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# Simple groups acting two-transitively on the set of generators of a finite elation Laguerre plane

Günter F. Steinke, Markus J. Stroppel*

**Abstract**

We show that finite elation Laguerre planes with a group of automorphisms acting two-transitively on the set of generators are Miquelian. In this paper we discuss in detail four series of groups of (twisted) Lie type that may possibly occur as socles of the two-transitive group induced on the set of generators; all other cases have been treated in a separate paper.

**MSC 2010:** 51E25, 51B15, 20B20.

**Keywords:** Laguerre plane, elation group, two-transitive group, socle, simple group of Lie type, unitary group, Ree group, Suzuki group.

## Introduction

Elation Laguerre planes were introduced in [27] and [22]. They seem to play a role similar to translation planes among the projective planes; at least in the infinite case where several constructions of non-classical (i.e., non-Miquelian) elation Laguerre planes are known. All finite elation Laguerre planes (in fact, all finite Laguerre planes) known to date are ovoidal; they are even Miquelian if the order is odd. However, it is not clear whether this situation is due to the fact that no other examples exist, or to the fact that the appropriate constructions have not yet been found.

We refer the reader to the more detailed introduction in [25], and collect in the sequel just briefly some basic facts that we need in the present paper.

Recall that a finite Laguerre plane $\mathcal{L} = (P, \mathcal{C}, \mathcal{G})$ of order $n$ consists of a set $P$ of $n(n+1)$ points, a set $\mathcal{C}$ of $n^3$ circles and a set $\mathcal{G}$ of $n+1$ generators (or parallel classes), where circles and generators are both subsets of $P$, such that the following three axioms are satisfied.

**Axioms for Laguerre planes.**

(G) $\mathcal{G}$ partitions $P$, each generator contains $n$ points, and there are $n+1$ generators.

(C) Each circle intersects each generator in precisely one point.

(J) Three points no two of which are on the same generator are joined by a unique circle.

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The *internal incidence structure* $\mathbb{A}_x$ at a point $x \in P$ has the collection of all points not on the generator $[x]$ through $x$ as point set and, as lines, all circles passing through $x$ (without the point $x$) and all generators apart from $[x]$. The axioms for a Laguerre plane yield easily that $\mathbb{A}_x$ is an affine plane, the *derived affine plane at x*. Circles through $x$ are called *touching in x* if they are equal or have no other point in common. Circles that touch each other in $x$ give parallel lines in $\mathbb{A}_x$.

The group $\mathrm{Aut}(\mathcal{L})$ of a Laguerre plane $\mathcal{L}$ acts on the set $\mathcal{G}$ of generators; the kernel $\Delta$ of that action is a normal subgroup. We call $\mathcal{L}$ an *elation Laguerre plane* if $\Delta$ acts transitively on the set $C$ of circles. It is known (see [22, 1.3]) that in every finite elation Laguerre plane the group $\Delta$ has a (unique) regular normal subgroup $E$; this group will be called the *elation group*. It is also known that $E$ is elementary abelian; the derivation $\mathbb{A}_x$ of an elation Laguerre plane is a dual translation plane with the stabilizer $E_x$ inducing the full group of translations on the dual. In particular, the order $n$ of a finite elation Laguerre plane is a power of a prime.

While it seems hopeless in the present situation to prove that all finite Laguerre planes are ovoidal, it appears sensible to study Laguerre planes under additional homogeneity assumptions. Doubly transitive groups of automorphisms have been investigated for various classes of geometries, see for example [6], [21], [12], [11], [17], [10]. In the present note we complete the proof that finite elation Laguerre planes with automorphism groups acting two-transitively on the set of generators are Miquelian.

In [25] we treat the cases where some two-transitive subgroup $\Gamma$ either contains a normal subgroup isomorphic to $\mathrm{A}_{q+1}$ (for $q \geq 5$), or a regular abelian normal subgroup. The cases of two-transitive actions that do not fit arithmetically are also discussed (and disposed of) there. The focus in the present paper is on the four infinite series of two-transitive groups where the socle is isomorphic either to a linear group $\mathrm{PSL}(2, q)$ (of Lie type $\mathrm{A}_1(q)$), or a unitary group $\mathrm{PSU}(3, f^2)$ (of Lie type ${}^2\mathrm{A}_2(f^2)$), a Suzuki group $\mathrm{Sz}(2^{2a+1})$ (of Lie type ${}^2\mathrm{B}_2(2^{2a+1})$), a Ree group $\mathrm{R}(3^{2a+1})$ (of Lie type ${}^2\mathrm{G}_2(3^{2a+1})$. The commutator group $\mathrm{R}(3)' \cong \mathrm{PSL}(2, 8)$ of the smallest (and non-simple) Ree group is also included here. It turns out (see 9.1 below) that the only case that actually occurs is the first one (of linear groups); the Laguerre plane has to be the Miquelian plane.

## 1 Notation and basic facts

We consider a finite elation Laguerre plane $\mathcal{L} = (P, C, \mathcal{G})$ of order $q$, and assume that $\mathrm{Aut}(\mathcal{L})$ acts two-transitively on the set $\mathcal{G}$ of generators.

**1.1 Notation.** We fix names for objects that will play their roles in the discussion.

- $\Delta := \mathrm{Aut}(\mathcal{L})_{[\mathcal{G}]}$ denotes the kernel of the action of $\mathrm{Aut}(\mathcal{L})$ on $\mathcal{G}$. We assume that $\Delta$ acts transitively on the set $C$ of circles.
- $E$ is the elation group of the Laguerre plane, i.e., the (unique, normal, and abelian) subgroup of $\Delta$ that acts regularly on $C$, see [22, 1.3]. Note that $\mathrm{Aut}(\mathcal{L}) = \mathrm{Aut}(\mathcal{L})_K E$ holds for each circle $K \in C$.
- The order of $\mathcal{L}$ is a power of some prime $r$; we write $q = r^e$.
- The generator containing $c$ will be denoted by $[c]$. (Square brackets will also be used to denote various other objects; usually in the context of partitions/equivalence relations.)
- For any point $\infty$, we have the affine plane $\mathbb{A}_\infty$ obtained by derivation at $\infty$; we write $T < E_\infty$ for the group of translations along the generators (i.e., in the vertical direction) in $\mathbb{A}_\infty$.

We remark that two-transitivity of $\operatorname{Aut}(\mathcal{L})$ on $\mathcal{G}$ is equivalent to the existence of a circle $K$ such that some subgroup $\Gamma$ of the stabilizer $\operatorname{Aut}(\mathcal{L})_K$ acts two-transitively on $K$. Transitivity on $\mathcal{G}$ appears to be the more natural assumption as it avoids the choice of a circle. In the proofs, however, it will be convenient to study the actions of $\Gamma$ on $K$, on $P \setminus K$, on $C$ and on $\mathcal{G}$.

**1.2 Lemma.** *Let $\mathcal{L}$ be a Laguerre plane of finite order $q$, and let $\sigma$ be an automorphism of $\mathcal{L}$ fixing some circle $K$. If $\sigma$ is an involution, then one of the following holds.*

1. *$\sigma$ fixes every point on $K$;*
2. *The order $q$ is a square and $\sigma$ fixes precisely $1 + \sqrt{q}$ points on $K$;*
3. *$\sigma$ fixes at most two points on $K$.*

*Proof.* Suppose that $\sigma$ fixes at least three points on $K$ (so we exclude case 3), and let $\infty$ be one of them.

In the projective closure $\mathbb{P}_\infty$ of $\mathbb{A}_\infty$ the automorphism $\sigma$ induces an involutory collineation $\sigma'$. By a theorem of Baer [4], we know that either $\sigma'$ has an axis, or $q$ is a square and the fixed elements of $\sigma'$ form a (projective) subplane of order $\sqrt{q}$ in $\mathbb{P}_\infty$. In the former case the axis is the line induced by $K$, and we have case 1. In the latter case $\sigma'$ fixes precisely $1 + \sqrt{q}$ points on the line induced by $K$, and we have case 2. □

## 2 Covering the simple socle

Let $\mathcal{L}$ be an elation Laguerre plane of order $q$, and let $K$ be an arbitrary circle. The following information is taken from [22, Theorem 3], see also [27].

**2.1 Lemma.** *The stabilizer $\Delta_K$ is a subgroup of the multiplicative group of the kernel of the dual translation plane obtained as derivation at any point of $K$. This kernel is a finite field, whence $\Delta_K$ is cyclic of some order dividing $q - 1$. Moreover, the group $\Delta_K$ acts semi-regularly on $P \setminus K$.* □

Assume that $\Gamma \le \operatorname{Aut}(\mathcal{L})_K$ acts two-transitively on $K$, and induces a group $\pi(\Gamma)$ with simple socle $F$ on $K$. Let $\Phi$ denote the stationary term of the commutator series of the full pre-image of $F$ under the restriction map

$$\pi\colon \Gamma \to \mathrm{S}_K\colon \gamma \mapsto \pi(\gamma) := \gamma|_K\,;$$

here $\mathrm{S}_K \cong \mathrm{S}_{q+1}$ is the group of all permutations of the set $K$. Then $\Phi$ is perfect (i.e., coincides with its commutator group). As the simple group $F$ coincides with each term of its commutator series, we have $\pi(\Phi) = F$. The kernel $\Delta_K \cap \Gamma$ of $\pi$ is cyclic, its automorphism group is thus abelian, and the perfect group $\Phi$ centralizes $\Delta_K$. This means:

**2.2 Lemma.** *The group $\Phi \le \Gamma$ is a perfect central extension of the socle $F$, and its center $\Phi \cap \Delta_K$ is a quotient of the Schur multiplier of $F$.* □

**2.3 Remark.** In general, the group $\mathrm{SL}(2, q)$ is the universal covering of $\mathrm{PSL}(2, q)$; there are only two exceptions to this rule (see [14, 25.7], cf. [3, 3.3.6]):

1. $\mathrm{SL}(2, 4) = \mathrm{PSL}(2, 4) \cong \mathrm{A}_5 \cong \mathrm{PSL}(2, 5)$ has a double cover, namely $\mathrm{SL}(2, 5)$.
2. The Schur multiplier of $\mathrm{PSL}(2, 9) \cong \mathrm{A}_6$ is cyclic of order 6, see [3, 33.15].

In the present investigation, these two types of groups are interesting in their actions on Laguerre planes of order $q \le 9$. Such Laguerre planes are known to be ovoidal (and even Miquelian if $q \ne 8$), see [5], [23], [24]. However, we will study the coverings of $\mathrm{PSL}(2,4)$ and of $\mathrm{PSL}(2,9)$ anyway, because they can be excluded quite easily (see 8.3 and 8.10 below).

For $f > 2$, the group[1] $\mathrm{SU}(3, f^2)$ is the universal covering of $\mathrm{PSU}(3, f^2)$. The Schur multiplier is isomorphic to the center of $\mathrm{SU}(3, f^2)$, and thus cyclic of order $\gcd(3, f+1)$ in these cases; see [8, Thm. 2]. Recall that the group $\mathrm{PSU}(3, 2^2)$ is isomorphic to a subgroup of $\mathrm{AGL}(2, \mathbb{F}_3)$, and thus solvable (cf. [7, Ch. II, § 4]).

For $s > 1$, the Schur multiplier of $\mathrm{Sz}(2^{2s+1})$ is trivial (see [1], cf. [3, 4.2.4]). The multiplier of $\mathrm{Sz}(2^3)$ is elementary abelian of order 4, and $\mathrm{Sz}(2) \cong \mathrm{AGL}(1,5)$ is not perfect. (Incidentally, there is a two-transitive subgroup of $\mathrm{S}_5 \cong \mathrm{PGL}(2,4)$ isomorphic to $\mathrm{AGL}(1,5)$ acting two-transitively on a line of the Miquelian plane of order 4, see [25].) The Ree groups $\mathrm{R}(3^{2t+1})$ have trivial multiplier if $t > 1$, see [1]. The group $\mathrm{R}(3) \cong \mathrm{P\Gamma L}(2,8)$ is not perfect, it contains the perfect group $\mathrm{PSL}(2,8)$ as a normal subgroup of index 3. Note that the latter has trivial multiplier.

## 3 Zsigmondy groups

**3.1 Definition.** Zsigmondy [28] has proved the following. If $a, b$ are co-prime positive integers, then for any natural number $n > 1$ there is a prime number $z$ that divides $a^n - b^n$ but does not divide $a^k - b^k$ for any positive integer $k < n$, with the following exceptions:

- $\{a, b\} = \{1, 2\}$, and $n = 6$; or
- $a + b$ is a power of two, and $n = 2$.

We will use this in the case where $a = r$ and $b = 1$, for $n = e$ (so $q = a^n$). Then Zsigmondy's Theorem implies that there is a prime $z$ such that the multiplicative group of $\mathbb{F}_q$ contains a subgroup $Z$ of order $z$ but no proper subfield of $\mathbb{F}_q$ contains such a multiplicative subgroup — unless either $q = 2^6$, or $r = 2^m - 1$ is a Mersenne prime and $q = r^2$. Such a subgroup will be called a Zsigmondy subgroup, for short.

**3.2 Lemma.** *Let $Z$ be a Zsigmondy subgroup of the multiplicative group of $\mathbb{F}_{r^e}$ for some prime $r$. Then every non-trivial simple module $V$ of $Z$ over $\mathbb{F}_r$ has dimension $e$, and the centralizer of $Z$ acts semi-regularly on $V \setminus \{0\}$.*

*Proof.* Schur's Lemma [16, 3.5, p. 118] yields that there is a field $F$ (of characteristic $r$) such that the endomorphisms induced by elements of $Z$ on $V$ are multiplications by scalars from $F$. As no proper subfield of $\mathbb{F}_{r^e}$ contains a group isomorphic to $Z$, the field $F$ contains a copy of $\mathbb{F}_{r^e}$, and $V$ is a vector space over $\mathbb{F}_{r^e}$. Now $Z$ acts by scalars from $\mathbb{F}_{r^e}$, and the irreducible module $V$ has dimension 1 over $\mathbb{F}_{r^e}$. Thus the dimension over the prime field $\mathbb{F}_r$ is $e$. The assertion about the centralizer follows from Schur's Lemma, again. □

[1] In the symbol for the unitary group, we give the order $f^2$ of the quadratic extension field, rather than the order $f$ of the ground field.

## 4 Even order: regular action of the torus

Let $\Gamma \leq \mathrm{Aut}(\mathcal{L})_K$ be two-transitive on $K$, and assume that the socle $F$ of the group $\pi(\Gamma) \leq S_K$ is simple. We refer to the stabilizer $D$ of the two points $\infty$ and $o$ as a *torus* in $F$. The torus $D$ acts semi-regularly on $K \setminus \{\infty, o\}$. In its pre-image $\tilde{D}$, the stabilizer of any point outside $[\infty] \cup [o]$ is therefore contained in $\Delta$.

**4.1 Notation.** For any subgroup $X$ of $F$ we denote the preimage $\{\xi \in \Phi \mid \pi(\xi) \in X\}$ in $\Phi$ by $\tilde{X}$. In cases where we know that the restriction $\pi|_\Phi$ of $\pi$ to $\Phi$ is injective, we will suppress this notation, identifying $\tilde{X}$ with $X$.

Pick an element $\tilde{\iota} \in \Phi$ that induces an involution $\iota$ on $K$ interchanging the two fixed points of the torus. (If we know that $\pi|_\Phi$ is injective, we will also identify $\tilde{\iota}$ with $\iota$.)

In any one of the simple socles we will be dealing with in the sequel, the torus $D$ will be a cyclic group (see 5.1, 6.1, and 8.1), and often its full pre-image $\tilde{D}$ in $\Phi$ will also be cyclic. Then the following result applies.

**4.2 Lemma.** *Assume that the torus $D$ is cyclic. For any subgroup $B \leq D$, the pre-image $\tilde{B} \leq \Phi$ is normalized by $\tilde{\iota}$; in fact, the involution induces inversion on $D$. Consequently, the actions of $\tilde{D}$ on the two generators $[\infty]$ and $[o]$ are quasi-equivalent. If $\tilde{D}$ is also cyclic then $\tilde{\iota}$ normalizes any subgroup of $\tilde{D}$ (not only the full pre-images of subgroups of $D$).*

*Proof.* The assertion follows from the fact that subgroups of cyclic groups are invariant under arbitrary automorphisms (because they are characterized by their orders). □

**4.3 Corollary.** *Assume that $\tilde{D}$ is cyclic. If some element $\delta \in \tilde{D}$ fixes a point $u \in [\infty] \setminus \{\infty\}$ then $\langle\delta\rangle$ also fixes the circle $C$ through $u$ touching $K$ in $o$. Then $\langle\delta\rangle = \tilde{\iota}\langle\delta\rangle\tilde{\iota}$ also fixes the point $\iota(u) \in [o] \setminus \{o\}$ and the circle $\tilde{\iota}(C)$ which touches $K$ in $\infty$ (and thus occurs as a parallel to the horizontal line $[0,0]$ induced by $K$ in the affine plane $\mathbb{A}_\infty$).* □

The characteristic 2 case comes with a somewhat surprising feature:

**4.4 Lemma.** *Assume that $q$ is even. If $C \neq K$ is any circle touching $K$ in a point different from $\infty$ then $C$ meets every circle touching $K$ in $\infty$ (in other words, each parallel to the line $[0,0]$ induced by $K$ in the affine plane $\mathbb{A}_\infty$) in precisely one point.*

*Proof.* As the affine plane $\mathbb{A}_\infty$ has even order, every oval in its projective completion has a knot, i.e., a point such that the set of tangents to the oval is precisely the set of lines through that point, see [13, Lemma 12.10, p. 244]. The circle $C$ shows up as an oval in the completion of $\mathbb{A}_\infty$, one tangent is the line at infinity while $[0,0]$ is another tangent. Thus the knot is the intersection of these two lines in the projective completion of $\mathbb{A}_\infty$, and every parallel to $[0,0]$ is a tangent, as claimed. □

In all cases with even $q$ and a simple socle $F$ treated later on, the torus acts semi-regularly on the complement of its set of fixed points on $K$ (see 5.1, 6.1, and 8.1), and the following applies.

**4.5 Lemma.** *Assume that $q$ is even, and that $\tilde{D}$ acts semi-regularly on $K \setminus \{\infty, o\}$. For each $u \in [\infty] \setminus \{\infty\}$ the stabilizer $\tilde{D}_u$ is contained in $\Delta_K$. In particular, the order of that stabilizer divides $q-1$.*

*If $\Phi = F$ then the torus $D$ acts semi-regularly both on $[\infty] \setminus \{\infty\}$ and on $[o] \setminus \{o\}$.*

*Proof.* Let $u$ be any point in $[\infty]\setminus\{\infty\}$, and let $C$ be the circle through $u$ touching $K$ in $o$. From 4.4 we know that $C$ touches each parallel to $[0,0]$. In particular, it meets $\tilde\iota(C)$ in precisely one point $a \in P\setminus([\infty]\cup[o])$. Now the stabilizer $\tilde D_u$ will fix $C$, $\tilde\iota(C)$ and thus also $a$, the generator $[a]$, and the intersection $[a]\cap K$. As $D$ acts semi-regularly on $K\setminus\{\infty,o\}$, we obtain $\tilde D_u \le \Delta_K$. □

**4.6 Remarks.** We already know $\Phi = F$ if $q$ is even and $F$ is simple, unless $F \cong \mathrm{PSL}(2,4)$ or $F \cong \mathrm{Sz}(2^3)$, cf. 2.3.

# 5 Unitary groups

In this section, we study the case of a socle isomorphic to $\mathrm{PSU}(3,f^2)$. Recall that $\mathrm{PSU}(3,f^2)$ is simple if the prime power $f$ is greater than 2; the group $\mathrm{PSU}(3,2^2)$ is isomorphic to a subgroup of the automorphism group of the affine plane of order 3, and thus solvable (cf. [7, Ch. II, § 4]).

**5.1 Notation.** Assume that $f$ is a power of the prime $r$, and let $x \mapsto \bar x := x^f$ denote the generator of the Galois group $\mathrm{Gal}(\mathbb{F}_{f^2}/\mathbb{F}_f)$. We describe a hermitian form $h$ on $\mathbb{F}_{f^2}^3$ by using the Gram matrix $\left(\begin{smallmatrix} 0&0&1\\ 0&-1&0\\ 1&0&0\end{smallmatrix}\right)$, so

$$h(x,y) = \overline{x_1}y_3 - \overline{x_2}y_2 + \overline{x_3}y_1$$

where $x = (x_1,x_2,x_3)$ and $y = (y_1,y_2,y_3)$. Put[2]

$$\delta_a := \begin{bmatrix} a&0&0\\ 0&\bar a/a&0\\ 0&0&1/\bar a\end{bmatrix} = \begin{bmatrix} a&0&0\\ 0&a^{f-1}&0\\ 0&0&a^{-f}\end{bmatrix} = \begin{bmatrix} a^{f+1}&0&0\\ 0&a^{2f-1}&0\\ 0&0&1\end{bmatrix} \quad\text{and}\quad \tau\left(\begin{smallmatrix} x\\ z\end{smallmatrix}\right) := \begin{bmatrix} 1&\bar x&z\\ 0&1&x\\ 0&0&1\end{bmatrix}.$$

Then the following facts are checked easily (cf. [3, Ch. 7, Sect. 22 and Ex. 8]): The set $R := \{\tau\left(\begin{smallmatrix} x\\ z\end{smallmatrix}\right) \mid x,z\in\mathbb{F}_{f^2}, \bar z + z = \bar x x\}$ forms a Sylow $r$-subgroup of $\mathrm{PSU}(3,f^2)$, the set $D := \{\delta_a \mid a\in\mathbb{F}_{f^2}\setminus\{0\}\}$ forms a subgroup of $\mathrm{PSU}(3,f^2)$, and the (semidirect) product $RD$ is the normalizer of $R$ in $\mathrm{PSU}(3,f^2)$. Moreover, in its standard action on the unital

$$U := \{[x] \mid x\in\mathbb{F}_{f^2}, x\ne 0, h(x,x) = 0\}\ ,$$

the group $\mathrm{PSU}(3,f^2)$ acts two-transitively, the stabilizer of the point $\left[\begin{smallmatrix}1\\0\\0\end{smallmatrix}\right] \in U$ is $RD$, the group $R$ is a regular normal subgroup of the stabilizer, and $D$ is the stabilizer of the two points $\left[\begin{smallmatrix}1\\0\\0\end{smallmatrix}\right]$ and $\left[\begin{smallmatrix}0\\0\\1\end{smallmatrix}\right]$. Note that $|U| = f^3+1$, and that $D$ is cyclic of order $f^2-1$.

The center of $\mathrm{SU}(3,f^2)$ is cyclic, its order is the greatest common divisor $\gcd(3,f+1)$. In other words, we have $\mathrm{PSU}(3,f^2)\cong\mathrm{SU}(3,f^2)$ precisely if 3 does not divide $f+1$.

**5.2 Theorem.** *The case $F \cong \mathrm{PSU}(3,f^2)$ is impossible.*

*Proof.* If $F\cong\mathrm{PSU}(3,f^2)$ then $f>2$ (because $\mathrm{PSU}(3,2^2)$ is not simple) and either $\Phi\cong\mathrm{PSU}(3,f^2)$ or $\Phi\cong\mathrm{SU}(3,f^2)$, cf. 2.3 and 2.2.

Assume first that $f$ is odd. Let $\alpha\in\mathrm{SU}(3,f^2)$ be an involution. Then $\alpha$ does not belong to the center of $\mathrm{SU}(3,f^2)$. Therefore, it represents an involution $[\alpha]$ in $\mathrm{PSU}(3,f^2)$ and, in any case, involutions $\tilde\beta\in\Phi$ and $\beta\in F$.

[2] Square brackets around a matrix indicate that we pass to the projective transformation represented by that matrix, and square brackets around a vector indicate the subspace generated by that vector.

The involutions in $\mathrm{SU}(3, f^2)$ form a single orbit under conjugation in $\mathrm{U}(3, f^2)$. Thus $\beta$ acts on $K$ in the same way as $\delta_{-1}$ acts on the unital $U$, cf. 5.1. In particular, our involution $\tilde{\beta} \in \Phi$ fixes precisely $f + 1$ points on $K$. On the affine derivation in any one of the fixed points, the involution $\tilde{\beta}$ will then induce an involution fixing precisely $f$ (affine) points on a line. This is impossible in an affine plane of order $q = f^3$ by Baer's result [4].

It remains to discuss the case where $\Phi$ is isomorphic either to $\mathrm{PSU}(3, f^2)$ or to $\mathrm{SU}(3, f^2)$ with $r = 2$ and $f = 2^{e/3}$.

Assume that $\tilde{D} \subset \mathbb{F}_{f^2}$ contains a Zsigmondy subgroup $Z$ of order $z > 3$; then $Z$ is also embedded in $D$. From 4.5 we know that $Z$ acts semi-regularly on $[o] \setminus \{o\}$, and thus semi-regularly on the group $T$. By Maschke's Theorem, the group $T$ splits as a direct product of minimal $Z$-invariant subgroups and each of those has order $f^2$ by 3.2. Thus $f^3 = |T|$ must be a power of $f^2$, which is impossible.

So either $z = 3$, or $\tilde{D}$ does not contain a Zsigmondy subgroup of $\mathbb{F}_{2^{2e/3}}$. In the first case, the multiplicative group of $\mathbb{F}_{2^{2e/3}}$ contains a subgroup of order 3 which is not contained in a proper subfield. Then $2e/3 = 2$ and $f = 2$, contradicting $f > 2$.

The second case only occurs if $f^2 = 2^6$ (whence $q = 2^9$) because Mersenne primes are odd (cf. 3.2). Here 3 is not a divisor of $q - 1$, and $\tilde{D}$ acts semi-regularly on $[o] \setminus \{o\}$ by 4.5. Now we apply our previous argument with $\tilde{D}$ instead of the Zsigmondy group, and reach a contradiction, as before. □

## 6 Suzuki groups

Let $\mathrm{Sz}(2^m)$ be one of the simple groups discovered[3] by Suzuki [26]. Then $m = 2s + 1$ is an odd number greater than 1, and $\mathrm{Sz}(2^m)$ acts two-transitively on a set of size $2^{2m} + 1$.

**6.1 Remarks.** For the reader's convenience, we collect some information that will be relevant later. Put $\theta := 2^{s+1}$ then $x \mapsto x^\theta$ is an automorphism of $\mathbb{F}_{2^m}$ such that the square of that automorphism is the Frobenius automorphism (mapping $x$ to $x^2$).

The stabilizer of a point in said two-transitive action is the normalizer of a Sylow 2-subgroup $R$ of $\mathrm{Sz}(2^m)$, and the semidirect product $RD$ of $R$ with a cyclic group $D$ of order $2^m - 1$. We parametrize the group $D$ by the multiplicative group of the field $\mathbb{F}_{2^m}$ such that $D = \{\delta_a \mid a \in \mathbb{F}_{2^m} \setminus \{0\}\}$ and $\delta_a\delta_b = \delta_{ab}$. The Sylow 2-subgroup $R$ can be parametrized by $\mathbb{F}_{2^m}^2$ such that $R = \left\{\tau\left(\begin{smallmatrix} a \\ b \end{smallmatrix}\right) \,\middle|\, a, b \in \mathbb{F}_{2^m}\right\}$ with multiplication $\tau\left(\begin{smallmatrix} a \\ b \end{smallmatrix}\right)\tau\left(\begin{smallmatrix} c \\ d \end{smallmatrix}\right) = \tau\left(\begin{smallmatrix} a+c \\ b+d+ac^\theta \end{smallmatrix}\right)$. We find the commutators $[\tau\left(\begin{smallmatrix} a \\ b \end{smallmatrix}\right), \tau\left(\begin{smallmatrix} c \\ d \end{smallmatrix}\right)] = \tau\left(\begin{smallmatrix} 0 \\ a^{\theta+1}+c^{\theta+1} \end{smallmatrix}\right)$ and $[\delta_c, \tau\left(\begin{smallmatrix} a \\ b \end{smallmatrix}\right)] = \tau\left(\begin{smallmatrix} (1+c)a \\ (1+c^{\theta+1})b+(1+c^\theta)a^{\theta+1} \end{smallmatrix}\right)$.

**6.2 Lemma.** *Every non-trivial normal subgroup of $RD$ either contains $R$, or is the commutator subgroup $R'$ of $R$.*

*Proof.* Assume that $N$ is a normal subgroup containing a non-trivial element $h = \tau\left(\begin{smallmatrix} x \\ y \end{smallmatrix}\right)\delta_a$. If $h \notin R$ then $a \neq 1$ and the commutator $[\tau\left(\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right), h]$ is an element of $R \setminus R'$ in $N$. If $h \in R \setminus R'$ then $a = 1$ and $x \neq 0$. In that case, the commutators $[\delta_c, h] = \tau\left(\begin{smallmatrix} (1+c)x \\ (1+c^{\theta+1})y+(1+c^\theta)x^{\theta+1} \end{smallmatrix}\right)$ with $c \neq 0$ run over a complete set of representatives of $R/R'$, and their squares fill $R'$. So the normal subgroup $N$ contains $R$ if it is not contained in $R'$.

[3] Actually, the Suzuki group $\mathrm{Sz}(2^m)$ is a group of Lie type ${}^2\mathrm{B}_2(s^m)$, but the twisted types were introduced after Suzuki's discovery.

It remains to consider the case where $h \in R'$; then $h = \tau\begin{pmatrix}0\\y\end{pmatrix}$ with $y \neq 0$ and the commutators $[\delta_c, h] = \tau\begin{pmatrix}0\\(1+c^{\theta+1})y\end{pmatrix}$ with $c \neq 0$ fill $R'$. □

**6.3 Theorem.** *The case $F \cong \mathrm{Sz}(2^m)$ is impossible.*

*Proof.* Assume $F \cong \mathrm{Sz}(2^m)$. If $\Phi \neq F$ then $m = 3$ and the center of $\Phi$ contains some involution $\zeta$, see 2.3. Then $\zeta$ fixes each generator and each point on $K$. Moreover, at least one more point is fixed on each generator because $q = 2^{2m}$ is even. Thus $\zeta$ induces on $\mathbb{A}_\infty$ an involutory collineation, fixing each point on the line induced by the circle $K$ but also further points not on that axis. This is impossible, see [4]. So we know $\Phi = F$.

Consider the group $V$ generated by $T$ and its conjugate $\iota T \iota$ in $E$. This is an elementary abelian 2-group; we will regard it as a vector space over $\mathbb{F}_2$ and use additive notation.

Choose a Zsigmondy subgroup $Z$ in $D$, see 3.1. The actions of $D$ on $T$ and on $[o]$ are equivalent, and so are those on $\iota T \iota$ and on $[\infty]$. From 4.5 we infer that $D$ acts semi-regularly on $V \setminus \{0\}$. Thus $V$ is a direct sum of four irreducible $Z$-modules, each of dimension $m$ over $\mathbb{F}_2$ by 3.2.

The subspace $T \leq V$ is invariant under $R$. On the quotient $W := V/T$ we have an induced action of $R$, with non-trivial space of fixed points $U := \mathrm{C}_W(R)$ because both $R$ and $W$ are 2-groups. The action of $D$ on $W$ is equivalent to the action on some $D$-invariant complement to $U$ in $W$. Thus $W$ is the direct sum of two irreducible $D$-modules of dimension $m$. If $U \neq W$ then $U$ is such a submodule, and $W/U$ is also an irreducible $D$-module. Now the set $\mathrm{C}_{W/U}(R)$ is again non-trivial, and a $D$-submodule of $W/U$. We find that $R$ acts trivially on $W/U$.

In any case, the commutator subgroup $R' = \{\rho^2 \mid \rho \in R\}$ therefore acts trivially on $W$. As $W = V/T$ acts regularly on $[\infty]$, we find that $R'$ acts trivially on $[\infty]$. The involution $\iota$ is conjugate to an involution in $R'$ (in fact, any two involutions in $\mathrm{Sz}(2^m)$ are conjugates, cf. [19, 24.2]), and we obtain that $\iota$ fixes some generator pointwise. As this generator is fixed by $\iota$, it is different from $[\infty]$ and $[o] = \iota([\infty])$. We thus infer that $\iota$ acts trivially on the set $C_{\infty,o}$ of circles which represents the set of lines through $o$ in the affine plane $\mathbb{A}_\infty$.

Now the commutator subgroup $D$ of the dihedral group generated by $\{\iota\} \cup D$ acts trivially on $C_{\infty,o}$ because it is generated by conjugates of $\iota$. This means that $D$ induces a group of homologies with axis $L_\infty$ on $\mathbb{A}_\infty$. This implies that the group $RD$ acts trivially on $L_\infty$, see 6.2. We obtain that $R$ consists of translations of $\mathbb{A}_\infty$, contradicting the fact that the group of all translations of $\mathbb{A}_\infty$ is commutative (because there are non-trivial translations with different centers, cf. [2, Satz 1] or [13, Th. 4.14]). □

**6.4 Remark.** One can extend the definition of the Suzuki group $\mathrm{Sz}(2^m)$ to the case $m = 1$, and still gets a two-transitive group $\mathrm{Sz}(2)$ on 5 points. However, the group $\mathrm{Sz}(2)$ is no longer a simple group; it is isomorphic to $\mathrm{AGL}(1,5)$, and has a normal Sylow 5-subgroup. The torus becomes trivial. This group acts on the Miquelian Laguerre plane of order 4, fixing a circle and permuting the points on that circle two-transitively. This action forms a remarkable exception to our results, see 9.1.

## 7 Ree groups

For each integer $a \geq 0$, the *Ree group* $\mathrm{R}(3^{2a+1})$ acts two-transitively on a set of size $3^{6a+3} + 1$, cf. [20]. In fact, the Ree group $\mathrm{R}(3^{2a+1})$ may also be interpreted as the twisted group of Lie type ${}^2\mathrm{G}_2(3^{2a+1})$. The orbit carries a nice geometry known as the *Ree unital*; see [18], [9].

The Ree group $R(3^{2a+1})$ is simple whenever $a > 0$, but R(3) has a normal subgroup isomorphic to PSL(2, 8), of index 3. This is a case where the socle of a two-transitive group is transitive but not two-transitive. In any case, the socle of $R(3^{2a+1})$ is the commutator subgroup.

**7.1 Proposition.** *Let $\mathcal{L}$ be a finite elation Laguerre plane of order $q$, assume that the automorphism group $\Gamma$ of $\mathcal{L}$ acts two-transitively on the set of generators, and let $F$ denote the socle of the group induced on the set of generators. Then $F$ is not the commutator subgroup of a Ree group.*

*Proof.* Aiming at a contradiction, we assume that $F$ is the commutator subgroup of $R(3^{2a+1})$; then $q = 3^{6a+3}$.

The involutions in $F$ form a single conjugacy class; see [15, Theorem XI.13.2] or [18, p. 258]. We pick a Sylow 2-subgroup $\Sigma$ of $\Phi$ and an involution $\tau \in S := \pi(\Sigma)$; then $\tau$ fixes $3^{2a+1} + 1 \geq 4$ points on $K$; cf. [18]. The Sylow 2-subgroup $S$ of $F$ is elementary abelian of order 8; cf. [18, p. 258] again.

Let $\infty$ be a point fixed by $\tau$. Then the group $A := \{\sigma \in \Sigma \mid \pi(\sigma) \in \langle\tau\rangle\}$ also fixes $\infty$, and acts on $[\infty] \setminus \{\infty\}$. As $S$ is elementary abelian, we have $\varphi^2 \in \Sigma \cap \Delta_K$. If $A$ would contain an element $\varphi$ of order 4 then $\langle\varphi\rangle$ could not act semi-regularly on $[\infty] \setminus \{\infty\}$ because the size $q - 1$ of that set is not divisible by 4. Thus $\varphi^2 \in \Delta_K$ fixes some point in $[\infty] \setminus \{\infty\}$. This contradicts 2.1, and we infer that $\tilde{S}$ is elementary abelian, as well.

Pick an involution $\sigma \in \Sigma \setminus \Delta$. Then $\sigma$ fixes at least 4 points on $K$. Since $q \equiv 3 \pmod 4$, the order of $\mathcal{L}$ is not a square and we have a contradiction to 1.2.2. □

# 8 The PSL case

**8.1 Notation.** We fix some notation regarding the groups SL(2, $q$) and PSL(2, $q$), for $q = r^e$.

The stabilizer of a point in the natural two-transitive action of SL(2, $q$) on the projective line is the normalizer of a Sylow $r$-subgroup, and isomorphic to a semidirect product $R\tilde{D}$ where $R = \{\tau(x) \mid x \in \mathbb{F}_q\} \cong C_r^e$ is the Sylow $r$-subgroup (with multiplication $\tau(x)\tau(y) = \tau(x+y)$) and the two-point stabilizer $\tilde{D} = \{\delta_a \mid a \in \mathbb{F}_q \setminus \{0\}\}$ (with multiplication $\delta_a\delta_b = \delta_{ab}$) which is cyclic of order $q - 1$. Here $\tau(x) = \left(\begin{smallmatrix} 1 & x \\ 0 & 1 \end{smallmatrix}\right)$ and $\delta_c = \left(\begin{smallmatrix} c & 0 \\ 0 & c^{-1} \end{smallmatrix}\right)$. The group $\tilde{D}$ acts on $R$ via $\delta_a \tau(x) \delta_a^{-1} = \tau\left(a^2 x\right)$. Thus commutators are given by $[\delta_a, \tau(x)] = \tau\left((a^2 - 1)x\right)$.

When passing to PSL(2, $q$), we factor out the group generated by $\delta_{-1}$ (which is trivial if $r = 2$, and an involution otherwise). The Sylow $r$-subgroup $R$ will be unharmed by this process, and we do not change its name. The image $D$ of $\tilde{D}$ in the quotient PSL(2, $q$) will have order $(q - 1)/2$ if $r$ is odd.

**8.2 Lemma.** *Every non-trivial normal subgroup of $RD$ contains $R$.*

*Proof.* Let $N$ be a normal subgroup of $RD$, and assume that $N$ contains a non-trivial element $h = \tau(x)\delta_a$. If $h \notin R$ then $a \neq \pm 1$ and the commutator $[h, \tau(1)] = [\delta_a, \tau(1)] = \tau\left((a^2 - 1)\right)$ is a non-trivial element of $R$ in $N$. For any non-trivial element $\tau(x) \in N \cap R$, the commutators $[\delta_a, \tau(x)]$ generate $R$ because the set $\{a^2 - 1 \mid a \in \mathbb{F}_q\}$ additively generates $\mathbb{F}_q$ (it coincides with $\mathbb{F}_q$ if $r = 2$). □

## Characteristic two

We treat the characteristic two case separately. As the center of $\mathrm{SL}(2,q)$ is trivial in the characteristic 2 case, we will identify $\mathrm{SL}(2,q)$ and $\mathrm{PSL}(2,q)$.

**8.3 Lemma.** *If $F \cong \mathrm{PSL}(2,2^e)$ then $\Phi = F$.*

*Proof.* The group $\mathrm{PSL}(2,2^e)$ has trivial Schur multiplier if $e > 2$, see 2.3. The Schur multiplier of $\mathrm{PSL}(2,4) \cong \mathrm{PSL}(2,5)$ is not trivial but cyclic of order 2. If $\Phi \cong \mathrm{SL}(2,5)$ (on a Laguerre plane of order 4) then the central involution would act as an involution on $\mathbb{A}_\infty$, fixing each point on the line induced by $K$. Thus it would fix each generator and at least one more point on each generator. This is impossible as this set of fixed points generates the affine plane. □

**8.4 Proposition.** *If $F \cong \mathrm{PSL}(2,2^e)$ then each element of order 2 in $F$ is a translation.*

*Proof.* Every involutory automorphism of a projective plane has either an axis (and then also a center), or is a Baer involution (see [4]). In the latter case, it fixes at least three points on any fixed line.

Under our present assumptions, the involution in question fixes the line induced by $K$ and induces a non-trivial element of $R$ on that line. Therefore, it does not fix any affine point on $K$, and then no affine point at all. The involution thus induces an axial collineation of the projective completion of $\mathbb{A}_\infty$; the axis is the line at infinity and the center is the point at infinity for the line induced by $K$. □

**8.5 Corollary.** *If $F \cong \mathrm{PSL}(2,2^e)$ then the projective closure of $\mathbb{A}_\infty$ has Lenz type V.* □

**8.6 Lemma.** *If $F \cong \mathrm{PSL}(2,2^e)$ then $R$ acts trivially on $[\infty]$. The involution $\iota \in F$ interchanging the generators $[\infty]$ and $[o]$ fixes some generator $[c]$ pointwise, and acts trivially on the pencil $\mathcal{C}_{\infty,o}$.*

*Proof.* We already know that $R$ fixes a point $u \in [\infty] \setminus \{\infty\}$. As $D$ normalizes $R$, the orbit $D(u)$ consists of fixed points of $R$. From 4.5 we know $D(u) = [\infty] \setminus \{\infty\}$.

The involution $\iota$ fixes at least one point $c \in K$. Thus $\iota$ is a conjugate of some involution in $F_\infty$, and acts trivially on $[c]$ by the previous paragraph. Every circle in $\mathcal{C}_{\infty,o}$ is obtained by joining $\{\infty, o, v\}$ for some $v \in [c]$. As the automorphism $\iota$ leaves the set $\{\infty, o, v\}$ invariant, it fixes that circle. □

**8.7 Theorem.** *If $F \cong \mathrm{PSL}(2,2^e)$ then $\mathcal{L}$ is isomorphic to the Miquelian Laguerre plane over the field with $2^e$ elements.*

*Proof.* Let $\iota \in F$ be the involution interchanging the generators $[\infty]$ and $[o]$. Then $\{\iota\} \cup D$ generates a dihedral group $G$ of order $2(2^e-1)$. This group $G$ acts on $\mathcal{C}_{\infty,o}$ with a kernel that contains $\iota$. As the conjugates of $\iota$ in $G$ generate $G$, we have that $D \le G$ acts trivially on $\mathcal{C}_{\infty,o}$. This set of circles induces the pencil of lines through $o$ in $\mathbb{A}_\infty$, and it turns out that $D$ is a transitive group of homologies in the translation plane $\mathbb{A}_\infty$. This means that the plane $\mathbb{A}_\infty$ is Desarguesian, isomorphic to the plane over the field with $2^e$ elements.

The action of $\iota$ on $P \setminus ([\infty] \cup [o])$ is determined, as follows. We use the group $R$ of translations to introduce coordinates on the horizontal line $[0,0]$ induced by $K$, then the involution $\iota$ acts on that line via $\binom{x}{0} \leftrightarrow \binom{1/x}{0}$ for $x \neq 0$, and interchanges $o = \binom{0}{0}$ with $\infty$. Using the fact that $\iota$ stabilizes the line $[y/x, 0]$ for each $x \neq 0$ and $y \in X$, we then obtain $\iota\binom{x}{y} = \binom{1/x}{y/x^2}$ for each $\binom{x}{y} \in X^2 \setminus [o]$.

In order to understand the circles, we first study the circle $C_u$ through $u \in [\infty] \setminus \{\infty\}$ touching $K$ in $o$. From 4.4 we infer that the intersection of $C_u$ with $\mathbb{A}_\infty$ is the graph of some bijection $f\colon X \to X$. As $C_u$ touches $K$ in $o$, its image $\iota(C_u)$ is a circle touching $K$ in $\infty$. Thus $\iota(C_u)$ induces a horizontal line in $\mathbb{A}_\infty$, and there is $b_u \in X$ such that $[0, b_u] \setminus \{o\} = \iota(C_u \setminus \{o\}) = \left\{ \iota\left(\begin{smallmatrix} x \\ f(x) \end{smallmatrix}\right) \,\middle|\, x \in X \setminus \{0\} \right\} = \left\{ \left(\begin{smallmatrix} 1/x \\ f(x)/x^2 \end{smallmatrix}\right) \,\middle|\, x \in X \setminus \{0\} \right\}$. This means $f(x)/x^2 = b_u$ or $f(x) = b_u x^2$, for all $x \neq 0$.

Applying $E_\infty$ to $C$ we obtain all other circles; in fact their traces on $\mathbb{A}_\infty$ have the form $\left\{ \left(\begin{smallmatrix} x \\ b_u x^2 + zx + w \end{smallmatrix}\right) \,\middle|\, x \in X \setminus \{0\} \right\}$ for arbitrary $b_u, w, z \in X$. Note that the value of $b_u$ is determined by the point $u$; the value $b_u = 0$ corresponds to $u = \infty$.

We have thus determined all the circles, and see that $\mathcal{L}$ is the Miquelian plane over the field with $2^e$ elements. □

## Odd characteristic

**8.8 Lemma** ([5]). *A finite Laguerre plane of odd order with at least one Desarguesian affine derivation is Miquelian.* □

**8.9 Lemma** ([22, Thm. 2 d)]). *If $\mathcal{L}$ is a finite elation Laguerre plane of odd order then at each circle $C$ there exists a unique reflection, i.e., an involutory automorphism of $\mathcal{L}$ fixing each point on $C$ (and no others).* □

**8.10 Lemma.** *If $F \cong \mathrm{PSL}(2, q)$ then $\Phi = F$.*

*Proof.* For even $q$, the assertion has been established in 8.3. So if $\Phi$ is not isomorphic to $\mathrm{PSL}(2, q)$ then $q$ is odd, and either $q = 9$ and $\Phi$ contains a central element of order 3, or $\Phi \cong \mathrm{SL}(2, q)$; see 2.3.

A central element $\delta$ of order 3 in $\Phi$ would act as a collineation of order 3 on $\mathbb{A}_\infty$, fixing each point on the line induced by $K$, and fixing at least two points in $[x] \setminus \{x\}$ for each $x \in K$. This is impossible because the set of fixed points would generate $\mathbb{A}_\infty$.

It remains to exclude $\Phi \cong \mathrm{SL}(2, q)$ for odd $q$. Pick a generator $a$ of the Sylow 2-subgroup of the multiplicative group of $\mathbb{F}_q$, and consider the group $Q := \left\langle \left(\begin{smallmatrix} a & 0 \\ 0 & a^{-1} \end{smallmatrix}\right), \left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right) \right\rangle$. This group has order $2s$, where $s$ is the maximal power of 2 dividing $q - 1$. Clearly, it fixes two points on $K$, and acts on $\mathcal{C}_{\infty,o} \setminus \{K\}$. The latter set has $q - 1$ elements, so the stabilizer $Q_C$ of any element $C$ in that set is not trivial, and contains an involution. However, there is only one involution in $\Phi$, corresponding to $\left(\begin{smallmatrix} -1 & 0 \\ 0 & -1 \end{smallmatrix}\right) \in \mathrm{SL}(2, q)$. As this involution acts trivially on $K$, it is the unique reflection at $K$, see 8.9. If this involution lies in $Q_C$ then it fixes each point on $C$, contradicting the fact that it acts semi-regularly outside $K$. □

**8.11 Theorem.** *If $F \cong \mathrm{PSL}(2, q)$ then $\Phi = F$ and $\mathcal{L}$ is the Miquelian Laguerre plane of order $q$.*

*Proof.* If $q$ is even then the assertion has been proved in 8.7.

It remains to treat the cases where $q$ is odd; then $\Phi = F$ by 8.10. Let $\iota \in F$ be an involution interchanging $\infty$ and $o$. Our arguments will be quite different in the cases where $\iota$ fixes points on $K$ or does not fix any points. These cases are distinguished by the residue of $q - 1$ modulo 4 because one needs square roots of $-1$ for eigenvalues in order to get fixed points for the involutions in $\mathrm{PSL}(2, q)$.

(A) Assume first that $q \equiv 1 \pmod 4$.

A look at the standard action of $F$ on $\mathrm{PG}(1,q)$ shows that each involution in $F$ fixes precisely two points on $K$. Consider an involution $\eta$ which fixes the points $\infty$ and $o$. We note that each involution $\iota \in F$ interchanging $o$ with $\infty$ will centralize $\eta$ because the involution $\eta$ is determined by its fixed points; in fact, we have $\eta = \delta_j$ for some square root $j$ of $-1$ in $\mathbb{F}_q$.

The involution $\hat\eta$ induced by $\eta$ on the projective closure of $\mathbb{A}_\infty$ is either axial, or a Baer involution. In the latter case, more than two points would be fixed on each fixed line. As $\eta \in F$ fixes just two points on the circle $K$ and this circle induces one of the lines of $\mathbb{A}_\infty$, the case of a Baer involution is excluded. Thus $\hat\eta$ is axial. The affine fixed point $o$ either is the center of $\hat\eta$ (then the axis is the line $L_\infty$ at infinity) or it lies on the axis (then that axis is the completion of the vertical line $[o]$ formed by the generator through $o$, and the center is the point $v$ at infinity belonging to the parallel class in $\mathbb{A}_\infty$ formed by all circles touching $K$ in $\infty$).

The set $\eta^R$ forms a conjugacy class in the generalized dihedral group $B$ generated by $\{\eta\} \cup R$, and $B$ is generated by $\eta^R$. As $R$ fixes the line $L_\infty$ and the point $v$, we either have that each element of $B$ induces a collineation with axis $L_\infty$ (if $\hat\eta$ has axis $L_\infty$) or we have that $B$ induces a group of collineations with center $v$. In any case, the group $R$ induces a group of collineations with axis $L_\infty$ and center $v$. Together with the group $E_{\infty,o}$ of vertical translations, we obtain a group $E_{\infty,o}R$ of translations that is transitive on the points of $\mathbb{A}_\infty$. Therefore, the projective completion of $\mathbb{A}_\infty$ is a plane of Lenz type V, coordinatized by a semifield.

If $\hat\eta$ has axis $[o]$ we choose a conjugate $\iota$ of $\eta$ interchanging the points $\infty$ and $o$. Then $\iota$ fixes some generator pointwise, and thus acts trivially on the set $\mathcal{C}_{\infty,o}$ of circles that pass through $\infty$ and $o$. Now the commutator subgroup of the dihedral group generated by $\{\iota\} \cup D$ acts trivially on $\mathcal{C}_{\infty,o}$ because it is generated by conjugates of $\iota$. This commutator group is a cyclic group of order $(q-1)/4$. If $q \neq 9$ then $\mathbb{F}_q$ has no proper subfield of order greater than $(q-1)/4$. Thus the kernel of the semifield coordinatizing $\mathbb{A}_\infty$ is $\mathbb{F}_q$, and $\mathbb{A}_\infty$ is Desarguesian. The Laguerre plane $\mathcal{L}$ is then Miquelian, see 8.8. For $q = 9$ the order of the multiplicative group of the subfield $\mathbb{F}_3$ has order $2 = (q-1)/4$. However, one knows that each Laguerre plane of order 9 is Miquelian, see [5], [23], [24].

If $\hat\eta$ has axis $L_\infty$ then $o$ is the center of $\hat\eta$. Let $\zeta$ be the unique reflection at $K$, see 8.9. Then $\zeta$ centralizes $F$. The product $\sigma := \eta\zeta = \zeta\eta$ is an involution fixing $\infty$ and $o$. This involution induces an automorphism $\hat\sigma$ of $\mathbb{A}_\infty$ that acts on $L_\infty$ just like $\zeta$ and on the line $[0,0]$ induced by $K$ just like $\hat\eta$. Thus $\hat\sigma$ has axis $[o]$. Now $\iota\zeta$ is a conjugate of $\sigma$, interchanges $\infty$ with $o$, fixes a generator pointwise, and thus fixes each circle in $\mathcal{C}_{\infty,o}$. Conjugation by $\iota\zeta$ induces inversion on $D$, and again the commutator subgroup of the dihedral group generated by $\{\iota\zeta\} \cup D$ acts trivially on $\mathcal{C}_{\infty,o}$. As above, we find that the Laguerre plane $\mathcal{L}$ is Miquelian.

(B) Now consider the case where $q \equiv 3 \pmod 4$; then the order $(q-1)/2$ of the torus is odd.

We show first that $D$ acts faithfully on the generators $[o]$ and $[\infty]$. Let $N$ be the kernel of the action of $D$ on $[\infty]$. Consider an involution $\iota \in F$ interchanging $\infty$ with $o$. Then conjugation by $\iota$ normalizes every subgroup of the cyclic subgroup $D$, so $N$ acts trivially on $[o]$, as well. Now $N$ fixes every circle touching $K$ in $\infty$ or $o$. Any point $x \notin [\infty] \cup [o] \cup K$ lies on two of these circles, and its orbit under $N$ has length at most 2. As the order of $D$ is odd, we find that $N$ fixes $x$ and $[x]$, and is thus trivial.

The action on $[o]$ is equivalent to the action via conjugation on the group $T$ of translations. That action is a linear representation over the prime field $\mathbb{F}_r$, and completely reducible by Maschke's Theorem. We will show that this action is in fact irreducible, and quasi-equivalent to the natural representation of $D$ as a subgroup of $\mathbb{F}_q$ (namely, the group of squares).

Our assumption on $q$ implies that $q$ is neither a square, nor even. Therefore, the exceptional cases in Zsigmondy's Theorem are excluded, cf. 3.1. So consider a Zsigmondy subgroup of $\mathbb{F}_q$, and let $Z$ be the corresponding subgroup of $D$. As $Z$ acts completely reducibly on $T$, we find a minimal $Z$-invariant subgroup $W$ of $T$ such that $Z$ acts faithfully on $W$. By minimality, this action is irreducible, and 3.2 yields that the action of $D$ on $W$ is semi-regular, and $|W| = q$. We obtain $W = T$, the action of $Z$ is irreducible, and the endomorphisms induced by elements of $D$ on $T$ are contained in the centralizer of this irreducible action, which has already been identified as the scalar action of a field (with $q$ elements).

The group $RD$ induces a group of collineations of the affine plane $\mathbb{A}_\infty$. This group fixes the line induced by $K$, and the action on the set of parallels to that line is equivalent to the action by conjugation on the group $T$ of translations. As both $R$ and $T$ are $r$-groups, the set $U$ of fixed points of $R$ in $T$ is a non-trivial subgroup of $T$. The group $D$ normalizes both $R$ and $T$ and thus leaves $U$ invariant. As $D$ acts irreducibly on $T$, this implies $U = T$. Thus $R$ fixes each horizontal line and induces a group of translations. Now the group $RT$ acts as a transitive group of translations of $\mathbb{A}_\infty$, and $\mathbb{A}_\infty$ is a translation plane.

The group $\Psi := \langle \iota, \zeta \rangle$ will not act semi-regularly on $\mathcal{C}_{\infty,o}$ because $4 = |\Psi|$ is not a divisor of $q-1 = |\mathcal{C}_{\infty,o} \setminus \{K\}|$. So there exists a circle $C \in \mathcal{C}_{\infty,o} \setminus \{K\}$ such that the stabilizer $\Psi_C$ contains some involution $\alpha$. Clearly $\alpha \neq \zeta$, so $\alpha$ induces inversion on the cyclic group $D$.

The action of the group $\Psi$ on the set $\mathcal{C}_{\infty,o}$ of circles is equivalent to the action by conjugation on the group $E_{\infty,o}$. The latter is elementary abelian of order $q = r^e$ and induces on $\mathbb{A}_\infty$ the group of shears with axis $[o]$. The involution $\zeta$ acts trivially on $K$ and induces inversion on $E_{\infty,o}$.

If $\alpha$ acts trivially on $E_{\infty,o}$ then the whole dihedral group $\langle \{\alpha\} \cup D \rangle$ acts trivially on $\mathcal{C}_{\infty,o}$. Then $D$ induces a group of homologies on the translation plane $\mathbb{A}_\infty$. Arguments as those used at the end of the proof of part (A) now show that $\mathbb{A}_\infty$ is Desarguesian, and $\mathcal{L}$ is Miquelian.

So assume that $\alpha$ does not act trivially on $\mathcal{C}_{\infty,o}$. The group $E_{\infty,o}$ splits as a direct product of $\mathrm{Fix}(\alpha)$ and $\mathrm{Fix}(\alpha\zeta)$ because $r$ is odd and $\zeta$ induces inversion, and both factors are non-trivial by our assumption on $\alpha$. Let $\beta \in \{\alpha, \zeta\alpha\}$ be an involution with at least $\sqrt{q}$ fixed elements in $E_{\infty,o}$, then $\beta$ fixes at least $\sqrt{q}$ circles in $\mathcal{C}_{\infty,o}$.

We consider a circle $C \in \mathcal{C}_{\infty,o} \setminus \{K\}$ fixed by $\beta$. Then $\beta$ also fixes $\zeta(C)$. The orbits $D(C)$ and $D(\zeta(C)) = \zeta(D(C))$ form a partition of $\mathcal{C}_{\infty,o} \setminus \{K\}$ because $\zeta$ acts semi-regularly on $\mathcal{C}_{\infty,o} \setminus \{K\}$ and cannot leave a subset of odd order invariant. On each one of these orbits, the action of the dihedral group $\langle \{\beta\} \cup D \rangle$ is equivalent to the usual action of the dihedral group on $(q-1)/2$ points because the normal subgroup $D$ acts regularly and the stabilizer of some element is generated by the involution $\beta$. This implies that $\beta$ fixes precisely one element in each one of the orbits (recall that $(q-1)/2$ is odd). Thus the set of fixed elements of $\beta$ in $E_{\infty,o}$ has 3 elements while there should be at least $\sqrt{q}$. This yields $q \leq 9$; but one knows that (elation) Laguerre planes of such small odd order are Miquelian (see [5], [23], [24]). If $\mathcal{L}$ is Miquelian of odd order then $\iota\zeta$ in fact acts trivially on $\mathcal{C}_{\infty,o}$. □

## 9 Main Theorem

**9.1 Main Theorem.** *If the automorphism group of an elation Laguerre plane of order $q$ contains a subgroup $\Gamma$ fixing a circle and acting two-transitively on that circle, then the Laguerre plane is Miquelian.*

*The socle of the group induced on the fixed circle is either isomorphic to* PSL$(2, q)$*, or we have* $q = 4$ *and the socle is isomorphic to* AGL$(1, 5)$*.*

*Proof.* After the results in [25], it only remains to discuss the cases where the socle of the group induced on the circle belongs to one of the following series (and has been treated at the location given):

- PSL$(2, q)$ (see 8.11),
- PSU$(3, f^2)$ (see 5.2),
- Sz$(2^{2a+1})$ (see 6.3),
- R$(3^{2a+1})$ (see 7.1),
- and the commutator group R$(3)' \cong$ PSL$(2, 8)$ with its transitive action on 28 points (also treated in 7.1). □

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