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Abstract

We classify parallel submanifolds of the Grassmannian $\mathrm{G}_2^+(\mathbb{R}^{n+2})$ which parameterizes the oriented 2-planes of the Euclidean space $\mathbb{R}^{n+2}$. Our main result states that every complete parallel submanifold of $\mathrm{G}_2^+(\mathbb{R}^{n+2})$, which is not a curve, is contained in some totally geodesic submanifold as a symmetric submanifold. This result holds also if the ambient space is the non-compact dual of $\mathrm{G}_2^+(\mathbb{R}^{n+2})$.

1 Introduction

[1] Let N be a Riemannian symmetric space. A submanifold of N is called *parallel* if the second fundamental form is parallel. D. Ferus [6] has shown that every compact parallel submanifold of a Euclidean space is a special orbit of some s-representation, called a *symmetric R-space*. In particular, such a submanifold is invariant under the reflections in its affine normal spaces, i.e. it is *(extrinsically) symmetric*. More generally, every complete parallel submanifold of a space form has this property (see [2, 7, 23, 24]). Note, this fact should be seen as an extrinsic analog of a well known result from the classification of Riemannian manifolds: every complete and simply connected Riemannian manifold with parallel curvature tensor is a symmetric space.

More generally, symmetric submanifolds of Riemannian symmetric spaces were studied and classified by H. Naitoh and others, see [1, Ch. 9.3]. These submanifolds are parallel and intrinsically symmetric (in particular, the induced Riemannian metric is complete), but not every complete parallel submanifold is extrinsically symmetric unless the ambient space is a space form. Nevertheless, in the other simply connected rank-one spaces (i.e. the projective spaces over the complex numbers or the quaternions, the Cayley plane, and their non-compact duals), there is still a close correspondence between parallel and symmetric submanifolds. Namely, it turns out that every complete parallel submanifold, which is not a curve, is contained in some totally geodesic submanifold as a symmetric submanifold (see [1, Ch. 9.4]). Further, recall that a submanifold is called *full* if it is not contained in any proper totally geodesic submanifold. In particular, in a simply connected rank-one space, the previous result implies that every full complete parallel submanifold, which is not a curve, is a symmetric submanifold.

However, in symmetric spaces of higher rank, parallel submanifolds are not well understood yet. Note, here the situation becomes more involved, since already the classification of the totally geodesic submanifolds is a non-trivial problem. Hence, it is an interesting fact that at least for the rank-two symmetric spaces the totally geodesic submanifolds are well known due to B.-Y. Chen/T. Nagano [3, 4][2] and S. Klein [14, 15, 16, 17, 18] using different methods. Thus, it is natural to ask, more generally, for the classification of parallel submanifolds in these ambient spaces.

In this article, we consider parallel submanifolds of the Grassmannian $\mathrm{G}_2^+(\mathbb{R}^{n+2})$ – which parameterizes the oriented 2-planes of the Euclidean space $\mathbb{R}^{n+2}$ – and its non-compact dual, the symmetric space $\mathrm{G}_2^+(\mathbb{R}^{n+2})^*$, i.e. the Grassmannian of time-like 2-planes in the pseudo Euclidean space $\mathbb{R}^{n,2}$ equipped with the indefinite metric $dx_1^2 + \cdots + dx_n^2 - dx_{n+1}^2 - dx_{n+2}^2$. Note, these are simply connected symmetric spaces of rank two if $n \geq 2$.

[1]Mathematics Subject Classication (2010): 53C35, 53C40, 53C42

[2]However, the claimed classification of totally geodesic submanifolds of $\mathrm{G}_2^+(\mathbb{R}^{n+2})$ from [3] is incomplete.

Theorem 1 (Main Theorem)**.** *If M is a complete parallel submanifold of the Grassmannian $\mathrm{G}_2^+(\mathbb{R}^{n+2})$ with $\dim(M) \geq 2$, then there exists a totally geodesic submanifold $\bar{M} \subset \mathrm{G}_2^+(\mathbb{R}^{n+2})$ such that M is a symmetric submanifold of $\bar{M}$. In particular, every full complete parallel submanifold of $\mathrm{G}_2^+(\mathbb{R}^{n+2})$, which is not a curve, is a symmetric submanifold. The analogous result holds for ambient space $\mathrm{G}_2^+(\mathbb{R}^{n+2})^*$.*

We also obtain the classification of higher-dimensional parallel submanifolds in a product of two Euclidean spheres or two real hyperbolic spaces (see Corollary 1 and Remark 1). Further, we conclude that every higher-dimensional complete parallel submanifold of $\mathrm{G}_2^+(\mathbb{R}^{n+2})$ is extrinsically homogeneous (see Corollary 2).

Here, we focus our attention on the real Grassmannian $\mathrm{G}_2^+(\mathbb{R}^{n+2})$ and its non-compact dual. For this, we first develop some general theory on the existence of parallel submanifolds in Riemannian symmetric spaces which is applicable, in particular, to the other simply connected rank-two spaces (e.g. the Grassmannians of complex or quaternionic 2-planes). Hence, one may hope that it is also possible to classify the parallel submanifolds of these ambient spaces by means of similar ideas. However, for the proof of Theorem 1 we use a "case by case" strategy and it is by no means clear whether the analogue of Theorem 1 remains true then.

1.1 Overview

We give an overview on the results presented in this article, an outline of the proof of Theorem 1 included. For a Riemannian symmetric space N and a submanifold[3] $M \subset N$, let TM, $\bot M$, $h : TM \times TM \to \bot M$ and $S : TM \times \bot M \to TM$ denote the tangent bundle, the normal bundle, the second fundamental form and the shape operator of M, respectively. Let ∇^M and ∇^N denote the Levi Civita connection of M and N, respectively, and $\nabla^\bot$ be the usual connection on $\bot M$ (obtained by orthogonal projection of $\nabla^N \xi$ along TM for every section ξ of $\bot M$). Let $\mathrm{Sym}^2(TM, \bot M)$ denote the vector bundle whose sections are $\bot M$-valued symmetric bilinear maps on TM. Then there is a linear connection on $\mathrm{Sym}^2(TM, \bot M)$ induced by ∇^M and $\nabla^\bot$ in a natural way, often called *Van der Waerden-Bortolotti connection.*

Definition 1. A submanifold $M \subset N$ is called *parallel* if h is a parallel section of $\mathrm{Sym}^2(TM, \bot M)$.

Example 1. A unit speed curve $c : J \to N$ is parallel if and only if it satisfies the equation

$$\nabla^N_\partial \nabla^N_\partial \dot{c} = -\kappa^2 \dot{c} \tag{1}$$

for some constant $\kappa \in \mathbb{R}$. For $\kappa = 0$ these curves are geodesics; otherwise, due to K. Nomizu and K. Yano [22], c is called an (extrinsic) circle.

Example 2. Let $\bar{M}$ be a totally geodesic submanifold of N (i.e. $h^{\bar{M}} = 0$). A submanifold of $\bar{M}$ is parallel if and only if it is parallel in N.

Definition 2. A submanifold $M \subset N$ is called *(extrinsically) symmetric* if M is a symmetric space (whose geodesic symmetries are denoted by σ_p^M, where p ranges over M) and for every point $p \in M$ there exists an involutive isometry $\sigma_p^\bot$ of N such that

- $\sigma_p^\bot(M) = M$;
- $\sigma_p^\bot | M = \sigma_p^M$;
- the differential $T_p\sigma_p^\bot$ is the linear reflection in the normal space $\bot_p M$.

As mentioned already before, every symmetric submanifold is parallel. However, in the situation of Example 2, we do not necessarily obtain a symmetric submanifold of N even if M is symmetric in $\bar{M}$.

[3]The notion "submanifold" comprizes all connected but possibly only immersed (in particular, not necessarily regular) submanifolds $M \subset N$, i.e. we are implicitly dealing with isometric immersions $f : M \to N$ defined from a connected Riemannian manifold.

Let M be a parallel submanifold of the symmetric space N and consider the linear space $\perp^1_p M := \{h(x,y)|x,y \in W\}_{\mathbb{R}}$ called the *first normal space* at p.

Question. Given a pair of linear spaces (W,U) both contained in T_pN and such that $W \perp U$, does there exist some parallel submanifold M through p with $W = T_pM$ and $U = \perp^1_p M$? In particular, are there natural obstructions against the existence of such a submanifold?

Let R^N denote the curvature tensor of N and recall that a linear subspace $V \subset T_pN$ is called *curvature invariant* if $R^N(V \times V \times V) \subset V$ holds. It is well known that T_pM is a curvature invariant subspace of T_pN for every parallel submanifold M. In Section 2.2, we will show that also $\perp^1_p M$ is curvature invariant. Moreover, the curvature endomorphisms of T_pN generated by T_pM leave $\perp^1_p M$ invariant and vice versa. This means that $(T_pM, \perp^1_p M)$ is an *orthogonal curvature invariant pair*, see Definition 4 and Proposition 1. As a first illustration of this concept, we classify the orthogonal curvature invariant pairs (W,U) of the complex projective space $\mathbb{C}\mathrm{P}^n$, see Example 3. We observe that here the linear space $W \oplus U$ is complex or totally real (in particular, curvature invariant) unless $\dim(W) = 1$. Hence, following the proof of Theorem 1 given below, we obtain the well known result that the analogue of Theorem 1 is true for ambient space $\mathbb{C}\mathrm{P}^n$.

In Section 3.1, we will determine the orthogonal curvature invariant pairs of $N = \mathrm{G}^+_2(\mathbb{R}^{n+2})$. Our result is summarized in Table 1. Note, even if we assume additionally that $\dim(W) \geq 2$, there do exist certain orthogonal curvature invariant pairs (W,U) for which the linear space $W \oplus U$ is not curvature invariant (in contrast to the situation where the ambient space is $\mathbb{C}\mathrm{P}^n$, see above). Hence, at least at the level of curvature invariant pairs, we can not yet give the proof of Theorem 1.

Therefore, it still remains to decide whether there actually exists some parallel submanifold $M \subset \mathrm{G}^+_2(\mathbb{R}^{n+2})$ such that $(W,U) = (T_pM, \perp^1_p M)$ in which case the orthogonal curvature invariant pair (W,U) will be called *integrable*. In Section 3.2, by means of a case by case analysis, we will show that if (W,U) is integrable and $\dim(W) \geq 2$, then the linear space $W \oplus U$ is curvature invariant. For this, we will need some more intrinsic properties of the second fundamental form of a parallel submanifold of a symmetric space which are derived in Section 2. Further, one can easily show that all arguments remain valid for ambient space $\mathrm{G}^+_2(\mathbb{R}^{n+2})^*$.

Proof of Theorem 1. We can assume that $n \geq 2$. Fix some $p \in M$. Using the results from Section 3.2 mentioned before, we conclude that the *second osculating space* $\mathcal{O}_pM := T_pM \oplus \perp^1_p M$ is a curvature invariant subspace of T_pN. Let $\exp^N : TN \to N$ denote the exponential spray. It follows from a result of P. Dombrowski [5] that $\bar{M} := \exp^N(\mathcal{O}_pM)$ is a totally geodesic submanifold of N such that $M \subset \bar{M}$ ("reduction of the codimension"). By construction, $\perp^1_q M = \perp_q M$ for all $q \in M$ where the normal spaces are taken in $T\bar{M}$, i.e. M is a *1-full* complete parallel submanifold of $\bar{M}$. Thus we conclude from Corollary 3 (see below) that M is even a symmetric submanifold of $\bar{M}$. The same arguments apply to ambient space N^*. $\square$

Consider the Riemannian product $\mathrm{S}^k \times \mathrm{S}^l$ of two Euclidean unit-spheres with $k + l = n$ and $k \leq l$. The map $\mathrm{S}^k \times \mathrm{S}^l \to \mathrm{G}^+_2(\mathbb{R}^{n+2}), (p,q) \mapsto \{(p, 0_{l+1}), (0_{k+1}, q)\}_{\mathbb{R}}$ (with $0_l := (0, \cdots, 0) \in \mathbb{R}^l$) defines a 2-fold isometric covering onto a totally geodesic submanifold of $\mathrm{G}^+_2(\mathbb{R}^{n+2})$, see [14],[17]. Hence every parallel submanifold of $\mathrm{S}^k \times \mathrm{S}^l$ is also parallel in $\mathrm{G}^+_2(\mathbb{R}^{n+2})$. Further, consider the totally geodesic embedding $\iota_{k,l} : \mathrm{S}^k \to \mathrm{S}^k \times \mathrm{S}^l, p \mapsto (p,p)$ which is a homothety onto its image by a factor $\sqrt{2}$.

Corollary 1 (Parallel submanifolds of $\mathrm{S}^k \times \mathrm{S}^l$)**.** *Every complete parallel submanifold $M \subset \mathrm{S}^k \times \mathrm{S}^l$ with $\dim(M) \geq 2$ is a product, $M = M_1 \times M_2$, of two symmetric submanifolds $M_1 \subset \mathrm{S}^k$ and $M_2 \subset \mathrm{S}^l$, or is conjugate to a symmetric submanifold of $\iota_{k,l}(\mathrm{S}^k)$ via some isometry of $\mathrm{S}^k \times \mathrm{S}^l$. In the first case, M is a symmetric submanifold of $\mathrm{S}^k \times \mathrm{S}^l$. In the second case, M is not symmetric in $\mathrm{S}^k \times \mathrm{S}^l$ unless $k = l$ and $M \cong \iota_{k,l}(\mathrm{S}^k)$. The analogous result holds for ambient space $\mathrm{H}^k \times \mathrm{H}^l$, the Riemannian product of two real hyperbolic spaces of sectional curvature -1.*

Proof. Let M be a parallel submanifold of $\tilde{N} := \mathrm{S}^k \times \mathrm{S}^l$. Then M is also parallel in $N := \mathrm{G}^+_2(\mathbb{R}^{n+2})$. Hence, according to Theorem 1 and its proof, the second osculating space $V := T_pM \oplus \perp^1_p M$ is a curvature invariant subspace of both T_pN and $T_p\tilde{N}$. Using the classification of curvature invariant subspaces of T_pN (see Theorem 5), we obtain that there are only two possibilities: we have $V = W_1 \oplus W_2$ where W_1 and W_2 are subspaces of the first and second factor of $\tilde{N}$, respectively (Type $(tr_{i,j})$), or $V = \{ (v, gv) \mid v \in W_0' \}$ for some $W_0' \subset T_p\iota_{k,l}(\mathrm{S}^k)$ and where g is a linear isometry of W_0' (Type (tr_i')). Further, then M is contained in the totally geodesic submanifold $\exp^N(V)$ as a symmetric submanifold.

In the first case, M is contained in the totally geodesic submanifold $\bar{M} := \exp^N(W_1) \times \exp^N(W_2)$ where, of course, each factor $\exp^N(W_i)$ is a Euclidean unit-sphere, too. If $\bar{M}$ is the product of two great circles in S^k and S^l, respectively, then $\dim(\bar{M}) = 2$ and $M = \bar{M}$. Otherwise, at least one of the factors of $\bar{M}$ is a higher-dimensonal Euclidean sphere. It follows from a result of Naitoh (see Theorem 4) that $M = M' \times M''$ where $M' \subset \exp^N(W_1)$ and $M'' \subset \exp^N(W_2)$ are symmetric submanifolds. Anyway, we obtain that $M = M' \times M''$ where $M' \subset \mathrm{S}^k$ and $M'' \subset \mathrm{S}^l$ are symmetric submanifolds. Therefore, the product $M' \times M''$ is symmetric in $\tilde{N}$.

In the second case, there exists some isometry $\tilde{g}$ on $\tilde{N}$ such that $\tilde{g}(M) \subset \iota_{k,l}(\mathrm{S}^k)$. Then $\tilde{g}(M)$ is a complete parallel submanifold of $\iota_{k,l}(\mathrm{S}^k)$, i.e. a symmetric submanifold since $\iota_{k,l}(\mathrm{S}^k)$ is a space form. It follows from Theorem 4 that M is not symmetric in $\tilde{N}$ unless M is totally geodesic. Moreover, a totally geodesic submanifold of $\iota_{k,l}(\mathrm{S}^k)$ is symmetric in $\tilde{N}$ if and only if the normal spaces of $\iota_{k,l}(\mathrm{S}^k)$ are curvature invariant (cf. [1, Ch. 9.3]) which is given only for $M \cong \iota_{k,l}(\mathrm{S}^k)$ and $k = l$. The result follows. □

Remark 1. The analogue of Corollary 1 is true also for ambient space $\mathrm{S}^k_r \times \mathrm{S}^l_s$, the product of two Euclidean spheres of arbitrary radii r and s, respectively, and $\mathrm{H}^k_r \times \mathrm{H}^l_s$, the product of two Hyperbolic spaces of sectional curvature $-1/r^2$ and $-1/s^2$, respectively.[4]

A proof of this remark requires similar arguments as presented in this paper and is omitted.

Recall that a submanifold $M \subset N$ is called *extrinsically homogeneous* if a suitable subgroup of the isometry group $\mathrm{I}(N)$ acts transitively on M. In [11, 12], we dealed with the question whether a complete parallel submanifold of a symmetric space of compact or non-compact type is automatically extrinsically homogeneous. It follows a priori from [12, Corollary 1.4] that every complete parallel submanifold M of a simply connected compact or non-compact rank-two symmetric space N without Euclidean factor (e.g. $N = \mathrm{G}_2^+(\mathbb{R}^{n+2})$ or $N = \mathrm{G}_2^+(\mathbb{R}^{n+2})^*$) is extrinsically homogeneous provided that the Riemannian space M does not split of (not even locally) a factor of dimension one or two (e.g. M is locally irreducible and $\dim(M) \geq 3$). Moreover, then M has even *extrinsically homogeneous holonomy bundle*. The latter means the following: there exists a subgroup $G \subset \mathrm{I}(N)$ such that $g(M) = M$ for every $g \in G$ and $G|_M$ is the group which is generated by the *transvections* of M. Using Theorem 1, we can now prove a stronger result for $N = \mathrm{G}_2^+(\mathbb{R}^{n+2})$.

Corollary 2 (Homogeneity of parallel submanifolds)**.** *Every complete parallel submanifold of* $\mathrm{G}_2^+(\mathbb{R}^{n+2})$*, which is not a curve, has extrinsically homogeneous holonomy bundle. In particular, every such submanifold is extrinsically homogeneous in* $\mathrm{G}_2^+(\mathbb{R}^{n+2})$*. This result holds also for ambient space* $\mathrm{G}_2^+(\mathbb{R}^{n+2})^*$*.*

Proof. Let M be a complete parallel submanifold of $N := \mathrm{G}_2^+(\mathbb{R}^{n+2})$ with $\dim(M) \geq 2$. Then there exists a totally geodesic submanifold $\bar{M} \subset N$ such that M is a symmetric submanifold of $\bar{M}$. In particular, $\bar{M}$ is intrinsically a symmetric space. Furthermore, since the rank of N is two, the rank of $\bar{M}$ is less than or equal to two. It follows immediately that there are no more than the following possibilities: $\bar{M}$ is the two-dimensional flat torus, locally a product $\mathbb{R} \times \tilde{M}$ where $\tilde{M}$ is a higher dimensional locally irreducible symmetric space, a higher dimensional locally irreducible symmetric space, or locally a product of two higher dimensional locally irreducible symmetric spaces (of course, this can also be explicitely seen from the classification of the totally geodesic submanifolds of N given in [14]).

In the first case, we automatically have $M = \bar{M}$ (since $\dim(M) \geq 2$). Hence, we have to show that the totally geodesic flat $\bar{M}$ has extrinsically homogeneous holonomy bundle: let $\bar{\mathfrak{i}} = \bar{\mathfrak{k}} \oplus \bar{\mathfrak{p}}$ and $\mathfrak{i} = \mathfrak{k} \oplus \mathfrak{p}$ denote the Cartan decompositions of the Lie algebras of $\mathrm{I}(\bar{M})$ and $\mathrm{I}(N)$, respectively. Then $[\bar{\mathfrak{p}}, \bar{\mathfrak{p}}] = \{0\}$, since $\bar{M}$ is flat. Let $\bar{G} \subset \mathrm{I}(\bar{M})$ denote the connected subgroup whose Lie algebra is $\bar{\mathfrak{p}}$. Then $\bar{G}$ is the transvection group of $\bar{M}$. Moreover, $\bar{\mathfrak{p}} \subset \mathfrak{p}$, because $\bar{M}$ is totally geodesic. Hence, we may take G as the connected subgroup of $\mathrm{I}(N)$ whose Lie algebra is $\bar{\mathfrak{p}}$.

[4]Note, totally geodesic submanifolds of $\mathrm{S}^k_r \times \mathrm{S}^l_s$ and $\mathrm{S}^k_1 \times \mathrm{S}^l_1$ are the same, but it is not a priory clear that both spaces admit the same higher dimensional parallel submanifolds.

The remaining cases are handled as follows: since $M \subset \bar{M}$ is symmetric, there exists a distinguished reflection $\sigma_p^\perp$ of $\bar{M}$ whose restriction to M is the geodesic reflection in p for every $p \in M$, see Definition 2. Therefore, these reflections generate a subgroup of $\mathrm{I}(\bar{M})$ whose connected component acts transitively on M and gives the full transvection group of M. Thus, it suffices to show that there exists a suitable subgroup of $\mathrm{I}(N)$ whose restriction to $\bar{M}$ is the connected component of $\mathrm{I}(\bar{M})$:

In the second case, let $\bar{\mathfrak{i}} = \bar{\mathfrak{k}} \oplus \bar{\mathfrak{p}}$, $\tilde{\mathfrak{i}} = \tilde{\mathfrak{k}} \oplus \tilde{\mathfrak{p}}$ and $\mathfrak{i} = \mathfrak{k} \oplus \mathfrak{p}$ denote the Cartan decompositions of the Lie algebras of $\mathrm{I}(\bar{M})$, $\mathrm{I}(\tilde{M})$ and $\mathrm{I}(N)$, respectively. Then $\bar{\mathfrak{k}} = \tilde{\mathfrak{k}} = [\tilde{\mathfrak{p}}, \tilde{\mathfrak{p}}] = [\bar{\mathfrak{p}}, \bar{\mathfrak{p}}]$, where the first and the last equality are related to the special product structure of $\bar{M}$ and the second one uses the fact that the Killing form of $\tilde{\mathfrak{i}}$ is non-degenerate. It follows that $\bar{\mathfrak{i}} = [\bar{\mathfrak{p}}, \bar{\mathfrak{p}}] \oplus \bar{\mathfrak{p}}$. Moreover, $\bar{\mathfrak{p}} \subset \mathfrak{p}$, see above. Hence, every Killing vector field of $\bar{M}$ is the restriction of some Killing vector field of N. This proves the second case. In the last two cases, a similar conclusion as in the second one can be made. The proof works also for the non-compact dual N^*. The result follows. $\square$

Note, in the previous theorems, the condition $\dim(M) \geq 2$ can not be ignored: consider the ambient space $\mathrm{G}_2^+(\mathbb{R}^4)$ which is isometric to $\mathrm{S}^2_{1/\sqrt{2}} \times \mathrm{S}^2_{1/\sqrt{2}}$. Here, a "generic" circle is full but not extrinsically homogeneous (in particular, not a symmetric submanifold), see [11], Example 1.9.

2 Parallel submanifolds of symmetric spaces

We solve the existence problem for parallel submanifolds of symmetric spaces by means of giving necessary and sufficient tensorial "integrability conditions" on the 2-jet.[5] From this, we derive the fact (already mentioned before) that $(T_pM, \perp^1_pM)$ is a curvature invariant pair for every parallel submanifold M. Then we establish a more intrinsic necessary integrability condition on the 2-jet which involves also the linearized isotropy representation of the ambient space, see Theorem 3. From this, we easily derive Corollary 4, which, under some additional assumption on the image of the linearized isotropy representation, gives another obstruction against the existence of a parallel submanifold with prescribed tangent and first normal space through p. Some of the results mentioned so far were already obtained in [10, 11], however, for readers convenience, here we will derive them directly from the integrability conditions mentioned before.

Further, we deal with parallel submanifolds with one-dimensional first normal spaces (see Proposition 2) and parallel "curved flats" of rank-two spaces (see Proposition 4). Finally, we recall a result of H. Naitoh on symmetric submanifolds of product spaces (see Theorem 4), which was mentioned already before.

2.1 Existence of parallel submanifolds in symmetric spaces

It was first shown by W. Strübing [23] that a parallel submanifold M of an arbitrary Riemannian manifold is uniquely determined by its *2-jet* (T_pM, h_p) at some point $p \in M$. Conversely, let a prescribed 2-jet (W, h) at p be given (i.e. $W \subset T_pN$ is a subspace and $h : W \times W \to W^\perp$ is a symmetric bilinear map). If there exists a parallel submanifold $M \subset N$ through p such that (W, h) is the 2-jet of M, then (W, h) will be called *integrable*. Note, according to [13, Theorem 7], for every integrable 2-jet, the corresponding parallel submanifold can be assumed to be complete.

Let U be the subspace of $W^\perp$ which is spanned by the image of h and set $V := W \oplus U$, i.e. U and V play the roles of the "first normal space" and the "second osculating space", respectively. Then the orthogonal splitting $V := W \oplus U$ turns $\mathfrak{so}(V)$ into a naturally $\mathbb{Z}_2$-graded algebra $\mathfrak{so}(V) = \mathfrak{so}(V)_+ \oplus \mathfrak{so}(V)_-$ where $A \in \mathfrak{so}(V)_+$ or $A \in \mathfrak{so}(V)_-$ according to whether A respects the splitting $V = W \oplus U$ or $A(W) \subset U$ and $A(U) \subset W$. Further, consider the linear map $\boldsymbol{h} : W \to \mathfrak{so}(T_pN)$ given by

$$\forall x, y \in W, \xi \in W^\perp : \boldsymbol{h}_x(y + \xi) = -S_\xi x + h(x, y) \tag{2}$$

(where S_ξ denotes the shape operator associated with h for every $\xi \in U$ in the usual way). Since $S_\xi = 0$ holds for

[5] Note, such conditions were already claimed in [13]. However, the tensorial conditions stated in [13, Theorem 2] are quite redundant.

every $\xi \in W^\perp$ which is orthogonal to U, we actually have

$$\forall x \in W : \boldsymbol{h}_x \in \mathfrak{so}(V)_- . \tag{3}$$

Definition 3. Let a curvature like tensor R on T_pN and an R-invariant subspace W of T_pN (i.e. $R(W \times W \times W) \subset W$) be given. A symmetric bilinear map $h : W \times W \to W^\perp$ will be called *R-semiparallel* if

$$\boldsymbol{h}_{R_{x,y}z - [\boldsymbol{h}_x, \boldsymbol{h}_y]\, z}\, v = [R_{x,y} - [\boldsymbol{h}_x, \boldsymbol{h}_y], \boldsymbol{h}_z]\, v \tag{4}$$

holds for all $x, y, z \in W$ and $v \in T_pN$. Here $R_{u,v} : T_pN \to T_pN$ denotes the *curvature endomorphism* $R(u, v, \cdot)$ for all $u, v \in T_pN$. If W is a curvature invariant subspace of T_pN and (4) holds for $R = R^N_p$, then h is simply called *semiparallel*.

In the situation of Definition 3, is easy to see that h is R-semiparallel if and only if (4) holds for all $x, y, z \in W$ and $v \in V$.

Clearly, each linear map A on V induces an endomorphism $A\cdot$ on $\Lambda^2 V$ by means of the usual rule of derivation, i.e. $A \cdot u \wedge v = Au \wedge v + u \wedge Av$. Let $(A\cdot)^k$ denotes the k-th power of $A\cdot$ on $\Lambda^2 V$. Similarly, $[A, \cdot]$ defines an endomorphism on $\mathfrak{so}(V)$ whose k-th power will be denoted by $[A, \cdot]^k$. Furthermore, every curvature like tensor $R : T_pN \times T_pN \times T_pN \to T_pN$ can be seen as a linear map $R : \Lambda^2 T_pN \to \mathfrak{so}(V)$ characterized by $R(u \wedge v) = R_{u,v}$. The following theorem states the necessary and sufficient "integrability conditions":[6]

Theorem 2. *Let N be a symmetric space. The 2-jet (W, h) is integrable if and only if*

- *W is a curvature invariant subspace of T_pN;*
- *h is semiparallel;*
- *we have*
$$[\boldsymbol{h}_x, \cdot]^k R^N_{y,z} v = R^N((\boldsymbol{h}_x\cdot)^k y \wedge z) v \tag{5}$$
for all $x, y, z \in W$, $k = 1, 2, 3, 4$ and each $v \in V$.

Proof. In order to apply the main result of [13], consider the space $\mathfrak{C}$ of all curvature like tensors on T_pN and the affine subspace $\tilde{\mathfrak{C}} \subset \mathfrak{C}$ which consists, by definition, of all curvature like tensors R on T_pN such that W is R-invariant and h is R-semiparallel. Then we define the one-parameter subgroup $R_x(t)$ of curvature like tensor on T_pN characterized by

$$\exp(t\boldsymbol{h}_x) R_x(t)(u, v, w) = R^N(\exp(t\boldsymbol{h}_x)u, \exp(t\boldsymbol{h}_x)v, \exp(t\boldsymbol{h}_x)w) \tag{6}$$

for all $u, v, w \in T_pN$ and $x \in W$. According to [13, Theorem 1 and Remark 2], (W, h) is integrable if and only if $R_x(t) \in \tilde{\mathfrak{C}}$ for all $x \in W$ and $t \in \mathbb{R}$ (since R^N is a parallel tensor). Moreover, if (W, h) is integrable, then one can show that the function $t \mapsto R_x(t)(y, z, v)$ is constant for all $x, y, z \in W$ and $v \in V$ (see [10, Example 3.7 (a) and Lemma 3.8]). Conversely, if $R^N_p \in \tilde{\mathfrak{C}}$ and $R_x(t)(y, z, v)$ is constant in t for all $x, y, z \in W$ and $v \in V$, then $R_x(t)$ in $\tilde{\mathfrak{C}}$ for all t by straightforward arguments.

Let us assume that (W, h) is integrable. Then the previous implies that

$$\exp(t\boldsymbol{h}_x) R^N_{y,z} \exp(-t\boldsymbol{h}_x) v = R^N_{\exp(t\boldsymbol{h}_x)y, \exp(t\boldsymbol{h}_x)z} v \tag{7}$$

Taking the derivatives of (7) with respect to t, we now see that (5) holds for all $k \geq 1$.

Conversely, suppose that $R^N_p \in \tilde{\mathfrak{C}}$ holds. It suffices to show that (5) implies that the function $t \mapsto R_x(t)(y, z, v)$ is constant for all $x, y, z \in W$ and $v \in V$:

[6]This result was also obtained in an unpublished paper by E. Heintze.

Put $A := \boldsymbol{h}_x$, set $\Sigma := \sum_{i=0}^{3}(A\cdot)^i(\Lambda^2 W)$ and note that

$$A \cdot y \wedge z = Ay \wedge z + y \wedge Az, \tag{8}$$
$$(A\cdot)^2 y \wedge z = A^2 y \wedge z + 2Ay \wedge Az + y \wedge A^2 z, \tag{9}$$
$$(A\cdot)^3 y \wedge z = A^3 y \wedge z + 3A^2 y \wedge Az + 3Ay \wedge A^2 z + y \wedge A^3 z, \tag{10}$$
$$(A\cdot)^4 y \wedge z = A^4 y \wedge z + 4A^3 y \wedge Az + 6A^2 y \wedge A^2 z + 4Ay \wedge A^3 z + y \wedge A^4 z \tag{11}$$

for all $y, z \in W$. Since $A^2 W \subset U$, we hence see that $(A\cdot)^4(\Lambda^2 W) \subset \Lambda^2 W + (A\cdot)^2(\Lambda^2 W)$. Therefore, $A \cdot \Sigma \subset \Sigma$ and, furthermore, since (5) holds for $k = 1, 2, 3, 4$, the natural map $\Lambda^2 T_pN \to \mathfrak{so}(T_pN), u \wedge v \mapsto R^N_{u,v}$ induces a linear map $\Sigma \to \mathfrak{so}(V), \omega \mapsto R^N(\omega)|_V$ which is equivariant with respect to the linear actions of the one-dimensional Lie algebra $\mathbb{R}$ induced by $A\cdot$ and $[A, \cdot]$ on Σ and $\mathfrak{so}(V)$, respectively. Switching to the level of one-parameter subgroups, we obtain that $R_x(t)(\omega)v$ is constant in t for all $\omega \in \Sigma$ and $v \in V$, in particular $R_x(t)(y, z, v)$ is constant in t for all $x, y, z \in W$, $v \in V$. □

Remark 2. In the situation of Theorem 2, suppose that (W, h) is integrable. Then we have

$$[\boldsymbol{h}_{x_1}, \ldots [\boldsymbol{h}_{x_k}, R^N_{y,z}] \ldots]|_V = R^N_{\boldsymbol{h}_{x_1}\cdot \ldots \cdot \boldsymbol{h}_{x_k} \cdot y \wedge z}|_V \tag{12}$$

for all $x_1, \ldots, x_k, y, z \in W$ with $k = 1, 2, \ldots$. Note, here $x_i \neq x_j$ is possible.

Proof. For Equation (12) with $k = 1, 2$ see [10, Lemma 3.9]. The proof for $k \geq 3$ is done in a similar fashion. □

2.2 Curvature invariant pairs

Suppose that (W, h) is an integrable 2-jet at p, set $U := \{h(x, y)\big|x, y \in W\}_{\mathbb{R}}$ and $V := W \oplus U$. Then W is a curvature invariant subspace of T_pN and $h : W \times W \to W^\perp$ is a semiparallel symmetric bilinear map, hence

$$R^N(W \times W \times W) \subset W \text{ and } R^N(W \times W \times U) \subset U\ . \tag{13}$$

In other words, $R^N_{x,y}(V) \subset V$ and $R^N_{x,y}|_V \in \mathfrak{so}(V)_+$ for all $x, y \in W$.

Moreover, using (12) with $k = 2$, we obtain that

$$R^N_{h(x,x),h(y,y)}|_V = [\boldsymbol{h}_x, [\boldsymbol{h}_y, R^N_{x,y}]]|_V + R^N_{S_{h(x,y)}x,y}|_V + R^N_{x,S_{h(y,y)}x}|_V \tag{14}$$

for all $x, y \in W$. Since r.h.s. of (14) leaves V invariant, the same is true for l.h.s. of (14). Furthermore, using that $R^N_{x,y}|_V \in \mathfrak{so}(V)_+$, Eq. 3 and the rules for $\mathbb{Z}_2$-graded Lie algebras, we see that r.h.s. of (14) defines an element of $\mathfrak{so}(V)_+$. Hence the same is true for l.h.s of (14), too. Finally, because h is symmetric, $\Lambda^2(U) = \{h(x, x) \wedge h(y, y)\big|x, y \in W\}_{\mathbb{R}}$ holds. We conclude that (13) holds also with the roles of W and U interchanged, i.e. we have

$$R^N(U \times U \times U) \subset U \text{ and } R^N(U \times U \times W) \subset W\ . \tag{15}$$

Definition 4. Let subspaces W, U of T_pN be given. We will call (W, U) a curvature invariant pair if both (13) and (15) hold. In particular, then W and U both are curvature invariant subspaces of T_pN. If additionally $W \perp U$, then (W, U) is called an orthogonal curvature invariant pair.

We obtain the first criterion matching on the question posed in Section 1.1 (cf. [10, Corollary 13]):

Proposition 1. *Let (W, h) be an integrable 2-jet. Set $U := \{\, h(x, y)\,|\, x, y \in W\,\}_{\mathbb{R}}$. Then (W, U) is an orthogonal curvature invariant pair.*

An (orthogonal) curvature invariant pair (W, U) which is induced by an integrable 2-jet as in Proposition 1 will be called *integrable.*

Furthermore, it is known that every complete parallel submanifold of a simply connected symmetric space whose normal spaces are curvature invariant is even a symmetric submanifold (cf. [1, Proposition 9.3]). Hence we obtain a result, which was already proved in [10]:

Corollary 3. *Every 1-full complete parallel submanifold of a simply connected symmetric space is a symmetric submanifold.*

If W is a curvature invariant subspace of T_pN, then

$$\mathfrak{h}_W := \{R^N_{x,y}|x, y \in W\}_{\mathbb{R}}. \tag{16}$$

is a Lie subalgebra of $\mathfrak{so}(T_pN)$. Further, there exist natural representations of $\mathfrak{h}_W$ on both W and $W^\perp$ (obtained by restriction, respectively). We are interested in the $\mathfrak{h}_W$-invariant subspaces of U. For this, we recall the following result, which is a simple consequence of Schur's Lemma.

Let $W^\perp = U_1 \oplus \cdots \oplus U_k$ be a decomposition into $\mathfrak{h}_W$-irreducible subspaces. After a permutation of the indices, there exists some $r \geq 1$ and a sequence $1 = k_1 < k_2 < \cdots < k_{r+1} = k+1$ such that $U_{k_i} \cong U_{k_i+1} \cong \cdots \cong U_{k_{i+1}-1}$ for $i = 1, \ldots, r$ but U_{k_i} is not isomorphic to U_{k_j} for $i \neq j$. Hence, there is also the decomposition $W^\perp = \oplus_{i=1}^r \mathbf{U}_i$ with $\mathbf{U}_i := U_{k_i} + U_{k_i+1} \cdots + U_{k_{i+1}-1}$. Then every irreducible $\mathfrak{h}_W$-invariant subspace U of $W^\perp$ is contained in some $\mathbf{U}_i$. Furthermore, the irreducible $\mathfrak{h}_W$-invariant subspaces of $\mathbf{U}_i$ are parameterized by the real projective space $\mathbb{R}\mathrm{P}^{k_{i+1}-k_i-1}$ (if U_{k_i} is irreducible even over $\mathbb{C}$) or the complex projective space $\mathbb{C}\mathrm{P}^{k_{i+1}-k_i-1}$ (otherwise) for $i = 1, \ldots, r$. More precisely, let $\lambda_j : U_{k_i} \to U_{k_i+j}$ be an $\mathfrak{h}_W$-isomorphism ($j = 1, \ldots, k_{i+1} - k_i - 1$). Further, set $\lambda_0 := \mathrm{Id}_{U_{k_i}}$ and $\lambda_c := \sum_{j=0}^{k_{i+1}-k_i-1} c_j \lambda_j$ for every $c = (c_0, \ldots, c_{k_{i+1}-k_i-1}) \in \mathbb{R}^{k_{i+1}-k_i}$. Then $U := \lambda(U_{k_i})$ is an irreducible $\mathfrak{h}_W$-invariant subspace of $\mathbf{U}_i$. This gives the claimed parameterization in case U_{k_i} is irreducible even over $\mathbb{C}$. The other case is handled similarly.

Example 3 (Curvature invariant pairs of $\mathbb{C}\mathrm{P}^n$). Consider the complex projective space $N := \mathbb{C}\mathrm{P}^n$. Its curvature tensor is given by $R^N_{u,v} = -u \wedge v - Ju \wedge Jv - 2\omega(u, v)J$ for all $u, v \in T_p\mathbb{C}\mathrm{P}^n$ (where J denotes the complex structure of T_pN and $\omega(u, v) := \langle Ju, v\rangle$ is the Kähler form). The curvature invariant subspaces of T_pN are known to be precisely the totally real and the complex subspaces. Let us determine the orthogonal curvature invariant pairs (W, U):

If W is totally real, then $R^N_{x,y} = -x \wedge y - Jx \wedge Jy$ for all $x, y \in W$. Hence the Lie algebra $\mathfrak{h}_W$ (see (16)) is given by the linear space $\{x \wedge y + Jx \wedge Jy|x, y \in W\}_{\mathbb{R}}$. In the following, we assume that $\dim(W) \geq 2$. Consider the decomposition $W^\perp = JW \oplus (\mathbb{C}W)^\perp$ (here $(\mathbb{C}W)^\perp$ means the orthogonal complement of $\mathbb{C}W$ in T_pN). Then $\mathfrak{h}_W$ acts irreducibly on $J(W)$ and trivially on $(\mathbb{C}W)^\perp$. Further, Eq. 13 shows that U is $\mathfrak{h}_W$-invariant. It follows that either $J(W) \subset U$ or $U \subset (\mathbb{C}W)^\perp$ (cf. [19, Proposition 2.3]). In the first case, we claim that actually $U = J(W)$ (and hence $V := W \oplus U$ is a complex subspace of T_pN, cf. [19, Lemma 4.1]):

Let $\tilde{U} \subset (\mathbb{C}W)^\perp$ be chosen such that $U = JW \oplus \tilde{U}$. Clearly, U is not complex, thus U is necessarily totally real, because U is curvature invariant. Moreover, we have $\dim(U) \geq 2$, thus $\mathfrak{h}_U$ (defined as above) acts irreducibly on $J(U) = W \oplus J(\tilde{U})$. Since W is $\mathfrak{h}_U$-invariant (see (15)), we see that this is not possible unless $J(\tilde{U}) = \{0\}$. The claim follows.

In the second case, we claim that U is totally real (and thus V is totally real, too, cf. [19, Lemma 3.2])):

In fact, otherwise U would be a complex subspace of $(\mathbb{C}W)^\perp$. Then the Lie algebra $\mathfrak{h}_U$ is given by $\mathbb{R}J \oplus \{x \wedge y + Jx \wedge Jy|x, y \in W\}_{\mathbb{R}}$. Thus $\mathfrak{h}_U$ acts on $U^\perp$ via $\mathbb{R}J$. Further, W is invariant under the action of $\mathfrak{h}_U$ according to (15) implying that W is complex, a contradiction. The claim follows.

Anyway, the linear space V is curvature invariant unless $\dim(W) = 1$. Therefore, by means of arguments given in the proof of Theorem 1, we see that every higher dimensional totally real parallel submanifold of $\mathbb{C}\mathrm{P}^n$ is a Lagrangian symmetric submanifold of some totally geodesically embedded $\mathbb{C}\mathrm{P}^k$ or a symmetric submanifold of some totally geodesically embedded $\mathbb{R}\mathrm{P}^k$.

If W is a complex subspace of $T_p\mathbb{C}\mathrm{P}^n$, then $\mathfrak{h}_W|_{W^\perp} = \mathrm{I\!R}J|_{W^\perp}$. Hence, if (W,U) is an orthogonal curvature invariant pair, then both U and $V := W \oplus U$ are complex subspaces, too. This shows that every complex parallel submanifold of $\mathbb{C}\mathrm{P}^n$ is a complex symmetric submanifold of some totally geodesically embedded $\mathbb{C}\mathrm{P}^k$.

2.3 Further necessary integrability conditions

Let N be a symmetric space, $K \subset \mathrm{I}(N)$ denote the isotropy subgroup at p, $\mathfrak{k}$ denote its Lie algebra and $\rho : \mathfrak{k} \to \mathfrak{so}(T_pN)$ be the linearized isotropy representation. Recall that

$$R^N_{u,v} \in \rho(\mathfrak{k}) \tag{17}$$

for all $u, v \in T_pN$ (since N is a symmetric space).

Given a 2-jet (W,h) at p, we set $U := \{h(x,y)\big|x,y \in W\}_{\mathrm{I\!R}}$, $V := W \oplus U$ and

$$\mathfrak{k}_V := \{\, X \in \mathfrak{k} \,|\, \rho(X)(V) \subset V \,\}. \tag{18}$$

Then there is an induced representation of $\mathfrak{k}_V$ on V. Further, consider the endomorphisms of T_pN given by

$$[\boldsymbol{h}_{x_1}, \ldots [\boldsymbol{h}_{x_k}, R^N_{y,z}] \ldots] \tag{19}$$

with $x_1, \ldots, x_k, y, z \in W$ and $k \geq 0$. Furthermore, recall that the centralizer of a subalgebra $\mathfrak{g} \subset \mathfrak{so}(V)$ is given by

$$Z(\mathfrak{g}) := \{\, A \in \mathfrak{so}(V) \,|\, \forall B \in \mathfrak{g} : [A,B] = 0 \,\}. \tag{20}$$

Using Theorem 2, we will now derive the following necessary integrability condition which also involves the linearized isotropy representation of the ambient space:

Theorem 3. *Let an integrable 2-jet (W,h) be given and set $U := \{h(x,y)\big|x,y \in W\}_{\mathrm{I\!R}}$. The endomorphisms* (19) *leave $V := W \oplus U$ invariant and hence generate a subalgebra $\mathfrak{g} \subset \mathfrak{so}(V)$ (by restriction to V). Further, for every $x \in W$ there exist $A_x \in \rho(\mathfrak{k}_V)|_V \cap \mathfrak{so}(V)_-$, $B_x \in Z(\mathfrak{g}) \cap \mathfrak{so}(V)_-$ such that $\boldsymbol{h}_x = A_x + B_x$.*

Proof. Since (W,U) is a curvature invariant pair, we have $R^N_{x,y}(V) \subset V$ for all $x, y \in W$ according to (13). Thus (19) leaves V invariant also for $k > 0$, see (2). Further, note that applying $[\boldsymbol{h}_x, \cdot\,]$ to (19) leaves the form of (19) invariant with the natural number k increased by one for every $x \in W$. Hence $[\boldsymbol{h}_x, \mathfrak{g}] \subset \mathfrak{g}$. Furthermore, the restriction of (19) to V belongs to $\mathfrak{so}(V)_+$ or $\mathfrak{so}(V)_-$ according to whether k is even or odd, see (3) and (13). Therefore, $\mathfrak{g}$ is a graded Lie subalgebra of $\mathfrak{so}(V)$, i.e. $\mathfrak{g} = \mathfrak{g}_+ \oplus \mathfrak{g}_-$ with $\mathfrak{g}_+ := \mathfrak{g} \cap \mathfrak{so}(V)_+$ and $\mathfrak{g}_- := \mathfrak{g} \cap \mathfrak{so}(V)_-$.

Let A_x denote the orthogonal projection of $\boldsymbol{h}_x$ onto $\mathfrak{g}$ with respect to the positive definite symmetric bilinear form on $\mathfrak{so}(V)$ which is given by $-\mathrm{trace}(A \circ B)$ for all $A, B \in \mathfrak{so}(V)$. Since there is the orthogonal splitting $\mathfrak{g} = \mathfrak{g}_+ \oplus \mathfrak{g}_-$ and $\boldsymbol{h}_x \in \mathfrak{so}(V)_-$ holds, we immediately see that $A_x \in \mathfrak{so}(V)_-$ (cf. [11, Lemma 4.19]). Furthermore, using the invariance property of the trace form (i.e. $\mathrm{trace}([A,B] \circ C) = \mathrm{trace}(A \circ [B,C])$), we conclude from $[\boldsymbol{h}_x, \mathfrak{g}] \subset \mathfrak{g}$ that $B_x := \boldsymbol{h}_x - A_x$ centralizes $\mathfrak{g}$. Further, we have $B_x \in \mathfrak{so}(V)_-$. It remains to show that $\mathfrak{g} \subset \rho(\mathfrak{k}_V)|_V$:

For this, it suffices to show that the restriction to V of (19) belongs to $\rho(\mathfrak{k}_V)|_V$ for every k: because of (17), r.h.s. of (12) belongs to $\rho(\mathfrak{k}_V)|_V$ and so does l.h.s. This proves the theorem. □

Given an orthogonal curvature invariant pair (W,U), we set $V := W \oplus U$. Then

$$\mathfrak{h} := \mathfrak{h}_W|_V + \mathfrak{h}_U|_V \tag{21}$$

is a Lie subalgebra of $\mathfrak{so}(V)_+$. Therefore, restricting the elements of $\mathfrak{h}$ to W or U defines representations of $\mathfrak{h}$ on W and U, respectively. Hence, we introduce the linear spaces of homomorphisms

$$\mathrm{Hom}(W,U) := \{\, \ell : W \to U \,|\, \ell \text{ is linear} \,\}; \tag{22}$$

$$\mathrm{Hom}_{\mathfrak{h}}(W,U) := \{\, \ell \in \mathrm{Hom}(W,U) \,|\, \forall A \in \mathfrak{h} : \ell \circ A|_W = A \circ \ell \,\}. \tag{23}$$

Note that the natural map

$$\mathfrak{so}(V)_- \to \mathrm{Hom}(W,U), A \mapsto A|_W \tag{24}$$

is actually a linear isomorphism inducing an equivalence

$$Z(\mathfrak{h}) \cap \mathfrak{so}(V)_- \cong \mathrm{Hom}_{\mathfrak{h}}(W,U), \tag{25}$$

where $Z(\mathfrak{h})$ denotes the centralizer of $\mathfrak{h}$ in $\mathfrak{so}(V)$. As a corollary of Theorem 3, we derive the following obstruction against the integrability of curvature invariant pairs:

Corollary 4. *Let an integrable curvature invariant pair (W,U) be given. Set $V := W \oplus U$ and suppose additionally that $\rho(\mathfrak{k}_V)|_V \cap \mathfrak{so}(V)_- = \{0\}$. Let $\mathfrak{h}$ be the Lie algebra (21). Then there exists a symmetric bilinear map $h : W \times W \to U$ with*

$$U = \{h(x,y)\big|x, y \in W\}_{\mathrm{IR}}; \tag{26}$$

$$\forall x \in W : h(x,\cdot) \in \mathrm{Hom}_{\mathfrak{h}}(W,U). \tag{27}$$

Proof. Let M be a parallel submanifold through p such that $W = T_pM$ and $U = \perp^1_p M$. Let h be the second fundamental form at p which defines the subalgebra $\mathfrak{g} \subset \mathfrak{so}(V)$ described in Theorem 3. First, we claim that $\mathfrak{h}$ is a subalgebra of $\mathfrak{g}$ (this is actually true for every integrable 2-jet):

Since (19) with $k = 0$ leaves V invariant and its restriction to V belongs to $\mathfrak{g}$, we have $A(V) \subset V$ and $A|_V \in \mathfrak{g}$ for all $A \in \mathfrak{h}_W$. Further, we have seen in the proof of Theorem 3 that $\mathfrak{g}$ is normalized by $\boldsymbol{h}_x$ for every $x \in W$. Furthermore, because h is a symmetric bilinear map whose image spans U, the linear space $\Lambda^2 U$ is spanned by the 2-wedges $h(x,x) \wedge h(y,y)$ with $x, y \in W$. Thus (14) implies that also $A(V) \subset V$ and $A|_V \in \mathfrak{g}$ for all $A \in \mathfrak{h}_U$ holds. The claim follows.

Consider the decomposition $\boldsymbol{h}_x = A_x + B_x$ given by Theorem 3. Then A_x vanishes, by the strength our assumption, and hence $\boldsymbol{h}_x = B_x \in Z(\mathfrak{g})$. Since $\mathfrak{h} \subset \mathfrak{g}$, we obtain, in particular, that $\boldsymbol{h}_x \in Z(\mathfrak{h}) \cap \mathfrak{so}(V)_-$ for all $x \in W$, i.e. $h(x,\cdot) \in \mathrm{Hom}_{\mathfrak{h}}(W,U)$ according to (25). This finishes our proof. ☐

For a higher-dimensional *extrinsic sphere*, it is known that the second osculating spaces are curvature invariant, cf. [1, Theorem 9.2.2]. More generally, we have:

Proposition 2. *Let N be a symmetric space, (W,h) be an integrable 2-jet and $U := \{h(x,y)\big|x, y \in W\}_{\mathrm{IR}}$. Assume that $\dim(U) = 1$ and $\dim(W) \geq 2$. Choose a unit vector $\eta \in U$ and suppose also that $\tilde{h}(x,y) := \langle h(x,y), \eta\rangle$ defines a non-degenerate bilinear form on W. Then $V := W \oplus U$ is a curvature invariant subspace of T_pN.*

Proof. In view of Proposition 1, it remains to show that $R^N_{x,\eta}(V) \subset V$ holds. For this, we may proceed as in the proof of [1, Theorem 9.2.2]:

We can assume that $x \neq 0$ in which case there exist $y, z \in W$ with $h(x,z) = \eta$ and $h(y,z) = 0$ (since $\tilde{h}$ is non-degenerate and $\dim(W) \geq 2$). Hence, using (5) with $k = 1$, we see that $R^N_{x,\eta} = [\boldsymbol{h}_z, R^N_{x,y}]$ holds on V. The result follows immediately. ☐

Given a 2-jet (W,h), we set $\mathrm{Kern}(h) := \{\, x \in W \mid h(x,y) = 0 \text{ for all } y \in W \,\}$. Thus $\mathrm{Kern}(h) = \{\, x \in W \mid \boldsymbol{h}(x) = 0 \,\}$, see (24). Further, let $\mathfrak{h}_W$ be the Lie algebra defined by (16).

Proposition 3. *Let an integrable 2-jet (W,h) be given. Then* $\mathrm{Kern}(h)$ *is invariant under the action of* $\mathfrak{h}_W$ *on* W.

Proof. The last assertion follows from the curvature invariance of W, the symmetry of h and (4) (cf. [20, Proof of Lemma 5.1]). ☐

Therefore, in the situation of Proposition 2, the symmetric bilinear form $\tilde{h}$ will be non-degenerate provided that W is an irreducible $\mathfrak{h}_W$-module and $h \neq 0$.

2.4 Two-dimensional parallel "curved flats"

Let N be a symmetric space, $\mathrm{I}(N)$ denote the isometry group, $\mathfrak{i}$ be its Lie algebra and $\mathfrak{i} = \mathfrak{k} \oplus \mathfrak{p}$ be the Cartan decomposition. Recall that a Cartan algebra is a maximal Abelian subalgebra of $\mathfrak{p}$ whose elements are semisimple (cf. [8, Remark 1]) and that any two Cartan algebras are conjugate in $\mathfrak{p}$ via some isometry from the connected component of $\mathrm{I}(N)$. The rank of N is, by definition, the dimension of a Cartan subalgebra of $\mathfrak{p}$. If N is of compact or non-compact type, then every maximal Abelian subalgebra of $\mathfrak{p}$ is already a Cartan subalgebra. The following is well known:

Lemma 1. *Suppose that N is of compact or non-compact type. Let a linear subspace $W \subset T_pN$ be given. The following is equivalent:*

(a) W is a curvature isotropic subspace of T_pN .

(b) The totally geodesic submanifold $\exp^N(W)$ is a flat of N .

(c) W is contained in a Cartan subalgebra of $\mathfrak{p}$.

(d) The sectional curvature of N vanishes on every 2-plane of W , i.e. $\langle R^N(u,v,v),u\rangle = 0$ for all $u, v \in W$.

Proposition 4. *Suppose that N is a rank-two symmetric space of compact or non-compact type. Let a parallel submanifold $M \subset N$ be given. If $\dim(M) = 2$ and the sectional curvature of N vanishes on T_pM for some $p \in M$, then there exists an orthonormal basis $\{e_1, e_2\}$ of T_pM such that $h(e_1, e_2) = 0$. Moreover, we have $\perp^1_p M = \{\eta_1, \eta_2\}_{\mathbb{R}}$ with $\eta_i := h(e_i, e_i)$ and*

$$\langle \eta_1, \eta_2\rangle = 0, \tag{28}$$

$$R^N_{e_1,\eta_2} = R^N_{\eta_1,e_2} = R^N_{e_1,e_2} = R^N_{\eta_1,\eta_2} = 0. \tag{29}$$

In particular, both T_pM and $\perp^1_p M$ are curvature isotropic.

Proof. By means of Lemma 1, we have $\dim(M) = 2$ and $R^N_{x,y} = 0$ for all $x, y \in T_pM$. Furthermore, it is known that in this situation the sectional curvature of N vanishes identically along the parallel submanifold M (see [10, Proposition 3.14]). It follows that $R^N_{x,y} = 0$ for all $x, y \in T_pM$ and all $p \in M$, i.e. M is a "curved flat" in the sense of Ferus/Petit. Therefore, since $\dim(M) = \mathrm{rank}(N) = 2$, the Riemannian space M is intrinsically flat according to a result of [8]. Furthermore, Equation (4) shows that $R^\perp_{x,y}\xi = 0$ for all $\xi \in \perp^1_p M$. Using the Equations of Gauß, Codazzi and Ricci for a parallel submanifold, i.e.

$$\forall x, y \in T_pM : R^N_{x,y} = R^M_{x,y} \oplus R^\perp_{x,y} + [\boldsymbol{h}_x, \boldsymbol{h}_y] \ , \tag{30}$$

we obtain that $[\boldsymbol{h}_x, \boldsymbol{h}_y] = 0$ for all $x, y \in W$. Therefore, as an immediate consequence of Theorem 2, we see that there exists an intrinsically flat parallel submanifold $\tilde{M}$ of the Euclidean space $V := \mathcal{O}_pM$ with $0 \in \tilde{M}$, $T_0\tilde{M} = T_pM$ and $\tilde{h}_0 = h_p$. It is known that such $\tilde{M}$ is an (extrinsic) product of either two plane circles (in case $\dim \perp^1_p M = 2$) or a plane circle and a straight line (in case $\dim \perp^1_p M = 1$). Let $\tilde{M}_1 \times \tilde{M}_2$ be the induced product structure of $\tilde{M}$. Choose orthonormal vectors e_i of $T_0\tilde{M}_i$ for $i = 1, 2$ and put $\eta_i := h(e_i, e_i)$. Then $h(e_1, e_2) = 0$, hence $R^N_{e_1,\eta_2}|_V = R^N_{e_1,h(e_2,e_2)}|_V = -R^N_{h(e_2,e_1),e_2}|_V = 0$ where the second equality uses (5) with $k = 1$). Applying Lemma 1 again, we obtain that the curvature endomorphism $R^N_{e_1,\eta_2}$ vanishes. Similarly, we can show that $R^N_{e_2,\eta_1} = 0$. Furthermore, (14) implies that also $R^N_{\eta_1,\eta_2}$ vanishes on V. Using Lemma 1 once more, the result now follows. ☐

2.5 Symmetric submanifolds of product spaces

We recall the following special case of [21, Theorem 2.2]:

Theorem 4 (H. Naitoh)**.** *Suppose that N is a simply connected symmetric space and that the de Rham decomposition of N has precisely two factors, $N = N_1 \times N_2$. If $M \subset N$ is a symmetric submanifold, then either $N_1 = N_2$ and $M = \{ (p, g(p)) \,|\, p \in N_1 \}$ where g is an isometry of N_1 (in particular, then M is totally geodesic) or M is a product $M_1 \times M_2$ of symmetric submanifolds $M_i \subset N_i$ for $i = 1, 2$.*

Proof. In fact, in case both factors of N are of compact type, we can immediately apply [21, Theorem 2.2]. In case both factors of N are of non-compact type, we use the duality between compact and non-compact spaces to pass to the previous case (note that the results of [21] are mainly based on [21, Lemma 3.1] which is preserved under duality). In the general case, we decompose $N \cong N_c \times N_{nc} \times N_e$ into its compact, non-compact and Euclidean factor (where one or more factors may be trivial) and show as in [21, p.562/563] that M splits as a product $M = M_c \times M_{nc} \times M_e$ of symmetric submanifolds $M_c \subset N_c$, $M_{nc} \subset N_{nc}$ and $M_e \subset N_e$, which finally establishes Theorem 4. $\square$

3 Parallel submanifolds of $\mathrm{G}_2^+(\mathbb{R}^{n+2})$

Let $n \geq 2$ and consider the simply connected compact Hermitian symmetric space $N := \mathrm{G}_2^+(\mathbb{R}^{n+2})$ of rank two which is given by the oriented 2-planes of $\mathbb{R}^{n+2}$. In standard notation, we have $N \cong \mathrm{SO}(n+2)/\mathrm{SO}(2) \times \mathrm{SO}(n)$. Let $\{e_1, \ldots, e_{n+2}\}$ be the standard orthonormal basis of $\mathbb{R}^{n+2}$ and set $p := \{e_{n+1}, e_{n+2}\}$. Then p is an oriented 2-plane in $\mathbb{R}^{n+2}$ and $T_pN = \mathrm{Hom}(\mathbb{R}^2, \mathbb{R}^n)$ (here and in the following we identify $\mathbb{R}^2 \cong \{e_{n+1}, e_{n+2}\}_{\mathbb{R}}$ and $\mathbb{R}^n \cong \{e_1, \ldots, e_n\}_{\mathbb{R}}$).

The Hermitian structure on T_pN is given by

$$J^N(\ell) := \ell \circ e_{n+1} \wedge e_{n+2} \tag{31}$$

for all $\ell \in \mathrm{Hom}(\mathbb{R}^2, \mathbb{R}^n)$ (here we use the natural isomorphism $\Lambda^2(\mathbb{R}^2) \cong \mathfrak{so}(2)$ such that $e_{n+1} \wedge e_{n+2}$ is the rotation in the positive sense by an angle of 90 degree in $\mathbb{R}^2$). Thus T_pN is also an n-dimensional complex vector space where multiplication with the imaginary unit i is given by J^N. Further, for every $\varphi \in \mathbb{R}$ set

$$\Re(\varphi) := \{ \ell \in \mathrm{Hom}(\mathbb{R}^2, \mathbb{R}^n) \,|\, \cos(\varphi)\ell(e_{n+1}) = -\sin(\varphi)\ell(e_{n+2}) \}. \tag{32}$$

Then $\mathcal{U} := \{ \Re(\varphi) \,|\, \varphi \in \mathbb{R} \}$ is a family of *real forms* of T_pN (i.e. maximal totally real subspaces of T_pN) and $\mathcal{U} = \{ \mathrm{e}^{\mathrm{i}\varphi}\Re \,|\, \varphi \in \mathbb{R} \}$ for every $\Re \in \mathcal{U}$. Following the notation from [14], we thus see that $\mathcal{U}$ is a "circle" of real forms.

Let $\mathfrak{so}(n+2) = \mathfrak{k} \oplus \mathfrak{p}$ be the Cartan decomposition of $\mathfrak{so}(n+2)$, i.e. $\mathfrak{k} = \mathfrak{so}(2) \oplus \mathfrak{so}(n)$ and $\mathfrak{p}$ is the orthogonal complement of $\mathfrak{k}$ with respect to the positive definite invariant form defined by $-\mathrm{trace}(A \circ B)$ for all $A, B \in \mathfrak{so}(n+2)$. Then $\mathfrak{p} = \{ A \in \mathfrak{so}(n+2) \,|\, A(\mathbb{R}^2) \subset \mathbb{R}^n,\ A(\mathbb{R}^n) \subset \mathbb{R}^2 \}$ and $\mathfrak{k} = [\mathfrak{p}, \mathfrak{p}]$. Using the natural isomorphism $\mathfrak{p} \to T_pN, A \mapsto A|_{\mathbb{R}^2}$, the linearized isotropy representation $\rho : \mathfrak{k} \to \mathfrak{so}(T_pN)$ is given by $\rho(A)B = [A, B]$ for all $A \in \mathfrak{k}$ and $B \in \mathfrak{p}$. Further, then we have $R^N(A, B, C) = -[[A, B], C]$ for all $A, B, C \in \mathfrak{p}$ (since N is a symmetric space). Thus, we obtain that $\rho(\mathfrak{k}) = \mathbb{R}J^N \oplus \mathfrak{so}(\Re)$ and

$$\forall u, v \in T_pN : R^N_{u,v} = (\langle \Re(v), \Im(u) \rangle - \langle \Re(u), \Im(v) \rangle) J^N - \Re(u) \wedge \Re(v) - \Im(u) \wedge \Im(v) \tag{33}$$

for every $\Re \in \mathcal{U}$ if the scalar product $\langle A, B \rangle$ is chosen as $-1/2\,\mathrm{trace}(A \circ B)$ for all $A, B \in \mathfrak{p}$. Here $v = \Re(v) + \mathrm{i}\Im(v)$ denotes the splitting with respect to the decomposition $T_pN = \Re \oplus \mathrm{i}\Re$ and the Lie algebra $\mathfrak{so}(\Re)$ acts on T_pN via $Av = A\Re(v) + \mathrm{i}A\Im(v)$ for all $A \in \mathfrak{so}(\Re)$ and $v \in T_pN$. For an equivalent description of R^N, see [14, p.84, Eq. (16)] (note that there our metric gets scaled by a factor 1/2).[7]

Recall that a subspace $W \subset T_pN$ is called curvature invariant if $R^N(x, y, z) \in W$ for all $x, y, z \in W$. This property is equivalent to W being a *Lie triple system* in $\mathfrak{p}$, i.e. $[[W, W], W] \subset W$. For the following result see [14, Theorem 4.1]:

Theorem 5 (S. Klein)**.** *For $N := \mathrm{G}_2^+(\mathbb{R}^{n+2})$, there are precisely the following curvature invariant subspaces of T_pN:*

- *Type (c_k): Let $\Re \in \mathcal{U}$ and a k-dimensional subspace $W_0 \subset \Re$ be given. Then $W := \mathbb{C}W_0$ is curvature invariant. Here we assume that $k \geq 1$.*

[7] Clearly, the curvature tensor itself does not change if one scales the metric by a constant factor, but r.h.s. of (33) depends on the chosen scaling.

- *Type* $(tr_{k,l})$*: Let* $\mathfrak{R} \in \mathcal{U}$ *and a pair of orthogonal subspaces* W_1, W_2 *of* $\mathfrak{R}$ *be given. Then* $W := W_1 \oplus \mathrm{i}W_2$ *is curvature invariant. Here the dimensions* k *and* l *of* W_1 *and* W_2*, respectively, are supposed to satisfy* $0 \le k \le l$ *and* $k + l \ge 2$*.*

- *Type* (c'_k)*: Let* $\mathfrak{R} \in \mathcal{U}$ *and a subspace* $W' \subset \mathfrak{R}$ *which is equipped with a Hermitian structure* I' *be given. Then* $W = \{\, v - \mathrm{i}I'v) \mid v \in W' \,\}$ *is curvature invariant. Here* $k \ge 1$ *denotes the complex dimension of* W'*.*

- *Type* (tr'_k)*: Let* $\mathfrak{R} \in \mathcal{U}$*, a subspace* $W' \subset \mathfrak{R}$ *which is equipped with a Hermitian structure* I' *and a real form* W'_0 *of the complex vector space* (W', I') *be given. Then* $W = \{\, v - \mathrm{i}I'v) \mid v \in W'_0 \,\}$ *is curvature invariant. Here* $k \ge 2$ *denotes the dimension of* W'_0*.*

- *Type* (ex_3)*: Let* $\mathfrak{R} \in \mathcal{U}$ *and an orthonormal system* $\{e_1, e_2\} \subset \mathfrak{R}$ *be given. The three-dimensional space* $W := \{e_1 - \mathrm{i}e_2, e_2 + \mathrm{i}e_1, e_1 + \mathrm{i}e_2\}_{\mathbb{R}}$ *is curvature invariant.*

- *Type* (ex_2)*: Suppose that* $n \ge 3$*. Let* $\mathfrak{R} \in \mathcal{U}$ *and an orthonormal system* $\{e_1, e_2, e_3\} \subset \mathfrak{R}$ *be given. The two-dimensional space* $W := \{2e_1 + \mathrm{i}e_2, e_2 + \mathrm{i}(e_1 + \sqrt{3}\, e_3)\}_{\mathbb{R}}$ *is curvature invariant.*

- *Type* (tr_1)*: Let* u *be a unit vector of* T_pN*. The one-dimensional space* $\mathbb{R}u$ *is curvature invariant.*

Our notation emphasizes that spaces of Type (c_k) and (c'_k) both are complex of dimension k over $\mathbb{C}$ and those of Type $(tr_{k,l})$ and (tr'_k) are totally real of dimension $k + l$ and k, respectively. The spaces of Type (ex_2) and (ex_3) are "exceptional" (in the sense that they do not occur in a series).

As was mentioned already at the beginning of this paper, the totally geodesic submanifolds of N were also classified in [3]. However, there the totally geodesic submanifolds which are associated with curvature invariant subspaces of Types (ex_2) and (ex_3) do not occur. For an explicit description of these submanifolds, see [17].

3.1 Curvature invariant pairs of $\mathrm{G}_2^+(\mathbb{R}^{n+2})$

In this section, we determine the orthogonal curvature invariant pairs of T_pN. Note that (W, U) is a curvature invariant pair if and only if (U, W) has this property. Since Theorem 5 provides seven types of curvature invariant subspaces of T_pN, there are, roughly said, $7 \cdot 8/2 = 28$ possibilities to consider.

Our approach is roughly explained as follows: given a curvature invariant subspace W, we will first determine the Lie algebra $\mathfrak{h}_W$ (see (16)) and the $\mathfrak{h}_W$-invariant subspaces of $W^\perp$. Second, we will also determine those skew-symmetric endomorphisms which belong to $\rho(\mathfrak{k})$ and leave W invariant, see (17). Once this information is available for curvature invariant subspaces of Type x and y, it will enable us to determine all curvature invariant pairs of Type (x,y).

Lemma 2. *Let* W *be of Type* (c_k) *defined by the data* $(\mathfrak{R}, W_0)$*.*

(a) We have $\mathfrak{h}_W = \mathbb{R}J^N \oplus \{\, u \wedge v \mid u, v \in W_0 \,\}_{\mathbb{R}}$*.*

(b) A subspace of $W^\perp$ *is* $\mathfrak{h}_W$*-invariant if and only if it is a complex subspace.*

(c) Let $A \in \mathfrak{so}(\mathfrak{R})$ *and* $c \in \mathbb{R}$*. The endomorphism of* T_pN *which is given by* $cJ^N + A$ *leaves* W *invariant if and only if* $A = \sum_{i \in I} u_i \wedge v_i + \tilde{u}_i \wedge \tilde{v}_i$ *with* $u_i, v_i \in W_0$ *and* $\tilde{u}_i, \tilde{v}_i \in W_0^\perp$ *(where* $W_0^\perp$ *denotes the orthogonal complement of* W_0 *in* $\mathfrak{R}$*).*

Proof. By means of (33), the curvature endomorphism $R^N_{\mathrm{i}v,v}$ is given by J^N for every unit vector $v \in W_0$. Further, $R^N_{u,v} = R^N_{\mathrm{i}u,\mathrm{i}v} = v \wedge u$ for all $u, v \in W_0$ and $R^N_{u,\mathrm{i}v} = 0$ if $u, v \in W_0$ with $\langle u, v \rangle = 0$. Part (a) follows. For (b), note that $\mathfrak{h}_W|_{W^\perp} = \mathbb{R}J^N|_{W^\perp}$. Part (c) is obvious. ☐

Corollary 5. *Suppose that* W *and* U *are curvature invariant subspaces of Type* (c_k) *and* (c_l) *with* $k, l \ge 1$ *defined by the data* $(\mathfrak{R}, W_0)$ *and* $(\mathfrak{R}^*, U_0)$*, respectively. If* $\mathfrak{R} = \mathfrak{R}^*$ *and* $W_0 \perp U_0$*, then* (W, U) *is an orthogonal curvature invariant pair. Moreover, every orthogonal curvature invariant pair of Type* (c_k, c_l) *is obtained in this way.*

Table 1: Orthogonal curvature invariant pairs of $\mathrm{G}_2^+(\mathbb{R}^{n+2})$

Type	Data	Conditions	Remarks
(c_k, c_l)	$(\mathfrak{R}, W_0; \mathfrak{R}^*, U_0)$	$\mathfrak{R} = \mathfrak{R}^*$, $W_0 \perp U_0$	—
$(tr_{i,j}, tr_{k,l})$	$(\mathfrak{R}, W_1, W_2; \mathfrak{R}^*, U_1, U_2)$	$\mathfrak{R} = e^{\mathrm{i}\varphi}\mathfrak{R}^*$, $W_1 \oplus W_2 \perp e^{\mathrm{i}\varphi}(U_1 \oplus U_2)$	$\varphi \neq 0 \mod \pi/2$
$(tr_{i,j}, tr_{k,l})$	$(\mathfrak{R}, W_1, W_2; \mathfrak{R}^*, U_1, U_2)$	$\mathfrak{R} = \mathfrak{R}^*$, $W_1 \oplus W_2 \perp U_1 \oplus U_2$	$(i,l), (j,k) \neq (1,1)$
$(tr_{i,j}, tr_{k,i})$	$(\mathfrak{R}, W_1, W_2; \mathfrak{R}^*, U_1, U_2)$	$\mathfrak{R} = \mathfrak{R}^*$, $W_1 = U_2$, $W_2 \perp U_1$	$i \geq 2$, $(j,k) \neq (1,1)$
$(tr_{i,j}, tr_{j,i})$	$(\mathfrak{R}, W_1, W_2; \mathfrak{R}^*, U_1, U_2)$	$\mathfrak{R} = \mathfrak{R}^*$, $W_1 = U_2$, $W_2 = U_1$	$i, j \geq 2$
$(tr_{1,j}, tr_{k,1})$	$(\mathfrak{R}, W_1, W_2; \mathfrak{R}^*, U_1, U_2)$	$\mathfrak{R} = \mathfrak{R}^*$, $W_1 \perp U_1$, $W_2 \perp U_1$, $W_2 \perp U_2$	$(j,k) \neq (1,1)$,
$(tr_{1,j}, tr_{j,1})$	$(\mathfrak{R}, W_1, W_2; \mathfrak{R}^*, U_1, U_2)$	$\mathfrak{R} = \mathfrak{R}^*$, $W_2 = U_1$	$j \geq 2$
$(tr_{1,1}, tr_{1,1})$	$(\mathfrak{R}, W_1, W_2; \mathfrak{R}^*, U_2, U_2)$	$\mathfrak{R} = \mathfrak{R}^*$, $W_1 \perp U_1$, $W_2 \perp U_2$	—
$(tr_{i,j}, tr_1)$	$(\mathfrak{R}, W_1, W_2; u)$	$u \in \mathbb{C}(W_1 \oplus W_2)^\perp$	$i, j \neq 1$
(tr_{1j}, tr_1)	$(\mathfrak{R}, W_1, W_2; u)$	$\Re(u) \in W_1^\perp$, $u \in \mathbb{C}W_2^\perp$	$j \geq 2$
$(tr_{1,1}, tr_1)$	$(\mathfrak{R}, W_1, W_2; u)$	$\Re(u) \in W_1^\perp$, $\Im(u) \in W_2^\perp$	—
(c_i, c'_j)	$(\mathfrak{R}, W_0; \mathfrak{R}^*, U', I')$	$\mathfrak{R} = \mathfrak{R}^*$, $W_0 \perp U'$	—
(c'_i, c'_j)	$(\mathfrak{R}, W', I'; \mathfrak{R}^*, U', J')$	$\mathfrak{R} = \mathfrak{R}^*$, $W' \perp U'$	—
(c'_i, c'_i)	$(\mathfrak{R}, W', I'; \mathfrak{R}^*, U', J')$	$\mathfrak{R} = \mathfrak{R}^*$, $W' = U'$, $I' = -J'$	—
$(tr'_k, tr_{i,j})$	$(\mathfrak{R}, W', I', W'_0; \mathfrak{R}^*, U_1, U_2)$	$\mathfrak{R} = \mathfrak{R}^*$, $W' \perp U_1 \oplus U_2$	—
(tr'_i, tr_1)	$(\mathfrak{R}, W', I', W'_0; u)$	$u \in \mathbb{C}W'^\perp$	—
(tr'_i, tr'_j)	$(\mathfrak{R}, W', I', W'_0; \mathfrak{R}^*, U', J', U'_0)$	$\mathfrak{R} = \mathfrak{R}^*$, $W' \perp U'$	—
(tr'_i, tr'_i)	$(\mathfrak{R}, W', I', W'_0; \mathfrak{R}^*, U', J', U'_0)$	$\mathfrak{R} = \mathfrak{R}^*$, $W' = U'$, $U'_0 = I'(W'_0)$, $J' = I'$	$i \geq 3$
(tr'_i, tr'_i)	$(\mathfrak{R}, W', I', W'_0; \mathfrak{R}^*, U', J', U'_0)$	$\mathfrak{R} = \mathfrak{R}^*$, $W' = U'$, $U'_0 = \exp(\varphi I')(W'_0)$, $J' = -I'$	$i \geq 3$
(tr'_2, tr'_2)	$(\mathfrak{R}, W', I', W'_0; \mathfrak{R}^*, U', J', U'_0)$	$\mathfrak{R} = \mathfrak{R}^*$, $W' = U'$, $U'_0 = \tilde{J}(W'_0)$, $J' = I'$	$\tilde{J} \in \mathrm{SU}(W', \tilde{I}) \cap \mathfrak{so}(W')$
(ex_3, tr_1)	$(\mathfrak{R}, \{e_1, e_2\}; u)$	$u = \pm 1/\sqrt{2}\,(e_2 - \mathrm{i}e_1)$	—
(tr_1, tr_1)	$(u; v)$	$u \perp v$	—

Here we use the notation from Theorem 5. Note, if W is of Type tr'_2 defined by $(\mathfrak{R}, W', I', W'_0)$, a second Hermitian structure on W' is given by $\tilde{I} := e_1 \wedge e_2 + I'e_1 \wedge I'e_2$ for some orthonormal basis $\{e_1, e_2\}$ of W'_0.

Proof. Using Lemma 2, the first part of the corollary is obvious. For the last assertion, since the linear space W is determined also by the tuple $(e^{\mathrm{i}\varphi}\mathfrak{R}, e^{\mathrm{i}\varphi}W_0)$ for all $\varphi \in \mathbb{R}$, we can assume that $\mathfrak{R} = \mathfrak{R}^*$. Thus the condition $W \perp U$ implies that $W_0 \perp U_0$. □

Corollary 6. *There are no orthogonal curvature invariant pairs (W, U) of Type $(c_k, tr_{i,j})$ or (c_k, tr_1).*

Proof. If W is of Type (c_k), then any $\mathfrak{h}_W$-invariant subspace of $W^\perp$ is complex, according to Lemma 2 (b). On the other hand, if U is of Type $(tr_{i,j})$ or (tr_1), then U is totally real. This gives the claim. □

Lemma 3. *Suppose that W is of Type $(tr_{k,l})$ defined by the data $(\mathfrak{R}, W_1, W_2)$.*

(a) *We have*

$$\mathfrak{h}_W = \{u_1 \wedge v_1 + u_2 \wedge v_2 \,|\, u_1, v_1 \in W_1,\ u_2, v_2 \in W_2\}_{\mathbb{R}}. \tag{34}$$

(b) *If $k, l \geq 2$, then a subspace of $W^\perp$ is $\mathfrak{h}_W$-invariant if and only if it is equal to $\mathrm{i}W_1$, W_2, a subspace of $\mathbb{C}(W_1 \oplus W_2)^\perp$ or a sum of such spaces (where $(W_1 \oplus W_2)^\perp$ denotes the orthogonal complement of $W_1 \oplus W_2$ in $\mathfrak{R}$). If $k = 1$ and $l \geq 2$, then a subspace of $W^\perp$ is $\mathfrak{h}_W$-invariant if and only if it is equal to W_2, a subspace of $\mathrm{i}W_1 \oplus \mathbb{C}(W_1 \oplus W_2)^\perp$ or a sum of such spaces. If $k = l = 1$, then any subspace of $W^\perp$ is $\mathfrak{h}_W$-invariant.*

(c) *Let $A \in \mathfrak{so}(\mathfrak{R})$ and $c \in \mathbb{R}$. The endomorphism $cJ^N + A$ leaves W invariant if and only if $c = 0$ and $A = \sum_{i \in I} u_1^i \wedge v_1^i + u_2^i \wedge v_2^i + \tilde{u}_i \wedge \tilde{v}_i$ with $u_1^i, v_1^i \in W_1$, $u_2^i, v_2^i \in W_2$ and $\tilde{u}_i, \tilde{v}_i \in (W_1 \oplus W_2)^\perp$.*

Proof. For Part (a), see the proof of Lemma 2. For (b), consider the decomposition $W^\perp = \mathrm{i}W_1 \oplus W_2 \oplus \mathbb{C}(W_1 \oplus W_2)^\perp$ into $\mathfrak{h}_W$-invariant subspaces. Then $\mathfrak{h}_W$ acts trivially on $(W_1 \oplus W_2)^\perp$ and irreducible on both $\mathrm{i}W_1$ and W_2. In particular, $\mathrm{i}W_1$ or W_2 is a trivial $\mathfrak{h}_W$-module only if $k = 1$ or $l = 1$, respectively. Moreover, $\mathrm{i}W_1$ and W_2 are non-isomorphic $\mathfrak{h}_W$ modules unless $k = l = 1$. The result follows. Part (c) is straightforward. □

Corollary 7. *Let W and U be curvature invariant subspaces of Type $(tr_{i,j})$ and $(tr_{k,l})$ defined by the data $(\Re, W_1, W_2)$ and $(\Re^*, U_1, U_2)$, respectively. If one of the following conditions holds, then (W, U) is an orthogonal curvature invariant pair:*

- *The real number φ is chosen such that $\Re = e^{\mathrm{i}\varphi}\Re^*$ and $e^{\mathrm{i}\varphi}(U_1 \oplus U_2)$ belongs to the orthogonal complement of $W_1 \oplus W_2$;*
- *$\Re = \Re^*$, $W_2 = U_1$ and $W_1 = U_2$;*
- *$\Re = \Re^*$, $W_2 \perp U_1$ and $W_1 = U_2$;*
- *$i = l = 1$, $\Re = \Re^*$, $W_1 \perp U_1$, $W_2 \perp U_1$ and $W_2 \perp U_2$;*
- *we have $i = l = 1$, $\Re = \Re^*$ and $W_2 = U_1$;*
- *we have $(i, j) = (k, l) = (1, 1)$, $\Re = \Re^*$, $W_1 \perp U_1$ and $W_2 \perp U_2$.*

Moreover, every orthogonal curvature invariant pair of Type $(tr_{i,j}, tr_{k,l})$ can be obtained in this way.

Proof. Obviously, the pairs (W, U) mentioned above satisfy $W \perp U$. Further, the fact that these are curvature invariant pairs is verified by means of Lemma 3. Conversely, let us see that these conditions are also necessary:

We have

$$\begin{aligned} u_1 &= e^{-\mathrm{i}\varphi} e^{\mathrm{i}\varphi} u_1 = \cos(\varphi) e^{\mathrm{i}\varphi} u_1 - \mathrm{i}\sin(\varphi) e^{\mathrm{i}\varphi} u_1, \\ \mathrm{i}u_2 &= e^{-\mathrm{i}\varphi} e^{\mathrm{i}\varphi} \mathrm{i}u_2 = \sin(\varphi) e^{\mathrm{i}\varphi} u_2 + \mathrm{i}\cos(\varphi) e^{\mathrm{i}\varphi} u_2. \end{aligned}$$

Further, $e^{\mathrm{i}\varphi} u_1 \in \Re$ and $\mathrm{i}e^{\mathrm{i}\varphi} u_2 \in \mathrm{i}\Re$ for all $(u_1, u_2) \in U_1 \oplus U_2$. Thus, the condition $U \perp W$ implies that

$$\begin{aligned} 0 &= \langle v_1, u_1 \rangle = \cos(\varphi) \langle v_1, e^{\mathrm{i}\varphi} u_1 \rangle, \\ 0 &= \langle v_1, \mathrm{i}u_2 \rangle = \sin(\varphi) \langle v_1, e^{\mathrm{i}\varphi} u_2 \rangle, \\ 0 &= \langle \mathrm{i}v_2, u_1 \rangle = -\sin(\varphi) \langle v_2, e^{\mathrm{i}\varphi} u_1 \rangle, \\ 0 &= \langle \mathrm{i}v_2, \mathrm{i}u_2 \rangle = \cos(\varphi) \langle v_2, e^{\mathrm{i}\varphi} u_2 \rangle \end{aligned}$$

for all $(v_1, v_2) \in W_1 \oplus W_2$ and $(u_1, u_2) \in e^{\mathrm{i}\varphi}(U_1 \oplus U_2)$. Hence, in case $\varphi \notin \pi/2\,\mathbb{Z}$, we necessarily have $e^{\mathrm{i}\varphi}(U_1 \oplus U_2) \perp W_1 \oplus W_2$.

Suppose that $\varphi \in \pi/2\,\mathbb{Z}$. Interchanging, if necessary, U_1 and U_2, we can even assume that $\varphi = 0$, i.e. $\Re = \Re^*$. From $W \perp U$ it follows that $W_1 \perp U_1$ and $W_2 \perp U_2$. By means of (13),(34), then U_2 is an $\mathfrak{h}_W$-invariant subspace of $\Re$ which is contained in $W_2^\perp$. Suppose first that $i \geq 2$. Then W_1 is a non-trivial irreducible $\mathfrak{h}_W$-module. Using Lemma 3 (b), we have $U_2 \subset (W_1 \oplus W_2)^\perp$ or $U_2 = W_1 \oplus \tilde{U}$ for some $\tilde{U} \subset (W_1 \oplus W_2)^\perp$. We claim that the latter is not possible unless $\tilde{U} = \{0\}$:

Since (W, U) is a curvature invariant pair, we know from (15),(34) that W_1 is an $\mathfrak{h}_U$-invariant subspace of $U_1^\perp$. Moreover, the condition $U_2 = W_1 \oplus \tilde{U}$ implies that $l \geq i \geq 2$. Therefore, by means of Lemma 3 (b), we have $W_1 \perp U_2$ or $W_1 = U_2 \oplus \tilde{W}$ for some $\tilde{W} \subset (U_1 \oplus U_2)^\perp$. We immediately see that this is not possible unless $\tilde{U} = \{0\}$.

Thus, we have $U_2 \perp W_1$ or $U_2 = W_1$. Clearly, this conclusion is true also for $i = 0$.

Similarly, in case $j \neq 1$, we can show that $U_1 \perp W_2$ or $U_1 = W_2$ (by means of passing from $\Re$ to $\mathrm{i}\Re$). In case $l \neq 1$ or $k \neq 1$, we obtain the same conclusions, respectively (by interchanging W and U). This finishes the proof. □

Corollary 8. *Let W and U be curvature invariant subspaces of Type $(tr_{i,j})$ and (tr_1) defined by the data $(\Re, W_1, W_2)$ and a unit-vector $u \in T_pN$, respectively. If one of the following conditions holds, then (W, U) is an orthogonal curvature invariant pair:*

- $i, j \geq 2$ *and* $u \in \mathbb{C}(W_1 \oplus W_2)^\perp$;
- $i = 1$, $j \geq 2$, $\Re(u) \perp W_1$ *and* $u \in \mathbb{C}W_2^\perp$;
- $i = j = 1$, $\Re(u) \perp W_1$ *and* $\Im(u) \perp W_2$.

Moreover, every orthogonal curvature invariant pair of Type $(tr_{i,j}, tr_1)$ can be obtained in this way.

Proof. Note, the pair (W, U) is an orthogonal curvature invariant pair if and only if $u \in W^\perp$ and $\mathfrak{h}_W$ annihilates the vector u. If $i, j \geq 2$, this is equivalent to $u \in \mathbb{C}(W_1 \oplus W_2)^\perp$ according to Lemma 3 (b). If $i = 1$ and $j \geq 2$, we use the same argument as before; however, now it is allowed that $\Im(u)$ has a component in W_1. The case $i \geq 2$, $j = 1$ also follows (by passing from $\Re$ to $\mathrm{i}\Re$). In case $i = j = 1$, the Lie algebra $\mathfrak{h}_W$ is trivial and the only condition is $u \in W^\perp$. □

Lemma 4. *Suppose that W is of Type (c'_k) determined by the data $(\Re, W', I')$.*

(a) We have $\mathfrak{h}_W = \{-2\langle I'u, v\rangle J^N - u \wedge v - I'u \wedge I'v \,|\, u, v \in W'\}_{\mathbb{R}}$.

(b) A subspace of $W^\perp$ is $\mathfrak{h}_W$-invariant if and only if it is equal to $\bar{W}$, a complex subspace of $(\mathbb{C}W')^\perp$ or a sum of such spaces (where $\bar{W}$ denotes the complex conjugate of W in T_pN and $(\mathbb{C}W')^\perp$ denotes the orthogonal complement of $\mathbb{C}W'$ in T_pN). In particular, any such space is complex, too.

(c) Let $c \in \mathbb{R}$ and $A \in \mathfrak{so}(\Re)$. The endomorphism $cJ^N + A$ leaves W invariant if and only if $A = \sum_{i\in I} u_i \wedge v_i + I'u_i \wedge I'v_i + \tilde{u}_i \wedge \tilde{v}_i$ with $u_i, v_i \in W'$ and $\tilde{u}_i, \tilde{v}_i \in W'^\perp$.

Proof. Part (a) is straightforward using (33). For (b), note that $\bar{W}$ is a complex subspace of T_pN, i.e. $J^N(W) \subset W$. Further, we have $A(\bar{W}) \subset \bar{W}$ for all $A \in \mathfrak{so}(W')$, hence $R^N_{u,v}(\bar{W}) \subset \bar{W}$ for all $u, v \in W$. Thus $\bar{W}$ is a $\mathfrak{h}$-submodule of T_pN. Furthermore, note that complex conjugation defines an isomorphism $W \to \bar{W}$ of $\mathfrak{h}_W$-modules and that the action of $\mathfrak{h}_W$ on W is irreducible (since it is the linearized isotropy representation of the complex projective space $\mathbb{C}\mathrm{P}^k$). Therefore, also $\bar{W}$ is an irreducible $\mathfrak{h}_W$-module. Moreover, $\mathbb{C}W' = W \oplus \bar{W}$ and $\mathfrak{h}_W$ acts on $(\mathbb{C}W')^\perp$ via multiples of J^N. Now (b) follows. Part (c) is straightforward. □

Corollary 9. *Suppose that W is of Type (c_k) determined by the data $(\Re, W_0)$ and that U is of Type (c'_l) determined by $(\Re^*, U', I')$. If $\Re = \Re^*$ and $W_0 \perp U'$, then (W, U) is an orthogonal curvature invariant pair. Moreover, every orthogonal curvature invariant pair of Type (c_k, c'_l) can be obtained in this way.*

Proof. Obviously, the pairs (W, U) mentioned above satisfy $W \perp U$. Further, the fact that these are curvature invariant pairs is verified by means of Lemmas 2 and 4, Parts (a) and (c). Conversely, let us see that these conditions are also necessary:

Here we can assume that $\Re = \Re^*$ (cf. the proof of Corollary 5). Further, since $\mathfrak{h}_W(U) \subset U$, it follows from Lemma 2 (b) that $U \subset \mathbb{C}W_0^\perp$. Then

$$0 = \langle u - \mathrm{i}I'u, v\rangle = \langle u, v\rangle \tag{35}$$

for all $u \in U'$, $v \in W_0$, i.e. $W_0 \perp U'$. □

Corollary 10. *Suppose that W and U are of Type (c'_k) and (c'_l) determined by the data $(\Re, W', I')$ and $(\Re^*, U', J')$, respectively. If one of the following conditions holds, then (W, U) is an orthogonal curvature invariant pair:*

- $\Re = \Re^*$, $U' = W'$ *and* $I' = -J'$;
- $\Re = \Re^*$ *and* $U' \perp W'$.

Moreover, every orthogonal curvature invariant pair of Type (c'_k, c'_l) *can be obtained in this way.*

Proof. It is straightforward that the pairs (W, U) mentioned above satisfy $W \perp U$. Further, the fact that these are curvature invariant pairs is verified by means of Lemma 4, Parts (a) and (c).

Conversely, the Hermitian structure I' extends to $W' \oplus \mathrm{i}W'$ (via complexification) and the linear space W is determined also by the data $(e^{\mathrm{i}\varphi}\Re, e^{\mathrm{i}\varphi}W', I'|_{e^{\mathrm{i}\varphi}W'})$. Hence, we can assume that $\Re = \Re^*$. Further, by means of Lemma 4, either $U \subset \mathbb{C}W'^{\perp}$ or $U = \bar{W} \oplus \tilde{U}$ for some subspace $\tilde{U} \subset \mathbb{C}W'^{\perp}$. In the first case, the condition $U \perp W$ implies that $W' \perp U'$ (see (35)). In the second case, we claim that $\tilde{U} = \{0\}$:

Otherwise, l is strictly greater than k. Hence, because $\mathfrak{h}_U$ leaves W invariant, Lemma 4 (b) shows that $W \subset \mathbb{C}U'^{\perp}$. As above, this implies that $W' \perp U'$. Hence $U = \bar{W}$, i.e. $W' = U'$ and $I' = -J'$, which finishes the proof. □

On the analogy of Corollary 6, we have

Corollary 11. *There are no orthogonal curvature invariant pairs of Type* $(c'_k, tr_{i,j})$ *or* (c'_k, tr_1). □

Lemma 5. *Suppose that* W *is of Type* (tr'_k) *determined by the data* $(\Re, W', I', W'_0)$.

(a) The curvature endomorphism $R^N_{u-\mathrm{i}I'u, v-\mathrm{i}I'v}$ *is given by* $-u \wedge v - I'u \wedge I'v$ *for all* $u, v \in W'_0$. *Hence* $\mathfrak{h}_W = \{u \wedge v + I'u \wedge I'v \,|\, u, v \in W'_0\}_{\mathbb{R}}$.

(b) An $\mathfrak{h}_W$*-invariant subspace of* $W^{\perp}$ *is contained in the orthogonal complement of the complex linear space* $\mathbb{C}W'$*, belongs to a distinguished family* $\mathcal{F}$ *of* k*-dimensional totally real subspaces of* $\mathbb{C}W' \cap W^{\perp}$ *which can be parameterized by the real projective space* $\mathbb{R}\mathrm{P}^2$ *(for* $k \geq 3$*) or the complex projective space* $\mathbb{C}\mathrm{P}^2$ *(for* $k = 2$*), or is a direct sum of such spaces.*

(c) Let $A \in \mathfrak{so}(\Re)$ *and* $c \in \mathbb{R}$ *be given. Then* $cJ^N + A$ *leaves the subspace* W *invariant if and only if* $A = -cI' + \sum_{i \in I} u_i \wedge v_i + I'u_i \wedge I'v_i + \tilde{u}_i \wedge \tilde{v}_i$ *with* $u_i, v_i \in W'_0$ *and* $\tilde{u}_i, \tilde{v}_i \in W'^{\perp}$ *(were* $W'^{\perp}$ *denotes the orthogonal complement of* W' *in* $\Re$*).*

Proof. Part (a) is straightforward. For (b), note that $\mathbb{C}W' \cap W^{\perp} = \mathrm{i}W \oplus \bar{W} \oplus \mathrm{i}\bar{W}$ is a decomposition into irreducible, pairwise equivalent $\mathfrak{h}_W$-modules. Moreover, if $k \geq 3$, then W is an irreducible $\mathfrak{h}_W$-module even over $\mathbb{C}$ whereas W is reducible over $\mathbb{C}$ for $k = 2$. Part (b) follows

For (c): We have $\mathrm{i}(v - \mathrm{i}I'v) = I'v + \mathrm{i}v = I'v - \mathrm{i}I'(I'v)$ for all $v \in W'$ and hence $J^N|_W = I'$. In particular, $J^N|_W - I'$ leaves W invariant. This reduces the question to the case $c = 0$ in which case we have to determine those A which leave the linear space W'_0 invariant and $AI'v = I'Av$ holds for all $v \in W'_0$, i.e. $A = A_1 \oplus A_2 \oplus \tilde{A}$ with $A_1 = A_2 \in \mathfrak{so}(W'_0)$ and $\tilde{A} \in \mathfrak{so}(W'^{\perp})$. This proves the result. □

Using Lemma 5 we have (cf. the proof of Corollary 8):

Corollary 12. *Let* W *and* U *be of Type* (tr'_k) *and* (tr_1) *defined by the data* $(\Re, W', I', W'_0)$ *and a unit vector* u *of* T_pN*, respectively. The pair* (W, U) *is an orthogonal curvature invariant pair if and only if* u *belongs to* $\mathbb{C}W'^{\perp}$. □

Clearly, subspaces of Type (tr'_k) are totally real. Hence Lemma 2 (b) combined with Lemma 4 (b) implies:

Corollary 13. *There are no orthogonal curvature invariant pairs* (W, U) *of Type* (tr'_i, c_j) *or* (tr'_i, c'_j). □

Corollary 14. *Let W and U be of Type $(\mathfrak{tr}_i')$ and $(\mathfrak{tr}_{k,l})$ defined by the data $(\Re, W', I', W_0')$ and $(\Re^*, U_1, U_2)$, respectively. If $\Re = \Re^*$ and the linear space $U_1 \oplus U_2$ belongs to the orthogonal complement of W' in $\Re$, then (W, U) is an orthogonal curvature invariant pair. Every orthogonal curvature invariant pair of Type $(\mathfrak{tr}_i', \mathfrak{tr}_{k,l})$ can be obtained in this way.*

Proof. Obviously, the pairs (W, U) mentioned above satisfy $W \perp U$. Further, the fact that these are curvature invariant pairs is verified by means of Lemmas 3 and 5, Parts (a) and (c). Conversely, let us see that the conditions are also necessary:

For the first assertion, note that W is defined also by the data $(e^{\mathrm{i}\varphi}\Re, e^{\mathrm{i}\varphi}W', I', e^{\mathrm{i}\varphi}W_0'(-\varphi))$ with $W_0'(-\varphi) := \{\cos(\varphi)v - \sin(\varphi)I'v \,|\, v \in W_0'\}$ for every $\varphi \in \mathbb{R}$. Hence we can assume that $\Re = \Re^*$. Now suppose that (W, U) is an orthogonal curvature invariant pair. Since U is $\mathfrak{h}_W$-invariant, there exists a decomposition $U = U^\# \oplus \tilde{U}$ into $\mathfrak{h}_W$-invariant subspaces $U^\# \subset \mathbb{C}W'$ and $\tilde{U} \subset \mathbb{C}W'^\perp$. We claim that the only possibilities are $U^\# = \{0\}$, $U^\# = \mathrm{i}W_0'$, $U^\# = I'(W_0')$ or $U^\# = I'(W_0') \oplus \mathrm{i}W_0'$:

First, the condition $\langle u, v\rangle = 0$ for all $u \in U$ and $v \in V$ implies that

$$0 = \langle u, v - \mathrm{i}I'v\rangle = \langle u, v\rangle$$

for all $u \in U_1$ and $v \in W_0'$. Hence $U_1 \subset W_0'^\perp$, thus $U_1 \cap W' \subset I'(W_0')$. Similarly, we can prove that $U_2 \cap W' \subset W_0'$. Further, we have

$$U^\# = U \cap \mathbb{C}W' = U_1 \cap W' \oplus \mathrm{i}(U_2 \cap W').$$

Thus, the $\mathfrak{h}_W$-invariance of U_1 implies that both summands are invariant under the $\mathfrak{h}_W$ action (note, $\mathfrak{h}_W \subset \mathfrak{so}(\Re)$, see Lemma 5 (a)). Since this action is irreducible on both W_0' and $I'(W_0')$, it follows from Schur's Lemma that $U_1 \cap W' \in \{\{0\}, I'(W_0')\}$ and $U_2 \cap W' \in \{\{0\}, W_0'\}$. Our claim follows.

Next, we claim that $U^\# = \{0\}$. Assume, by contradiction, that $I'(W_0') \subset U$. Since $\dim(W_0') \geq 2$, there exists a pair of orthonormal vectors $u, v \in W_0'$. Then $\{I'u, I'v\} \subset U \cap \Re = U_1$, hence $R^N_{I'u, I'v} = -I'u \wedge I'v$ leaves W invariant since (W, U) is a curvature invariant pair. Applying Lemma 5 (c) (with $c = 0$), we see that this is not possible. By means of a similar argument, we conclude that $\mathrm{i}W_0'$ is not contained in U. It follows that $U^\# = \{0\}$, i.e. $U \subset \mathbb{C}W'^\perp$. Clearly, this shows that $U_1 \oplus U_2 \perp W'$. This finishes our proof. ☐

Corollary 15. *Let W and U be of Type $(\mathfrak{tr}_k')$ and $(\mathfrak{tr}_l')$ defined by the data $(\Re, W', I', W_0')$ and $(\Re^*, U', J', U_0')$, respectively. Further, in case $k = 2$, let $\{e_1, e_2\}$ be an orthonormal basis of W_0' and let $\tilde{I}$ be the Hermitian structure of W' defined by $e_1 \wedge e_2 + I'e_1 \wedge I'e_2$.*

If $\Re = \Re^*$ and one of the following conditions holds, then (W, U) is an orthogonal curvature invariant pair:

- $\Re = \Re^*$ *and* $U' \perp W'$*;*
- $k \geq 3$, $\Re = \Re^*$, $U' = W'$, $I' = J'$ *and* $U_0' = I'(W_0')$*;*
- $k \geq 3$, $U' = W'$, $I' = -J'$ *and* $U_0' = \exp(\varphi I')(W_0')$ *for some* $\varphi \in \mathbb{R}$*;*
- $k = 2$, $U' = W'$ *and there exists some* $\tilde{J} \in \mathrm{SU}(W', \tilde{I}) \cap \mathfrak{su}(W')$ *such that* $U = \tilde{J}(W)$*.*

Moreover, every orthogonal curvature invariant pair of Type $(\mathfrak{tr}_k', \mathfrak{tr}_l')$ can be obtained in this way.

Proof. In the one direction, we first verify that the given pairs (W, U) satisfy $U \perp W$. This is straightforward in the first case. In the second and the third case, we have $U = \mathrm{i}W$ and $U = e^{\mathrm{i}\varphi}\bar{W}$, respectively, and the result follows. In the last case, note that $f_1 := e_1$ and $f_2 := I'e_1$ define a Hermitian basis of $(W', \tilde{I})$. Consider the complex matrix (g_{ij}) defined by

$$g_{ij} := \langle f_i, \tilde{J}f_j\rangle + \mathrm{i}\langle \tilde{I}f_i, \tilde{J}f_j\rangle \tag{36}$$

for $i, j = 1, 2$. Then (g_{ij}) belongs to $\mathrm{SU}(2) \cap \mathfrak{su}(2)$, hence there exist $t \in \mathbb{R}$ and $w \in \mathbb{C}$ with $t^2 + |w|^2 = 1$ such that

$$\begin{pmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{pmatrix} = \begin{pmatrix} \mathrm{i}t & -\bar{w} \\ w & -\mathrm{i}t \end{pmatrix} \tag{37}$$

Using the skew-symmetry of $\tilde{J}$ and (37), we calculate

$$\langle e_i - \mathrm{i}I'e_i, \tilde{J}(e_i - \mathrm{i}I'e_i)\rangle = \langle e_i, \tilde{J}e_i\rangle + \langle I'e_i, \tilde{J}I'e_i\rangle = 0 \text{ for } i = 1, 2, \tag{38}$$

$$\langle e_2 - \mathrm{i}I'e_2, \tilde{J}(e_1 - \mathrm{i}I'e_1)\rangle = \langle \tilde{I}f_1, \tilde{J}f_1\rangle + \langle \tilde{I}f_2, \tilde{J}f_2\rangle = \Im(g_{11} + g_{22}) = 0, \tag{39}$$

$$\langle e_1 - \mathrm{i}I'e_1, \tilde{J}(e_2 - \mathrm{i}I'e_2)\rangle = -\langle e_2 - \mathrm{i}I'e_2, \tilde{J}(e_1 - \mathrm{i}I'e_1)\rangle = 0. \tag{40}$$

This shows that $W \perp \tilde{J}(W)$.

Further, in order to see that the given pairs (W, U) are actually curvature invariant pairs, we proceed as follows: the first case is handled by means of Lemma 4, (a) and (c). In the next two cases, it easy to see that $\mathfrak{h}_W = \mathfrak{h}_U$ holds. The latter condition is actually true also in the last case, which is due to the fact that here $\mathfrak{h}_W = \mathbb{R}\tilde{I}$ holds and $\mathfrak{h}_U = \{\, \tilde{J} \circ A \circ \tilde{J}^{-1} \mid A \in \mathfrak{h}_W \,\}$ (since $\tilde{J} \in \mathrm{SU}(W', \tilde{I})$). Clearly, this implies that (W, U) is a curvature invariant pair.

In the other direction, let (W, U) be an orthogonal curvature invariant pair of Type $(\mathfrak{tr}'_k, \mathfrak{tr}'_l)$ defined by the data $(\Re, W', I', W'_0; \Re^*, U', J', U'_0)$. We will show that it can be obtained in one of the four ways described before:

Here we can assume that $\Re = \Re^*$ (cf. the proof of Corollary 14). Since U is $\mathfrak{h}_W$-invariant, we have $U = U^\sharp \oplus \tilde{U}$ with $U^\sharp \subset \mathbb{C}W'$ and $\tilde{U} \subset \mathbb{C}W'^\perp$, according to Lemma 5 (b). Moreover, the linear space $U^\#$ is the direct sum of (at most three) linear spaces which belong to a distinguished family $\mathcal{F}$ of k-dimensional subspaces of $\mathbb{C}W'$. First, we claim that $U^\# = \{0\}$ or $k = l$:

Suppose that $U^\# \neq \{0\}$. In particular, then $l \geq k$. Therefore, since W is an $\mathfrak{h}_U$-invariant subspace of $U^\perp$ with $\dim(W) = k \leq l$, it follows from Lemma 5 (b) that $W \subset \mathbb{C}U'^\perp$ or $k = l$. If we assume, by contradiction, that $W \subset \mathbb{C}U'^\perp$, then

$$0 = \langle v - \mathrm{i}I'v, u\rangle = \langle v, u\rangle, \tag{41}$$

$$0 = \langle v - \mathrm{i}I'v, \mathrm{i}u\rangle = -\langle I'v, u\rangle \tag{42}$$

for all $u \in U'$ and $v \in W'_0$, i.e. we obtain that $W' \perp U'$. Hence $\mathbb{C}W' \perp \mathbb{C}U'$ and, in particular, $U^\# = U \cap \mathbb{C}W' = \{0\}$, a contradiction. We conclude that $k = l$.

We thus see that either $U \in \mathcal{F}$ or $U \subset \mathbb{C}W'^\perp$. In the latter case, we even have $U' \perp W'$ (see Eqs. (41),(42)). In the first case, we have, in particular, $U'_0 \subset W'$ and $I'(U'_0) \subset W'$, hence $U' = W'$ (since $k = l$).

Furthermore, we claim that $\mathfrak{h}_W = \mathfrak{h}_U$:

For this, it suffices (by means of a symmetry argument) to show that $\mathfrak{h}_U \subset \mathfrak{h}_W$. Let $A \in \mathfrak{h}_U$ be given. We can assume, by means of Lemma 5 (a), that $A = u_1 \wedge u_2 + I'u_1 \wedge I'u_2$ with $u_1, u_2 \in U'_0$. Further, $A(W) \subset W$ by definition of a curvature invariant pair. Since $W' = U'$, we obtain from Lemma 5 (c), applied to W with $c = 0$, that $A = \sum_{i \in I} u_i \wedge v_i + I'u_i \wedge I'v_i$ with $u_i, v_i \in W'_0$, i.e. $A \in \mathfrak{h}_W$. This proves our claim.

For $k \geq 3$, set $\ell_0(v) := I'v + \mathrm{i}v$, $\ell_1(v) := v + \mathrm{i}I'v$ and $\ell_2(v) := I'v - \mathrm{i}v$ for all $v \in W'_0$. Then ℓ_i is an isomorphism onto $\mathrm{i}W$, $\bar{W}$ and $\mathrm{i}\bar{W}$, respectively. It follows from Schur's Lemma that there exists $(c_0 : c_1 : c_2) \in \mathbb{R}\mathrm{P}^2$ such that $U = \lambda(W'_0)$ with $\lambda := c_0\lambda_0 + c_1\lambda_1 + c_2\lambda_2$. Note $\lambda(v) = c_1 v + (c_0 + c_2)I'v + (c_0 - c_2)\mathrm{i}v + c_1\mathrm{i}I'v$, hence it can not happen that $a := c_1$ and $b := c_0 + c_2$ both vanish (since otherwise $U \subset \mathrm{i}\Re$, which is not possible). Thus, we can assume that $a^2 + b^2 = 1$, i.e. there exists some $\varphi \in [0, \pi[$ such that

$$U'_0 = \{\, \cos(\varphi)v + \sin(\varphi)I'v \mid v \in W'_0 \,\}. \tag{43}$$

Further, recall that $U' = W'$ and that this space is equipped with the two complex structures I', J' such that U'_0 is a real form with respect to J'. Furthermore, it follows from (43) that U'_0 is a real form with respect to I', too. Since moreover $k \geq 3$, an application of Schur's Lemma shows that therefore $I' = \pm J'$ holds. We obtain that (W, U) is a curvature invariant pair if and only if $U = \mathrm{e}^{\mathrm{i}\varphi} W$ or $U = \mathrm{e}^{\mathrm{i}\varphi}\bar{W}$ for some $\varphi \in \mathbb{R}$. It remains to determine under which conditions the linear spaces $W = \{\, v - \mathrm{i}I'v \,|\, v \in W_0' \,\}$ and $U = \mathrm{e}^{\mathrm{i}\varphi} W$ or $U = \mathrm{e}^{\mathrm{i}\varphi}\bar{W}$ are orthogonal to each other:

Set $W_0'(\varphi) := \{\, \cos(\varphi)v + \sin(\varphi)\mathrm{i}I'v \,|\, v \in W_0' \,\}$ and note that $\mathrm{e}^{\mathrm{i}\varphi}W = \{\, v - \mathrm{i}I'v \,|\, v \in W_0'(\varphi) \,\}$ and $\mathrm{e}^{\mathrm{i}\varphi}\bar{W} = \{\, v + \mathrm{i}I'v \,|\, v \in W(\varphi) \,\}$. We obtain that

$$\begin{aligned}
&\langle v - \mathrm{i}I'v, \cos(\varphi)u + \sin(\varphi)I'u \mp \mathrm{i}I'(\cos(\varphi)u + \sin(\varphi)I'u)\rangle \\
&= \langle v, \cos(\varphi)u + \sin(\varphi)I'u\rangle \pm \langle I'v, I'(\cos(\varphi)u + \sin(\varphi)I'u)\rangle \\
&= \cos(\varphi)\langle v, u\rangle \pm \cos(\varphi)\langle I'v, I'u\rangle = \cos(\varphi)\langle v, u\rangle \pm \cos(\varphi)\langle v, u\rangle
\end{aligned}$$

for all $u, v \in W_0'$, i.e. $U = \mathrm{e}^{\mathrm{i}\varphi}W$ is orthogonal to W only if $\varphi = \pi/2 \mod \mathbb{Z}\pi/2$ and $U = \mathrm{e}^{\mathrm{i}\varphi}\bar{W}$ is always orthogonal to W. This proves the corollary for $k \geq 3$.

For $k = 2$, we observe that $\tilde{I}$ equips W' with another Hermitian structure such that $[I', \tilde{I}] = 0$. Further, W_0' and $I'(W_0')$ both become complex subspaces of $(W', \tilde{I})$. We recall that any two Hermitian structures of a Euclidean vector space are conjugate by some orthogonal transformation and any two real forms of a unitary vector space are conjugated by some unitary transformation, hence there exists some $g \in \mathrm{SO}(W')$ such that $\tilde{J} \circ I' \circ g^{-1} = J'$ and $g(W_0') = U_0'$. Clearly, then $\mathfrak{h}_U = \{\, g \circ A \circ g^{-1} \,|\, A \in \mathfrak{h}_W \,\}$. Thus, the condition $\mathfrak{h}_W = \mathfrak{h}_U$ derived above shows that g normalizes $\mathfrak{h}_W$. Since furthermore $\mathfrak{h}_W = \mathbb{R}\tilde{I}$, it follows that g commutes or anti-commutes with $\tilde{I}$. This shows (W, U) is a curvature invariant pair if and only if there exists some $g \in \mathrm{U}(W', \tilde{I})$ such that either $U = g(W)$ or $U = g(\bar{W})$. However, the second case can be omitted since the linear map g_0 defined by $g_0|_{W_0'} := \tilde{I}|_{W_0'}$ and $g_0|_{I'(W_0')} := -\tilde{I}|_{I'(W_0')}$ belongs to $\mathrm{U}(W', \tilde{I})$ and maps W to $\bar{W}$. Moreover, since W is invariant under $\tilde{I}$, we can even assume that g belongs to the special unitary group $\mathrm{SU}(W', \tilde{I})$.

It follows that $U = g(W)$ for some $g \in \mathrm{SU}(W', \tilde{I})$. It remains to show that $g \in \mathfrak{su}(W')$. Let (g_{ij}) be the matrix defined by (36). Since $(g_{ij}) \in \mathrm{SU}(2)$, there exist $z, w \in \mathbb{C}$ with $|z|^2 + |w|^2 = 1$ such that (37) holds. Using (39), we conclude from $W \perp g(W)$ that $\Re(z) = 0$, hence $(g_{ij}) \in \mathfrak{su}(2)$, i.e. $\tilde{J} := g \in \mathrm{SU}(W', \tilde{I}) \cap \mathfrak{su}(W')$. This finishes the proof. □

Lemma 6. *Suppose there exists some $\mathfrak{R} \in \mathcal{U}$ and an orthonormal basis $\{e_1, \ldots, e_n\}$ of $\mathfrak{R}$ such that W is spanned by $w_1 := e_1 - \mathrm{i}e_2$, $w_2 := e_2 + \mathrm{i}e_1$ and $w_3 := e_1 + \mathrm{i}e_2$.*

(a) $\mathfrak{h}_W$ is the one-dimensional Lie algebra which is generated by $J^N + e_1 \wedge e_2$.

(b) A subspace of $W^\perp$ is $\mathfrak{h}_W$-invariant if and only if it is the one-dimensional space $\mathbb{R}(e_2 - \mathrm{i}e_1)$, a complex subspace of $\mathbb{C}\{e_1, e_2\}^\perp$ or a sum of such spaces..

(c) Let $A \in \mathfrak{so}(\mathfrak{R})$ and $c \in \mathbb{R}$ be given. Then $cJ^N + A$ leaves the subspace W invariant if and only if there exist real numbers c_{ij} such that $A = c(e_1 \wedge e_2) + \sum_{3 \leq i < j \leq n} c_{ij} e_i \wedge e_j$.

Proof. Part (a) follows from $R^N_{w_1,w_2} = -2(J^N + e_1 \wedge e_2)$ and $R^N_{w_1,w_3} = R^N_{w_2,w_3} = 0$. Clearly, $W^\perp = \mathbb{R}(e_2 - \mathrm{i}e_1) \oplus \{e_1, e_2\}^\perp_{\mathbb{C}}$ and $\mathfrak{h}_W$ acts trivially on the first factor and by means of J^N on the second factor. This proves (b). For (c), the fact that $J^N + e_1 \wedge e_2$ leaves W is invariant reduces the problem to the case $c = 0$. If A leaves W invariant, then $Aw_1 = Ae_1 - \mathrm{i}Ae_2$ must be a linear combination of w_2 and w_3, say $Aw_1 = \lambda w_2 + \mu w_3$. It follows that

$$\mu = \langle \lambda w_2 + \mu w_3, e_1\rangle = \langle Aw_1, e_1\rangle = \langle Ae_1 - \mathrm{i}Ae_2, e_1\rangle = \langle Ae_1, e_1\rangle = 0,$$

hence $Ae_1 = \lambda e_2$ and $Ae_2 = -\lambda e_1$. Thus $Aw_3 = Ae_1 + \mathrm{i}Ae_2 = \lambda(e_2 - \mathrm{i}e_1) \in W^\perp \cap W = \{0\}$. It follows that $\lambda = 0$. This implies that $Ae_1 = Ae_2 = 0$, i.e. $A \in \mathfrak{so}(\{e_3, \ldots, e_n\}_{\mathbb{R}})$. This proves our claim. □

We immediately see:

Corollary 16. *Let W and U be of Type (ex_3) and (tr_1) defined by the data $(\mathfrak{R}, \{e_1, e_2\})$ and a unit vector u of T_pN, respectively. Then (W, U) is an orthogonal curvature invariant pair if and only if $u = \pm 1/\sqrt{2}\,(\mathrm{e}_2 - \mathrm{i}\mathrm{e}_1)$.* □

Corollary 17. *There do not exist any orthogonal curvature invariant pairs of Type* (ex_3, c_k), (ex_3, c_k'), (ex_3, tr_k'), $(ex_3, tr_{i,j})$ *or* (ex_3, ex_3).

Proof. Suppose that W is of Type (ex_3) defined by the data $(\Re, \{e_1, e_2\})$. If U is of Type (c_k) or (c_k') $(k \geq 1)$ and W is a subspace of $U^\perp$ such that $\mathfrak{h}_U(W) \subset W$, then W is necessarily a complex subspace of T_pN (cf. Lemmas 2 and 4). However, Type (ex_3) is not complex according to Lemma 6 (c). This shows that (W, U) is not an orthogonal curvature invariant pair.

Further, if U is of Type (tr_k') $(k \geq 2)$ or $(tr_{i,j})$ $(i + j \geq 2)$, then U does neither contain a complex subspace of T_pN nor U is one-dimensional. The result follows from Lemma 6 (b).

Suppose that U is of Type (ex_3), too, defined by $(\Re^*, \{f_1, f_2\})$, and, by contradiction, that (W, U) is a curvature invariant pair. Then U is defined also by $(e^{\mathrm{i}\varphi}\Re^*, \{f_1(\varphi), f_2(\varphi)\})$ with $f_1(\varphi) := e^{\mathrm{i}\varphi}(\cos(\varphi)f_1 + \sin(\varphi)f_2)$ and $f_2(\varphi) := e^{\mathrm{i}\varphi}(-\sin(\varphi)f_1 + \cos(\varphi)f_2)$. Hence we can assume that $\Re = \Re^*$.

By means of Lemma 6 (b) and since $\dim(U) = 3$, there exists the orthogonal decomposition $U = \mathbb{R}(e_2 - \mathrm{i}e_1) \oplus \tilde{U}$ where $\tilde{U} \subset \mathbb{C}\{e_1, e_2\}^\perp$ is complex 1-dimensional. Further, an orthogonal decomposition $U = U^\# \oplus \tilde{U}$ of a 3-dimensional subspace $U \subset T_pN$ into a real 1-dimensional space $U^\#$ and a complex 1-dimensional space $\tilde{U}$ is unique (if it exists). Another such decomposition is given by $\tilde{U}^\sharp := \mathbb{R}(\mathrm{i}f_2 + f_1)$ and $\tilde{U} := \{f_2 + \mathrm{i}f_1, \mathrm{i}f_2 - f_1\}_{\mathbb{R}}$. Thus, on the one hand, we conclude that $\{f_1, f_2\} \subset \{e_1, e_2\}^\perp$. On the other hand, $\mathrm{i}f_2 + f_1 = \pm e_2 - \mathrm{i}e_1$, a contradiction. □

Consider Type (ex_2):

Lemma 7. *Let* $\Re \in \mathcal{U}$ *and an orthonormal basis* $\{e_1, \ldots, e_n\}$ *of* $\Re$ *be given such that* W *is spanned by* $w_1 := 2e_1 + \mathrm{i}e_2$ *and* $w_2 := e_2 + \mathrm{i}(e_1 + \sqrt{3}\, e_3)$.

(a) The curvature endomorphism $R_{1,2} := R^N_{w_1, w_2}$ *is given by* $-J^N - e_1 \wedge e_2 - \sqrt{3}\, e_2 \wedge e_3$.

(b) A subspace U *of* $W^\perp$ *is invariant under* $R_{1,2}$ *if and only if* U *is either the complex space* $\mathbb{C}(-e_1 + \sqrt{3}\, e_3 + \mathrm{i}2e_2)$, *belongs to a distinguished family of (real) 2-dimensional subspaces of the linear space*

$$\{2e_2 + \mathrm{i}(-3e_1 + 1/\sqrt{3}\, e_3), e_1 + 5/\sqrt{3}\, e_3 - 2\mathrm{i}e_2\}_{\mathbb{R}} \oplus \{e_4, \ldots, e_n\}_{\mathbb{C}}, \tag{44}$$

or is a sum of such spaces.

(c) Let $A \in \mathfrak{so}(\Re)$ *and* $c \in \mathbb{R}$. *Then* $cJ^N + A$ *leaves the subspace* W *invariant if and only if there exist real numbers* c_{ij} *such that* $A = c(e_1 \wedge e_2 + \sqrt{3}\, e_2 \wedge e_3) + \sum_{4 \leq i < j \leq n} c_{ij} e_i \wedge e_j$.

Proof. Part (a) is straightforward. For (b), we first verify that the eigenvalues of $A := R^N_{w_1, w_2}$ (seen as a complex-linear endomorphism of T_pN) are given by $\{\mathrm{i}, -\mathrm{i}, -3\mathrm{i}\}$. The complex eigenspace for the eigenvalue $-3\mathrm{i}$ is a subspace of $W^\perp$, given by $\mathbb{C}(-e_1 + \sqrt{3}\, e_3 + \mathrm{i}2e_2)$. Furthermore, we have $A^2 = -\mathrm{Id}$ on the $(2n - 4)$-dimensional subspace of T_pN which is given by (44), i.e. A defines a second complex structure on (44). This proves (b).

For (c): Since W is curvature invariant, the endomorphism $J^N + e_1 \wedge e_2 + \sqrt{3}\, e_2 \wedge e_3$ leaves W invariant. This reduces the problem to $c = 0$. If $A(W) \subset W$, then $Aw_1 = dw_2$ and $Aw_2 = -dw_1$ for some $d \in \mathbb{R}$ (since A is skew-symmetric and the length of w_1 is given by $\sqrt{5}$ which is also the length of w_2). Considering the action of A on the real parts of w_1 and w_2, this implies that $2Ae_1 = de_2$, $Ae_2 = -2de_1$. Taking also the imaginary parts into account, we see that $Ae_2 = d(e_1 + \sqrt{3}\, e_3)$, which gives a contradiction unless $d = 0$. Then $Ae_1 = Ae_2 = 0$ and hence $Aw_2 = \mathrm{i}\sqrt{3}\, Ae_3$ must be a multiple of w_1, which is not possible unless also $Ae_3 = 0$, i.e. $A|_{\{e_1, e, e_3\}_{\mathbb{R}}} = 0$. This implies the result. □

Corollary 18. *Suppose that* W *is of Type* (ex_2). *Then there are no orthogonal curvature invariant pairs* (W, U) *at all.*

Proof. Suppose that $\Re \in \mathcal{U}$, $\{e_1, e_2, e_3\}$ is an orthogonal system of $\Re$ and W is spanned by $w_1 := 2e_1 + \mathrm{i}e_2$ and $w_2 := e_2 + \mathrm{i}(e_1 + \sqrt{3}\, e_3)$. Suppose furthermore, by contradiction, that there exists some curvature invariant subspace U of T_pN such that (W, U) is an orthogonal curvature invariant pair.

If U is of Type (c_k), (c'_k) or (ex_3), then W is a 2-dimensional $\mathfrak{h}_U$-invariant subspace of $W^\perp$ but not a complex subspace of T_pN according to Lemma 6 (c). However, this is not possible, because of Part (b) of Lemmas 2, 4 and 6.

Now suppose that U is of Type $(tr_{i,j})$ determined by the data $(\Re^*, U_1, U_2)$. Using Lemmas 3 (c) and 7 (a), we see that $\mathfrak{h}_W(U) \subset U$ does not hold.

Similarly, the case that U is of Type (tr_1) can not occur.

Suppose that U is of Type (tr'_k) $(k \geq 2)$ determined by $(\Re^*, U', I', U'_0)$. Then we can assume that $\Re = \Re^*$. Using Lemma 5 (b), the fact that W is 2-dimensional linear subspace of T_pN which is invariant under $\mathfrak{h}_U$ implies that either $W \subset \mathbb{C}U'^\perp$ or W is a two-dimensional $\mathfrak{h}_U$-invariant subspace of $\mathbb{C}U'$.

In the first case, $\langle \Re(w_i), u\rangle = \langle \Im(w_i), u\rangle = 0$ for all $u \in U'$ and $i = 1, 2$. With $i = 1$, it follows that $\langle e_1, u\rangle = \langle e_2, u\rangle = 0$, then the previous with $i = 2$ implies that also $\langle e_3, u\rangle = 0$ for all $u \in U'$. Thus Lemma 7 (a) and the fact that $\mathfrak{h}_W(U) \subset U$ show that U is a complex subspace of T_pN, a contradiction.

In the second case, we have $\dim(U) = \dim(U'_0) = 2$ and both $\Re(w_i)$ and $\Im(w_i)$ belong to U' for $i = 1, 2$. Thus we conclude that $\{e_1, e_2, e_3\} \subset U'$. Hence we can extend this orthogonal system to an orthonormal basis $\{e_1, \ldots e_n\}$ of $\Re$ such that $e_4 \in U'$. Let $\{u_1, u_2\}$ be an orthonormal basis of U'_0. According to Lemma 5, the curvature endomorphism $R^N_{u_1 - \mathrm{i}I'u_1, u_2 - \mathrm{i}I'u_2}$ is given by $A := -u_1 \wedge u_2 - I'u_1 \wedge I'u_2$. Hence, since (W, U) is a curvature invariant pair, we obtain that $A(W) \subset W$. Using Lemma 7 (c) (with $c = 0$), this fact implies that there exist real numbers c_{ij} such that $u_1 \wedge u_2 + I'u_1 \wedge I'u_2 = \sum_{4 \leq i < j \leq n} c_{ij} e_i \wedge e_j$. On the other hand, there exist real numbers d_{ij} with $u_1 \wedge u_2 + I'u_1 \wedge I'u_2 = \sum_{1 \leq i < j \leq 4} d_{ij} e_i \wedge e_j$. Since $\{e_i \wedge e_j\}_{i<j}$ is a linear independent subset of $\mathfrak{so}(\Re)$, we obtain that $c_{ij} = d_{ij} = 0$, hence $u_1 \wedge u_2 + I'u_1 \wedge I'u_2 = 0$ which is not possible.

Consider the case that U is of Type (ex_2), too. Then there exists some $\Re^* \in \mathcal{U}$ and an orthonormal system $\{f_1, f_2, f_3\}$ of $\Re^*$ such that U is spanned by $u_1 := 2f_1 + \mathrm{i}f_2$ and $u_2 := f_2 + \mathrm{i}(f_1 + \sqrt{3}\, f_3)$. Let φ be chosen such that $e^{\mathrm{i}\varphi}\Re^* = \Re$.

In accordance with Lemma 7, the curvature endomorphism $R_{1,2} := R^N_{u_1, u_2}$ is given by $-J^N + A$ with $A := -f_1 \wedge f_2 - \sqrt{3}\, f_2 \wedge f_3 \in \mathfrak{so}(\Re^*) = \mathfrak{so}(\Re)$. We decompose $f_i = f_i^\top \oplus f_i^\perp$ with respect to the splitting $\mathcal{R}' = e^{-\mathrm{i}\varphi}\{e_1, e_2, e_3\}_{\mathbb{R}} \oplus e^{-\mathrm{i}\varphi}\{e_4, \ldots, e_n\}_{\mathbb{R}}$. Since $R_{1,2}(W) \subset W$, Lemma 7 (c) (with $c = -1$) implies that

$$e_1 \wedge e_2 + \sqrt{3}\, e_2 \wedge e_3 = f_1^\top \wedge f_2^\top + \sqrt{3}\, f_2^\top \wedge f_3^\top.$$

Comparing the length of the tensors on the left and right hand side above, we see that

$$\langle f_1^\top, f_1^\top\rangle = \langle f_2^\top, f_2^\top\rangle = \langle f_3^\top, f_3^\top\rangle = 1,$$

i.e. $e^{\mathrm{i}\varphi} f_i \in \{e_1, e_2, e_3\}_{\mathbb{R}}$ for $i = 1, 2, 3$. Hence we can assume that $n = 3$. Since spaces of Type (ex_2) are not complex, it follows from Lemma 7 (c) that U is spanned also by $\tilde{u}_1 := 2e_2 + \mathrm{i}(-3e_1 + 1/\sqrt{3}\, e_3)$ and $\tilde{u}_2 := e_1 + 5/\sqrt{3}\, e_3 - 2\mathrm{i}e_2$. A short calculation shows that $\tilde{R}_{1,2} := R^N_{\tilde{u}_1, \tilde{u}_2}$ is given by $8/3\, J^N - 4(e_1 \wedge e_2 + \sqrt{3}\, e_2 \wedge e_3)$. Thus we obtain that $\tilde{R}_{1,2}$ does not leave W invariant. Hence (W, U) is not a curvature invariant pair. □

3.2 Integrability of the curvature invariant pairs of $\mathrm{G}_2^+(\mathbb{R}^{n+2})$

Let (W, U) be an orthogonal curvature invariant pair of $\mathrm{G}_2^+(\mathbb{R}^{n+2})$ and set $V := W \oplus U$. We will assume that $\dim(W) \geq 2$. It remains the question whether (W, U) is integrable. By means of a case by case analysis, we will show that the answer is “no” unless V is curvature invariant.

Let $\mathfrak{k}$ denote the isotropy Lie algebra of $N := \mathrm{G}_2^+(\mathbb{R}^{n+2})$ and $\rho : \mathfrak{k} \to \mathfrak{so}(T_pN)$ be the linearized isotropy representation. Recall that $\rho(\mathfrak{k}) = \mathbb{R}J^N \oplus \mathfrak{so}(\Re)$. Further, by definition, the Lie algebra $\mathfrak{k}_V$ is the maximal subalgebra of $\mathfrak{k}$ such that $\rho(\mathfrak{k}_V)|_V$ is a subalgebra of $\mathfrak{so}(V)$, see (18). Recall also the definition of the Lie algebra $\mathfrak{h} \subset \mathfrak{so}(V)$, see (21).

Type $(\mathbf{c_i}, \mathbf{c_j})$ Suppose that W and U are of Type (c_i) and (c_j) defined by the data $(\Re, W_0)$ and $(\Re^*, U_0)$, respectively, with $\Re = \Re^*$ and $W_0 \perp U_0$. Hence V is curvature invariant of Type (c_{i+j}) defined by the data $(\Re, W_0 \oplus U_0)$.

Type $(\mathbf{tr_{i,j}}, \mathbf{tr_{k,l}})$ Let W and U be of Type $(tr_{i,j})$ and $(tr_{k,l})$ defined by the data $(\Re, W_1, W_2)$ and $(\Re^*, U_1, U_2)$, respectively. Let φ be chosen such that $\Re = e^{\mathrm{i}\varphi}\Re^*$. Substituting, if necessary, $\mathrm{i}\Re^*$ for $\Re^*$, we can assume that $\varphi \in [-\pi/4, \pi/4]$.

- Case $i = j = k = l = 1$: let v_1, v_2, u_1 and u_2 be unit vectors in W_1, W_2, U_1 and U_2, respectively. Here we have $\langle v_1, v_2\rangle = \langle u_1, u_2\rangle = \langle u_1, v_1\rangle = \langle u_2, v_2\rangle = 0$. Further, suppose that (W, U) is integrable. Since W and U both are two-dimensional curvature isotropic subspaces of T_pN, Proposition 4 implies the existence of an orthonormal basis $\{e_1, e_2\}$ of W such that $\{\eta_i := h(e_i, e_i)\}_{i=1,2}$ is an orthogonal basis of U. Further, $R^N_{e_1,\eta_2} = 0 = R^N_{e_2,\eta_1}$, see (29). Thus, there exist real numbers q, r, s, t with $q^2 + r^2 \neq 0$ and $s^2 + t^2 \neq 0$ such that $e_1 = qv_1 + r\mathrm{i}v_2$ and $\eta_2 = su_1 + t\mathrm{i}u_2$. Hence,

$$\Re(e_1) = qv_1, \tag{45}$$
$$\Im(e_1) = rv_2, \tag{46}$$
$$\Re(\eta_2) = s\cos(\varphi)e^{\mathrm{i}\varphi}u_1 + t\sin(\varphi)e^{\mathrm{i}\varphi}u_2, \tag{47}$$
$$\Im(\eta_2) = -s\sin(\varphi)e^{\mathrm{i}\varphi}u_1 + t\cos(\varphi)e^{\mathrm{i}\varphi}u_2. \tag{48}$$

  Further, Eq. 33 shows that

$$\langle \Re(e_1), \Im(\eta_2)\rangle = \langle \Re(\eta_2), \Im(e_1)\rangle, \tag{49}$$
$$\Im(e_1) \wedge \Im(\eta_2) = -\Re(e_1) \wedge \Re(\eta_2), \tag{50}$$
$$\langle \Re(e_2), \Im(\eta_1)\rangle = \langle \Re(\eta_1), \Im(e_2)\rangle, \tag{51}$$
$$\Im(e_2) \wedge \Im(\eta_1) = -\Re(e_2) \wedge \Re(\eta_1), \tag{52}$$

  where real and imaginary parts may be taken with respect to $\Re$. We claim that the case $\varphi \neq 0$ can not occur: Otherwise, we have $\varphi \neq 0 \mod \pi/2$, hence Corollary 7 shows that $\{v_1, v_2, e^{\mathrm{i}\varphi}u_1, e^{\mathrm{i}\varphi}u_2\}$ is an orthonormal basis of $\Re$. Thus, $\{\Re(\eta_2), \Im(\eta_2)\} \subset \{e^{\mathrm{i}\varphi}u_1, e^{\mathrm{i}\varphi}u_2\}_{\mathbb{R}}$ according to (47), (48) and neither $\Im(\eta_2)$ nor $\Re(\eta_2)$ vanishes (since otherwise $t\sin(\varphi) = 0 = s\cos(\varphi)$ or $s\sin(\varphi) = 0 = t\cos(\varphi)$, respectively, i.e. $s = t = 0$ anyway which is not possible). Therefore, if, say, $\Re(e_1)$ does not vanish, then r.h.s. of (50) is non-zero, i.e. $\{\Re(e_1), \Re(\eta_2)\}_{\mathbb{R}} = \{\Im(e_1), \Im(\eta_2)\}_{\mathbb{R}}$ is a two-dimensional space in contradiction to (45)-(48).

  Hence, $\varphi = 0$. Let us assume that $\Re(e_1) = 0$. Then $e_1 = \pm\mathrm{i}v_2$, $e_2 = \pm v_1$ and r.h.s. of (50) vanishes. It follows that

$$0 = \langle e_1, \eta_2\rangle = \langle \Im(e_1), \Im(\eta_2)\rangle.$$

  Since l.h.s. of (50) vanishes, too, we thus we obtain that $\Im(\eta_2) = 0$, i.e. there exist $\lambda, \mu \neq 0$ with $\eta_2 = \mu u_1$ and $\eta_1 = \lambda\mathrm{i}u_2$. Therefore, $\langle v_2, u_1\rangle = 0 = \langle v_1, u_2\rangle$ according to (49),(51). Thus V is curvature invariant of Type $(tr_{2,2})$ defined by $(\mathcal{R}, \{v_1, u_1\}_{\mathbb{R}}, \{v_2, u_2\}_{\mathbb{R}})$. Similarly, if $\Im(e_1) = 0$, then V is curvature invariant of Type $(tr_{2,2})$, too.

  If neither $\Re(e_1) = 0$ nor $\Im(e_1) = 0$, and, say, $\Re(\eta_2) \neq 0$, then there exist $\lambda, \mu \neq 0$ with $\Re(\eta_2) = \lambda u_1$ and $\Re(e_1) = \mu v_1$. Hence r.h.s. of (50) is given by $\lambda\mu\, u_1 \wedge v_1$ which is non-zero by the condition $\langle u_1, v_1\rangle = 0$. Using (50), we obtain that $\Im(\eta_2) \neq 0$ and $\{v_1, u_1\}_{\mathbb{R}} = \{v_2, u_2\}_{\mathbb{R}}$, hence $\{v_1, v_2, u_1, u_2\}_{\mathbb{R}} = \{v_1, v_2\}_{\mathbb{R}} = \{u_1, u_2\}_{\mathbb{R}}$. The conditions $\langle u_1, v_1\rangle = \langle u_2, v_2\rangle = 0$ imply that $u_2 = \pm v_1$ and $u_1 = \pm v_2$. In this case, $V := W \oplus U$ is curvature invariant of Type (c_2) defined by $(\Re, \{v_1, v_2\}_{\mathbb{R}})$. The possibility $\Im(\eta_2) \neq 0$ leads to the same conclusion. □

- Case $\Re = \Re^*$, $j = k \geq 2$, $W_2 = U_1$: then $W_1 \perp U_2$ or $W_1 = U_2$ unless $i = l = 1$. In case $W_1 = U_2$, the linear space V is curvature invariant of Type c_{k+l} defined by $(\Re, U_1 \oplus W_1)$. Otherwise, we claim that $\rho(\mathfrak{k}_V)|_V \cap \mathfrak{so}(V)_- = \{0\}$:

  Let $c \in \mathbb{R}$, $B \in \mathfrak{so}(\Re)$, set $A := cJ^N + B$ and suppose that $A(V) \subset V$ and $A|_V \in \mathfrak{so}(V)_-$ holds. Then $A(W) \subset U$ and $A(U) \subset W$. We aim to show that $A = 0$. Let $v \in W_1$. Thus $Av \in U$. It follows that $c\mathrm{i}v \in \mathrm{i}U_2$. In the same way, $c\mathrm{i}u \in \mathrm{i}W_1$ for all $u \in U_2$. Hence $c = 0$ or $W_1 = U_2$ which is different case. Hence $A(U_1) = B(U_1) \subset W \cap \Re = W_1$ and $A(W_1) \subset U_1$. In other words, setting $V_1 := W_1 \oplus U_1$, we have $A|_{V_1} \in \mathfrak{so}(V_1)_-$.

For the same reason, with $V_2 := W_2 \oplus U_2$, we have $A|_{V_2} \in \mathfrak{so}(V_2)_-$. Since 24 is a linear isomorphism, for the vanishing of A it suffices to show that $A|_{W_2} = 0$ and $A|_{U_1} = 0$:

On the one hand, $A(W_2) = A(U_1) \subset W_1$. On the other hand, $A(W_2) \subset U_2$. Hence $A(W_2) \subset W_1 \cap U_2$. Further, the linear space $W_1 \cap U_2$ is trivial if $W_1 \perp U_2$ or if $i = l = 1$ and $W_1 \neq U_2$. Therefore, $A|_{W_2} = 0$ unless $W_1 = U_2$. Similar considerations show that also $A|_{U_1} = 0$ unless $W_1 = U_2$. This establishes our claim.

Assume, by contradiction, that (W, U) is integrable but $W_1 \neq U_2$. Then $\rho(\mathfrak{k}_V)|_V \cap \mathfrak{so}(V)_- = \{0\}$ as was shown above. Thus there exists a symmetric bilinear map $h : W \times W \to U$ satisfying (26),(27) according to Corollary 4. Note, the Lie algebra $\mathfrak{h}$ (21) is given by $\mathfrak{so}(W_1) \oplus \mathfrak{so}(W_2) \oplus \mathfrak{so}(U_2)$ such that $\mathfrak{so}(W_1) \oplus \mathfrak{so}(U_2)$ acts as a direct sum representation on $W_1 \oplus \mathrm{i}U_2$ whereas $\mathfrak{so}(W_2)$ acts diagonally on $\mathrm{i}W_2 \oplus U_1$ (i.e. $A(\mathrm{i}v + u) = \mathrm{i}Av + Au$ for all $A \in \mathfrak{so}(W_2)$ and $(v, u) \in W_2 \times U_1$). Schur's Lemma implies that $\mathrm{Hom}_{\mathfrak{h}}(W, U) \subset \mathbb{R}J^N|_{\mathrm{i}W_2} \oplus \mathrm{Hom}(W_1, \mathrm{i}U_2)$ (where the second summand is non-trivial only if $i = l = 1$). In particular, there exists a linear function $\lambda : W \to \mathbb{R}$ such that $h(x, \mathrm{i}y) = \lambda_x y$ for all $x \in W$ and $y \in W_2$. Further, we have $h(x, y) \in \mathrm{i}U_2$ for all $x \in W$ and $y \in W_1$. Thus, we conclude by the symmetry of h that

$$h(\mathrm{i}x, \mathrm{i}y) = \lambda_x y = h(\mathrm{i}y, \mathrm{i}x) = \lambda_y x$$

for all $x, y \in W_2$. Since $\dim(W_2) = j \geq 2$, we hence see that $\lambda_x = 0$ for all $x \in \mathrm{i}W_2$, i.e. $h|_{\mathrm{i}W_2 \times \mathrm{i}W_2} = 0$. Furthermore, we conclude that

$$h(W_1 \times \mathrm{i}W_2) = h(\mathrm{i}W_2 \times W_1) \in U_1 \cap \mathrm{i}U_2 = \{0\}.$$

Therefore, $h(W \times W) = h(W_1 \times W_1) \subset \mathrm{i}U_2$, a contradiction, since the image of h spans U and $U_1 \neq \{0\}$.

The case $i = l \geq 2$, $W_2 = U_1$ also follows (by means of passing from $\Re$ to $\mathrm{i}\Re$). $\square$

- In the remaining cases, we have $W_1 \perp e^{\mathrm{i}\varphi} U_2$ or $i = l = 1$ and, also $W_2 \perp e^{\mathrm{i}\varphi} U_1$ or $j = k = 1$. Furthermore, at least one of the indices $\{i, j, k, l\}$ is (strictly) greater than 1. In case $\varphi = 0$, $W_1 \perp U_2$ and $W_2 \perp U_1$, we obtain that V is curvature invariant of Type $tr_{i+k,j+l}$ defined by $(\Re, W_1 \oplus U_1, W_2 \oplus U_2)$. Otherwise, we claim that $\rho(\mathfrak{k}_V)|_V \cap \mathfrak{so}(V)_- = \{0\}$:

Let $c \in \mathbb{R}$, $B \in \mathfrak{so}(\Re)$ be given such that $A := cJ^N + B$ satisfies $A(V) \subset V$ and $A|_V \in \mathfrak{so}(V)_-$. Thus $A(W) \subset U$ and $A(U) \subset W$, hence $W_1 \perp e^{\mathrm{i}\varphi} U_2$ and $W_2 \perp e^{\mathrm{i}\varphi} U_1$. Further, substituting, if necessary, $\mathrm{i}\Re$ for $\Re$, we can assume that $i \geq 2$. Let $v \in W_1$ be a unit vector. Then $Av \in U$, i.e. $Av = u_1 + \mathrm{i}u_2$ for suitable $u_1 \in U_1$ and $u_2 \in U_2$. Since

$$e^{\mathrm{i}\varphi}(Av) = e^{\mathrm{i}\varphi}(c\mathrm{i}v + Bv) = c\mathrm{i}\cos(\varphi)v - c\sin(\varphi)v + \cos(\varphi)Bv + \mathrm{i}\sin(\varphi)Bv,$$

we see that

$$e^{\mathrm{i}\varphi}u_1 = \Re(e^{\mathrm{i}\varphi}(Av)) = -c\sin(\varphi)v + \cos(\varphi)Bv$$

and

$$e^{\mathrm{i}\varphi}u_2 = \Im(e^{\mathrm{i}\varphi}(Av)) = c\cos(\varphi)v + \sin(\varphi)Bv.$$

The condition $W_1 \perp e^{\mathrm{i}\varphi} U_2$ implies that

$$0 = \langle v, e^{\mathrm{i}\varphi}u_2 \rangle = c\cos(\varphi),$$

thus $c = 0$ since $\varphi \in [-\pi/4, \pi/4]$. Therefore, $A \in \mathfrak{so}(\Re)$ anyway, in particular, $Av \in \Re$ for all $v \in W_1$.

Therefore, we have $\Re(e^{\mathrm{i}\varphi}Av) = \cos(\varphi)Av = e^{\mathrm{i}\varphi}u_1$ and $\Im(e^{\mathrm{i}\varphi}Av) = \sin(\varphi)Av = e^{\mathrm{i}\varphi}u_2$. We conclude that

$$0 = \langle u_1, u_2 \rangle = \langle e^{\mathrm{i}\varphi}u_1, e^{\mathrm{i}\varphi}u_2 \rangle = \sin(\varphi)\cos(\varphi)\langle Av, Av \rangle,$$

hence $\varphi = 0$ or $Av = 0$ for all $v \in W_1$. In the same way, we can show that $\varphi = 0$ or $Av = 0$ for all $v \in W_2$. By means of Eq. 24, we conclude that $A|_V = 0$ unless $\varphi = 0$. Suppose now that $\varphi = 0$. Set $V_1 := W_1 \oplus U_1$ and $V_2 := W_2 \oplus U_2$. Note that $A|_{V_1} \in \mathfrak{so}(V_1)_-$ and $A|_{V_2} \in \mathfrak{so}(V_2)_-$. Assume that, say, the condition $W_1 \perp U_2$ fails. Then we necessarily have $i = l = 1$ and $W_2 \perp U_1$ holds. In particular, there exists $v_1 \in W_1$ such that the linear form $\langle v_1, \cdot \rangle$ defines an isomorphism $U_2 \to \mathbb{R}$. Then,

$$\langle Av, v_1 \rangle = -\langle v, Av_1 \rangle = 0.$$

for all $v \in W_2$. We conclude that $A|_{W_2} = 0$ and hence $A|_{V_2} = 0$ since (24) is a linear isomorphism. For the same reason, $A|_{U_1} = 0$ and hence $A|_{V_1} = 0$. We conclude that $A|_V = 0$. This establishes our claim.

Assume, by contradiction, that (W, U) is integrable but at least one of the conditions $\varphi = 0$, $W_1 \perp U_2$ or $W_2 \perp U_1$ fails. Then $\rho(\mathfrak{k}_V)|_V \cap \mathfrak{so}(V)_- = \{0\}$. Thus, there exists a symmetric bilinear map $h : W \times W \to U$ satisfying (26),(27). Note, the Lie algebra $\mathfrak{h}$ from (21) is given by $\mathfrak{so}(W_1) \oplus \mathfrak{so}(W_2) \oplus \mathfrak{so}(U_1) \oplus \mathfrak{so}(U_2)$ acting as a direct sum representation on $W_1 \oplus \mathrm{i}W_2 \oplus U_1 \oplus \mathrm{i}U_2$ (where one or more summands may be trivial).

First, assume that $j \geq 2$. Hence, by means of Schur's Lemma, $\mathrm{Hom}_{\mathfrak{h}}(W, U) \subset \mathrm{Hom}(W_1, U)$. If $i \neq 1$, then we even have $\mathrm{Hom}_{\mathfrak{h}}(W, U) = \{0\}$. Otherwise, if $i = 1$, we thus see that $h(x, y) = h(y, x) = 0$ for all $x \in \mathrm{i}W_2$ and $y \in W$, i.e. $h(W \times W) = h(W_1 \times W_1)$ which spans a one-dimensional space, a contradiction (since $\dim(U) \geq 2$). The case $i \geq 2$ is handled by the same arguments. Note, the case $i = j = 1$ can actually not occur (since then $k = l = 1$ because of Proposition 4). □

Type $(\mathbf{tr_{i,j}}, \mathbf{tr_1})$ Suppose that W is defined by the data $(\Re, W_1, W_2)$ and U is one-dimensional spanned by a unit-vector u.

- Case $i = j = 1$: Let e_1 and e_2 be unit vectors of $\Re$ which generate W_1 and W_2, respectively. Then $W = \mathbb{R}e_1 + \mathrm{i}\mathbb{R}e_2$ is a curvature isotropic subspace of T_pN. Suppose that (W, U) is integrable. According to Proposition 4, there exists some unit vector $v \in W$ such that $R^N_{v,u} = 0$. Then $\tilde{W} := \{v, u\}_{\mathbb{R}}$ is a curvature isotropic subspace of T_pN, too. It follows from Theorem 5 that there exists some $\Re^* \in \mathcal{U}$ and an orthonormal system $\{e_1', e_2'\}$ of $\Re^*$ with $\tilde{W} = \{e_1', \mathrm{i}e_2'\}_{\mathbb{R}}$. In particular, $\langle \Re^*(v), \Im^*(v) \rangle = 0$, where $\Re^*$ and $\Im*$ denote taking the real and imaginary parts with respect to $\Re^*$. Further, let φ be chosen such that $e^{\mathrm{i}\varphi}\Re^* = \Re$. Then we have $\Re(e^{\mathrm{i}\varphi}v) = e^{\mathrm{i}\varphi}\Re^*(v)$ and $\Im(e^{\mathrm{i}\varphi}v) = e^{\mathrm{i}\varphi}\Im^*(v)$. Hence, on the one hand, $\langle \Re(e^{\mathrm{i}\varphi}v), \Im(e^{\mathrm{i}\varphi}v) \rangle = \langle \Re^*(v), \Im^*(v) \rangle = 0$. On the other hand, since $v \in W$, we may write $v = \cos\phi e_1 + \sin\phi \mathrm{i}e_2$. This gives

$$\cos(\varphi)\sin(\varphi)(\cos^2\phi - \sin^2\phi) = 0, \tag{53}$$

  i.e. $\phi = 0 \mod \pi/4\,\mathbb{Z}$ or $\varphi = 0 \mod \pi/2\,\mathbb{Z}$.

  In the first case, we can assume that $\phi = \pi/4$, i.e. $v = 1/\sqrt{2}(e_1 + \mathrm{i}e_2)$. It remains to determine the possibilities for u. First, we have $u \in W^\perp$, i.e.

$$\langle \Re(u), e_1 \rangle = \langle \Im(u), e_2 \rangle = 0. \tag{54}$$

  Hence, we may write $\Re(u) = \lambda_1 e_2 + u_1$ and $\Im(u) = \lambda_2 e_1 + u_2$ with $u_1, u_2 \in \{e_1, e_2\}^\perp$. Second, using (33), we obtain from $R^N_{v,u} = 0$ that

$$\langle \Re(v), \Im(u) \rangle = \langle \Re(u), \Im(v) \rangle, \tag{55}$$

$$\Re(v) \wedge \Re(u) = \Im(v) \wedge \Im(u). \tag{56}$$

  It follows that $\lambda_1 = \lambda_2$ and $\lambda_1 e_1 \wedge e_2 + e_1 \wedge u_1 = -\lambda_1 e_2 \wedge e_1 - e_2 \wedge u_2$, i.e. $u_1 = u_2 = 0$. We obtain that $u = 1/\sqrt{2}(e_2 + \mathrm{i}e_1)$. Therefore, the linear space $W \oplus \mathbb{R}u$ is curvature invariant of Type (ex_3), determined by the data $(\Re, \{e_1, e_2\})$.

  In the second case, we can assume that $\varphi = 0$, i.e. $\Re = \Re^*$. Then $\{e_1', e_2'\}$ is a second orthonormal system of $\Re$, and there exists a linear combination $u = \lambda_1 e_1' + \lambda_2 \mathrm{i}e_2'$ with $\lambda_1^2 + \lambda_2^2 = 1$. If, say, $\lambda_2 \neq 0$, then the condition $0 = \langle u, \mathrm{i}e_2 \rangle$ implies that $\langle e_2', e_2 \rangle = 0$. Further, note that $W \cap \tilde{W} = \mathbb{R}v$. Therefore, there exist $\mu_1, \mu_1', \mu_2, \mu_2' \in \mathbb{R}$ such that $v = \mu_1 e_1 + \mathrm{i}\mu_2 e_2 = \mu_1' e_1' + \mathrm{i}\mu_2' e_2'$. Hence $\Im(v) = \mu_2 e_2 = \mu_2' e_2'$ which together with $\langle e_2', e_2 \rangle = 0$ implies that $\mu_2 = \mu_2' = 0$, i.e. $v = \pm e_1$ and $e_1 = \pm e_1'$. Thus the condition $u \perp W$ shows that $u = \pm \mathrm{i}e_2'$. Therefore, $W \oplus \mathbb{R}u$ is curvature invariant of Type $(tr_{1,2})$, determined by $(\Re, \mathbb{R}e_1, \{e_2, e_2'\}_{\mathbb{R}})$. □

- Case $i = 1$, $j \geq 2$: here we have $\Re(u) \in W_1^\perp$ and $u \in \mathbb{C}W_2^\perp$ (where $W_2^\perp$ denotes the orthogonal complement of W_2 in $\Re$). Further, if $\Im(u) = 0$, or if $\Re(u) = 0$ and $u \perp \mathrm{i}W_1$, then $W \oplus U$ is curvature invariant of Type $(tr_{2,j})$ or $(tr_{1,j+1})$ defined by the triple $(\Re, W_1 \oplus U, W_2)$ or $(\Re, W_1, W_2 \oplus U)$, respectively. Otherwise, we claim that the linear space $\rho(\mathfrak{k}_V)|_V \cap \mathfrak{so}(V)_-$ is trivial:

  Let $c \in \mathbb{R}$ and $B \in \mathfrak{so}(\Re)$ be given and suppose that $A := cJ^N \oplus B$ satisfies $A(V) \subset V$ and $A|_V \in \mathfrak{so}(V)_-$. Let us write $u = u_1 + \mathrm{i}u_2$ with $u_1, u_2 \in \Re$. Further, let an orthogonal pair of unit-vector $v_1'', v_2'' \in W_2$ be given (such a pair exists since $j \geq 2$). Then there exist $\lambda_i \in \mathbb{R}$ such that $A\mathrm{i}v_i'' = -cv_i'' + \mathrm{i}Bv_i'' = \lambda_i u$. Comparing the real parts of the last equation, we obtain that $-cv_i'' = \lambda_i u_1$ for $i = 1, 2$, i.e. $c = 0$ (since $v_1'' \sim v_2''$ is not given).

Suppose, by contradiction, that $B|_{W_1} \neq 0$. Let v_1 be a unit vector of W_1. Then $Bv_1 = \lambda u$ for some $\lambda \neq 0$ and hence $u_2 = 0$, which is a case not in consideration. Hence we can assume that $B|_{W_1} = 0$. Suppose now, by contradiction, that $B|_{W_2} \neq 0$. Let v_2 be a unit vector of W_2 such that $Bv_2 \neq 0$. Then $u_1 = 0$ and there exists $\lambda \neq 0$ such that $Bv_2 = \lambda u_2$. Hence, using that $B|_{W_1} = 0$,

$$0 = \langle v_2, Bv_1\rangle = -\langle Bv_2, v_1\rangle = -\lambda\langle u_2, v_1\rangle$$

for all $v_1 \in W_1$. Since $\lambda \neq 0$, we obtain that u_2 belongs to the orthogonal complement of W_1, i.e. we have shown that $\Re(u) = 0$ and $u \perp \mathrm{i}W_1$, which is different case. This proves our claim.

Therefore, if we assume that neither the case $\Im(u) = 0$ nor the case $\Re(u) = 0$ and $u \perp \mathrm{i}W_1$ holds but, by contradiction, that (W, U) is integrable, then there exists a symmetric bilinear map $h : W \times W \to U$ satisfying (26),(27). Note, the Lie algebra $\mathfrak{h}$ is given by $\mathfrak{so}(W_2)$ acting irreducibly on $\mathrm{i}W_2$ and trivially on both W_1 and U. Thus, $\mathrm{Hom}_{\mathfrak{h}}(W, U) = \mathrm{Hom}(W_1, U)$. We obtain that $h(W_1, \mathrm{i}W_2) = h(\mathrm{i}W_2, W_1) = h(\mathrm{i}W_2, \mathrm{i}W_2) = 0$. Let e_1 and e_2 be unit vectors in W_1 and W_2, respectively. Thus $h(e_1, \mathrm{i}e_2) = 0$ and there exists $\lambda \neq 0$ such that $h(e_1, e_1) = \lambda u$. Then the linear space $\mathbb{R}e_1 \oplus \mathrm{i}\mathbb{R}e_2$ is of Type $(tr_{1,1})$, i.e. $R^N_{e_1,\mathrm{i}e_2} = 0$. Therefore, on the one hand, Eq. 5 (with $k = 1$) in combination with Lemma 1 yields $R^N_{h(e_1,e_1),\mathrm{i}e_2} = 0$. It follows that $R^N_{\mathrm{i}e_2,u} = 0$, thus $0 = \Im(u) \wedge \Im(\mathrm{i}e_2) + \Re(u) \wedge \Re(\mathrm{i}e_2) = \Im(u) \wedge e_2$, therefore $\Im(u) \sim e_2$. On the other hand, sine $\Im(u) \in W_2^{\perp}$, this implies that $\Im(u) = 0$, which is not given. □

- $i = 0$, $j \geq 2$ or $i \geq 2$, $j = 0$. Since W is an irreducible $\mathfrak{h}_W$-module (see Lemma 5), Proposition 2 implies that here the linear space $W \oplus U$ is curvature invariant. □
- Case $i, j \geq 2$ and $u \perp \mathbb{C}(W_1 \oplus W_2)$: Then $\mathrm{Hom}_{\mathfrak{h}}(W, U) = \{0\}$. Hence, if we assume, by contradiction, that (W, U) is integrable, then Corollary 4 implies that $\rho(\mathfrak{k}_V)|_V \cap \mathfrak{so}(V)_-$ is non-trivial. Using similar calculations as in the previous cases, we see that this implies that $\Re(u) = 0$ or $\Im(u) = 0$, in which case V is curvature invariant of Type $(tr_{i,j+1})$ or $(tr_{i+1,j})$, respectively. □

Type $(\mathbf{c'_i}, \mathbf{c'_j})$ Let W and U be of Type (c'_i) and (c'_j) defined by the data $(\Re, W', I')$ and $(\Re^*, U', J')$ with $\Re = \Re^*$. If $U' = W'$ and $J' = -I'$, then $U = \bar{W}$ and $V = W \oplus \bar{W} = \mathbb{C}W'$ is curvature invariant of Type (c_i). If $W' \perp U'$, then V is curvature invariant of Type (c'_{i+j}) defined by $(\Re, W' \oplus U', I' \oplus J')$. □

Type $(\mathbf{c_i}, \mathbf{c'_j})$ Suppose that W is of Type (c_i) determined by the data $(\Re, W_0)$ and U is of Type (c'_j) determined by $(\Re^*, U', I')$. Suppose further that $\Re = \Re^*$ and $W_0 \perp U'$ holds. Let $u \in U'$ be a unit vector and consider $A := J^N + u \wedge I'u \in \mathfrak{so}(T_pN)$. We observe that $A \in \mathfrak{h}_U$ according to Lemma 4. Further, A^2 acts by the scalars -2 and -1 on U and W, respectively, hence $\mathrm{Hom}_{\mathfrak{h}}(W, U) = \{0\}$. Furthermore, we claim that $\rho(\mathfrak{k}_V)|_V \cap \mathfrak{so}(V)_- = \{0\}$:

Let $c \in \mathbb{R}$ and $B \in \mathfrak{so}(\Re)$ be given, set $A := cJ^N \oplus B$ and suppose that $A(V) \subset V$ and $A|_V \in \mathfrak{so}(V)_-$. If v is a unit vector of W_0, then $v, \mathrm{i}v \in W$ and thus

$$0 = \langle Av, \mathrm{i}v\rangle = c\langle \mathrm{i}v, \mathrm{i}v\rangle,$$

i.e. $c = 0$. Hence $A \in \mathfrak{so}(\Re)$ and Av belongs to $U \cap \Re = \{0\}$, i.e. $A\mathrm{i}v = \mathrm{i}Av = 0 = Av$ for all $v \in W_0$. Thus, $A|_V = 0$.

Therefore, Corollary 4 implies that neither (W, U) nor (U, W) is integrable. □

Type $(\mathbf{tr_{i,j}}, \mathbf{tr'_k})$ Suppose that W and U are of Type $(tr_{i,j})$ and (tr'_k) determined by the data $(\Re, W_1, W_2)$ and $(\Re^*, U', I', U'_0)$, respectively. By means of Corollary 14, we can assume that $\Re = \Re^*$ and that $U' \oplus U'_0$ belongs to the orthogonal complement of W_1 in $\Re$. We claim that the linear space $\rho(\mathfrak{k}_V)|_V \cap \mathfrak{so}(V)_-$ is trivial:

Let $c \in \mathbb{R}$ and $B \in \mathfrak{so}(\Re)$ be given, set $A := cJ^N \oplus B$ and suppose that $A(V) \subset V$ and $A|_V \in \mathfrak{so}(V)_-$ holds. If $v_1 \in W_1$, then $c\mathrm{i}v_1$ is the imaginary part of Av_1. Since $Av_1 \in U$, we see that $Av_1 = c(I'v_1 + \mathrm{i}v_1)$. In particular, $cI'v_1 \in U'_0 \subset U'$. Because $W_1 \cap U' = \{0\}$, this implies $c = 0$, i.e. A vanishes on W_1. In the same way, we can show that A vanishes on $\mathrm{i}W_2$, too. Hence, we see that $A|_V = 0$, since (24) is a linear isomorphism. This establishes our claim.

Further, according to Lemma 5, $\mathfrak{h}_U$ acts irreducibly on U and trivially on W. Hence $\mathrm{Hom}_{\mathfrak{h}}(W,U) = \{0\}$. Therefore, Corollary 4 implies that neither (W,U) nor (U,W) is integrable. $\square$

Type $(\mathbf{tr'_i}, \mathbf{tr_1})$ Suppose that (W,U) is an integrable orthogonal curvature invariant pair with W and U of Type (tr'_i) and (tr_1), respectively. Since W is an irreducible $\mathfrak{h}_W$-module (see Lemma 5), Proposition 2 shows that here the linear space $W \oplus U$ is curvature invariant. $\square$

Type $(\mathbf{tr'_k}, \mathbf{tr'_l})$ Let W and U be of Type (tr'_k) and (tr'_l) defined by the data $(\Re, W', I', W'_0)$ and $(\Re^*, U', J', U'_0)$, respectively $(k, l \geq 2)$. We can assume that $\Re = \Re^*$.

Suppose that W' is orthogonal to U'. It follows that $W \oplus U$ is curvature invariant of Type $(\mathfrak{tr}_{k+l})$ defined by $(\Re, W' \oplus U', I' \oplus J', W'_0 \oplus U'_0)$.

Suppose that $k = l \geq 3$, $U' = W'$, $U'_0 = \exp(\varphi I')(W'_0)$ and $J' = -I'$, i.e. $U = e^{-\mathrm{i}\varphi}\bar{W}$ for some $\varphi \in \mathbb{R}$. We claim that (W,U) is not integrable. In order to explain the idea of our proof, first consider the case $\varphi = 0$. Then, the linear space V is curvature invariant and the totally geodesic submanifold $\exp^N(V)$ is isometric to a product $\mathrm{S}^k \times \mathrm{S}^k$ such that $p = (o,o)$ (where o is some origin of S^k) and, moreover, the linear space W is given by $\{\, (x,x) \,|\, x \in T_o\mathrm{S}^k \,\}$. If we assume, by contradiction, that (W,U) is integrable, then the corresponding complete parallel submanifold through p would be contained in $\mathrm{S}^k \times \mathrm{S}^k$ by means of reduction of the codimension and, moreover, it would even be a symmetric submanifold of $\mathrm{S}^k \times \mathrm{S}^k$ according to Corollary 3. But this is not possible, since a symmetric submanifold $M \subset \mathrm{S}^k \times \mathrm{S}^k$ through the point $p = (o,o)$ with $T_pM = \{\, (x,x) \,|\, x \in T_o\mathrm{S}^k \,\}$ is totally geodesic according to Theorem 4. In the general case, the linear space V is not curvature invariant, but a similar idea shows that (W,U) is not integrable, as follows.

Definition 5. Let $A \in \mathfrak{so}(W')$ be given. We say that A is real, holomorphic or anti-holomorphic if $A(W'_0) \subset W'_0$, $A \circ I' = I' \circ A$ or $A \circ I' = -I' \circ A$, respectively.

Consider the linear map J_φ on $W' \oplus \mathrm{i}W'$ which is given on $W'_0 \oplus \mathrm{i}I'(W'_0)$ by $J_\varphi(v - \mathrm{i}I'v) := e^{-\mathrm{i}\varphi}(v + \mathrm{i}I'v)$ and $J_\varphi(v + \mathrm{i}I'v) := -e^{\mathrm{i}\varphi}(v - \mathrm{i}I'v)$ for all $v \in W'_0$ and which is extended to $W' \oplus \mathrm{i}W'$ by $\mathbb{C}$-linearity (note, $W'_0 \oplus \mathrm{i}I'(W'_0)$ is a real form of $W' \oplus \mathrm{i}W'$).

Lemma 8. *Let W be of Type (tr'_k) defined by the data $(\Re, W', I', W'_0)$. Set $U := e^{-\mathrm{i}\varphi}\bar{W}$ and $V := W \oplus U$.*

(a) J_φ is a Hermitian structure on $W' \oplus \mathrm{i}W'$ such that W gets mapped onto U and vice versa. In particular, V is a complex subspace of $W' \oplus \mathrm{i}W'$ and $J_\varphi|_V$ belongs to $\mathfrak{so}(V)_-$.

(b) Let $A \in \mathfrak{so}(W')$ and suppose that A is real. We extend A and I' to a complex linear maps on $W' \oplus \mathrm{i}W'$. rIf A is holomorphic, then A commutes with J_φ for all $\varphi \in \mathbb{R}$. If A is anti-holomorphic, then $\exp(\varphi I') \circ A$ anti-commutes with J_φ for all $\varphi \in \mathbb{R}$.

Proof. Let $\{v_1, \ldots, v_k\}$ be an orthonormal basis of W'_0 and set $e_i := 1/\sqrt{2}(v_i - \mathrm{i}I'v_i)$. Then $\{e_1, \ldots, e_k, \bar{e}_1, \ldots \bar{e}_k\}$ is a Hermitian basis of $W' \oplus \mathrm{i}W'$. We define unitary maps I and J on $W' \oplus \mathrm{i}W'$ via $I(e_i) = \mathrm{i}e_i$ and $I(\bar{e}_i) = -\mathrm{i}\bar{e}_i$, $J(e_i) = \bar{e}_i$ and $J(\bar{e}_i) = -e_i$. Further, set $K := I \circ J$. Since $I^2 = J^2 = -\mathrm{Id}$ and $I \circ J = -J \circ I$, the usual quaternionic relations hold and $Kv = -\mathrm{i}\bar{v}$ for all $v \in W' \oplus \mathrm{i}W'$.

Note that

$$J_\varphi = \exp(\varphi I) \circ J = J \circ \exp(-\varphi I) = \exp(\varphi/2I) \circ J \circ \exp(-\varphi/2I). \tag{57}$$

It follows that J_φ defines another complex structure on $W' \oplus \mathrm{i}W'$. Since $W = \{e_1, \ldots, e_k\}_{\mathbb{R}}$, we see that $J_\varphi(W) = e^{-\mathrm{i}\varphi}\bar{W} = U$. Similarly, since $\bar{W} \subset \{\bar{e}_1, \ldots, \bar{e}_k\}_{\mathbb{R}}$, we obtain that $J_\varphi(\bar{W}) = e^{\mathrm{i}\varphi}W$ and hence $J_\varphi(U) = W$. This proves the first part of the lemma. Note that I is the $\mathbb{C}$-linear extension of I' to $W' \oplus \mathrm{i}W'$. Hence, if $A \in \mathfrak{so}(W')$ is holomorphic or anti-holomorphic, then A commutes or anti-commutes with I on $W' \oplus \mathrm{i}W'$, respectively. If A is additionally real, then the same is true for J instead of I:

In fact, since A is real, we have $A(\bar{v}) = \overline{Av}$ for all $v \in W'$, hence $A \circ K = K \circ A$ on $W' \oplus \mathrm{i}W'$ and thus

$$A \circ J = A \circ K \circ I = K \circ A \circ I = \pm K \circ I \circ A = \pm J \circ A$$

where the sign is chosen according to whether A is holomorphic or anti-holomorphic. Our claim follows.

Therefore, if A is real and holomorphic, then A commutes with both J and I, hence A commutes also with J_φ according to (57). Suppose that $A \in \mathfrak{so}(W')$ is real and anti-holomorphic. Using (57), $A \circ J = -J \circ A$ and the fact that A is anti-holomorphic, we have

$$\begin{aligned} J_\varphi \circ \exp(\varphi I) \circ A &= J \circ \exp(-\varphi I) \circ \exp(\varphi I) \circ A = J \circ A \\ &= -A \circ J = -A \circ \exp(-\varphi I) \circ J_\varphi = -\exp(\varphi I) \circ A \circ J_\varphi . \end{aligned}$$

Since $I = I'$ on $W' \oplus \mathrm{i}W'$, we see that $\exp(\varphi I') \circ A$ anti-commutes with J_φ. □

Lemma 9. *Let W be of Type (tr'_k) defined by the data $(\Re, W', I', W'_0)$. Set $U := e^{-\mathrm{i}\varphi}\bar{W}$ for some $\varphi \in \mathbb{R}$ and $V := W \oplus U$.*

(a) The linear map

$$F : W'_0 \oplus W'_0 \to V, (v,u) \mapsto 1/2[v - \mathrm{i}I'v + J_\varphi(v - \mathrm{i}I'v) + u - \mathrm{i}I'u - J_\varphi(u - \mathrm{i}I'u)]. \tag{58}$$

is an isometry such that the linear spaces $\{ (v,v) \,|\, v \in W'_0 \}$ and $\{ (v,-v) \,|\, v \in W'_0 \}$ get identified with W and U, respectively.

(b) By means of the equivalence $W'_0 \oplus W'_0 \cong V$ from Part (a), the direct sum Lie algebra $\mathfrak{so}(W'_0) \oplus \mathfrak{so}(W'_0)$ gets identified with the Lie algebra $\rho(\mathfrak{k}_V)|_V$ such that $(A,A) \in \rho(\mathfrak{k}_V)|_V \cap \mathfrak{so}(V)_+$ and $(A,-A) \in \rho(\mathfrak{k}_V)|_V \cap \mathfrak{so}(V)_-$ for every $A \in \mathfrak{so}(W'_0)$.

(c) The complex structure $J_\varphi|_V$ commutes with every element of $\rho(\mathfrak{k}_V)|_V \cap \mathfrak{so}(V)_+$ and anti-commutes with every element of $\rho(\mathfrak{k}_V)|_V \cap \mathfrak{so}(V)_-$.

Proof. For (a): We have $F(v,v) = v - \mathrm{i}I'v \in W$ and $F(v,-v) = J_\varphi(v - \mathrm{i}I'v) = e^{-\mathrm{i}\varphi}(v + \mathrm{i}I'v) \in \mathrm{e}^{-\mathrm{i}\varphi}\bar{W} = U$. Since $\dim(W) = \dim(U) = \dim(W'_0)$, we conclude that F is actually a linear isometry onto V which has the properties described above.

For (b): Given $A \in \mathfrak{so}(W'_0)$, we associate therewith linear maps $\hat{A}$ and $\tilde{A}$ on W' defined by $\hat{A}(v+I'v) := Av + I'Av$ and $\tilde{A}(v+I'v) := Av - I'Av$. By definition, both $\hat{A}$ and $\tilde{A}$ are real, further, $\hat{A}$ is holomorphic whereas $\tilde{A}$ is anti-holomorphic. Furthermore, we consider the second splitting $V = V_1 \oplus V_2$ with $V_1 = \{ v - \mathrm{i}I'v + J_\varphi(v - \mathrm{i}I'v) \,|\, v \in W'_0 \}$ and $V_2 = \{ v - \mathrm{i}I'v - e^{-\mathrm{i}\varphi}(v - \mathrm{i}I'v) \,|\, v \in W'_0 \}$. This splitting induces a monomorphism of Lie algebras $\mathfrak{so}(W'_0) \oplus \mathfrak{so}(W'_0) \hookrightarrow \mathfrak{so}(V)$. We claim that this monomorphism is explicitely given by

$$(A,B) \mapsto 1/2[(\widehat{A+B}) + \exp(\varphi I') \circ (\widetilde{A-B})] \; . \tag{59}$$

Let $A \in \mathfrak{so}(W'_0)$ be given. We have

$$\begin{aligned} &1/2(\hat{A} + \exp(\varphi I') \circ \tilde{A})(v - \mathrm{i}I'v + J_\varphi(v - \mathrm{i}I'v) = 1/2[(Av - \mathrm{i}I'Av) + e^{-\mathrm{i}\varphi}(Av + \mathrm{i}I'Av) \\ &+ e^{-\mathrm{i}\varphi}(Av + \mathrm{i}I'Av) + e^{\mathrm{i}\varphi}e^{-\mathrm{i}\varphi}(Av - \mathrm{i}I'Av)] = (Av - \mathrm{i}I'Av) + e^{-\mathrm{i}\varphi}(Av + \mathrm{i}I'Av); \\ &1/2(\hat{A} + \exp(\varphi I') \circ \tilde{A})(v - \mathrm{i}I'v - J_\varphi(v - \mathrm{i}I'v) = 1/2[(Av - \mathrm{i}I'Av) - e^{-\mathrm{i}\varphi}(Av + \mathrm{i}I'Av) + e^{-\mathrm{i}\varphi}(Av + \mathrm{i}I'Av) \\ &- e^{\mathrm{i}\varphi}e^{-\mathrm{i}\varphi}(Av - \mathrm{i}I'Av)] = 0. \end{aligned}$$

This establishes our claim in case $B = 0$. For $A = 0$, a similar calculation works.

Further, we claim that in this way $\mathfrak{so}(W'_0) \oplus \mathfrak{so}(W'_0) \cong \rho(\mathfrak{k}_V)|_V$ such that $(A,A) \cong \hat{A} \in \rho(\mathfrak{k}_V)|_V \cap \mathfrak{so}(V)_+$ and $(A,-A) \cong \exp(\varphi I') \circ \tilde{A} \in \rho(\mathfrak{k}_V)|_V \cap \mathfrak{so}(V)_+$:

For "$\subseteq$": We have $\hat{A}(v \pm \mathrm{i}I'v) = Av \pm \mathrm{i}I'Av$ for all $v \in W'_0$ and $A \in \mathfrak{so}(W'_0)$, thus $\hat{A}$ maps W to W and U to U (since $\hat{A}$ is $\mathbb{C}$-linear). Further, $\hat{A} \in \mathfrak{so}(\Re)$. This shows that $\hat{A} \in \rho(\mathfrak{k}_V)|_V \cap \mathfrak{so}(V)_+$.

Furthermore, $\tilde{A}(v \pm \mathrm{i}I'v) = Av \mp \mathrm{i}I'Av$ and $I'(v \pm \mathrm{i}I'v) = \mp\mathrm{i}(v \pm \mathrm{i}I'v)$ for all $v \in W_0'$, thus $\exp(\varphi I') \circ \tilde{A}$ maps W to U and vice versa. Finally, note that $\exp(\varphi I') \circ \tilde{A}$ is in fact self-adjoint since

$$(\exp(\varphi I') \circ \tilde{A})^* = \tilde{A}^* \circ \exp(\varphi I')^* = -\tilde{A} \circ \exp(-\varphi I') = -\exp(\varphi I') \circ \tilde{A}\,.$$

Hence $\exp(\varphi I') \circ \tilde{A} \in \rho(\mathfrak{k}_V)|_V \cap \mathfrak{so}(V)_-$.

For "$\supseteq$": Conversely, let some $A \in \rho(\mathfrak{k}_V)|_V$ be given. We will distinguish the cases $A \in \mathfrak{so}(V)_+$ and $A \in \mathfrak{so}(V)_-$. Anyway, we may write $A = cJ^N|_V + B|_V$ with $B \in \mathfrak{so}(W')$ and $c \in \mathrm{I\!R}$. Set $B' := cI'|_V + B|_V$. Note that $I' = J^N$ on $W + \mathrm{i}W$ and $I' = -J^N$ on $\bar{W} + \mathrm{i}\bar{W}$, hence

$$A = B' \text{ on } W + \mathrm{i}W \tag{60}$$

$$A = B' + 2cJ^N \text{ on both } \bar{W} \text{ and } U. \tag{61}$$

If $A \in \mathfrak{so}(V)_+$, then $A(W) \subset W$ and hence we conclude from (60) that B' is real and holomorphic; thus $B' = C$ with $C := B'|_{W_0'}$. In particular, $B'(\bar{W}) \subset \bar{W}$ and hence $B'(U) \subset U$. Thus $c = 0$ because of (61) and since $J^N(U) \subset U^\perp$ where $U^\perp$ denotes the orthogonal complement of U in T_pN.

If $A \in \mathfrak{so}(V)_-$, then A maps W to U and vice versa, hence $B'(W) \subset U$ because of (60), thus $e^{\mathrm{i}\varphi}B'(W) \subset \bar{W}$ which shows that the linear endomorphism $C := \exp(-\varphi I') \circ B'$ is real and anti-holomorphic on W'. Hence B' is anti-holomorphic, too. Therefore, also $\exp(-\varphi I')(B'(\bar{W})) \subset W$, thus $B'(U) \subset W$. We conclude that $c = 0$ according to (61) (since $J^N(U) \subset \bar{W} + \mathrm{i}\bar{W} \subset W^\perp$). This establishes our claim. Part (a) follows.

For (c), recall that $\hat{A}$ commutes with $J_\varphi|_V$ whereas $\tilde{A}$ anti-commutes with $J_\varphi|_V$ according to Lemma 8 for every $A \in \mathfrak{so}(W_0')$. Hence, the result is a consequence of Part (b) and (59). $\square$

Proposition 5. *Suppose that W is of Type (tr_k') with $k \geq 3$. The curvature invariant pair $(W, \mathrm{e}^{-\mathrm{i}\varphi}\bar{W})$ is not integrable.*

Proof. Let W be defined by the data $(\Re, W', I', W_0')$, set $U := e^{-\mathrm{i}\varphi}\bar{W}$ and $V := W \oplus U$. Let $\mathfrak{g}$ be the subalgebra of $\mathfrak{so}(V)$ described in Theorem 3 and recall that there exist $A_x \in \rho(\mathfrak{k}_V)|_V \cap \mathfrak{so}(V)_-$, $B_x \in Z(\mathfrak{g}) \cap \mathfrak{so}(V)_-$ such that $\boldsymbol{h}_x = A_x + B_x$ for every $x \in W$. We claim that here $B_x = 0$:

For this, recall that $\mathfrak{h}_W|_V$ is a subalgebra of $\rho(\mathfrak{k}_V)|_V \cap \mathfrak{so}(V)_+$ (see (13) and (17)). Therefore $[J_\varphi|_V, R^N_{x,y}|_V] = 0$ for all $x, y \in W$ and $A_x \circ J_\varphi|_V = -J_\varphi|_V \circ A_x$ according to Lemma 9 (c). It follows, on the one hand, that $J_\varphi|_V$ anti-commutes with $[R^N_{x,y}|_V, A_z]$ for all $x, y, z \in W$. Assume, by contradiction, that $B_x \neq 0$ for some $x \in W$. We will show that this implies, on the other hand, that $J_\varphi|_V$ commutes with $[R^N_{x,y}|_V, A_z]$ (which is not possible unless $[R^N_{x,y}|_V, A_z] = 0$, since $J_\varphi|_V$ is a complex structure):

Consider the Lie algebra $\mathfrak{h}$ (21). By means of Lemma 5 (a), we have

$$\mathfrak{h} = \mathfrak{h}_W|_V = \mathfrak{h}_U|_V = \{x \wedge y + I'x \wedge I'y | x, y \in W_0'\}_{\mathrm{I\!R}}. \tag{62}$$

Thus W and U are irreducible $\mathfrak{h}$-modules and, moreover, since $k \geq 3$, Schur's Lemma shows that $\mathrm{Hom}_{\mathfrak{h}}(W, U)$ is a one-dimensional space. Hence, because of (25), the linear space $Z(\mathfrak{h}) \cap \mathfrak{so}(V)_-$ is one-dimensional, too. Further, since $\mathfrak{h} \subset \mathfrak{g}$ (cf. the proof of Corollary 4), we have $B_x \in Z(\mathfrak{g}) \cap \mathfrak{so}(V)_- \subset Z(\mathfrak{h}) \cap \mathfrak{so}(V)_-$. Furthermore, we have $J_\varphi|_V \in Z(\mathfrak{h}) \cap \mathfrak{so}(V)_-$, too, according to Lemma 8 (a) and Lemma 9 (c). Hence there exists $\lambda \neq 0$ with $J_\varphi|_V = \lambda B_x \in Z(\mathfrak{g})$. Therefore, since $[R^N_{x,y}|_V, A_z] \in \mathfrak{g}$, we see that $J_\varphi|_V$ commutes with $[R^N_{x,y}|_V, A_z]$.

Hence, we conclude that $[R^N_{x,y}|_V, \boldsymbol{h}_z] = [R^N_{x,y}|_V, \boldsymbol{h}_z - B_z] = [R^N_{x,y}|_V, A_z] = 0$ for all $x, y, z \in W$. As remarked above, this shows that $\boldsymbol{h}_z \in \mathrm{I\!R}J_\varphi$ for all $z \in W$. But this would imply that $\boldsymbol{h} = 0$ since $\boldsymbol{h}$ is injective or zero according to Proposition 3 (because W is an irreducible $\mathfrak{h}_W$-module), a contradiction.

Thus $B_x = 0$, i.e. $\boldsymbol{h}_x = A_x \in \rho(\mathfrak{k}_V)|_V$ for all $x \in W$. Let us choose some $o \in \mathrm{S}^k$, a linear isometry $f : T_o\mathrm{S}^k \to W_0'$ and consider the Riemannian product $\tilde{N} := \mathrm{S}^k \times \mathrm{S}^k$ whose curvature tensor will be denoted by $\tilde{R}$. On the analogy of (58),

$$F : T_{(o,o)}\tilde{N} \to T_pN, (x, y) \mapsto 1/2[f(x) + f(y) - \mathrm{i}I'(f(x) + f(y)) + J_\varphi(f(x) - f(y) - \mathrm{i}I'(f(x) - f(y)))]$$

is an isometry onto V such that $\{\, F^{-1} \circ A|_V \circ F \mid A \in \rho(\mathfrak{k}_V) \,\}$ is the direct sum Lie algebra $\mathfrak{so}(T_o\mathrm{S}^k) \oplus \mathfrak{so}(T_o\mathrm{S}^k)$. Note, the latter is the image $\tilde{\rho}(\tilde{\mathfrak{k}})$ of the linearized isotropy representation of $\tilde{N}$. Put $\tilde{W} := F^{-1}(W)$, $\tilde{h} := F^{-1} \circ h \circ F \times F$ and $\tilde{U} := \{\tilde{h}(u,v) \big| u,v \in \tilde{W}\}_{\mathrm{I\!R}}$. Then $\tilde{U} = F^{-1}(U)$ and hence $T_{(o,o)}\tilde{N} = \tilde{W} \oplus \tilde{U}$ holds. Furthermore, we claim that $(\tilde{W}, \tilde{h})$ is an integrable 2-jet in $T_{(o,o)}\tilde{N}$:

Let $\iota : \mathrm{S}^k \to \mathrm{S}^k \times \mathrm{S}^k, p \mapsto (p,p)$. Then $T_{(o,o)}\iota(\mathrm{S}^k) = \{\, (x,x) \mid x \in T_o\mathrm{S}^k \,\}$, hence $F(T_{(o,o)}\iota(\mathrm{S}^k)) = W$, i.e. $T_{(o,o)}\iota(\mathrm{S}^k) = \tilde{W}$. Further, on the one hand, we have

$$
\begin{aligned}
&R^N(F(x,x), F(y,y), F(z,z)) = R^N(f(x) - \mathrm{i}I'f(x), f(y) - \mathrm{i}I'f(y), f(z) - \mathrm{i}I'f(z)) \\
&= -f(x) \wedge f(y)\, f(z) + \mathrm{i}(I'f(x) \wedge I'f(y))I'f(z)
\end{aligned}
$$

for all $x, y, z \in T_o\mathrm{S}^k$ according to Lemma 5 (a). On the other hand,

$$
\begin{aligned}
&F(x \wedge y\, z, x \wedge y\, z) = f(x) \wedge f(y)\, f(z) - \mathrm{i}I'(f(x) \wedge f(y) f(z)) \\
&= f(x) \wedge f(y)\, f(z) - \mathrm{i}(I'f(x) \wedge I'f(y))I'f(z).
\end{aligned}
$$

This shows that $F \circ \tilde{R}^N_{(x,x),(y,y)}|_{\tilde{W}} = R^N_{F(x,x),F(y,y)} \circ F|_{\tilde{W}}$. Furthermore, (13), (17) and Lemma 9 (b) show that $\tilde{R}^N_{(x,x),(y,y)}$ and $F^{-1} \circ R^N_{F(x,x),F(y,y)} \circ F$ both belong to $\tilde{\rho}(\tilde{\mathfrak{k}})_+$, i.e. there exist $A, B \in \mathfrak{so}(T_o\mathrm{S}^k)$ with $\tilde{R}^N_{(x,x),(y,y)} = A \oplus A$ and $F^{-1} \circ R^N_{F(x,x),F(y,y)} \circ F = B \oplus B$. Thus, since the direct sum endomorphism $A \oplus A$ is uniquely determined by its restriction to $\tilde{W}$ for every $A \in \mathfrak{so}(T_o\mathrm{S}^k)$, we conclude that $F \circ \tilde{R}^N_{(x,x),(y,y)} = R^N_{F(x,x),F(y,y)} \circ F$. Therefore, $\tilde{W}$ is curvature invariant and $\tilde{h}$ is semiparallel in $\tilde{N}$. Moreover, since $\boldsymbol{h}_x \in \rho(\mathfrak{k}_V)|_V$ for all $x \in W$, we have $\tilde{\boldsymbol{h}}_x \in \tilde{\rho}(\tilde{\mathfrak{k}})$ for all $x \in \tilde{W}$ which shows that Eq. 5 for $(\tilde{W}, \tilde{h})$ is implicitly given for all k. Hence, by means of Theorem 2, we obtain that $(\tilde{W}, \tilde{h})$ is an integrable 2-jet in $\tilde{N}$.

Thus, there exists a complete parallel submanifold $\tilde{M} \subset \tilde{N}$ through (o,o) whose 2-jet is given by $(\tilde{W}, \tilde{h})$. The fact that $T_{(o,o)}\tilde{N} = \tilde{W} \oplus \tilde{U}$ holds implies that $\tilde{M}$ is 1-full in $\tilde{N}$, i.e. extrinsically symmetric according to Corollary 3. Further, since $\tilde{M}$ is tangent to $\iota(\mathrm{S}^k)$ at (o,o), there do not exist submanifolds $\tilde{M}_1 \subset \mathrm{S}^k$ and $\tilde{M}_2 \subset \mathrm{S}^l$ such that $\tilde{M} = \tilde{M}_1 \times \tilde{M}_2$. Therefore, by means of Theorem 4, $\tilde{M}$ is totally geodesic, i.e. $h = 0$, a contradiction. $\square$

Lemma 10. *Suppose that $k = 2$ and W is of Type (tr'_k) defined by the data $(\Re, W', I', W'_0)$. Let $\{e_1, e_2\}$ be an orthonormal basis of W'_0 and $\tilde{I} := e_1 \wedge e_2 + I'e_1 \wedge I'e$. Further, let $\tilde{J} \in \mathrm{SU}(W', \tilde{I}) \cap \mathfrak{su}(W')$ be given and set $U := \tilde{J}(W)$. If $V := W \oplus U$ is not curvature invariant, then $\rho(\mathfrak{k}_V)|_V \cap \mathfrak{so}(V)_- = \mathrm{I\!R}\tilde{J}$.*

Proof. First, note that $\tilde{J}^2 = -\mathrm{Id}$, hence $\tilde{J}|_V \in \mathfrak{so}(V)_-$. Further, $\tilde{J} \in \mathrm{SU}(W', \tilde{I}) \subset \mathfrak{so}(W') \subset \mathfrak{so}(\Re)$, thus $\tilde{J}|_V \in \rho(\mathfrak{k}_V)|_V \cap \mathfrak{so}(V)_-$. Conversely, let $A \in \rho(\mathfrak{k}_V)$ be given. Then there exist some $c \in \mathrm{I\!R}$ and $A \in \mathfrak{so}(\Re)$ such that $A = cJ^N + B$ and $A(V) \subset V$. Suppose further that $A|_V \in \mathfrak{so}(V)_-$. With $\tilde{A} := cI' + B|_{W'}$, we have $\tilde{A} \in \mathfrak{so}(W')$ and $A|_W = \tilde{A}|_W$, hence $\tilde{A}(W) = A(W) \subset \tilde{J}(W)$, thus $\tilde{J}(\tilde{A}(W)) \subset W$ (since $\tilde{J}^2 = -1$). In particular, we have $\tilde{A}(W'_0) \subset \tilde{J}(W'_0) \subset W'$ and $\tilde{A}(I'(W'_0)) \subset \tilde{J}(I'(W'_0)) \subset W'$, thus $\tilde{A}|_{W'} \in \mathfrak{so}(W')$ and $C := \tilde{J} \circ \tilde{A}$ is real and holomorphic on W' (see Definition 5). Further, $\tilde{A}^* = -\tilde{A}$ which implies that

$$
\tilde{J} \circ C = C^* \circ \tilde{J}. \tag{63}
$$

We claim that $C = \lambda\mathrm{Id}$ for some $\lambda \in \mathrm{I\!R}$ or $\tilde{J} = \pm I'$:

For this, let RH denote the algebra of real and holomorphic maps on W'. Note, $\tilde{I}$ is real and holomorphic, hence there is the splitting $\mathrm{RH} = \mathrm{RH}_+ \oplus \mathrm{RH}_-$ with

$$
\begin{aligned}
\mathrm{RH}_+ &:= \{\, A \in \mathrm{RH} \mid A \circ \tilde{I} = \tilde{I} \circ A \,\}, \\
\mathrm{RH}_- &:= \{\, A \in \mathrm{RH} \mid A \circ \tilde{I} = -\tilde{I} \circ A \,\}.
\end{aligned}
$$

Then $\mathrm{RH}_+ = \{\mathrm{Id}, \tilde{I}\}_{\mathrm{I\!R}}$ and $\mathrm{RH}_- = \{r, r \circ \tilde{I}\}_{\mathrm{I\!R}}$, where r denotes the linear reflection of W' in the subspace which is spanned by $\{e_1, I'e_1\}$. Further, consider the involution on $\mathrm{End}(W')$ defined by $C \mapsto -\tilde{J} \circ C^* \circ \tilde{J}$. This map preserves both RH_+ and RH_- and, furthermore, its fixed points in RH are the solutions to (63). It follows that a solution to (63) with $C \in \mathrm{RH}$ decomposes as $C = C_+ + C_-$ such that $C_\pm \in \mathrm{RH}_\pm$ and $C_\pm$ is a solution to (63).

Further, we have $\tilde{J}\circ\tilde{I}=\tilde{I}\circ\tilde{J}=-\tilde{I}^*\circ\tilde{J}$ since $\tilde{I}$ is skew-symmetric and commutes with $\tilde{J}$. Hence a solution to (63) with $C\in \mathrm{RH}_+$ is given only if C is a multiple of Id. If $C\in\mathrm{RH}_-$ is a solution to (63), then $C\circ\tilde{I}$ is a solution to this equation, too, since

$$\tilde{J}\circ C\circ\tilde{I}=C^*\circ\tilde{J}\circ\tilde{I}=C^*\circ\tilde{I}\circ\tilde{J}=(I^*\circ C)^*\circ\tilde{J}=(-\tilde{I}\circ C)\circ\tilde{J}=C\circ\tilde{I}\circ\tilde{J}.$$

Thus, since RH_- is invariant under multiplication from the right by $\tilde{I}$, the intersection of the solution space to (63) with RH_- is either trivial or all of RH_-. Hence, to finish the proof of our claim, it suffices to show that $C:=r$ is not a solution to (63) unless $\tilde{J}=\pm I'$:

For this, recall that there exist $t\in\mathrm{I\!R}$ and $w\in\mathbb{C}$ with $t^2+|w|^2=1$ such that the matrix of $\tilde{J}$ with respect to the $\tilde{I}$-Hermitian basis $\{e_1,I'e_1\}$ is given by Eqs. (36)-(37). Further, r is the conjugation of W' with respect to the complex structure $\tilde{I}$ and the real form $\{e_1,I'e_1\}_{\mathrm{I\!R}}$. In particular, r is a selfadjoint involution on W'. Hence, if (63) holds for $C:=r$, then $r\circ\tilde{J}\circ r=\tilde{J}$, i.e.

$$\begin{pmatrix}\mathrm{i}t & -\bar{w}\\ w & -\mathrm{i}t\end{pmatrix}=\begin{pmatrix}-\mathrm{i}t & -w\\ \bar{w} & \mathrm{i}t\end{pmatrix}. \tag{64}$$

Clearly, this implies that $t=0$ and $w=\pm1$, i.e. $\tilde{J}=\pm I'$.

This proves our claim. Further, if $\tilde{J}=\pm I'$, then $U=I'(W)=\mathrm{i}W$ in which case $V=W+\mathrm{i}W$ is curvature invariant of Type (c_2') defined by $(\Re,W',I')$. Otherwise, it follows that there exists $a\in\mathrm{I\!R}$ with $\tilde{A}=a\tilde{J}$. Hence $A|_V=a\tilde{J}|_V+c(J^N|_V-I'|_V)$. It remains to show that $c(J^N|_V-I'|_V)=0$: we have $J^N|_W=I'|_W$ and $c(J^N|_V-I'|_V)=A|_V-a\tilde{J}|_V\in\mathfrak{so}(V)_-$. Therefore, since $\mathfrak{so}(V)_-\to\mathrm{Hom}(W,U), A\mapsto A|_W$ is an isomorphism, $c(J^N|_V-I'|_V)=0$. This finishes our proof. ☐

Proposition 6. *In the situation of Lemma 10, the curvature invariant pair (W,U) is not integrable unless V is a curvature invariant subspace of T_pN.*

Proof. Suppose that (W,U) is integrable. By definition, this means that there exists a symmetric bilinear map $h:W\times W\to U$ such that (W,h) is an integrable 2-jet with $U=\{h(x,y)\big|x,y\in W\}_{\mathrm{I\!R}}$. Consider the Liealgebra $\mathfrak{g}$ from Theorem 3. Clearly, $\tilde{I}=R^N_{e_1,e_2}|_V$ is of the form (19) (with $k=0$), hence $\tilde{I}\in\mathfrak{g}$. Therefore, according to Theorem 3, there exists $A_x\in\rho(\mathfrak{k}_V)|_V\cap\mathfrak{so}(V)_-$ and $B_x\in\mathfrak{su}(V,\tilde{I})\cap\mathfrak{so}(V)_-$ with $\boldsymbol{h}_x=A_x+B_x$. We assume, by contradiction, that V is not curvature invariant. Hence $\rho(\mathfrak{k}_V)|_V\cap\mathfrak{so}(V)_-=\mathrm{I\!R}\tilde{J}$ as a consequence of Lemma 10. We obtain that $A_x\circ\tilde{I}=\tilde{I}\circ A_x$, hence $[\boldsymbol{h}_x,\tilde{I}]=[A_x,\tilde{I}]+[B_x,\tilde{I}]=0$. In other words, we conclude that $\boldsymbol{h}_x$ belongs to $\mathfrak{su}(V,\tilde{I})\cap\mathfrak{so}(V)_-$ for all $x\in W$.

Let $\{I,J,K\}$ be a quaternionic basis of $\mathfrak{su}(V,\tilde{I})$ defined as follows: set $I|_W:=\tilde{I}|_W$, $I|_U=-\tilde{I}|_U$, $J:=\tilde{J}|_V$ and $K:=I\circ J$. Since $\tilde{I}$ commutes with $\tilde{J}$, we have $I\circ J=-J\circ I$ and then the usual quaternionic relations hold, i.e. $I^2=J^2=K^2=-\mathrm{Id}$, $J\circ K=-K\circ J=I$ and $K\circ I=-I\circ K=J$. Note that $I\in\mathfrak{so}(V)_+$ whereas $J,K\in\mathfrak{so}(V)_-$. Hence $\boldsymbol{h}_x\in\{J,K\}_{\mathrm{I\!R}}$ for all $x\in W$. Further, $\mathfrak{h}_W$ acts irreducibly on W which implies that $\boldsymbol{h}:W\to\mathfrak{so}(V)_-$ is injective according to Proposition 3. Hence there exists some $x\in W$ with $\boldsymbol{h}_x=K$. Further, set $y:=e_1-\mathrm{i}I'e_1$, $z:=e_2-\mathrm{i}I'e_2$ and recall that $R^N_{y,z}=-e_1\wedge e_2-I'e_1\wedge I'e_2=-\tilde{I}$. Therefore, with x,y,z chosen as above, Eq. (5) with $k=1$ means that

$$[K,\tilde{I}]=-R^N(Ky\wedge z+y\wedge Kz)|_V\ .$$

Note that l.h.s. of the last Equation vanishes (since $K\in\mathrm{SU}(V,\tilde{I})$). We claim that r.h.s. does not vanish unless $\tilde{J}=\pm I'$:

For this, let r.h.s. be writen as $cJ|_V+A|_V$ for some $c\in\mathrm{I\!R}$. We claim that A does not vanish on V: we have $z=\tilde{I}y$, hence

$$Ky=IJy=-\tilde{I}\tilde{J}y=-\tilde{J}\tilde{I}y=-\tilde{J}z=-Jz,$$

hence $z=JKy=-KJy$ and $Kz=Jy=\tilde{J}y$, thus

$$Ky\wedge z+y\wedge Kz=z\wedge\tilde{J}z+y\wedge\tilde{J}y.$$

Therefore, using Eq. (33), we obtain

$$A = \tilde{J}e_1 \wedge e_1 + \tilde{J}\mathrm{j}e_1 \wedge \mathrm{j}e_1 + \tilde{J}e_2 \wedge e_2 + \tilde{J}\mathrm{j}e_2 \wedge \mathrm{j}e_2,$$

i.e. $A = -2\tilde{J} \neq 0$ on V. This proves the claim.

In particular, the curvature endomorphism $R^N(Ky \wedge z + y \wedge Kz)$ is non-trivial on V which implies, by the previous, that (5) with $k = 1$ does not hold for all $x, y, z \in W$ and $v \in V$. This shows that the curvature invariant pair (W, U) is not integrable. □

Type $(\mathbf{ex_3}, \mathbf{tr_1})$ Let W and U be curvature invariant subspaces of Type (ex_3) and (tr_1) defined by the data $(\Re, \{e_1, e_2\})$ and a unit vector $u \in T_pN$, respectively. Then $u = \pm 1/\sqrt{2}\,(e_2 - \mathrm{i}e_1)$ and the linear space $W \oplus U$ is curvature invariant of Type (c_2) defined by the data $(\Re, \{e_1, e_2\}_{\mathbb{R}})$. □

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