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Preprint 2012/003

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ISSN **1613-8309**

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LaTeX-Style: Winfried Geis, Thomas Merkle

REPRESENTING HOMOLOGY CLASSES BY SYMPLECTIC SURFACES

ABSTRACT. We derive an obstruction to representing a homology class of a symplectic 4-manifold by an embedded, possibly disconnected, symplectic surface.

A natural question concerning symplectic 4-manifolds is the following: Given a closed, symplectic 4-manifold (M, ω) and a homology class $B \in H_2(M; \mathbb{Z})$, determine whether there exists an embedded, possibly disconnected, closed symplectic surface representing the class B. This question has been studied by H.-V. Lê and T.-J. Li [8, 9]. One necessary condition is, of course, that $\langle [\omega], B \rangle > 0$. Among other things, it is shown in [9] that a class B with $\langle [\omega], B \rangle > 0$ in a symplectic 4-manifold is always represented by a symplectic *immersion* of a connected surface. It is also noted that an obstruction to representing a homology class B by an embedded, *connected*, symplectic surface comes from the adjunction formula: The integer

$$B^2 + K_M B + 2,$$

where K_M denotes the canonical class of the symplectic 4-manifold (M, ω), has to be non-negative. This obstruction, however, disappears, if the number of components of the symplectic surface is allowed to grow large. Note that there are examples of classes in symplectic 4-manifolds which are represented by an embedded disconnected symplectic surface, but not by a connected symplectic surface: For example in the two-fold blow-up $X \# 2\overline{\mathbb{CP}^2}$ of a symplectic 4-manifold X the sum of the classes of the exceptional spheres is not represented by a connected embedded symplectic surface according to the adjunction formula. It is the purpose of this article to derive an obstruction to representing a homology class by an embedded, possibly disconnected, symplectic surface.

In [9] it is also shown that for symplectic manifolds M of dimension at least six, every class in $H_2(M; \mathbb{Z})$ on which the symplectic class evaluates positively is represented by a connected embedded symplectic surface. In [8] there is a conjecture which in the case of symplectic 4-manifolds M says that if α is a class in $H_2(M; \mathbb{Z})$ on which the symplectic class evaluates positively, then there exists a positive integer N depending on α such that $N\alpha$ is represented by an embedded, not necessarily connected, symplectic surface. In the examples at the end of this article we give counterexamples to this conjecture in the 4-dimensional case.

The non-existence of an embedded symplectic surface in the class B has the following consequence for the Seiberg-Witten invariants, which we only state in the case $b_2^+ > 1$.

Proposition 1. *Let M be a symplectic 4-manifold with $b_2^+(M) > 1$ and $B \neq 0$ an integral second homology class which cannot be represented by an embedded, possibly disconnected, symplectic surface. Then the Seiberg-Witten invariant of the $Spin^c$-structure $s_{K_M^{-1}} \otimes PD(B)$ is zero, where $s_{K_M^{-1}}$ denotes the $Spin^c$-structure with determinant line bundle K_M^{-1} induced by a compatible almost complex structure.*

Note that if $H_1(M; \mathbb{Z})$ has no 2-torsion (and hence $Spin^c$-structures are determined by their determinant line bundles), this means that $-K_M + 2PD(B)$ is not a basic class. Proposition 1 is a

Date: September 23, 2011.
2000 *Mathematics Subject Classification.* 57R17, 57N13.
Key words and phrases. 4-manifold, symplectic, branched covering.

consequence of a theorem of Taubes, relating classes with non-zero Seiberg-Witten invariants to embedded symplectic surfaces [14].

In the following, let (M,ω) denote a closed, symplectic 4-manifold and $\Sigma\subset M$ an embedded, possibly disconnected, closed symplectic surface representing a class $B\in H_2(M;\mathbb{Z})$. We always assume that the orientation of M is given by the symplectic form ($\omega\wedge\omega>0$). If the class B is divisible by an integer $d>1$, in the sense that there exists a class $A\in H_2(M;\mathbb{Z})$ such that $B=dA$, then there exists a d-fold cyclic ramified covering $\phi:\overline{M}\rightarrow M$, branched along Σ. The branched covering is again a closed, symplectic 4-manifold. This is a well-known fact (the pull-back of the symplectic form ω plus t times a Thom form for the preimage $\overline{\Sigma}$ of the branch locus is for small positive t a symplectic form on $\overline{M}$; see [3, 11] for a careful discussion). The invariants of $\overline{M}$ are given by the following formulas [4, p. 243], [5]:

$$
\begin{aligned}
K_{\overline{M}} &= \phi^*(K_M+(d-1)PD(A))\\
K_{\overline{M}}^2 &= d(K_M+(d-1)PD(A))^2\\
w_2(\overline{M}) &= \phi^*(w_2(M)+(d-1)PD(A)_2)\\
\sigma(\overline{M}) &= d(\sigma(M)-\frac{d^2-1}{3}A^2)
\end{aligned}
$$

Here PD denotes the Poincaré dual and $PD(A)_2\in H^2(M;\mathbb{Z}_2)$ is the mod 2 reduction of $PD(A)$. The second equation follows from the first because the branched covering map has degree d.

Suppose that the branched covering $\overline{M}$ is minimal and not a ruled surface over a curve of genus greater than 1. Then theorems of C. H. Taubes and A.-K. Liu [10, 13] imply that $K_{\overline{M}}^2\geq 0$. With the formula above, we get the following obstruction on the class A.

Theorem 2. *Let (M,ω) be a symplectic 4-manifold, $\Sigma\subset M$ an embedded, possibly disconnected, closed symplectic surface and $d>1$ an integer such that $dA=[\Sigma]$ for a class $A\in H_2(M;\mathbb{Z})$. Consider the d-fold cyclic branched cover $\overline{M}$, branched along Σ. If $\overline{M}$ is minimal and not a ruled surface over a curve of genus greater than 1, then*

$$(K_M+(d-1)PD(A))^2\geq 0.$$

It is therefore important to ensure that the branched covering $\overline{M}$ is minimal and not a ruled surface. First, we have the following lemma.

Lemma 3. *Let $\phi:\overline{M}\rightarrow M$ be a cyclic d-fold branched covering of closed, oriented 4-manifolds. Then $b_2^+(\overline{M})\geq b_2^+(M)$.*

Proof. With our choice of orientations, the map $\phi:\overline{M}\rightarrow M$ has positive degree. By Poincaré duality, the induced map $\phi^*:H^*(M;\mathbb{R})\rightarrow H^*(\overline{M};\mathbb{R})$ is injective. It maps classes in the second cohomology of positive square to classes of positive square. This implies the claim. □

Proposition 4. *In the notation of Theorem 2, each of the following two conditions imply that $\overline{M}$ is minimal and has $b_2^+(\overline{M})>1$ and hence is not a ruled surface:*

(a) *If d is odd assume that M is spin and if d is even assume that $PD(A)$ is characteristic. Also assume that $3\sigma(M)\neq(d^2-1)A^2$.*
(b) *Assume that $b_2^+(M)\geq 2$ and there exists an integer $k\geq 2$ such that the class $K_M+(d-1)PD(A)$ is divisible by k.*

Proof. Consider the d-fold branched covering $\overline{M}$, branched along Σ. The assumptions in case (a) imply that $\overline{M}$ is spin and that the signature $\sigma(\overline{M})$ is non-zero. According to a theorem of M. Furuta [2] we have $b_2^+(\overline{M}) \geq 3$. Also the symplectic manifold $\overline{M}$ is minimal, because it is spin. In case (b) the lemma implies that $b_2^+(\overline{M}) \geq 2$. In addition, the symplectic manifold $\overline{M}$ is minimal, because its canonical class is divisible by k (a non-minimal symplectic 4-manifold contains a symplectic sphere S with $KS = -1$). □

Example 5. Consider $M = K3$. Then we have $K_M = 0$. Let $d \geq 3$ be an integer and $A \in H_2(M;\mathbb{Z})$ a class with $A^2 < 0$. Theorem 2 together with Proposition 4 part (b) imply that dA is not represented by an embedded symplectic surface. Note that $K3$ contains indivisible classes of negative self-intersection which, for a suitable choice of symplectic structure, are represented by symplectic surfaces, for example symplectic (-2)-spheres. Let A be the homology class of such a sphere and $\alpha = 3A$. Then α is a counterexample to Conjecture 1.4 in [8].

Example 6. Let X be a symplectic spin 4-manifold with $b_2^+ > 1$ and M the blow-up $X \# \overline{\mathbb{CP}^2}$. Let E denote the class of the exceptional sphere in M. We have $K_M = K_X + PD(E)$. For every even integer d with $d^2 > K_X^2$, the class dE is not represented by a symplectic surface. Taking for example the blow-up of the $K3$ surface and $\alpha = 2E$, we get another counterexample to Lê's conjecture.

Note that with this method it is impossible to find a counterexample to Lê's conjecture under the additional assumption that $\alpha^2 > 0$.

In light of the second example, the following conjecture seems natural.

Conjecture. *Let M be the blow-up $X \# \overline{\mathbb{CP}^2}$ of a symplectic 4-manifold X and E the class of the exceptional sphere. Then dE is not represented by an embedded symplectic surface for all integers $d \geq 2$.*

This conjecture holds by a similar argument as above for X the $K3$ surface and the 4-torus T^4.

Remark 7. Branched covering arguments have been used in the past to find lower bounds on the genus of a connected surface representing a divisible homology class in a closed 4-manifold, see [1, 6, 7, 12].

Acknowledgements. I would like to thank D. Kotschick for very helpful comments.

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