1} - (A_\varnothing^{(j)} - \lambda)^{-1} = \Big((A_\Gamma - \lambda)^{-1} - (A_\varnothing - \lambda)^{-1}\Big)P^{(j)}$$

for $j = 2, 3, 4$. Following Birman [2] we choose a smooth closed line Γ_0 in Ω around the crack and we introduce two operators $B_\varnothing$ resp. B_Γ representing the elasticity operator on Ω resp. $\Omega\backslash\Gamma$ with additional Dirichlet conditions at Γ_0. Let $\lambda < 0$. As in [2] the operators $(A_\Gamma - \lambda)^{-1} - (B_\Gamma - \lambda)^{-1}$ and $(A_\varnothing - \lambda)^{-1} - (B_\varnothing - \lambda)^{-1}$ are compact. Due to the compactness of the embedding of $H^1(\Omega_1)$ and $H^1(\Omega_1\backslash\Gamma)$ in $L_2(\Omega_1)$, where Ω_1 denotes the interior domain with respect to Γ_0, the operator $(B_\Gamma - \lambda)^{-1} - (B_\varnothing - \lambda)^{-1}$ is compact and the statement follows. □

We point out that this lemma is essential for what follows. Our aim is now to construct a test function, contained in $d[a_\Gamma] \cap H_4$ or $d[a_\Gamma] \cap H_{4'}$, respectively, whose variational coefficient is below the threshold of the essential spectrum, which equals Λ in all considered cases. This gives rise to an eigenvalue below this threshold. Thus we can focus on the parameter Λ and the corresponding generalised eigenfunction ϕ of the operator without crack. This function will be modified such that the modified function v will be in $H^1(\Omega; \mathbb{C}^2)$ or $H^1(\Omega; \mathbb{C}^3)$, respectively. In a final step we will add a further perturbation αf, which is allowed to jump at the crack. The function $v + \alpha f$ will in a natural way be our test function.

3. The two-dimensional case

3.1. **General setting.** We obtain the parameter Λ and the generalised eigenfunctions by applying a Fourier transform in x_1-direction and solving the corresponding Sturm–Liouville problem. The operator $A_\varnothing$ is reduced by Fourier transform to an ordinary differential operator $A_\varnothing(\xi)$ parametrised by the frequency $\xi \in \mathbb{R}$. It is given by

$$A_\varnothing(\xi) = \begin{pmatrix} 2\xi^2 - \partial^2 & -\mathrm{i}\xi\partial \\ -\mathrm{i}\xi\partial & \xi^2 - 2\partial^2 \end{pmatrix} \tag{3.1}$$

where

$$D(A_\varnothing(\xi)) = \Big\{u \in H^2((-\pi/2, \pi/2); \mathbb{C}^2) : \mathrm{i}\xi u_2 + \partial u_1|_{x_2 = \pm\frac{\pi}{2}} = \partial u_2|_{x_2 = \pm\frac{\pi}{2}} = 0\Big\}. \tag{3.2}$$

The symmetries extend to the operator $A_\varnothing(\xi)$ via

$$h_j := \Big\{u \in D(A_\varnothing(\xi)) : u_l(x_2) = (-1)^{j+l}u_l(-x_2)\Big\}, \quad l = 1, 2.$$

The subspace h_3 consists of all functions, linearly dependent of the function $(1, 0)$, and the subspace h_4 is given by $h_4 := h_1 \ominus h_3$. Förster showed in [3] that the smallest eigenvalue of this family of Sturm-Liouville-problems in h_4 is Λ with $\Lambda = 1.887837 \pm 10^{-6}$ achieved for $\varkappa = \pm 0.632138 \pm 10^{-6}$. Searching for functions $\psi_\pm \in h_4$ fulfilling the eigenvalue equation $A_\varnothing(\pm\varkappa)\psi_\pm = \Lambda\psi_\pm$ together with the boundary conditions yields generalised eigenfunctions

$\phi_\pm(x_1, x_2) = \psi_\pm(x_2)e^{\pm ix_1\varkappa}$ of $A_\varnothing$. In particular, they satisfy the boundary conditions from (1.4). They can be combined to the real-valued function

$$\phi(x_1, x_2) := \begin{pmatrix} \psi_1(x_2)\cos(\varkappa x_1) \\ \psi_2(x_2)\sin(\varkappa x_1) \end{pmatrix}, \tag{3.3}$$

where

$$\psi(t) := \begin{pmatrix} \varkappa^2\beta\cos\left(\frac{\beta\pi}{2}\right)\cos(\gamma t) + \gamma^2\beta\cos\left(\frac{\gamma\pi}{2}\right)\cos(\beta t) \\ -\gamma\beta\varkappa\cos\left(\frac{\beta\pi}{2}\right)\sin(\gamma t) + \gamma^2\varkappa\cos\left(\frac{\gamma\pi}{2}\right)\sin(\beta t) \end{pmatrix} \tag{3.4}$$

and the parameters are to be chosen as

$$\beta := \sqrt{\Lambda - \varkappa^2}, \quad \gamma := \sqrt{\frac{\Lambda}{2} - \varkappa^2}. \tag{3.5}$$

The function ϕ fulfils $\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \phi_1(x_1, x_2)\, dx_2 = 0$ for all $x_1 \in \mathbb{R}$ and satisfies the boundary conditions.

Now we will focus on the elasticity operator for the strip with a crack Γ, where $\Gamma := \{0\}\times[-l, l]$ or $\Gamma := [-l, l]\times\{0\}$. In the latter case we assume $l < 1$. This introduces additional boundary conditions at the crack for the elasticity operator as well as more freedom for the choice of test functions. First we will multiply the gerenalised eigenfunction with a decaying function $\rho_\delta(x_1)$, so we get the approximate eigenfunction

$$v_\delta(x_1, x_2) := \rho_\delta(x_1)\phi(x_1, x_2)$$

The sequence of functions ρ_δ, $\delta \le 1$, will be chosen in $C_0^2(\mathbb{R})$ as

$$\rho_\delta(x_1) := \rho(\delta x_1)$$

for an even function $\rho \in C_0^2(\mathbb{R})$ being constant $\rho(x_1) = 1$ for $|x_1| \le 1$. We define our test function as perturbation of v_δ

$$u_{\alpha,\delta}(x_1, x_2) := v_\delta(x_1, x_2) + \alpha f(x_1, x_2) \tag{3.6}$$

with f supported near the crack Γ. We assume $\alpha \in \mathbb{R}$ and f real-valued to simplify the calculations. The functions ϕ, ρ_δ and therefore the function v_δ are also real.

We assume that the function f is supported inside $(-1, 1)\times(-\pi/2, \pi/2)$, i.e., in particular $\rho_\delta = 1$ on the support of f. In addition the function f has to be an element of H_4 or $H_{4'}$, resp., such that $u_{\alpha,\delta} \in d[a_\Gamma] \cap H_4$ or $u_{\alpha,\delta} \in d[a_\Gamma] \cap H_{4'}$, resp. Further conditions on f will arise during the calculations.

The value of the quadratic form a_Γ acting on the test function $u_{\alpha,\delta}$ is given by

$$\begin{aligned} a_\Gamma[u_{\alpha,\delta}, u_{\alpha,\delta}] &= a_\Gamma[v_\delta + \alpha f, v_\delta + \alpha f] \\ &= a_\Gamma[v_\delta, v_\delta] + 2\alpha \operatorname{Re} a_\Gamma[f, v_\delta] + \alpha^2 a_\Gamma[f, f]. \end{aligned}$$

In order to show existence of eigenvalues we consider the variational coefficient

$$\begin{aligned} \Xi(\alpha, \delta) &:= \frac{a_\Gamma\left[u_{\alpha,\delta}, u_{\alpha,\delta}\right] - \Lambda\left(u_{\alpha,\delta}, u_{\alpha,\delta}\right)}{\left(u_{\alpha,\delta}, u_{\alpha,\delta}\right)} \\ &= \frac{a_\Gamma[v_\delta, v_\delta] + 2\alpha a_\Gamma[f, v_\delta] + \alpha^2 a_\Gamma[f, f] - \Lambda\left[\|v_\delta\|^2 + 2\alpha\left(f, v_\delta\right) + \alpha^2\|f\|^2\right]}{\|v_\delta\|^2 + 2\alpha\left(f, v_\delta\right) + \alpha^2\|f\|^2}. \end{aligned}$$

If it takes negative values, the spectrum of $A_\Gamma^{(4)}$ or of $A_\Gamma^{(4')}$, resp., has to include points below Λ which have to be eigenvalues of the operator by Lemma 2.1 . For this we have to understand the influence of the choice of f on the bilinear expressions $a_\Gamma[f, v_\delta]$ and $a_\Gamma[f, f]$ and the appearing inner products.

For the denominator we obtain the following

Lemma 3.1. *Let $u_{\alpha,\delta}$ be defined by* (3.6) *and denote*

$$a_j := \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} |d_j(x_2)|^2 \, \mathrm{d}x_2.$$

Then for all $\alpha_0 > 0$

$$\|u_{\alpha,\delta}\|^2 \sim \frac{a_1 + a_2}{2\delta} \|\rho\|^2 + \mathcal{O}(1) \tag{3.7}$$

as $\delta \to 0$ uniform in $|\alpha| \le \alpha_0$.

Proof. The definition of $u_{\alpha,\delta}$ leads to

$$(u_{\alpha,\delta}, u_{\alpha,\delta}) = \|v_\delta\|^2 + 2\alpha\,(f, v_\delta) + \alpha^2 \|f\|^2. \tag{3.8}$$

For the first term $\|v_\delta\|$ we obtain

$$\begin{aligned}\|v_\delta\|^2 &= \int_\Omega \left[|d_1(x_2)|^2 \cos^2(\varkappa x_1) + |d_2(x_2)|^2 \sin^2(\varkappa x_1)\right] \rho_\delta^2(x_1) \,\mathrm{d}x \\ &= \frac{a_1 + a_2}{2} \|\rho_\delta\|^2 + \frac{a_1 - a_2}{2} \int_{\mathbb{R}} \cos(2\varkappa x_1) \rho_\delta^2(x_1) \,\mathrm{d}x_1, \\ &\sim \frac{a_1 + a_2}{2\delta} \|\rho\|^2 + \mathcal{O}(1)\end{aligned}$$

based on the differentiability assumption $\rho \in C_0^1(\mathbb{R})$ to treat the second integral.

The second and third term in (3.8) are independent of δ due to the support assumption on f and are therefore also $\mathcal{O}(1)$ uniform in $|\alpha| \le \alpha_0$. □

Next, we will consider the numerator of the variational coefficient, i.e.,

$$a_\Gamma\left[u_{\alpha,\delta}, u_{\alpha,\delta}\right] - \Lambda\left(u_{\alpha,\delta}, u_{\alpha,\delta}\right) = P(v_\delta) + 2\alpha Q(v_\delta, f) + \alpha^2 R(f) \tag{3.9}$$

where we use the notation

$$P(v_\delta) := a_\Gamma[v_\delta, v_\delta] - \Lambda \left\|v_\delta\right\|^2, \tag{3.10}$$

$$Q(v_\delta, f) := a_\Gamma[v_\delta, f] - \Lambda\left(v_\delta, f\right), \tag{3.11}$$

$$R(f) := a_\Gamma[f, f] - \Lambda \left\|f\right\|^2. \tag{3.12}$$

All these terms are independent of α. The first term $P(v_\delta)$ is treated using the following lemma.

Lemma 3.2. *Let $P(v_\delta)$ be given by* (3.10)*, then*

$$P(v_\delta) = \int_\Omega (\partial_1 \rho_\delta)^2 \left(2\phi_1^2 + \phi_2^2\right) \,\mathrm{d}x.$$

In consequence,

$$P(v_\delta) \sim \left(a_1 + \frac{a_2}{2}\right) \delta \|\partial_1 \rho\|^2 + \mathcal{O}(\delta^2), \qquad \delta \to 0. \tag{3.13}$$

Proof. As the function ϕ is formally an eigenfunction of $A_\varnothing$, we have

$$\Lambda\phi_1 = -2\partial_1^2\phi_1 - \partial_2^2\phi_1 - \partial_1\partial_2\phi_2,$$
$$\Lambda\phi_2 = -\partial_1\partial_2\phi_1 - \partial_1^2\phi_2 - 2\partial_2^2\phi_2.$$

Therefore, we obtain $\Lambda \left\|v_\delta\right\|^2 = Z_1 + Z_2$ with

$$\begin{aligned} Z_1 &:= \int_\Omega \Lambda\phi_1^2\rho_\delta^2 \,\mathrm{d}x \\ &= \int_\Omega \Big[\Big(2\left(\partial_1\phi_1\right)^2 + \left(\partial_2\phi_1\right)^2 + \partial_1\phi_2\partial_2\phi_1\Big)\rho_\delta^2 + 4\phi_1\rho_\delta\partial_1\phi_1\partial_1\rho_\delta\Big] \,\mathrm{d}x \\ Z_2 &:= \int_\Omega \Lambda\phi_2^2\rho_\delta^2 \,\mathrm{d}x \\ &= \int_\Omega \Big[\Big(\partial_2\phi_1\partial_1\phi_2 + \left(\partial_1\phi_2\right)^2 + 2\left(\partial_2\phi_2\right)^2\Big)\rho_\delta^2 + 2\left(\partial_2\phi_1 + \partial_1\phi_2\right)\phi_2\rho_\delta\partial_1\rho_\delta\Big] \,\mathrm{d}x. \end{aligned}$$

Here we used integration by parts and in particular that the functions ϕ_j satisfy the boundary conditions from (1.4). On the other hand, we obtain for the quadratic form

$$\begin{aligned} a_\Gamma[v_\delta, v_\delta] &= \int_\Omega \Big(2\left(\partial_1 v_{\delta,1}\right)^2 + 2\left(\partial_2 v_{\delta,2}\right)^2 + \left(\partial_1 v_{\delta,2} + \partial_2 v_{\delta,1}\right)^2\Big) \,\mathrm{d}x \\ &= \int_\Omega \Big[\Big(2\left(\partial_1\phi_1\right)^2 + 2\left(\partial_2\phi_2\right)^2 + \left(\partial_1\phi_2 + \partial_2\phi_1\right)^2\Big)\rho_\delta^2\Big] \,\mathrm{d}x \\ &\quad + \int_\Omega \Big[\Big(4\left(\partial_1\phi_1\right)\phi_1 + 2\partial_1\phi_2\phi_2 + 2\phi_2\partial_2\phi_1\Big)\rho_\delta\partial_1\rho_\delta + \left(2\phi_1^2 + \phi_2\right)^2\left(\partial_1\rho_\delta\right)^2\Big] \,\mathrm{d}x. \end{aligned}$$

Combining both, we arrive at the first statement of Lemma 3.2

$$a_\Gamma[v_\delta, v_\delta] - \Lambda\left\|v_\delta\right\|^2 = \int_\Omega \left(\partial_1\rho_\delta\right)^2\left(2\phi_1^2 + \phi_2^2\right) \,\mathrm{d}x.$$

This can be treated as the first term in the proof of Lemma 3.1, except for an additional factor of δ^2 and that we need to control a second derivative for ρ. This proves the lemma. ☐

Next, we will focus the second term $Q(v_\delta, f)$. The choice of f has to be made in such a way that $u_{\alpha,\delta} \in d[a_\Gamma] \cap H_4$ respectively that $d[a_\Gamma] \cap H_{4'}$. If the crack is $\Gamma = \{0\} \times [-l, l]$ we choose the function f continuously differentiable in x_2-direction and f_2 is allowed to jump at the crack in x_1-direction. If $\Gamma = [-l, l] \times \{0\}$ the function f is assumed to be continuously differentiable in x_1-direction and f_2 is allowed to jump in x_2-direction along Γ. The symmetry requirements in both cases imply that f_1 is continuous.

Then the term $Q(v_\delta, f)$ can be calculated using integration by parts. For such functions f we use the notation

$$\left[f_j\right]_{x_1=0}(x_2) = \lim_{x_1\searrow 0} f_j(x_1, x_2) - \lim_{x_1\nearrow 0} f_j(x_1, x_2)$$

to denote their jumps in x_1-direction; similarly for jumps in x_2-direction.

Lemma 3.3. *Let $Q(v_\delta, f)$ defined by (3.11), then we obtain*

$$Q(v_\delta, f) = -\int_{-l}^{l} \left(\partial_1\phi_2(0, x_2) + \partial_2\phi_1(0, x_2)\right)\left[f_2\right]_{x_1=0}(x_2) \,\mathrm{d}x_2, \tag{3.14}$$

in the case $\Gamma = \{0\} \times [-l, l]$ *and*

$$Q(v_\delta, f) = -2\int_{-l}^{l} \partial_2\phi_2(x_1, 0)\,[f_2]_{x_2=0}(x_1)\;\mathrm{d}x_1, \tag{3.15}$$

for $\Gamma = [-l, l] \times \{0\}$. *Both are independent of* δ.

Proof. We consider the crack $\Gamma = \{0\} \times [-l, l]$ first. Due to the support assumption made for f we immediately get

$$Q(v_\delta, f) = a_\Gamma[v_\delta, f] - \Lambda\,(v_\delta, f) = a_\Gamma[f, \phi] - \Lambda\,(f, \phi)\,,$$

such that integration by parts and the properties of the function f (being continuously differentiable in x_2-direction and having a jump at the crack in x_1-direction) imply

$$\begin{aligned} Q(v_\delta, f) = &-\int_{-l}^{l} 2\partial_1\phi_1(0, x_2)\,[f_1]_{x_1=0}(x_2)\;\mathrm{d}x_2 \\ &+\int_{-l}^{l} (\partial_1\phi_2(0, x_2) + \partial_2\phi_1(0, x_2))\,[f_2]_{x_1=0}(x_2)\;\mathrm{d}x_2. \end{aligned}$$

Since $\partial_1\phi_1(0, x_2) = 0$ we obtain

$$Q(v_\delta, f) = -\int_{-l}^{l} (\partial_1\phi_2(0, x_2) + \partial_2\phi_1(0, x_2))\,[f_2]_{x_1=0}(x_2)\;\mathrm{d}x_2,$$

which is independent of f_1 and δ. The second case $\Gamma = [-l, l] \times \{0\}$ is treated analogously, except that there are only boundary terms arising from f_2. □

The proof of our first main result is reduced to making an appropriate choice for the auxiliary function f.

Lemma 3.4. *There exist functions* ρ_δ *and* f, *such that for* $u_{\alpha,\delta}$ *given by* (3.6) *the variational coefficient*

$$\Xi(\alpha, \delta) = \frac{a_\Gamma[u_{\alpha,\delta}, u_{\alpha,\delta}] - \Lambda(u_{\alpha,\delta}, u_{\alpha,\delta})}{(u_{\alpha,\delta}, u_{\alpha,\delta})}$$

is negative for some choice of α *and* δ.

This lemma leads immediately to

Theorem 3.5. *The two-dimensional elasticity operator* A_Γ *has an embedded eigenvalue in the interval* $[0, \Lambda)$ *for every crack* Γ *of the form* $\{0\} \times [-l, l]$ *or* $[-l, l] \times \{0\}$.

Proof. The function $u_{\alpha,\delta}$, given by (3.6) is an element of H_4 or $H_{4'}$, resp. Then the statement of Lemma 3.4 together with Lemma 2.1 in the form $\Lambda = \inf \sigma_{ess}(A_\Gamma^{(4)}) = \inf \sigma_{ess}(A_\Gamma^{(4')})$ yields the desired statement. □

It remains to prove Lemma 3.4.

Proof of Lemma 3.4. The variational coefficient is given by

$$\Xi(\alpha, \delta) = \frac{P(v_\delta) + 2\alpha Q(v_\delta, f) + \alpha^2 R(f)}{(u_{\alpha,\delta}, u_{\alpha,\delta})}.$$

The term $P(v_\delta)$ is positive and $R(f)$ will be also positive in general. Therefore we have to ensure that $Q(v_\delta, f)$ does not vanish. It holds for $\Gamma = \{0\} \times [-l, l]$

$$Q(v_\delta, f) = -\int_{-l}^{l} T(x_2) \, [f_2]_{x_1=0} \, (x_2) \; \mathrm{d}x_2 \tag{3.16}$$

and the first factor of the integrand is given by

$$\begin{aligned} T(x_2) &:= \partial_1 \phi_2(0, x_2) + \partial_2 \phi_1(0, x_2) = \varkappa \psi_2(x_2) + \psi_1'(x_2) \\ &= -2\varkappa^2 \gamma \beta \cos\left(\frac{\beta\pi}{2}\right) \sin(\gamma x_2) + \gamma^2(\varkappa^2 - \beta^2) \cos\left(\frac{\gamma\pi}{2}\right) \sin(\beta x_2), \end{aligned}$$

satisfies $T(x_2) \neq 0$ for $x_2 \neq 0$ small. So we can choose the jump $[f_2]_{x_1=0}$ in such a way that $Q(v_\delta, f) < 0$. In the second case $\Gamma = [-l, l] \times \{0\}$

$$Q(v_\delta, f) = -\int_{-l}^{l} S(x_1) \, [f_2]_{x_2=0} \, (x_1) \; \mathrm{d}x_1 \tag{3.17}$$

and the first factor

$$S(x_1) := 2\partial_2 \phi_2(x_1, 0) = 2\gamma^2 \varkappa \beta \left(\cos\left(\frac{\gamma\pi}{2}\right) - \cos\left(\frac{\beta\pi}{2}\right) \right) \sin(\varkappa x_1)$$

which is not identically vanishing for our choice of β and γ due to (3.5). Again we can choose the jump in such a way that $Q(v_\delta, f) < 0$. We fix one such choice for f for now.

The Lemmas 3.1 and 3.2 give the behaviour of the remaining terms as $\delta \to 0$ and therefore lead to

$$\Xi(\alpha, \delta) \sim \frac{d_1 \delta^2 \, \|\partial_1 \rho\|^2 + 2\alpha\delta Q(v_\delta, f) + \alpha^2 \delta R(f) + \mathcal{O}(\delta^3)}{c_1 \|\rho\|^2 + \mathcal{O}(\delta)} \tag{3.18}$$

with $d_1 = (a_1 + a_2/2)$ and $c_1 = (a_1 + a_2)/2$. Choosing α such that $2\alpha Q(v_\delta, f) + \alpha^2 R(f) < 0$ and then δ small enough, we see that $\Xi(\alpha, \delta)$ becomes negative for some choice of parameters. □

Remark 1. *The choice of ϕ was not unique, there is also the following choice*

$$\widetilde{\phi}(x_1, x_2) = \begin{pmatrix} -\psi_1(x_2) \sin(\varkappa x_1) \\ \psi_2(x_2) \cos(\varkappa x_1) \end{pmatrix} \tag{3.19}$$

or any linear combination of them. Note that the corresponding function $\widetilde{v}_\delta$ is still an element of $d[a_\Gamma] \cap H_4$ but no longer of $d[a_\Gamma] \cap H_{4'}$. We will comment, which changes have to be made for this choice in the case of a horizontal crack.

In Lemma 3.2 the leading coefficient changes to $(a_1/2 + a_2)$. In the proof of Lemma 3.4 we have to change the formula for S. For the above choice it is given as

$$\widetilde{S}(x_1) = 2\partial_2 \widetilde{\phi}_2(x_1, 0) = 2\psi_2'(0) \cos(\varkappa x_1).$$

3.2. **Asymptotic bounds.** Embedded eigenvalues exist only if $l > 0$, in the following we ask at what rate such eigenvalues are separated from the threshold Λ. The following gives an asymptotic lower bound for the distance of the eigenvalue to the threshold Λ. The minimum of $\Xi(\alpha, \delta)$ corresponds to an eigenvalue, thus any estimate, how small $\Xi(\alpha, \delta)$ can get, gives such a bound.

We construct a test function based on $f^{(l)}$, which scales like

$$f^{(l)}(x_1, x_2) := l \, f\left(\frac{x_1}{l}, \frac{x_2}{l}\right), \qquad l \ll 1, \tag{3.20}$$

for a given fixed function f. The scaling is chosen such that $\|f^{(l)}\| = \|f\|$. Based on the test function $u_{\alpha,\delta,l}$ defined in analogy to (3.6) we consider the variational coefficient $\Xi(\alpha,\delta,l)$. It remains to make optimal choices of parameters.

Now we have to distinguish between the different orientations of the crack.

<u>Horizontal crack:</u> $\Gamma = [-l,l]\times\{0\}$.
We choose $\widetilde{\phi}$ as in (3.19) and f_2 jumps in $x_2 = 0$ along $x_1 \in [-l,l]$. The optimal choice of the parameters depends on the asymptotic properties of $R(f^{(l)})$ and $Q(\widetilde{v}_\delta, f^{(l)})$ as $l \to 0$. As consequence of the definition of $f^{(l)}$, we obtain

$$R(f^{(l)}) = l^{-2} a_\Gamma[f,f] - \|f\|^2 \tag{3.21}$$

and (again independent of δ)

$$Q(\widetilde{v}_\delta, f^{(l)}) = -\int_{-1}^{1} \widetilde{S}(lx_1)[f_2]_{x_2=0}(x_1)\,\mathrm{d}x_1 \sim -\mu_0 + \mathcal{O}(l^2)$$

based on Taylor expansion of $\widetilde{S}$ and with

$$\mu_0 := \widetilde{S}(0)\int_{-1}^{1}[f_2]_{x_2=0}(x_1)\,\mathrm{d}x_1.$$

The choice of f is made such that $\mu_0 > 0$. Therefore, the choice of α in the proof of Lemma 3.4 is made to minimise $-2\alpha\mu_0 + \alpha^2 l^{-2} a_\Gamma[f,f] - \alpha^2\|f\|^2$. Thus

$$\alpha = \frac{\mu_0}{l^{-2}a_\Gamma[f,f] - \|f\|^2} \sim \frac{\mu_0}{a_\Gamma[f,f]} l^2 + \mathcal{O}(l^4).$$

In particular, we see that $2\alpha Q(\widetilde{v}_\delta, f^{(l)}) + \alpha^2 R(f^{(l)})$ is of the size l^2. It remains to choose δ. The optimal choice of δ makes $d_1\delta^2\|\partial_1\rho\|^2 + \delta(2\alpha Q(\widetilde{v}_\delta, f^{(l)}) + \alpha^2 R(f^{(l)}))$ minimal, see (3.18), and therefore

$$\delta = -\frac{2\alpha Q(\widetilde{v}_\delta, f^{(l)}) + \alpha^2 R(f^{(l)})}{2d_1\|\partial_1\rho\|^2} \sim \frac{\mu_0^2}{(a_1+2a_2)\|\partial_1\rho\|^2 a_\Gamma[f,f]} l^2 + \mathcal{O}(l^4).$$

This yields

$$\Xi(\alpha,\delta,l) \sim \nu l^4 + \mathcal{O}(l^5)$$

for a certain constant ν. Hence, we obtain the following statement on the location of the eigenvalue. In view of [8] this seems to be optimal.

Theorem 3.6. *Assume the crack is of the form $\Gamma^{(l)} = [-l,l]\times\{0\}$. Then there exists a constant C independent of l and an eigenvalue $\lambda(l) \in \sigma_p(A^{(4)}_{\Gamma^{(l)}})$ such that*

$$|\lambda(l) - \Lambda| \geq Cl^4. \tag{3.22}$$

<u>Vertical crack:</u> $\Gamma = \{0\}\times[-l,l]$.
We choose ϕ as in (3.3) and f_2 jumps in $x_1 = 0$ along $x_2 \in [-l,l]$. Main difference to the previous case is that now

$$Q(v_\delta, f^{(l)}) = -\int_{-1}^{1} T(lx_2)[f_2]_{x_1=0}(x_2)\,\mathrm{d}x_2 \sim -\mu_1 l + \mathcal{O}(l^3),$$

$$\mu_1 = T'(0)\int_{-1}^{1} x_2[f_2]_{x_1=0}(x_2)\,\mathrm{d}x_2.$$

Applying the same reasoning as above, this gives α of the size l^3 and $2\alpha Q(v_\delta, f^{(l)}) + \alpha^2 R(f^{(l)})$ is of the size l^4. This yields δ of the size l^4 and finally that $\Xi(\alpha, \delta, l)$ is of size l^8.

Theorem 3.7. *Assume the crack is of the form $\Gamma^{(l)} = \{0\} \times [-l, l]$. Then there exists a constant C independent of l and an eigenvalue $\lambda(l) \in \sigma_p(A^{(4)}_{\Gamma^{(l)}})$ such that*

$$|\lambda(l) - \Lambda| \geq Cl^8. \tag{3.23}$$

4. The three-dimensional case

4.1. **The generel setting.** The approach is the similar to the two-dimensional setting. Applying the Fourier transform with respect to the first two variables we obtain a decomposition of the operator $A_\varnothing$ into a family of operators $(A(\xi))_{\xi\in\mathbb{R}^2}$ acting on $L^2\left((-\pi/2, \pi/2)\,; \mathbb{C}^3\right)$. The same holds true for the operators $A^{(j)}_\varnothing$, $j = 2, 3, 4$. Searching for the spectral minimum of the operator $A^{(4)}_\varnothing$ we obtain the same constant Λ as in two-dimensional case. In contrast to the above considered setting we obtain a family of eigenfunctions $(\psi(\xi, \cdot))_{\xi\in S_\varkappa}$ of the operator $A^{(4)}(\xi)$ satisfying $A^{(4)}(\xi)\psi(\xi, \cdot) = \Lambda\psi(\xi, \cdot)$. Here and in the sequel $S_\varkappa$ denotes the circle of radius κ centred at the origin, where $\varkappa > 0$ is the same constant as in the two-dimensional case. An appropriate choice for the functions $\psi(\xi, \cdot)$ is for example

$$\psi(\xi, t) := \begin{pmatrix} \mathrm{i}\xi_1\varkappa\beta\cos\left(\frac{\beta\pi}{2}\right)\cos(\gamma t) + \mathrm{i}\xi_1\frac{\gamma^2\beta}{\varkappa}\cos\left(\frac{\gamma\pi}{2}\right)\cos(\beta t) \\ \mathrm{i}\xi_2\varkappa\beta\cos\left(\frac{\beta\pi}{2}\right)\cos(\gamma t) + \mathrm{i}\xi_2\frac{\gamma^2\beta}{\varkappa}\cos\left(\frac{\gamma\pi}{2}\right)\cos(\beta t) \\ -\gamma\beta\varkappa\cos\left(\frac{\beta\pi}{2}\right)\sin(\gamma t) + \gamma^2\varkappa\cos\left(\frac{\gamma\pi}{2}\right)\sin(\beta t) \end{pmatrix}, \tag{4.1}$$

chosen similar to (3.4), see [3] or [5] for further details. This introduces more freedom for the choice of the generalised eigenfunction ϕ. Let $k \in L^2(S_\varkappa)$, $k \neq 0$. We define:

$$\phi(\hat{x}, x_3) := \int_{S_\varkappa} \psi(\xi, x_3)k(\xi)\mathrm{e}^{\mathrm{i}\xi\hat{x}}\,\mathrm{d}\sigma(\xi), \qquad (\hat{x}, x_3) \in \Omega. \tag{4.2}$$

Let furthermore $\rho \in C_0^\infty(\mathbb{R}^2)$ be a real-valued, radial function with $\rho = 1$ on $B_1(0)$. For $\delta \in (0, 1)$ we define in analogy to the previous section

$$\rho_\delta(\hat{x}) := \rho(\delta\hat{x}), \qquad \hat{x} \in \mathbb{R}^2,$$
$$v_\delta(\hat{x}, x_3) := \phi(\hat{x}, x_3)\rho_\delta(\hat{x}), \qquad (\hat{x}, x_3) \in \Omega.$$

Note that v_δ is not supposed to be real in this consideration. Taking real parts will (with minor modifications) allow to move to real functions again. Then $v_\delta \in C^\infty(\Omega)$ and $v_\delta \in d[a_\Gamma] \cap H_4$. The next statement is the three-dimensional equivalent to Lemma 3.1

Lemma 4.1. *There exists $c > 0$ such that*

$$\lim_{\delta\to 0} \delta\|v_\delta\|^2 = c. \tag{4.3}$$

Proof. The proof consists of applying two results of Agmon and Hörmander [1]. By Theorem 3.1. in [1] we get for $i \in \{1, 2, 3\}$ and $x_3 \in (-\pi/2, \pi/2)$:

$$\lim_{\delta\to 0} \underbrace{\delta\int_{\mathbb{R}^2} |\phi_i(\hat{x}, x_3)|^2\tilde{\rho}_\delta(\hat{x})^2\,\mathrm{d}\hat{x}}_{=:\gamma_{i,\delta}(x_3)} = \int_{S_\varkappa} |k(\xi)\psi_i(\xi, x_3)|^2 \int_{N_\xi} \tilde{\rho}(x)^2\,\mathrm{d}\sigma(x)\,\mathrm{d}\sigma(\xi) > 0,$$

where N_ξ is the normal hyperplane at $\xi \in S_\varkappa$. Furthermore by Theorem 2.1. in [1] it holds that

$$\gamma_{i,\delta}(x_3) \leq \|\rho\|_\infty^2 \delta \int_{B_0(2/\delta)} |\phi_i(\hat{x}, x_3)|^2 \, d\hat{x} \leq 2C\|\rho\|_\infty^2 \int_{S_\varkappa} |k(\xi)\psi_i(\xi, x_3)|^2 \, \mathrm{d}\sigma(\xi),$$

The assertion follows now by applying the Lebesgue theorem. □

Let now U be an open subset of $\mathbb{R}^2$ and let $f \in C^\infty(\Omega \backslash \Gamma) \cap H_4$ have continuous limits on Γ. For $\alpha > 0$ and $\delta \in (0,1)$ we define as before

$$u_{\alpha,\delta} := v_\delta + \alpha f \tag{4.4}$$

and consider the variational coefficient

$$a_\Gamma[u_{\alpha,\delta}, u_{\alpha,\delta}] - \Lambda\|u_{\alpha,\delta}\|^2 = P(v_\delta) + 2\alpha Q(v_\delta, f) + \alpha^2 R(f), \tag{4.5}$$

where

$$P(v_\delta) := a_\Gamma[v_\delta, v_\delta] - \|v_\delta\|^2, \tag{4.6}$$

$$Q(v_\delta, f) := \mathrm{Re}\Big(a_\Gamma[v_\delta, f] - \Lambda(v_\delta, f)\Big), \tag{4.7}$$

$$R(f) := a_\Gamma[f, f] - \Lambda\|f\|^2. \tag{4.8}$$

The remainder term $R(f)$ will in general be non-negative. If not, the desired statement is already proven and we can disregard this case. A direct calculation gives

$$P(v_\delta) = \int_\Omega 2|\phi_1|^2(\partial_1\rho_\delta)^2 + 2|\phi_2|^2(\partial_2\rho_\delta)^2 + |\phi_3|^2\Big((\partial_1\rho_\delta)^2 + (\partial_2\rho_\delta)^2\Big) + |\partial_1\rho_\delta\phi_2 + \partial_2\rho_\delta\phi_1|^2 \, \mathrm{d}x.$$

and

$$Q(v_\delta, f) = -\,\mathrm{Re}\int_U 2\partial_3\phi_3(\hat{x}, 0)\overline{[f_3]_{x_3=0}(\hat{x})} \, d\hat{x}, \tag{4.9}$$

where $[f_3]_{x_3=0}$ denotes in analogy to the previous section the jump of f_3 in x_3 direction. We obtain the following theorem.

Theorem 4.2. *The three-dimensional elasticity operator A_Γ has an embedded eigenvalue in the interval* $[0, \Lambda)$.

Proof. By Lemma 4.1 and a small adaptation of its proof to estimate $P(v_\delta)$, there exist constants $c_1, c_2 > 0$ and $\delta_0 \in (0,1)$ such that for all $\alpha < 1$ und $\delta \in (0, \delta)$

$$P(v_\delta) \leq c_1\delta, \qquad \|u_{\alpha,\delta}\| \geq \frac{c_2}{\delta}$$

holds. Thus for $\alpha < 1$ and $\delta \in (0, \delta_0)$ we have

$$\frac{a_\Gamma[u_{\alpha,\delta}, u_{\alpha,\delta}] - \Lambda\|u_{\alpha,\delta}\|^2}{\|u_{\alpha,\delta}\|^2} \leq \frac{c_1\delta^2 + 2\alpha\delta Q(v_\delta, f) + \alpha^2\delta R(f)}{c_2}$$

and

$$\begin{aligned} 2\partial_3\phi_3(\hat{x}, 0) &= \int_{\partial B(0,\kappa)} \partial_3\psi_3(\xi, 0)k(\xi)e^{i\xi\hat{x}} \, \mathrm{d}\sigma(\xi) \\ &= 2\kappa\gamma^2\beta\left(\cos\left(\frac{\pi}{2}\beta\right) - \cos\left(\frac{\pi}{2}\gamma\right)\right)\int_{S_\varkappa} k(\xi)e^{i\xi\hat{x}} \, \mathrm{d}\sigma(\xi) \\ &=: T(\hat{x}) \end{aligned}$$

If $k(\xi) = 1$ ($\xi \in S_\varkappa$) then

$$T(0) = 2\kappa\gamma^2\beta\left(\cos\left(\frac{\pi}{2}\beta\right) - \cos\left(\frac{\pi}{2}\gamma\right)\right)\int_{S_\varkappa} k(\xi)\, \mathrm{d}\sigma(\xi) \neq 0$$

which ensures the existence of embedded eigenvalues as in the two-dimensional case, by choosing an appropriate function f such that $Q(v_\delta, f) < 0$. □

In addition we get the following asymptotic bound, which can be proven as in the previous section.

Theorem 4.3. *Let $U \subseteq \mathbb{R}^2$ open. For $l > 0$ we definie $\Gamma^{(l)} := lU \times \{0\}$. Then there exists a constant C independent of l and an eigenvalue $\lambda(l) \in \sigma_p(A^{(4)}_{\Gamma^{(l)}})$ such that*

$$|\lambda(l) - \Lambda| \geq Cl^6. \tag{4.10}$$

4.2. **Reduction by rotational symmetry.** By appropriate choice of the function k we can show the existence of embedded eigenvalues in smaller invariant subspaces. For this we will make use of the invariance of the elastic operator and of the underlying geometry with respect to rotations. We suppose that the crack is of the form $B_l(0) \times \{0\}$, for $l > 0$. For $\varphi \in (0, 2\pi)$ let M_φ denote the planar rotation matrix to the rotation angle φ and let $\mathbb{R}_+ := (0, \infty)$. Using

$$L^2(\mathbb{R}_+, r\mathrm{d}r; \mathbb{C}^3) := \left\{ u : \mathbb{R}_+ \to \mathbb{C}^3 \text{ measurable} : \int_0^\infty |u(r)|^2 r \,\mathrm{d}r < \infty \right\} \tag{4.11}$$

we define for $m \in \mathbb{Z}$ the spaces

$$Y_m := \Bigg\{ u \in L^2(\mathbb{R}^2; \mathbb{C}^3) : \exists v \in L^2(\mathbb{R}_+, r\mathrm{d}r; \mathbb{C}^3) : \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix}(r\cos\varphi, r\sin\varphi) = e^{\mathrm{i}m\varphi}\begin{pmatrix} M_\varphi \begin{pmatrix} v_1(r) \\ v_2(r) \end{pmatrix} \\ v_3(r) \end{pmatrix} \text{ for a.e. } r \text{ and } \varphi \Bigg\} \tag{4.12}$$

As Y_m is isometrically isomorphic to $L^2(\mathbb{R}_+, r\mathrm{d}r; \mathbb{C}^3)$, it is a closed subspace of $L^2(\mathbb{R}^2; \mathbb{C}^3)$. The orthogonal projection Q_m onto Y_m is denoted by

$$\begin{pmatrix} (Q_m u)_1 \\ (Q_m u)_2 \end{pmatrix}(r\cos\varphi, r\sin\varphi) = \mathrm{e}^{\mathrm{i}m\varphi} M_\varphi \frac{1}{2\pi} \underbrace{\int_0^{2\pi} \mathrm{e}^{-\mathrm{i}m\psi} M_{-\psi} \begin{pmatrix} u_1(r\cos\psi, r\sin\psi) \\ u_2(r\cos\psi, r\sin\psi) \end{pmatrix} \mathrm{d}\psi}_{=:\tilde{u}_m(r)}$$

$$(Q_m u)_3(r\cos\varphi, r\sin\varphi) = \frac{1}{2\pi}\mathrm{e}^{im\varphi}\int_0^{2\pi} \mathrm{e}^{-\mathrm{i}m\psi} u_3(r\cos\psi, r\sin\psi).\ \mathrm{d}\psi$$

A short calculation shows that Y_m reduces the Fourier transform, i.e., Q_m and the Fourier transform commute. As the spaces $L^2(\Omega)$ and $L^2\left((-\pi/2, \pi/2); L^2(\mathbb{R}^2)\right)$ can be canonically identified we define

$$X_m := \left\{ u \in L^2(\Omega; \mathbb{C}^3) \text{ for a.e. } x_3 \in (-\pi/2, \pi/2) \text{ we have: } u(x_3) \in Y_m \right\}. \tag{4.13}$$

We will list some properties of the spaces X_m which are not difficult to prove.

Lemma 4.4. *Let $m, \tilde{m} \in \mathbb{Z}$, $m \neq \tilde{m}$.*

(1) $\oplus_{l\in\mathbb{Z}} X_l = L^2(\Omega; \mathbb{C}^3)$.
(2) *The projection P_m on X_m is denoted by $(P_m u)(x_3) = Q_m(u(x_3))$.*
(3) *P_m commutes with the projections $P^{(j)}$ on the spaces H_j, $j = 1, 2, 3, 4$.*

(4) *If* $u \in H^1(\Omega;\mathbb{C}^3)$ *then also* $P_m u \in H^1(\Omega;\mathbb{C}^3)$ *and the series* $\sum_{j\in\mathbb{Z}} P_j u$ *converges in* $H^1(\Omega;\mathbb{C}^3)$ *to* u.
(5) *The trace operators* $\gamma_\pm : H^1(\Omega;\mathbb{C}^3) \to L^2(\mathbb{R}^2;\mathbb{C}^3)$ *on the boundaries* $\mathbb{R}^2 \times \{\pm\pi/2\}$ *satisfy* $\gamma_\pm(P_m u) = Q_m(\gamma_\pm u)$.
(6) *If* $u \in X_m$ *and* $v \in X_{\tilde m}$ *then* $a_\Gamma[u,v] = 0$.
(7) *If* $u \in X_m$ *then* $\overline{u} \in X_{-m}$.

Remark 2. *The fourth assertion follows from the fact that* $H^1(\mathbb{R}^2;\mathbb{C}^2)$ *is invariant under* Q_m. *Let the Fourier transform with respect to the first two variables be denoted by* Φ. *As* X_m *reduces the operator* Φ *we can easily prove the the sixth assertion by expressing* $a_\Gamma[u,v]$ *in terms of* Φu *and* Φv.

Let now $U := B_0(1)$, $\Gamma := U \times \{0\}$ and $m \in \mathbb{Z}$. The previous lemma ensures that the space $H^1(\Omega\backslash\Gamma;\mathbb{C}^3)$ is invariant under the projection P_m as we know that

$$u|_{\mathbb{R}^2\times(-\pi/2,0)} \in H^1\left(\mathbb{R}^2 \times (-\pi/2,0);\mathbb{C}^3\right),$$
$$u|_{\mathbb{R}^2\times(0,\pi/2)} \in H^1\left(\mathbb{R}^2 \times (0,\pi/2);\mathbb{C}^3\right)$$

and that the traces of the two functions coincide on $(\mathbb{R}^2\backslash\Gamma)\times\{0\}$. Likewise, we obtain for $u \in P_m H^1(\Omega\backslash\Gamma;\mathbb{C}^3)$ und $v \in P_{\tilde m} H^1(\Omega\backslash\Gamma;\mathbb{C}^3)$ that $a_\Gamma[u,v] = 0$. This implies that the operator $A_\Gamma^{(4)}$ can be further decomposed by the spaces X_m. Let

$$k_m(\kappa\cos\varphi,\kappa\sin\varphi) := \mathrm{e}^{\mathrm{i}\varphi m} \qquad (r,\varphi) \in \mathbb{R}_+ \times (0,2\pi). \tag{4.14}$$

The function ϕ_m defined as in (4.2) with k is replaced by k_m is an element of X_m as can be seen directly in connection with (4.1). Furthermore we obtain for $(r,\varphi) \in \mathbb{R}_+ \times (0,2\pi)$, $\hat{x} := (r\cos\varphi, r\sin\varphi)$:

$$\begin{aligned} T_m(\hat{x}) &:= 2\kappa\gamma^2\beta\left(\cos\left(\frac{\pi}{2}\beta\right) - \cos\left(\frac{\pi}{2}\gamma\right)\right)\int_{S_\varkappa} k(\xi)\mathrm{e}^{\mathrm{i}\xi\hat{x}}\,\mathrm{d}\sigma(\xi) \\ &= \mathrm{e}^{\mathrm{i}m\varphi}\underbrace{2\kappa^2\gamma^2\beta\left(\cos\left(\frac{\pi}{2}\beta\right) - \cos\left(\frac{\pi}{2}\gamma\right)\right)\int_0^{2\pi} \mathrm{e}^{\mathrm{i}m\theta}\mathrm{e}^{\mathrm{i}\kappa r\cos\theta}\,\mathrm{d}\theta}_{=:v_m(r)}. \end{aligned}$$

Note that $v_m|_{[0,\infty)}$ is smooth and that $v_m|_{[0,1]} \neq 0$. For the last statement we refer to the next section, where the derivatives of v_m at 0 are explicitly calculated. The perturbation f has to be chosen in X_m and its jump should be in the third component to minimise the variational coefficient, i.e.,

$$[f_3]_{x_3=0}(r\cos\varphi, r\sin\varphi) = e^{i\varphi m}w(r) \qquad (r,\varphi) \in \mathbb{R}_+ \times (0,2\pi) \tag{4.15}$$

for a certain $w \in C_0(\mathbb{R}_+)$. This implies

$$\begin{aligned} Q(v_\delta, f) &= -\operatorname{Re}\left(\int_0^1 r\int_0^{2\pi} T_m(r\cos\varphi, r\sin\varphi)\overline{[f_3]_{x_3=0}(r\cos\varphi, r\sin\varphi)}\,\mathrm{d}\varphi\,\mathrm{d}r\right) \\ &= -2\pi\operatorname{Re}\left(\int_0^1 r\, v_m(r)\overline{w}(r)\,\mathrm{d}r\right). \end{aligned}$$

This expression is clearly strictly negative for an appropriate choice of the jump w. Therefore, in analogy to the proof in the previous section we immediately obtain the following theorem.

Theorem 4.5. *Let $m \in \mathbb{Z}$. Then the elasticity operator $A_\Gamma^{(4)}$ restricted to X_m has an embedded eigenvalue in the interval $[0, \Lambda)$.*

As this is true for all subspaces X_m this implies

Corollary 4.6. (1) *The elasticity operator $A_\Gamma^{(4)}$ has infinitely many eigenvalues (counted by multiplicity) in the interval $[0, \Lambda)$.*
(2) *For $m \in \mathbb{N}$ there exists a real eigenfunction of the operator $A_\Gamma^{(4)}$ in $X_m \oplus X_{-m}$, for $m > 0$ the (real) eigenspace is at least two-dimensional.*

Proof. As the considered sesquilinear form $a_\Gamma^{(4)}$ is real, the operator $A_\Gamma^{(4)}$ is also real, i.e., for $u \in D(A_\Gamma^{(4)})$ we have $\overline{u} \in D(A_\Gamma^{(4)})$ and $\overline{A_\Gamma^{(4)} u} = A_\Gamma^{(4)} \overline{u}$. Let $m \in \mathbb{N}$ and $u \in X_m$ be an eigenfunction of the operator for the eigenvalue λ. Then $\overline{u}$ is also an eigenfunction to the same eigenvalue and therefore also $u \pm \overline{u}$. Due to the structure of the spaces X_m for $m \neq 0$, both of these are linearly independent. □

4.3. **Asymptotic bounds.** Let $\Gamma^{(l)} := B_0(l) \times \{0\}$ and $f \in X_m \cap d[a_\Gamma^{(4)}]$. We denote in analogy to the two-dimensional case

$$f^{(l)}(x) := l^{-3/2} f\left(\frac{x}{l}\right) \qquad (x \in \Omega). \tag{4.16}$$

Hence we immediately get

$$R(f^{(l)}) = l^{-2} a_{\Gamma^{(l)}}[f] - \|f\|^2$$

and using the the above notation we obtain

$$\begin{aligned} Q(v_\delta, f^{(l)}) &= -2\pi l^{-3/2} \operatorname{Re}\biggl(\int_0^l r v_m(r) \overline{w\left(\frac{r}{l}\right)} \,\mathrm{d}r \biggr) \\ &= -2\pi l^{1/2} \operatorname{Re}\biggl(\int_0^1 r v_m(rl) \overline{w(r)} \,\mathrm{d}r \biggr) \\ &= -2\pi l^{1/2} \biggl(\sum_{\alpha=0}^{|m|} \operatorname{Re}\left(l^\alpha v_m^{(\alpha)}(0) \int_0^1 r^{\alpha+1} \overline{w(r)} \,\mathrm{d}r \right) + \operatorname{Re} \int_0^1 r R_{|m|}\left(\frac{r}{l}\right) \overline{w(r)} \,\mathrm{d}r \biggr), \end{aligned}$$

based on a Taylor expansion of v_m. If $|m| \geq 1$, then we obtain for $0 \leq \alpha \leq |m|$

$$\begin{aligned} v_m^{(\alpha)}(0) &= 2\kappa^{2+\alpha}\gamma^2\beta \left(\cos\left(\frac{\pi}{2}\beta\right) - \cos\left(\frac{\pi}{2}\gamma\right) \right) i^\alpha \int_0^{2\pi} (\cos\theta)^\alpha e^{im\theta} \,\mathrm{d}\theta \\ &= 2\kappa^{2+\alpha}\gamma^2\beta \left(\cos\left(\frac{\pi}{2}\beta\right) - \cos\left(\frac{\pi}{2}\gamma\right) \right) i^\alpha \int_0^{2\pi} (\cos\theta)^\alpha \cos(m\theta) \,\mathrm{d}\theta \\ &= 2\kappa^{2+\alpha}\gamma^2\beta \left(\cos\left(\frac{\pi}{2}\beta\right) - \cos\left(\frac{\pi}{2}\gamma\right) \right) i^\alpha \cdot \begin{cases} \frac{1}{2^m}\pi & \text{, if } i = |m|, \\ 0 & \text{, if } i < |m|, \end{cases} \end{aligned}$$

while $m = 0$ yields

$$v_0(0) = 4\pi\kappa^2\gamma^2\beta \left(\cos\left(\frac{\pi}{2}\beta\right) - \cos\left(\frac{\pi}{2}\gamma\right) \right).$$

Hence $Q(v_\delta, f^{(l)}) = -\mu_0 l^{|m|+1/2} + O(l^{|m|+3/2})$, where

$$\mu_0 := 2\pi \operatorname{Re}\left(v_m^{(|m|)}(0) \int_0^1 r^{i+1} \overline{w(r)} \,\mathrm{d}r \right).$$

The choice of f and thus in particular w is made such that $\mu_0 > 0$. This implies $Q(v_\delta, f^{(l)})$ to be of order $l^{|m|+1/2}$, $\alpha \sim l^{|m|+1/2+2}$, $\delta \sim l^{2|m|+3}$ und finally implies an order of $l^{4|m|+6}$ for the size of the variational coefficient. We conclude the following asymptotic bound.

Theorem 4.7. *Let $m \in \mathbb{Z}$. There exists a constant C_m independent of l and an eigenvalue $\lambda(l,m) \in \sigma_p(A^{(4)}_{\Gamma^{(l)}}|_{X_m})$ such that*

$$|\lambda(l,m) - \Lambda| \geq C_m l^{4|m|+6}. \tag{4.17}$$

Acknowledgements The research and in particular AH and CS were supported by DFG grant WE-1964/4-1. CS is also grateful to Semion Wugalter for discussions and valuable hints on the subject.

References

[1] S. Agmon, L. Hörmander. Asymptotic properties of solutions of differential equations with simple characteristics *Journal d'analyse mathémathique* **30** (1976)

[2] M.Sh. Birman. Perturbations of continuous spectra of singular elliptic operators under the change of boundary and boundary conditions *Vestnik Leningrad Univ.* **17** (1962), 22–55. (in Russian)

[3] C. Förster, *Trapped modes in elastic waveguides.* Dissertation, Universität Stuttgart, 2007.

[4] C. Förster, T. Weidl. Trapped modes for an elastic strip with perturbation of the material properties. *Quart. J. Mech. Appl. Math.* **59** (2006), no. 3, 399–418.

[5] C. Förster, T. Weidl Trapped modes in an elastic plate with a hole (Russian.) *Algebra i Analiz* **23** (2011), (1), 255-288.

[6] W. McLean. *Strongly elliptic systems and boundary integral equations.* Cambridge University Press, Cambridge, 2000.

[7] I. Roitberg, D. Vassiliev and T. Weidl. Edge resonance in an elastic semistrip. *Quart. J. Mech. Appl. Math.* **51** (1998), no. 1, 1–13.

[8] P. Exner, S.A. Vugalter. Asymptotic estimates for bound states in wave guides coupled laterally through a narrow window. *Ann. Inst. H. Poincaré* **65** (1996), 109–123.

Institut für Analysis, Dynamik und Modellierung, Universität Stuttgart, Pfaffenwaldring 57, 70569 Stuttgart

E-mail address: jens.wirth@iadm.uni-stuttgart.de

E-mail address: andre.haenel@iadm.uni-stuttgart.de

Erschienene Preprints ab Nummer 2007/001
Komplette Liste: `http://www.mathematik.uni-stuttgart.de/preprints`

2011/018 *Hänel, A.; Schulz, C.; Wirth, J.:* Embedded eigenvalues for the elastic strip with cracks
2011/017 *Wirth, J.:* Thermo-elasticity for anisotropic media in higher dimensions
2011/016 *Höllig, K.; Hörner, J.:* Programming Multigrid Methods with B-Splines
2011/015 *Ferrario, P.:* Nonparametric Local Averaging Estimation of the Local Variance Function
2011/014 *Müller, S.; Dippon, J.:* k-NN Kernel Estimate for Nonparametric Functional Regression in Time Series Analysis
2011/013 *Knarr, N.; Stroppel, M.:* Unitals over composition algebras
2011/012 *Knarr, N.; Stroppel, M.:* Baer involutions and polarities in Moufang planes of characteristic two
2011/011 *Knarr, N.; Stroppel, M.:* Polarities and planar collineations of Moufang planes
2011/010 *Jentsch, T.; Moroianu, A.; Semmelmann, U.:* Extrinsic hyperspheres in manifolds with special holonomy
2011/009 *Wirth, J.:* Asymptotic Behaviour of Solutions to Hyperbolic Partial Differential Equations
2011/008 *Stroppel, M.:* Orthogonal polar spaces and unitals
2011/007 *Nagl, M.:* Charakterisierung der Symmetrischen Gruppen durch ihre komplexe Gruppenalgebra
2011/006 *Solanes, G.; Teufel, E.:* Horo-tightness and total (absolute) curvatures in hyperbolic spaces
2011/005 *Ginoux, N.; Semmelmann, U.:* Imaginary Kählerian Killing spinors I
2011/004 *Scherer, C.W.; Köse, I.E.:* Control Synthesis using Dynamic D-Scales: Part II — Gain-Scheduled Control
2011/003 *Scherer, C.W.; Köse, I.E.:* Control Synthesis using Dynamic D-Scales: Part I — Robust Control
2011/002 *Alexandrov, B.; Semmelmann, U.:* Deformations of nearly parallel G_2-structures
2011/001 *Geisinger, L.; Weidl, T.:* Sharp spectral estimates in domains of infinite volume
2010/018 *Kimmerle, W.; Konovalov, A.:* On integral-like units of modular group rings
2010/017 *Gauduchon, P.; Moroianu, A.; Semmelmann, U.:* Almost complex structures on quaternion-Kähler manifolds and inner symmetric spaces
2010/016 *Moroianu, A.; Semmelmann,U.:* Clifford structures on Riemannian manifolds
2010/015 *Grafarend, E.W.; Kühnel, W.:* A minimal atlas for the rotation group $SO(3)$
2010/014 *Weidl, T.:* Semiclassical Spectral Bounds and Beyond
2010/013 *Stroppel, M.:* Early explicit examples of non-desarguesian plane geometries
2010/012 *Effenberger, F.:* Stacked polytopes and tight triangulations of manifolds
2010/011 *Györfi, L.; Walk, H.:* Empirical portfolio selection strategies with proportional transaction costs
2010/010 *Kohler, M.; Krzyżak, A.; Walk, H.:* Estimation of the essential supremum of a regression function
2010/009 *Geisinger, L.; Laptev, A.; Weidl, T.:* Geometrical Versions of improved Berezin-Li-Yau Inequalities
2010/008 *Poppitz, S.; Stroppel, M.:* Polarities of Schellhammer Planes
2010/007 *Grundhöfer, T.; Krinn, B.; Stroppel, M.:* Non-existence of isomorphisms between certain unitals
2010/006 *Höllig, K.; Hörner, J.; Hoffacker, A.:* Finite Element Analysis with B-Splines: Weighted and Isogeometric Methods
2010/005 *Kaltenbacher, B.; Walk, H.:* On convergence of local averaging regression function estimates for the regularization of inverse problems
2010/004 *Kühnel, W.; Solanes, G.:* Tight surfaces with boundary

2010/003 *Kohler, M; Walk, H.:* On optimal exercising of American options in discrete time for stationary and ergodic data
2010/002 *Gulde, M.; Stroppel, M.:* Stabilizers of Subspaces under Similitudes of the Klein Quadric, and Automorphisms of Heisenberg Algebras
2010/001 *Leitner, F.:* Examples of almost Einstein structures on products and in cohomogeneity one
2009/008 *Griesemer, M.; Zenk, H.:* On the atomic photoeffect in non-relativistic QED
2009/007 *Griesemer, M.; Moeller, J.S.:* Bounds on the minimal energy of translation invariant n-polaron systems
2009/006 *Demirel, S.; Harrell II, E.M.:* On semiclassical and universal inequalities for eigenvalues of quantum graphs
2009/005 *Bächle, A, Kimmerle, W.:* Torsion subgroups in integral group rings of finite groups
2009/004 *Geisinger, L.; Weidl, T.:* Universal bounds for traces of the Dirichlet Laplace operator
2009/003 *Walk, H.:* Strong laws of large numbers and nonparametric estimation
2009/002 *Leitner, F.:* The collapsing sphere product of Poincaré-Einstein spaces
2009/001 *Brehm, U.; Kühnel, W.:* Lattice triangulations of E^3 and of the 3-torus
2008/006 *Kohler, M.; Krzyżak, A.; Walk, H.:* Upper bounds for Bermudan options on Markovian data using nonparametric regression and a reduced number of nested Monte Carlo steps
2008/005 *Kaltenbacher, B.; Schöpfer, F.; Schuster, T.:* Iterative methods for nonlinear ill-posed problems in Banach spaces: convergence and applications to parameter identification problems
2008/004 *Leitner, F.:* Conformally closed Poincaré-Einstein metrics with intersecting scale singularities
2008/003 *Effenberger, F.; Kühnel, W.:* Hamiltonian submanifolds of regular polytope
2008/002 *Hertweck, M.; Höfert, C.R.; Kimmerle, W.:* Finite groups of units and their composition factors in the integral group rings of the groups $PSL(2, q)$
2008/001 *Kovarik, H.; Vugalter, S.; Weidl, T.:* Two dimensional Berezin-Li-Yau inequalities with a correction term
2007/006 *Weidl, T.:* Improved Berezin-Li-Yau inequalities with a remainder term
2007/005 *Frank, R.L.; Loss, M.; Weidl, T.:* Polya's conjecture in the presence of a constant magnetic field
2007/004 *Ekholm, T.; Frank, R.L.; Kovarik, H.:* Eigenvalue estimates for Schrödinger operators on metric trees
2007/003 *Lesky, P.H.; Racke, R.:* Elastic and electro-magnetic waves in infinite waveguides
2007/002 *Teufel, E.:* Spherical transforms and Radon transforms in Moebius geometry
2007/001 *Meister, A.:* Deconvolution from Fourier-oscillating error densities under decay and smoothness restrictions