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Abstract

In this paper we consider partitioning estimators of the local variance function, based on the first and second nearest neighbors, given an independent and identically distributed sample. Moreover, we assume that only censored data are available. Consistency of the partitioning estimator is given by known survival functions and in the more general case of unknown survival functions, estimated via the well-known Kaplan-Meier estimators. In this more general case, also the rate of convergence of the local variance estimator is given.
Keywords: local variance, censoring, partitioning estimation, nearest neighbors, weak consistency, rate of convergence.

AMS Subject classification: 62G05, 62G20, 62G08, 62N01.

1 Introduction

In survival analysis, one is interested in techniques for analyzing non-negative random variables in the presence of censoring. For it, let $(X, Y, C),\ (X_1, Y_1, C_1),\ (X_2, Y_2, C_2), \ldots$ i.i.d. $R^d \times R_+ \times R_+$-valued random vectors. X is the random vector of covariates with distribution μ, which, e.g., in medical applications contains information about a human taking part in a medical study around an illness. Y represents the survival time of the patient. C represents the censoring time. Moreover, we introduce the variable T, defined as minimum of Y and C, and the variable δ, containing the information whether there is or not censoring. This yields a set of data

$$\{(X_1, T_1, \delta_1), \ldots, (X_n, T_n, \delta_n)\},$$

with

$$\begin{cases} \delta_i = 1 & \text{for } Y_i \leq C_i \\ \delta_i = 0 & \text{for } Y_i > C_i, \end{cases}$$

and

$$T_i = \min\{Y_i, C_i\},$$

for $i = 1, \ldots, n$. We introduce now the so-called survival functions

$$F(t) = \boldsymbol{P}(Y > t),$$

$$G(t) = \boldsymbol{P}(C > t),$$

and

$$K(t) = \boldsymbol{P}(T > t) = F(t)G(t).$$

Introduce also

$$F^*(t) := \boldsymbol{P}(Y^2 > t) = F(\sqrt{t}),$$
$$K^*(t) := \boldsymbol{P}(T^* > t) = F^*(t)G(t) = F(\sqrt{t})G(t),$$

where $T^* = \min\{Y^2, C\}$.
The survival functions map the event of survival onto time and are therefore monotone decreasing.
Define

$$T_F := \sup\{y : \ F(y) > 0\},$$

$$T_G := \sup\{y : \ G(y) > 0\}, \tag{1}$$

$$T_K := \sup\{y : \ K(y) > 0\} = \min\{T_F, T_G\},$$

and notice that

$$T_{F^*} := \sup\{y : \ F^*(y) > 0\} = T_F$$

and

$$T_{K^*} := \sup\{y : \ K^*(y) > 0\} = \min\{T_{F^*}, T_G\} = \min\{T_F, T_G\} = T_K.$$

In medical studies the observation of the survival time of the patient is sometimes incomplete due to right censoring formulated just before. It could, for example, happen that the patient is alive at the termination of a medical study, or that he dies by other causes than those under study, or, trivially, that the patient moves and the hospital loses information about him. For more details see for example [8], Chapter 26.

The regression function, $m(x) := \boldsymbol{E}\{Y|X = x\}$, is known as the function that minimizes the L_2 risk. The problem of the estimation of the regression function under randomly right censored data is already known, see for instance [5], [8], [15] and [12].

A related interesting problem is the estimation of the local variance (or conditional variance) under censoring, defined as

$$\sigma^2(x) := \boldsymbol{E}\{(Y - m(X))^2|X = x\} = \boldsymbol{E}\{Y^2|X = x\} - m^2(x). \tag{2}$$

In the literature many papers deal with nonparametric local variance estimation in the (uncensored) case of fixed design. See for instance [1], [9], [16, 17], [19], [26], [18] and [2].

[22], besides the treatment of the local variance under fixed design, introduces also the case of random design with density of X. Moreover the case of random design was treated by [6], [10], [20], [21] and [23]. [12] investigated as application heteroscedastic conditional variance estimation via plug-in by least squares methods and in the same article, he gave as another application, regression estimation in the case of censored data. Combining this two applications, we want to give an estimator of the local variance function under censored data. For that, instead of least squares methods, we modify the partitioning local variance estimator based on the first and second nearest neighbors, introduced by [7] and we generalize it, for the more complicated case of partially known observations, due to a censorship. The partitioning estimator based on the nearest neighbors represents itself a generalization of an estimator of the residual variance given by [3], [4], [14].

After some preliminary definitions and assumptions in section 2, we introduce a censored partitioning estimation of the local variance via nearest neighbors under known survival function, in Section 3. Later, in Section 4, we give the same estimator under unknown survival functions, that we estimate via Kaplan-Meier estimators.

Finally, in section 5, we give a rate of convergence for the more general estimator given in Section 4.

2 Some Definitions and Assumptions

We recall now the definitions of nearest neighbors.

For given $i \in \{1, \ldots, n\}$, the first nearest neighbor of X_i among $X_1, \ldots, X_{i-1}$, $X_{i+1}, \ldots, X_n$ is denoted by $X_{[N,1]}$ with

$$N[i, 1] := N_n[i, 1] := \underset{1\le j\le n,\ j\ne i}{\arg\min}\ \rho(X_i, X_j), \tag{3}$$

here ρ is a metric (typically the Euclidean one) in R^d. The k-th nearest neighbor of X_i among $X_1, \ldots, X_{i-1}, X_{i+1}, \ldots, X_n$ is defined as $X_{N[i,k]}$ via generalization of definition (3):

$$N[i, k] := N_n[i, k] := \underset{1\le j\le n,\ j\ne i,\ j\notin\{N[i,1],\ldots,N[i,k-1]\}}{\arg\min} \rho(X_i, X_j), \tag{4}$$

by removing the preceding neighbors. If ties occur, a possibility to break them is given by taking the minimal index or by adding independent components Z_i, uniformly distributed on $[0, 1]$, to the observation vectors X_i see pp. 86, 87 [8]. The latter possibility of tie-breaking allow us to assume throughout the paper that ties occur with probability zero.

Hence, we get a reorder of the data according to increasing values of the distance of the variable X_j ($j \in \{1, \ldots, n\} \setminus \{i\}$) from the variable X_i ($i = 1, \ldots, n$). Correspondingly to that, we get also a new order for the variables Y_j :

$$\left(X_{N[i,1]},\ Y_{N[i,1]}\right),\ \ldots, \left(X_{N[i,k]},\ Y_{N[i,k]}\right),\ \ldots, \left(X_{N[i,n-1]},\ Y_{N[i,n-1]}\right).$$

In Sections 3 and 4 we will give a local variance estimator based on $N[i,1]$ and $N[i,2]$.

Moreover, for our intents we require the following conditions:

(A1) C and (X,Y) are independent,

(A2) $\exists L>0$, such that $\boldsymbol{P}\{\max\{Y,Y^2\}\le L\}=1$ and $\boldsymbol{P}\{C>L\}>0$.
G is continuous.

(A3) $\forall 0<T'_K<T_K:\ \boldsymbol{P}\{0\le Y\le T'_K\}<1$, $\boldsymbol{P}\{0\le Y^2\le T'_K\}<1$
F is continuous in a neighborhood of T_K and in a neighborhood of $\sqrt{T_K}$.

As we already said, under censoring the information about the survival time of a patient are incomplete in the sense that sometimes we cannot observe Y_i but only C_i with the indication that it is not the real life time (by δ_i). Therefore the random triple (X,T,δ) does not identify anymore the conditional distribution of Y given X. To achieve it, we need an additional assumption, that is **(A1)**: the censoring time C is independent of the common distribution of the survival time Y and the patient data X. In the medical applications **(A1)** is fulfilled in the case the censoring takes place regardless the characteristics of the patients and depends only on external factors not related to the information represented by the covariate X. Examples of this situation are the (random) termination of a study, which does not depend on the person who participated to it or the interruption of the cooperation of the patient to the medical study, maybe because of luck of enthusiasm.
The first part of **(A2)** is obviously fulfilled because of the intrinsic boundedness of Y (survival time of a human being!). The second part of **(A2)**, the positivity of $\boldsymbol{P}\{C>L\}$ means that not the whole censoring process takes place in $[0,L]$. In practice, it means that there is the possibility to extend the medical study, so that, with positive probability, C is larger than the bound L of Y. The continuity of G will be necessary for the convergence of the estimator G_n of G, that we introduce in following. Moreover, for this estimator the assumption **(A3)** allows giving a rate of convergence on the whole interval $[0,T_K]$.
For unknown F and G, [11] proposed two estimates, F_n and G_n, respectively, the product-limit estimates (see for example [8], pp. 541, 542). In medical research, the Kaplan-Meier estimate is used to measure the fraction of patients living for a certain amount of time after treatment. Also in economics it is common, for measuring the length of time people remain unemployed after a job loss. In engineering, it can be used to measure the time until failure of machine parts.
Let F_n and G_n be the Kaplan-Meier estimates of F and G, respectively, which are defined as

$$F_n(t)=\begin{cases}\prod_{i=1,\dots,n\ T(i)\le t}\left(\frac{n-i}{n-i+1}\right)^{\delta_{(i)}} & t\le T(n)\\ 0 & \text{otherwise}\end{cases}$$

and

$$G_n(t)=\begin{cases}\prod_{i=1,\dots,n\ T(i)\le t}\left(\frac{n-i}{n-i+1}\right)^{1-\delta_{(i)}} & t\le T(n)\\ 0 & \text{otherwise,}\end{cases}$$

where $((T(1),\delta(1)),\dots,(T(n),\delta(n)))$ are the n pairs of observed (T_i,δ_i) set in increasing order.
[5] introduced a transformation $\widetilde{Y}$ of the variable T with

$$\widetilde{Y}:=\frac{\delta T}{G(T)}, \tag{5}$$

and correspondingly:

$$\widetilde{Y}_i=\frac{\delta_i T_i}{G(T_i)}, \tag{6}$$

under known survival function G, and finally

$$\widetilde{Y}_{n,i}=\frac{\delta_i T_i}{G_n(T_i)}, \tag{7}$$

where G is estimated by Kaplan-Meier estimator G_n in the case it is unknown.
Define then

$$\widetilde{Y^2} := \frac{\delta T^2}{G(T)} \tag{8}$$

and their observations (G is known)

$$\widetilde{Y_i^2} = \frac{\delta_i T_i^2}{G(T_i)}, \tag{9}$$

and, for unknown G,

$$\widetilde{Y_{n,i}^2} = \frac{\delta_i T_i^2}{G_n(T_i)}. \tag{10}$$

Notice that $\widetilde{Y^2} \neq \widetilde{Y}^2 = \left(\frac{\delta T}{G(T)}\right)^2$.
The first part of assumption **(A2)** is equivalent to $0 \leq Y \leq L$, $Y^2 \leq L$ a.s., and it imply $T_K \leq L$ a.s.
Because of $0 \leq T_i \leq T_K \leq L$ for $i = 1, \dots, n$ with $G(L) = \boldsymbol{P}\{C > L\} > 0$ we get

$$1 \geq G(T_{(1)}) \geq \cdots \geq G(T_{(n)}) \geq G(T_K) \geq G(L) > 0 \quad a.s. \tag{11}$$

For fixed n also G_n is monotone decreasing

$$1 \geq G_n(T_{(1)}) \geq \cdots \geq G_n(T_{(n)}) \geq G_n(T_K) \geq G_n(L) > 0 \quad a.s. \tag{12}$$

Therefore, because of the boundedness of Y from (12) and the convergence theorem of [25] follows

$$\widetilde{Y}_{n,i} < U < \infty \text{ and } \widetilde{Y_{n,i}^2} < U < \infty \quad a.s. \tag{13}$$

(13) follows from (12) and $G_n(L) \to G(L)$ a.s. (the latter because of [8], Theorem 26.1)
For the transformation $\widetilde{Y}$ and $\widetilde{Y^2}$ the following nice properties can be shown:

$$\begin{aligned}
&\boldsymbol{E}\left\{\widetilde{Y}|X\right\} \\
= \;&\boldsymbol{E}\left\{\frac{1_{\{Y<C\}}\min\{Y,C\}}{G(\min\{Y,C\})}\middle|X\right\} \\
= \;&\boldsymbol{E}\left\{\boldsymbol{E}\left\{1_{\{Y<C\}}\frac{Y}{G(Y)}\middle|X,Y\right\}\middle|X\right\} \\
= \;&\boldsymbol{E}\left\{\frac{Y}{G(Y)}\underbrace{\boldsymbol{E}\{1_{\{Y<C\}}|X,Y\}}_{=G(Y)\text{ by }\textbf{(A1)}}|X\right\} \\
= \;&\boldsymbol{E}\{Y|X\}
\end{aligned} \tag{14}$$

and

$$\begin{aligned}
&\boldsymbol{E}\left\{\widetilde{Y^2}|X\right\} \\
= \;&\boldsymbol{E}\left\{\frac{1_{\{Y<C\}}\min\{Y^2,C\}}{G(\min\{Y,C\})}\middle|X\right\} \\
= \;&\boldsymbol{E}\left\{\boldsymbol{E}\left\{1_{\{Y<C\}}\frac{Y^2}{G(Y)}\middle|X,Y\right\}\middle|X\right\} \\
= \;&\boldsymbol{E}\left\{\frac{Y^2}{G(Y)}\underbrace{\boldsymbol{E}\{1_{\{Y<C\}}|X,Y\}}_{=G(Y)\text{ by }\textbf{(A1)}}|X\right\} \\
= \;&\boldsymbol{E}\{Y^2|X\}.
\end{aligned} \tag{15}$$

(cf. [24]). (14) and (15) mean that the conditional expectation of the transformed censored variable with respect to X equals the conditional expectation of the uncensored variable with respect to X (under **(A1)**). This implies that under known G, in the case that only the pair (T_i, δ_i) instead of (Y_i) is available,

$$\frac{1}{n}\sum_{i=1}^{n}\frac{\delta_i T_i}{G(T_i)}$$

is an unbiased estimate of $\boldsymbol{E}\{Y\}$.
Observe now, that

$$\begin{aligned}
& \boldsymbol{Var}\{\widetilde{Y}|X\} = \boldsymbol{Var}\left\{\frac{\delta T}{G(T)}|X\right\} = \boldsymbol{E}\left\{\left(\frac{\delta T}{G(T)}\right)^2\middle|X\right\} \\
- \quad & \boldsymbol{E}^2\left\{\frac{\delta T}{G(T)}\middle|X\right\} = \boldsymbol{E}\left\{\frac{(\delta T)^2}{G^2(T)}\middle|X\right\} - \boldsymbol{E}^2\{Y|X\} \\
\overset{\delta^2=\delta}{\underset{\downarrow}{=}} \quad & \boldsymbol{E}\left\{\frac{1_{\{Y\le C\}}(\min\{Y,C\})^2}{G^2(\min\{Y,C\})}\middle|X\right\} - m^2(X) \\
= \quad & \boldsymbol{E}\left\{\boldsymbol{E}\left\{1_{\{Y\le C\}}\frac{Y^2}{G^2(Y)}\middle|X,Y\right\}\middle|X\right\} - m^2(X) \\
= \quad & \boldsymbol{E}\left\{\frac{Y}{G^2(Y)}\underbrace{\boldsymbol{E}\{1_{\{Y<C\}}|X,Y\}}_{=G(Y)\text{ by }\textbf{(A1)}}|X\right\} - m^2(X) \\
= \quad & \boldsymbol{E}\left\{\frac{Y^2}{G(Y)}\middle|X\right\} - m^2(X).
\end{aligned}$$

Under the assumptions **(A1)-(A3)** and the definition of $\widetilde{Y^2_{n,i}}$ we introduce in the following sections a property local variance estimator under censoring.

3 Censored Partitioning Estimation via Nearest Neighbors under Known Survival Functions

In this section, the aim is to discuss estimators of the local variance function with partitioning approach, based on the first and second nearest neighbors under censoring. We need some helpful lemmas, that we will present in short. Before them, recall the definitions of nearest neighbors, (3) and (4), and define, for $i = 1, \dots, n$

$$\delta_{N[i,k]} = 1_{\{Y_{N[i,k]} \le C_i\}}$$

and

$$T_{N[i,k]} = \min(Y_{N[i,k]}, C_i).$$

Finally, assume that ties occur with probability zero.
Now, tree helpful lemmas.

Lemma 3.1 *With the above definitions and Definition (6), it holds*

$$\boldsymbol{E}\left\{\frac{\delta_i T_i}{G(T_i)}\frac{\delta_{N[i,1]}T_{N[i,1]}}{G(T_{N[i,1]})}\middle|X_i\right\} = \boldsymbol{E}\left\{Y_i Y_{N[i,1]}|X_i\right\}. \qquad (16)$$

Proof Consider that

$$\begin{aligned}
& \boldsymbol{E}\left\{\left.\frac{\delta_i T_i}{G(T_i)}\frac{\delta_{N[i,1]}T_{N[i,1]}}{G(T_{N[i,1]})}\right| X_1,\ldots,X_n\right\} \\
= \; & \boldsymbol{E}\left\{\left.\sum_{l\in\{1,\ldots,n\}\setminus\{i\}}\frac{\delta_i T_i}{G(T_i)}\frac{\delta_l T_l}{G(T_l)}1_{\{N[i,1]=l\}}\right| X_1,\ldots,X_n\right\} \\
= \; & \sum_{l\in\{1,\ldots,n\}\setminus\{i\}}\boldsymbol{E}\left\{\left.\frac{\delta_i T_i}{G(T_i)}\frac{\delta_l T_l}{G(T_l)}1_{\{N[i,1]=l\}}\right| X_1,\ldots,X_n\right\} \\
= \; & \sum_{l\in\{1,\ldots,n\}\setminus\{i\}}\boldsymbol{E}\left\{\left.\frac{\delta_i T_i}{G(T_i)}\right| X_i\right\}\boldsymbol{E}\left\{\left.\frac{\delta_l T_l}{G(T_l)}\right| X_l\right\}1_{\{N[i,1]=l\}} \\
& \text{(by the independence assumption)} \\
= \; & \boldsymbol{E}\left\{\left.\frac{\delta_i T_i}{G(T_i)}\right| X_i\right\}\sum_{l\in\{1,\ldots,n\}\setminus\{i\}}\boldsymbol{E}\left\{\left.\frac{\delta_l T_l}{G(T_l)}\right| X_l\right\}1_{\{N[i,1]=l\}} \\
= \; & \boldsymbol{E}\{Y_i|X_i\}\sum_{l\in\{1,\ldots,n\}\setminus\{i\}}\boldsymbol{E}\{Y_l|X_l\}1_{\{N[i,1]=l\}},
\end{aligned}$$

the latter by (14).
Moreover

$$\begin{aligned}
& \boldsymbol{E}\left\{Y_i Y_{N[i,1]}|X_1,\ldots,X_n\right\} \\
= \; & \boldsymbol{E}\left\{\left.\sum_{l\in\{1,\ldots,n\}\setminus\{i\}} Y_i Y_l 1_{\{N[i,1]=l\}}\right| X_1,\ldots,X_n\right\} \\
= \; & \sum_{l\in\{1,\ldots,n\}\setminus\{i\}}\boldsymbol{E}\left\{Y_i Y_l 1_{\{N[i,1]=l\}}|X_1,\ldots,X_n\right\} \\
= \; & \sum_{l\in\{1,\ldots,n\}\setminus\{i\}}\boldsymbol{E}\left\{Y_i Y_l|X_1,\ldots,X_n\right\}1_{\{N[i,1]=l\}} \\
= \; & \sum_{l\in\{1,\ldots,n\}\setminus\{i\}}\boldsymbol{E}\left\{Y_i|X_i\right\}\boldsymbol{E}\left\{Y_l|X_l\right\}1_{\{N[i,1]=l\}} \\
& \text{(by independence)} \\
= \; & \boldsymbol{E}\left\{Y_i|X_i\right\}\sum_{l\in\{1,\ldots,n\}\setminus\{i\}}\boldsymbol{E}\left\{Y_l|X_l\right\}1_{\{N[i,1]=l\}}.
\end{aligned}$$

These results imply (16). ∎

Analogously to the above lemma one has

Lemma 3.2 *It holds*

$$\boldsymbol{E}\left\{\left.\frac{\delta_i T_i}{G(T_i)}\frac{\delta_{N[i,2]}T_{N[i,2]}}{G(T_{N[i,2]})}\right| X_i\right\} = \boldsymbol{E}\left\{Y_i Y_{N[i,2]}|X_i\right\}. \tag{17}$$

The proof is analogous to the proof of Lemma 3.1 and therefore omitted.
A similar argument yields the following

Lemma 3.3 *It holds*

$$\boldsymbol{E}\left\{\left.\frac{\delta_{N[i,1]}T_{N[i,1]}}{G(T_{N[i,1]})}\frac{\delta_{N[i,2]}T_{N[i,2]}}{G(T_{N[i,2]})}\right| X_i\right\} = \boldsymbol{E}\left\{Y_{N[i,1]}Y_{N[i,2]}|X_i\right\}. \tag{18}$$

Again, the proof is omitted.
Recall then the following known relation (see (15))

$$\boldsymbol{E}\left\{\frac{\delta_i T_i^2}{G(T_i)}\right\} = \boldsymbol{E}\{Y_i^2|X_i\}. \tag{19}$$

Set now

$$\begin{aligned} H_i &:= H_{n,i} \\ &:= \frac{\delta_i T_i^2}{G(T_i)} - \frac{\delta_i T_i}{G(T_i)}\frac{\delta_{N[i,1]}T_{N[i,1]}}{G(T_{N[i,1]})} - \frac{\delta_i T_i}{G(T_i)}\frac{\delta_{N[i,2]}T_{N[i,2]}}{G(T_{N[i,2]})} \\ &\quad + \frac{\delta_{N[i,1]}T_{N[i,1]}}{G(T_{N[i,1]})}\frac{\delta_{N[i,2]}T_{N[i,2]}}{G(T_{N[i,2]})} \end{aligned} \tag{20}$$

for $i \in \{1, \ldots, n\}$ and note

$$\begin{aligned} & \boldsymbol{E}\{H_i|X_i = x\} && (21) \\ = \ & \boldsymbol{E}\left\{Y_i^2 - Y_iY_{N[i,1]} - Y_iY_{N[i,2]} + Y_{N[i,1]}Y_{N[i,2]}|X_i = x\right\} \\ & \text{(the latter by lemmas 3.1, 3.2 and 3.3)} && (22) \\ = \ & \boldsymbol{E}\left\{(Y_i - Y_{N[i,1]})(Y_i - Y_{N[i,2]})|X_i = x\right\} = \boldsymbol{E}\left\{W_i|X_i = x\right\} && (23) \end{aligned}$$

with

$$W_i := (Y_i - m(X_i))^2 + (m(X_i) - m(X_{N[i,1]}))(m(X_i) - m(X_{N[i,2]})) \tag{24}$$

according to Liitiäinen at al. ([14], [13]).
Our proposal for an estimator of the local variance function under known survival function G is given by

$$\widehat{\sigma}_n^2(x) := \frac{\sum_{i=1}^n H_i 1_{A_n(x)}(X_i)}{\sum_{i=1}^n 1_{A_n(x)}(X_i)}. \tag{25}$$

The following theorem states consistency of this estimator.

Theorem 3.4 *Let Assumptions **(A1)-(A3)** hold. Let $\mathcal{P}_n = \{\boldsymbol{A}_{n,1}, \ldots, \boldsymbol{A}_{n,l_n}\}$ be a sequence of partitions on R^d such that for each sphere S centered at the origin*

$$\lim_{n\to\infty} \max_{j\in\{A_{n,j}\cap S\neq\emptyset\}} \operatorname{diam} \boldsymbol{A}_{n,j} = 0, \tag{26}$$

and

$$\lim_{n\to\infty} \frac{\#\{j: \ A_{n,j} \cap S \neq \emptyset\}}{n} = 0. \tag{27}$$

Then

$$\int |\widehat{\sigma}_n^2 - \sigma^2(x)|\mu(dx) \xrightarrow{P} 0.$$

Before giving the proof of this theorem, introduce the following modification of the estimation (25)

$$\widehat{\widehat{\sigma}}_n^2(x) := \frac{\sum_{i=1}^n H_i 1_{A_n(x)}(X_i)}{n\mu(A_n(x))}, \tag{28}$$

and the following lemma.

Lemma 3.5 *Under the conditions of Theorem 3.4, $\widehat{\widehat{\sigma}}_n^2(x)$ is consistent, i.e.,*

$$\int |\widehat{\widehat{\sigma}}_n^2(x) - \sigma^2(x)|\mu(dx) \xrightarrow{P} 0.$$

Proof Choose a sphere S centered at 0 which contains the support of μ. Set $J_n := \{j : A_{n,j} \cap S \neq \emptyset\}$ and $l_n := \# J_n$. The variance of the estimator can be bounded by

$$\boldsymbol{Var}\left\{\widehat{\sigma}_n^2(x)\right\} \leq 72 \frac{L^4}{G(L)^4} \frac{1}{n\mu(A_n(x))}.$$

It holds

$$\begin{aligned}
& \boldsymbol{Var}\{\widehat{\sigma}_n^2(x)\} \\
\leq \quad & \frac{4}{n^2 \mu(A_n(x))^2} \left[\boldsymbol{Var}\left\{ \sum_{i=1}^n \frac{\delta_i T_i^2 1_{A_n(x)}(X_i)}{G(T_i)} \right\} \right. \\
& + \boldsymbol{Var}\left\{ \sum_{i=1}^n \frac{\delta_i T_i}{G(T_i)} \frac{\delta_{N[i,1]} T_{N[i,1]} 1_{A_n(x)}(X_i)}{G(T_{N[i,1]})} \right\} \\
& + \boldsymbol{Var}\left\{ \sum_{i=1}^n \frac{\delta_1 T_1}{G(T_1)} \frac{\delta_{N[i,2]} T_{N[i,2]} 1_{A_n(x)}(X_i)}{G(T_{N[i,2]})} \right\} \\
& \left. + \boldsymbol{Var}\left\{ \sum_{i=1}^n \frac{\delta_{N[i,1]} T_{N[i,1]} 1_{A_n(x)}(X_i)}{G(T_{N[i,1]})} \cdot \frac{\delta_{N[i,2]} T_{N[i,2]} 1_{A_n(x)}(X_i)}{G(T_{N[i,2]})} \right\} \right]
\end{aligned}$$

Each of the four variances in the right-hand side is bounded by

$$18 \frac{L^4}{G(L)^4} n\mu(A_n(x)).$$

We show this only for the fourth variance because the other three variances can be treated in the same way. We apply the Efron-Stein inequality, following the argument in the proof of Equation (12) in [7].

Let $n \geq 2$ be fixed. Replacement of (X_j, Y_j, C_j) by (X'_j, Y'_j, C'_j) for fixed $j \in \{1, \ldots, n\}$ (where $(X_1, Y_1, C_1), \ldots, (X_n, Y_n, C_n), (X'_1, Y'_1, C'_1), \ldots, (X'_n, Y'_n, C'_n)$ are independent and identically distributed) leads, for fixed x, from

$$U_n := \sum_{i=1}^n \frac{\delta_{N[i,1]} T_{N[i,1]} 1_{A_n(x)}(X_i)}{G(T_{N[i,1]})} \cdot \frac{\delta_{N[i,2]} T_{N[i,2]} 1_{A_n(x)}(X_i)}{G(T_{N[i,2]})},$$

$N[j,1]$ and $N[j,2]$ to $U_{n,j}$, $N'[j,1]$ and $N'[j,2]$, respectively.

With $T'_j := \min\left\{Y'_j, C'_j\right\}$, $\delta'_j = 1_{\{Y'_j \leq C'_j\}}$ we obtain

$$|U_n - U_{n,j}| \leq A_{n,j} + B_{n,j} + C_{n,j} + D_{n,j} + E_{n,j} + F_{n,j}$$

where, with $Z_i = \frac{\delta_i T_i}{G(T_i)}$, $Z'_j = \frac{\delta'_j T'_j}{G(T'_j)}$ and (11)

$$\begin{aligned}
A_{n,j} &= \sum_{\substack{l,\ q \in \{1,\ldots,n\}\setminus\{j\} \\ l \neq q}} Z_l Z_q 1_{A_n(x)}(X_i) 1_{\{N[j,1]=l\}} 1_{\{N[j,2]=q\}} \\
&\leq \frac{L^2}{G(L)^2} 1_{A_n(x)}(X_i), \\
B_{n,j} &= \sum_{\substack{l,\ q \in \{1,\ldots,n\}\setminus\{j\} \\ l \neq q}} Z_l Z_q 1_{A_n(x)}(X'_i) 1_{\{N'[j,1]=l\}} 1_{\{N'[j,2]=q\}} \\
&\leq \frac{L^2}{G(L)^2} 1_{A_n(x)}(X'_i), \\
C_{n,j} &= \sum_{\substack{i,\ q \in \{1,\ldots,n\}\setminus\{j\} \\ i \neq q}} Z_j Z_q 1_{A_n(x)}(X_i) 1_{\{N[i,1]=j\}} 1_{\{N[i,2]=q\}}
\end{aligned}$$

$$
\begin{aligned}
&\leq \frac{L^2}{G(L)^2} \sum_{i\in\{1,\dots,n\}\setminus\{j\}} 1_{A_n(x)}(X_i) 1_{\{N[i,1]=j\}}, \\
D_{n,j} &= \sum_{\substack{i,\ q\in\{1,\dots,n\}\setminus\{j\} \\ i\neq q}} Z'_j Z_q 1_{A_n(x)}(X_i) 1_{\{N'[i,1]=j\}} 1_{\{N'[i,2]=q\}}, \\
&\leq \frac{L^2}{G(L)^2} \sum_{i\in\{1,\dots,n\}\setminus\{j\}} 1_{A_n(x)}(X_i) 1_{\{N'[i,1]=j\}}, \\
E_{n,j} &= \sum_{\substack{i,\ l\in\{1,\dots,n\}\setminus\{j\} \\ i\neq l}} Z_l Z_j 1_{A_n(x)}(X_i) 1_{\{N[i,1]=l\}} 1_{\{N[i,2]=j\}}, \\
&\leq \frac{L^2}{G(L)^2} \sum_{i\in\{1,\dots,n\}\setminus\{j\}} 1_{A_n(x)}(X_i) 1_{\{N[i,2]=j\}}, \\
F_{n,j} &= \sum_{\substack{i,\ l\in\{1,\dots,n\}\setminus\{j\} \\ i\neq l}} Z_l Z'_j 1_{A_n(x)}(X_i) 1_{\{N'[i,1]=l\}} 1_{\{N'[i,2]=j\}} \\
&\leq \frac{L^2}{G(L)^2} \sum_{i\in\{1,\dots,n\}\setminus\{j\}} 1_{A_n(x)}(X_i) 1_{\{N'[i,2]=j\}}.
\end{aligned}
$$

Now,

$$
\begin{aligned}
A^2_{n,j} &\leq \frac{L^4}{G(L)^4} 1_{A_n(x)}(X_i), \\
B^2_{n,j} &\leq \frac{L^4}{G(L)^4} 1_{A_n(x)}(X'_i), \\
C^2_{n,j} &\leq \frac{L^4}{G(L)^4} \sum_{i\in\{1,\dots,n\}\setminus\{j\}} 1_{A_n(x)}(X_i) 1_{\{N[i,1]=j\}}, \\
&\quad \text{(by the Cauchy-Schwarz inequality)} \\
D^2_{n,j} &\leq \frac{L^4}{G(L)^4} \sum_{i\in\{1,\dots,n\}\setminus\{j\}} 1_{A_n(x)}(X_i) 1_{\{N'[i,1]=j\}}, \\
E^2_{n,j} &\leq \frac{L^4}{G(L)^4} \sum_{i\in\{1,\dots,n\}\setminus\{j\}} 1_{A_n(x)}(X_i) 1_{\{N[i,2]=j\}}, \\
F^2_{n,j} &\leq \frac{L^4}{G(L)^4} \sum_{i\in\{1,\dots,n\}\setminus\{j\}} 1_{A_n(x)}(X_i) 1_{\{N'[i,2]=j\}}.
\end{aligned}
$$

Considering now the terms $\sum_{j=1}^n \boldsymbol{E}\left\{A^2_{n,j}\right\}$ and $\sum_{j=1}^n \boldsymbol{E}\left\{B^2_{n,j}\right\}$, we have for them an upper bound

$$
\frac{L^4}{G(L)^4} n\mu(A_n(x)),
$$

respectively. Analogously, considering the terms $\sum_{j=1}^n \boldsymbol{E}\left\{C^2_{n,j}\right\}$, $\sum_{j=1}^n \boldsymbol{E}\left\{D^2_{n,j}\right\}$, $\sum_{j=1}^n \boldsymbol{E}\left\{E^2_{n,j}\right\}$ and $\sum_{j=1}^n \boldsymbol{E}\left\{F^2_{n,j}\right\}$, by changing the order of summation, for each of these terms we have an upper bound

$$
\frac{L^4}{G(L)^4} \boldsymbol{E}\left\{ \sum_{i\in\{1,\dots,n\}} 1_{A_n(x)}(X_i) \right\} \leq \frac{L^4}{G(L)^4} n\mu(A_n(x)).
$$

Thus

$$
\begin{aligned}
& \boldsymbol{E}\left\{\sum_{j=1}^{n}|U_n-U_{n,j}|^2\right\} \\
\leq\ & 6\boldsymbol{E}\left(\sum_{j=1}^{n}\boldsymbol{E}\left\{A_{n,j}^2\right\}+\sum_{j=1}^{n}\boldsymbol{E}\left\{B_{n,j}^2\right\}+\sum_{j=1}^{n}\boldsymbol{E}\left\{C_{n,j}^2\right\}\right. \\
& \left.+\sum_{j=1}^{n}\boldsymbol{E}\left\{D_{n,j}^2\right\}+\sum_{j=1}^{n}\boldsymbol{E}\left\{E_{n,j}^2\right\}+\sum_{j=1}^{n}\boldsymbol{E}\left\{F_{n,j}^2\right\}\right) \\
\leq\ & 6\cdot 6\frac{L^4}{G(L)^4}n\mu(A_n(x)),
\end{aligned}
$$

which, by the Efron-Stein inequality, yields the above bound of the variance. Then, we have

$$
\boldsymbol{Var}\left\{\widehat{\widehat{\sigma}}_n^2(x)\right\}\leq\frac{72\cdot L^4}{G(L)^4}\frac{1}{n\mu(A_n(x))}.
$$

By the well known relation for the mean squared error we get

$$
\boldsymbol{E}\left\{|\widehat{\widehat{\sigma}}_n^2(x)-\boldsymbol{E}\widehat{\widehat{\sigma}}_n^2(x)|\right\}\leq\sqrt{\boldsymbol{Var}(\widehat{\widehat{\sigma}}_n^2(x))}\leq 6\sqrt{2}\frac{L^2}{G(L)^2}\frac{1}{\sqrt{n\mu(A_n(x))}}. \tag{29}
$$

By the triangle inequality

$$
\boldsymbol{E}|\widehat{\widehat{\sigma}}_n^2(x)-\sigma^2(x)|\leq\boldsymbol{E}|\widehat{\widehat{\sigma}}_n^2(x)-\boldsymbol{E}\widehat{\widehat{\sigma}}_n^2(x)|+|\boldsymbol{E}\widehat{\widehat{\sigma}}_n^2(x)-\sigma^2(x)|, \tag{30}
$$

and, with $l_n=\#\{j: A_{n,j}\cap S\neq\emptyset\}$ we note, with some constant c

$$
\begin{aligned}
& \int_S\boldsymbol{E}|\widehat{\widehat{\sigma}}_n^2(x)-\boldsymbol{E}\widehat{\widehat{\sigma}}_n^2(x)|\mu(dx)\leq c\frac{1}{\sqrt{n}}\int_S\frac{1}{\sqrt{\mu(A_n(x))}}\mu(dx) \\
\leq\ & c\frac{1}{\sqrt{n}}\sqrt{\int_S\frac{1}{\mu(A_n(x))}\mu(dx)}=O\left(\sqrt{\frac{l_n}{n}}\right)=o\left(\left(n_n^{-d}n^{-1}\right)^{\frac{1}{2}}\right).
\end{aligned} \tag{31}
$$

Further

$$
\begin{aligned}
\boldsymbol{E}\widehat{\widehat{\sigma}}_n^2(x) &= \frac{\boldsymbol{E}\left(H_1 1_{A_n(x)}(X_1)\right)}{\mu(A_n(x))} \\
& \text{(by symmetry)} \\
&= \frac{\boldsymbol{E}\left(\boldsymbol{E}\left(H_1 1_{A_n(x)}(X_1)\right)|X_1\right)}{\mu(A_n(x))} \\
&= \frac{\boldsymbol{E}\left(\boldsymbol{E}\left(H_1|X_1\right)1_{A_n(x)}(X_1)\right)}{\mu(A_n(x))} \\
&= \frac{\boldsymbol{E}\left(\boldsymbol{E}\left(W_1|X_1\right)1_{A_n(x)}(X_1)\right)}{\mu(A_n(x))} \\
& \text{(by 21)} \\
&= \int\frac{\boldsymbol{E}\{W_1|X_1=z\}1_{A_n(x)}(z)}{\mu(A_n(x))}\mu(dz) \qquad (32) \\
&= \boldsymbol{E}\sigma_n^{2*}(x) \qquad (33)
\end{aligned}
$$

(see the proof of the McDiarmid inequality [8], Appendix A, with $\sigma_n^{2*}(x)$ defined by

$$
\sigma_n^{2*}(x):=\frac{\sum_{i=1}^{n}(Y_i-Y_{N[i,1]})(Y_i-Y_{N[i,2]})1_{A_n(x)}(X_i)}{n\mu(A_n(x))}. \tag{34}
$$

(34)). Then, according to the proof of the McDiarmid inequality, we have

$$\int_S \left|\sigma^2(x) - \boldsymbol{E}\widehat{\widehat{\sigma}}_n^2(x)\right| \mu(dx) = K_n := \int_S \left|\sigma^2(x) - \boldsymbol{E}\widehat{\widehat{\sigma}}_n^{2*}(x)\right| \mu(dx) \to 0 \tag{35}$$

(30), together with (31) and (35) yield the assertion. ■

Proof of Theorem 3.4 We begin by the following extension

$$\begin{aligned}
& \int |\widehat{\sigma}_n^2(x) - \sigma^2(x)|\mu(dx) \\
\leq\ & \int |\widehat{\sigma}_n^2(x) - \widehat{\widehat{\sigma}}_n^2(x)|\mu(dx) + \int |\widehat{\widehat{\sigma}}_n^2(x) - \sigma^2(x)|\mu(dx) \\
\leq\ & L_n + D_n.
\end{aligned}$$

It holds $D_n \xrightarrow{P} 0$ because of Lemma 3.5.
Now, concerning L_n, arguing as in Györfi et al. [8], p. 465, compare also the end of the proof of Theorem 3.1 in [7]

$$\begin{aligned}
& \int |\widehat{\sigma}_n^2(x) - \widehat{\widehat{\sigma}}_n^2(x)|\mu(dx) \\
\leq\ & \int \left| \frac{\sum_{i=1}^n H_i 1_{A_n(x)}(X_i)}{\sum_{i=1}^n 1_{A_n(x)}(X_i)} \right. \\
& \left. - \frac{\sum_{i=1}^n H_i 1_{A_n(x)}(X_i)}{n\mu(A_n(x))} \right| \mu(dx) \\
\leq\ & const \int \sum_{i=1}^n 1_{A_n(x)}(X_i) \left| \frac{1}{n\mu(A_n(x))} - \frac{1}{\sum_{i=1}^n 1_{A_n(x)}(X_i)} \right| \mu(dx) \\
& \text{(for some finite constant, because of } \textbf{(A2)} \text{ and (11))} \\
\leq\ & const \int \left| \sum_{i=1}^n \frac{1_{A_n(x)}(X_i)}{n\mu(A_n(x))} - 1 \right| \mu(dx) \to 0
\end{aligned}$$

because of (26) and (27), which proves the theorem. ■

4 Censored Partitioning Estimation via Nearest Neighbors under Unknown Survival Functions

As already treated in the previous section the survival function G is typically unknown and has to be estimated, by the Kaplan-Meier estimator. We introduce now the final result, in order to show consistency of the partitioning estimator of the local variance based on the first and the second neighbor under censoring and unknown survival function.
Let

$$\widetilde{\sigma}_n^2(x) := \frac{\sum_{i=1}^n H_{i,G_n} 1_{A_n(x)}(X_i)}{\sum_{i=1}^n 1_{A_n(x)}(X_i)} \tag{36}$$

where

$$\begin{aligned}
H_{i,G_n} &:= H_{n,i,G_n} \\
&= \frac{\delta_i T_i^2}{G_n(T_i)} - \frac{\delta_i T_i}{G_n(T_i)} \frac{\delta_{N[i,1]} T_{N[i,1]}}{G_n(T_{N[i,1]})} - \frac{\delta_i T_i}{G_n(T_i)} \frac{\delta_{N[i,2]} T_{N[i,2]}}{G_n(T_{N[i,2]})} \\
&\quad + \frac{\delta_{N[i,1]} T_{N[i,1]}}{G_n(T_{N[i,1]})} \frac{\delta_{N[i,2]} T_{N[i,2]}}{G_n(T_{N[i,2]})}
\end{aligned}$$

Then for this estimator a consistency result holds; we prove this as follows.

Theorem 4.1 *Under the assumptions of Theorem 3.4,*

$$\int |\widetilde{\sigma}_n^2(x) - \sigma^2(x)|\mu(dx) \xrightarrow{P} 0.$$

Proof Introduce the following modification of the estimator (36)

$$\widehat{\widehat{\sigma}}_n^2(x) := \frac{\sum_{i=1}^n H_{i,G_n} 1_{A_n(x)}(X_i)}{n\mu(A_n(x))}. \tag{37}$$

We note

$$\begin{aligned}
& \int |\widetilde{\sigma}_n^2(x) - \sigma^2(x)|\mu(dx) \\
\leq \quad & \int |\widetilde{\sigma}_n^2(x) - \widehat{\widehat{\sigma}}_n^2(x)|\mu(dx) + \int |\widehat{\widehat{\sigma}}_n^2(x) - \widehat{\widehat{\sigma}}_n^2(x)|\mu(dx) \\
& + \int |\widehat{\widehat{\sigma}}_n^2(x) - \sigma^2(x)|\mu(dx) \\
= \quad & A_n + B_n + C_n,
\end{aligned}$$

with $\widehat{\widehat{\sigma}}_n^2$ defined by (28). But $C_n \xrightarrow{P} 0$ *a.s.* by Lemma 3.5.
Now, concerning A_n

$$\begin{aligned}
& \int |\widetilde{\sigma}_n^2(x) - \widehat{\widehat{\sigma}}_n^2(x)|\mu(dx) \\
\leq \quad & \frac{\sum_{i=1}^n H_{i,G_n} 1_{A_n(x)}(X_i)}{\sum_{i=1}^n 1_{A_n(x)}(X_i)} - \frac{\sum_{i=1}^n H_{i,G_n} 1_{A_n(x)}(X_i)}{n\mu(A_n(x))} \\
\leq \quad & U^* \int \left| \frac{\sum_{i=1}^n 1_{A_n(x)(X_i)}}{n\mu(A_n(x))} \right| \mu(dx) \to 0, \quad a.s.
\end{aligned}$$

for some random variable $U^* < \infty$ (see [8], p. 465, by (13) and the boundedness of C).
Finally, concerning B_n

$$\begin{aligned}
& \int |\widehat{\widehat{\sigma}}_n^2(x)^{(NN)} - \widehat{\widehat{\sigma}}_n^2(x)|\mu(dx) \\
= \quad & \int \left| \frac{\sum_{i=1}^n H_{i,G_n} 1_{A_n(x)}(X_i)}{n\mu(A_n(x))} - \frac{\sum_{i=1}^n H_i 1_{A_n(x)}(X_i)}{n\mu(A_n(x))} \right| \mu(dx) \\
= \quad & \int \left| \frac{1}{n} \frac{\sum_{i=1}^n \left[H_{i,G_n} - H_i\right] 1_{A_n(x)}(X_i)}{\mu(A_n(x))} \right| \mu(dx) \\
\leq \quad & \frac{1}{n}\sum_{i=1}^n |H_{i,G_n} - H_i| \qquad \underbrace{\int \frac{1_{A_n(x)}(X_i)}{\mu(A_n(x))}\mu(dx).}_{\leq 1 \text{ because of } \mu(A_n(x)) = \mu(A_n(X_i)) \text{ for } X_i \in A_n(x)}
\end{aligned} \tag{38}$$

But

$$\begin{aligned}
& \frac{1}{n}\sum_{i=1}^n |H_{i,G_n} - H_i| \\
\leq \quad & \frac{1}{n}\sum_{i=1}^n \Bigg\{ \left| \frac{\delta_i T_i^2}{G_n(T_i)} - \frac{\delta_i T_i^2}{G(T_i)} \right| \\
& + \left| \frac{\delta_i T_i}{G_n(T_i)} \frac{\delta_{N[i,1]} T_{N[i,1]}}{G_n(T_{N[i,1]})} - \frac{\delta_i T_i}{G(T_i)} \frac{\delta_{N[i,1]} T_{N[i,1]}}{G(T_{N[i,1]})} \right| \\
& + \left| \frac{\delta_i T_i}{G_n(T_i)} \frac{\delta_{N[i,2]} T_{N[i,2]}}{G_n(T_{N[i,2]})} - \frac{\delta_i T_i}{G(T_i)} \frac{\delta_{N[i,2]} T_{N[i,2]}}{G(T_{N[i,2]})} \right| \\
& + \left| \frac{\delta_{N[i,1]} T_{N[i,1]}}{G_n(T_{N[i,1]})} \frac{\delta_{N[i,2]} T_{N[i,2]}}{G_n(T_{N[i,2]})} - \frac{\delta_{N[i,1]} T_{N[i,1]}}{G(T_{N[i,1]})} \frac{\delta_{N[i,2]} T_{N[i,2]}}{G(T_{N[i,2]})} \right| \Bigg\}
\end{aligned}$$

$$=: \quad \frac{1}{n}\sum_{i=1}^{n}(P_{n,i}+O_{n,i}+I_{n,i}+U_{n,i}). \tag{39}$$

Now, concerning $P_{n,i}$,

$$\frac{1}{n}\sum_{i=1}^{n}P_{n,i} \leq \frac{L}{n}\sum_{i=1}^{n}\delta_i T_i \left|\frac{1}{G_n(T_i)}-\frac{1}{G(T_i)}\right| \to 0 \quad a.s.$$

due to Lemma 26.1 in [8].

Finally, concerning $U_{n,i}$, (and similarly, for $O_{n,i}$ and $I_{n,i}$) we recall (12) and notice that, for the ordered sequence of the variables of the first (and second) neighbors we get, with obvious meaning of the notation,

$$T_{N[(1),1]} \leq T_{N[(2),1]} \leq \cdots \leq T_{N[(n),1]} \leq T_K \leq L \quad a.s.,$$
$$T_{N[(1),2]} \leq T_{N[(2),2]} \leq \cdots \leq T_{N[(n),2]} \leq T_K \leq L \quad a.s.$$

and, with same positive random variable U^*

$$\begin{aligned}&1 \geq G_n(T_{N[(1),1]}) \geq G_n(T_{N[(2),1]}) \geq \cdots \geq G_n(T_{N[(n),1]})\\ &\geq G_n(T_K) \geq G_n(L) \geq U^* > 0 \quad a.s.,\end{aligned}$$

$$\begin{aligned}&1 \geq G_n(T_{N[(1),2]}) \geq G_n(T_{N[(2),2]}) \geq \cdots \geq G_n(T_{N[(n),2]})\\ &\geq G_n(T_K) \geq G_n(L) \geq U^* > 0 \quad a.s.,\end{aligned} \tag{40}$$

respectively, because of $G_n(L) \to G(L) > 0$ $a.s.$ by [8], Theorem 26.1. By this

$$H_{i,G_n} < U^{**} < \infty \quad (i=1,\ldots,n,\ n \in N) \quad a.s. \tag{41}$$

with some positive random variable U^{**}.

Now,

$$\begin{aligned}
\left|\frac{1}{n}\sum_{i=1}^{n}U_{n,i}\right| &= \left|\frac{1}{n}\sum_{i=1}^{n}\left[\frac{\delta_{N[i,1]}T_{N[i,1]}}{G_n(T_{N[i,1]})}\frac{\delta_{N[i,2]}T_{N[i,2]}}{G_n(T_{N[i,2]})}-\frac{\delta_{N[i,1]}T_{N[i,1]}}{G(T_{N[i,1]})}\frac{\delta_{N[i,2]}T_{N[i,2]}}{G(T_{N[i,2]})}\right]\right|\\
&\leq L^2\frac{1}{n}\sum_{i=1}^{n}\left|\frac{1}{G_n(T_{N[i,1]})G_n(T_{N[i,2]})}-\frac{1}{G(T_{N[i,1]})G(T_{N[i,2]})}\right|\\
&\leq L^2\frac{1}{n}\sum_{i=1}^{n}\frac{1}{G_n(T_{N[i,1]})}\left|\frac{1}{G_n(T_{N[i,2]})}-\frac{1}{G(T_{N[i,2]})}\right|\\
&\quad +L^2\frac{1}{n}\sum_{i=1}^{n}\frac{1}{G(T_{N[i,2]})}\left|\frac{1}{G_n(T_{N[i,1]})}-\frac{1}{G(T_{N[i,1]})}\right|\\
&\leq L^2\frac{1}{n}\frac{1}{G_n^2(L)G(L)}\sum_{i=1}^{n}\left|G_n(T_{N[i,2]})-G(T_{N[i,2]})\right|\\
&\quad +L^2\frac{1}{n}\frac{1}{G_n^2(L)G(L)}\sum_{i=1}^{n}\left|G_n(T_{N[i,1]})-G(T_{N[i,1]})\right|\\
&\leq 2L^2\frac{1}{G(L)G_n(L)}\sup_{0\leq t<\infty}|G_n(t)-G(t)|
\end{aligned} \tag{42}$$

$\to 0 \quad a.s.$ by (13) and because of the result $\sup_{0\leq t\leq T_K}|G_n(t)-G(t)| \to 0$ $a.s.$, due to [26] (compare [15], Theorem 10).

This completes the proof. ∎

5 Rate of Convergence

Finally, the following theorem states a rate of convergence for the estimator (36).

Theorem 5.1 *Let the assumptions **(A1)-(A3)** hold. Let the estimate $\widetilde{\sigma}^2$ be given by (36) with cubic partition of R^d with side length h_n of the cubes ($n \in N$). Moreover, assume the Lipschitz conditions*

$$|m(x) - m(t)| \leq \Gamma \|x - t\|^{\alpha},\ x, t \in R^d,$$

and

$$|\sigma^2(x) - \sigma^2(t)| \leq \Lambda \|x - t\|^{\beta},\ x, t \in R^d,$$

($0 < \alpha \leq 1,\ 0 < \beta \leq 1,\ \Gamma,\ \Lambda \in R_+,\ \|\ \|$ denoting the Euclidean norm).
Then, with

$$h_n \sim n^{-\frac{1}{d+2\beta}}$$

one gets

$$\int |\widetilde{\sigma}_n^2 - \sigma^2(x)|\mu(dx) = O_P\left(\left(\frac{\log n}{n}\right)^{\frac{1}{6}} + \max\left\{n^{-\frac{2\alpha}{d}}, n^{-\frac{\beta}{2\beta+d}}\right\}\right)$$

Proof We note, with (37) and (28),

$$\begin{aligned} & \int |\widetilde{\sigma}_n^2(x)^{(NN)} - \sigma^2(x)|\mu(dx) \\ \leq\ & \int |\widetilde{\sigma}_n^2(x)^{(NN)} - \widehat{\widetilde{\sigma}}_n^2(x)^{(NN)}|\mu(dx) + \int |\widehat{\widetilde{\sigma}}_n^2(x)^{(NN)} - \widehat{\widetilde{\sigma}}_n^2(x)|\mu(dx) \\ & + \int |\widehat{\widetilde{\sigma}}_n^2(x) - \sigma^2(x)|\mu(dx) \\ \leq\ & A_n + B_n + C_n. \end{aligned}$$

Now, concerning A_n

$$\begin{aligned} & \int |\widetilde{\sigma}_n^2(x)^{(NN)} - \widehat{\widetilde{\sigma}}_n^2(x)^{(NN)}|\mu(dx) \\ =\ & \int \left| \frac{\sum_{i=1}^n H_{i,G_n} 1_{A_n(x)}(X_i)}{\sum_{i=1}^n 1_{A_n(x)}(X_i)} - \frac{\sum_{i=1}^n H_{i,G_n} 1_{A_n(x)}(X_i)}{n\mu(A_n(x))} \right| \mu(dx) \\ \leq\ & U^{**} \int \sum_{i=1}^n 1_{A_n(x)}(X_i) \left| \frac{1}{\sum_{i=1}^n 1_{A_n(x)}(X_i)} - \frac{1}{n\mu(A_n(x))} \right| \mu(dx) \\ & \text{(a.s., with a random variable } U^{**} < \infty,\ \text{because of (41))} = O_P\left(n^{-\frac{1}{2}} h_n^{-\frac{d}{2}}\right) \end{aligned}$$

by the proof of Theorem 4.3 in [8].
Moreover

$$\begin{aligned} B_n &= \int |\widehat{\widetilde{\sigma}}_n^2(x)^{(NN)} - \widehat{\widetilde{\sigma}}_n^2(x)|\mu(dx) \\ &\leq \frac{1}{n}\sum_{i=1}^n |H_{i,G_n} - H_i| \\ &\quad \text{(see (38))} \\ &\leq \frac{1}{n}\sum_{i=1}^n (P_{n,i} + O_{n,i} + I_{n,i} + U_{n,i}) \\ &\quad \text{(see (39))} \end{aligned}$$

Now, concerning $P_{n,i}$,

$$\frac{1}{n}\sum_{i=1}^{n} P_{n,i} \leq \frac{L}{n}\sum_{i=1}^{n}\delta_i T_i \left|\frac{1}{G_n(T_i)} - \frac{1}{G(T_i)}\right|$$
$$\leq \quad \frac{L^2}{n}\sum_{i=1}^{n} \sup_{0\leq t\leq T_i} |G_n(t) - G(t)| = O_P\left(\left(\frac{\log n}{n}\right)^{\frac{1}{6}}\right),$$

the latter by the Cauchy-Schwarz inequality and the proof of Satz 4 in [15].
Finally, concerning $U_{n,i}$, (and similarly, for $O_{n,i}$ and $I_{n,i}$) we note that, instead of (42), one also obtains, by [8], Corollary 6.1, with a suitable constant γ_d,

$$\begin{aligned}
\left|\frac{1}{n}\sum_{i=1}^{n} U_i\right| &= \left|\frac{1}{n}\sum_{i=1}^{n}\left[\frac{\delta_{N[i,1]}T_{N[i,1]}}{G_n(T_{N[i,1]})}\frac{\delta_{N[i,2]}T_{N[i,2]}}{G_n(T_{N[i,2]})} - \frac{\delta_{N[i,1]}T_{N[i,1]}}{G(T_{N[i,1]})}\frac{\delta_{N[i,2]}T_{N[i,2]}}{G(T_{N[i,2]})}\right]\right| \\
&\leq 2\gamma_d L^2 \frac{1}{n}\frac{1}{G_n^2(L)G(L)}\sum_{i=1}^{n}|G_n(T_i) - G(T_i)| \\
&\leq 2\gamma_d L^2 \frac{1}{G_n^2(L)G(L)}\sup_{0\leq t\leq T_K}|G_n(t) - G(t)| \\
&= O_P\left(\left(\frac{\log n}{n}\right)^{\frac{1}{6}}\right),
\end{aligned}$$

the latter as before. It remains to give a rate for

$$C_n = \int |\widehat{\widehat{\sigma}}_n^2(x) - \sigma^2(x)|\mu(dx).$$

For that introduce again the expansion

$$\begin{aligned}
&\boldsymbol{E}\left\{\int \left|\widehat{\widehat{\sigma}}_n^2(x) - \sigma^2(x)\right|\mu(dx)\right\} \\
\leq\ &\boldsymbol{E}\left\{\int \left|\widehat{\widehat{\sigma}}_n^2(x) - \boldsymbol{E}\widehat{\widehat{\sigma}}_n^2(x)\right|\mu(dx)\right\} + \int \left|\boldsymbol{E}\widehat{\widehat{\sigma}}_n^2(x) - \sigma^2(x)\right|\mu(dx),
\end{aligned}$$

where

$$\boldsymbol{E}\left\{\int \left|\widehat{\widehat{\sigma}}_n^2(x) - \boldsymbol{E}\widehat{\widehat{\sigma}}_n^2(x)\right|\mu(dx)\right\} = O\left(\sqrt{\frac{l_n}{n}}\right) = O\left(\left(n_n^{-d}n^{-1}\right)^{\frac{1}{2}}\right),$$

as in (31).
Now,

$$\begin{aligned}
&\boldsymbol{E}\widehat{\widehat{\sigma}}_n^2(x) \\
= &\int \frac{\boldsymbol{E}\{W_1|X_1 = z\}1_{A_n(x)}(z)}{\mu(A_n(x))}\mu(dz) \\
&\text{(according to (32))} \\
&\int \frac{\boldsymbol{E}\{(Y_1 - m(X_1))^2|X_1 = z\}1_{A_n(x)}(z)}{\mu(A_n(x))}\mu(dz) \\
&+ \int \frac{\boldsymbol{E}\{(m(X_1) - m(X_{N[1,1]}))(m(X_1) - m(X_{N[1,2]}))|X_1 = z\}1_{A_n(x)}(z)}{\mu(A_n(x))}\mu(dz) \\
&\text{(as in (24)).}
\end{aligned}$$

Then

$$\int \left|\boldsymbol{E}\widehat{\widehat{\sigma}}_n^2(x) - \sigma^2(x)\right|\mu(dx)$$

$$
\begin{aligned}
&\leq \int \left| \frac{(Y_1 - m(X_1))^2 | X_1 = z\} 1_{A_n(x)}(z)}{\mu(A_n(x))} \mu(dz) - \sigma^2(x) \right| \mu(dx) \\
&\quad + \int \boldsymbol{E} \left\{ (m(X_1) - m(X_{N[1,1]}))(m(X_1) - m(X_{N[1,2]})) | X_1 = z \right\} \\
&\quad \text{(because of } \int \frac{1}{\mu(A_n(z))} 1_{A_n(z)} \mu(dx)\mu(dz) \leq 1) \\
&\leq \Lambda h_n + \boldsymbol{E} \left\{ (m(X_1) - m(X_{N[1,1]}))(m(X_1) - m(X_{N[1,2]})) \right\} \\
&\leq \Lambda h_n + \left(\boldsymbol{E} \| X_1 - X_{N[1,1]} \|^{2\alpha} \right)^{\frac{1}{2}} \left(\boldsymbol{E} \| X_1 - X_{N[1,2]} \|^{2\alpha} \right)^{\frac{1}{2}} \\
&= O\left(h_n + n^{-\frac{2\alpha}{d}} \right)
\end{aligned}
$$

Therefore

$$
\begin{aligned}
&\boldsymbol{E} \left\{ \int \left| \widehat{\widehat{\sigma}}_n^2(x) - \sigma^2(x) \right| \mu(dx) \right\} \\
&\leq O\left(h_n^{-\frac{d}{2}} n^{-\frac{1}{2}} \right) + O\left(n^{-\frac{2\alpha}{d}} + h_n \right).
\end{aligned}
$$

and finally,

$$
\begin{aligned}
&\int \left| \widehat{\widehat{\sigma}}_n^2(x) - \sigma^2(x) \right| \mu(dx) \\
&O_P\left(h_n^{-\frac{d}{2}} n^{-\frac{1}{2}} \right) + O\left(n^{-\frac{2\alpha}{d}} + h_n \right).
\end{aligned}
$$

Now, summarizing

$$
\begin{aligned}
&\int |\widetilde{\sigma}_n^{2\ (NN)} - \sigma^2(x)| \mu(dx) \\
&= O_P\left(n^{-\frac{1}{2}} h_n^{-\frac{d}{2}} \right) + O_P\left(\left(\frac{\log n}{n} \right)^{\frac{1}{6}} \right) \\
&\quad + O_P\left(h_n^{-\frac{d}{2}} n^{-\frac{1}{2}} + n^{-\frac{2\alpha}{d}} + \left(\frac{\log n}{n} \right)^{\frac{1}{6}} \right) \\
&= O_P\left(\left(\frac{\log n}{n} \right)^{\frac{1}{6}} + \max\left\{ n^{-\frac{2\alpha}{d}}, n^{-\frac{\beta}{2\beta+d}} \right\} \right)
\end{aligned}
$$

and hence assertion. ■

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