Universität
Stuttgart

Fachbereich
Mathematik

On the Prime Graph of the Unit Group of Integral Group Rings of Finite Groups II

Wolfgang Kimmerle, Alexander Konovalov

Preprint 2012/018

Universität Stuttgart

Fachbereich Mathematik

On the Prime Graph of the Unit Group of Integral Group Rings of Finite Groups II

Wolfgang Kimmerle, Alexander Konovalov

Preprint 2012/018

Fachbereich Mathematik
Fakultät Mathematik und Physik
Universität Stuttgart
Pfaffenwaldring 57
D-70 569 Stuttgart

E-Mail: preprints@mathematik.uni-stuttgart.de
WWW: http://www.mathematik.uni-stuttgart.de/preprints

ISSN **1613-8309**

On the Prime Graph of the Unit Group of Integral Group Rings of Finite Groups II

W. Kimmerle and A. Konovalov

1. Introduction

Let G be a group. The prime graph $\Pi(G)$ is defined as follows. The vertices of $\Pi(G)$ are the primes p for which G has an element of order p. Two different vertices p and q are joined by an edge provided there is an element of G of order pq.

The integral group ring of G is denoted by $\mathbb{Z}G$ and $V(\mathbb{Z}G)$ denotes the subgroup of the unit group $U(\mathbb{Z}G)$ consisting of all units with augmentation 1. The object of this article, which is a continuation of [14], is the prime graph of $V(\mathbb{Z}G)$ in the case when G is a finite group.

The question is whether the prime graph of $V(\mathbb{Z}G)$ coincides with that of G [13, Problem 21]. This question, known as the "Prime Graph Question" (PQ) may be regarded as a weak version of the first Zassenhaus conjecture ZC–1 which says that each torsion unit of $V(\mathbb{Z}G)$ is conjugate to an element of G – here G is considered in a natural way as subgroup of $V(\mathbb{Z}G)$ and its elements are then called trivial units of $\mathbb{Z}G$. The conjecture ZC–1 is certainly one of the major open questions for integral group rings and if it is valid for a specific group G it provides of course a positive answer to the prime graph question for $V(\mathbb{Z}G)$.

In Section 2 we reduce the study of the prime graph question to the study of nonabelian composition factors and their automorphism groups. This gives many calculations made in the last years with respect to simple groups (see e.g. [2, 3, 9]) a new value. These calculations are not only examples. In particular with respect to sporadic simple groups they might be the only way to a general result. We note that not only simple groups have to be checked but also their automorphism groups have to be examined which has been done up to now only very rarely.

In Section 3 we consider groups whose order is only divisible by three primes. The prime graph question has a positive answer for all simple groups G of this type and for all their groups of automorphisms sandwiched between G and $\mathrm{Aut}G$ except possibly the cases of M_{10} and $\mathrm{PGL}(2,9) \cong A6.2_2$. For these two groups we are not yet able to show that the normalized unit group of its integral group ring has no elements of order 6.

2. The reduction to almost simple groups

An almost simple group is a subgroup of the automorphism group of a finite non-abelian simple group which contains $\mathrm{Inn}G \cong G$. The object of this section is the following result which reduces the prime graph question to the study of almost simple groups.

Theorem 2.1. *Let G be a finite group. Assume that for each almost simple group X which occurs as image of G the prime graph question has a positive answer. Then the prime graph question has a positive answer for G.*

The first step for the proof is the following.

Proposition 2.2. *[14, Proposition 4.3] Let*

$$1 \longrightarrow A \longrightarrow E \xrightarrow{\phi} G \longrightarrow 1$$

be a short exact sequence of groups. Assume that A is a p - group. Then

$$\Pi(V(\mathbb{Z}G)) = \Pi(G) \Longrightarrow \Pi(V(\mathbb{Z}E)) = \Pi(E).$$

So it remains to study what happens under minimal perfect extensions.

Proposition 2.3. *Let*

$$1 \longrightarrow N \longrightarrow E \xrightarrow{\phi} G \longrightarrow 1$$

be a short exact sequence of groups. Assume that N is a perfect minimal normal subgroup of G. Let $q \in \pi(E) \setminus \pi(N)$ and $p \in \pi(E)$ with $q \neq p$. Then $V(\mathbb{Z}E))$ has elements of order $p \cdot q$ if and only if E has elements of this order.

Proof. Note first that if G has elements of order $p \cdot q$ then E has elements of order $p \cdot q$. Assume that $V(\mathbb{Z}E)$ has a unit u of order $p \cdot q$ but E does not have an element of this order. Then G has no elements of order $p \cdot q$ and it follows from the assumption that $\hat{\phi}(u^p) = 1$ or $\hat{\phi}(u^q) = 1$, where $\hat{\phi}$ denotes the homomorphism from $V(\mathbb{Z}E)$ to $V(\mathbb{Z}G)$ induced from ϕ.

Let v be a non-trivial torsion unit of $\mathbb{Z}E$. By [17, Theorem 2.7] the partial augmentations of v for a conjugacy class of elements whose order is divisible by a prime not dividing the order of V are zero. By S.D. Berman [1] and G. Higman [12, 18] the 1-coefficient of non-trivial torsion units with augmentation 1 has to be zero. It follows that torsion elements of prime order in the kernel of $\hat{\phi}$ divide the order of N. Because of the assumption that q does not divide $|N|$ we must have $\hat{\phi}(u^q) = 1$.

For a group X and a prime r denote by $X(r)$ the set of all elements of X which order is a positive power of r. Write u as

$$u = \sum_{g \in E(p)} a_g g + \sum_{g \in E(q)} b_g g + \sum_{g \in R} c_g g,$$

where $R = E \setminus (E(p) \cup E(q))$. Clearly

$$\hat{\phi}(u) = \sum_{g \in E(p)} a_g \phi(g) + \sum_{g \in E(q)} b_g \phi(g) + \sum_{g \in R} c_g \phi(g),$$

By the previous $\hat{\phi}(u)$ has order q. Thus by [17] u has on conjugacy classes of elements in R partial augmentation zero. If $x \in E(p)$ and $\phi(x) \neq 1$ then the partial augmentation of $\phi(x)$ is zero. Thus the sum of partial augmentations of $y \in E$ with $\phi(y) = x$ is zero. Note that y is either an element of $E(p)$ or of R. Finally, if $x \in E(q)$, then $\phi(x) \neq 1$ because q does not divide $|N|$. Thus the 1-coefficient of $\phi(u)$ is the sum of partial augmentations of elements of $E(p)$ and R. Again by Berman and Higman the 1-coefficient of $\phi(u)$ is zero. Summarizing, we get that

$$\sum_{g \in E(p)} a_g + \sum_{g \in R} c_g = 0.$$

(Note that the arguments even show that $\sum_{g \in E(p)} a_g = \sum_{g \in R} c_g = 0$. However this fact will not be required for the proof of the proposition.)
Because normalized units have augmentation 1 it follows that

$$(*) \qquad \sum_{g \in E(q)} b_g = 1.$$

Now u^q has order p and clearly

$$u^q \equiv \sum_{g \in E(p)} a_g g^q + \sum_{g \in E(q)} b_g g^q + \sum_{g \in R} c_g g^q \mod [\mathbb{Z}E, \mathbb{Z}E] + q\mathbb{Z}E.$$

Again by Berman and Higman and by [17] applied now to u^q it follows that the sum over the coefficients of u^q of all elements of order a positive power of q is also zero. Hence we see that

$$\sum_{g\in E(q)} b_g \equiv 0 \mod q.$$

This contradiction to $(*)$ completes the proof. $\square$

Remarks. a) Instead of [17, Theorem 2.7] we could have used the stronger result [9, Theorem 2.3] which says that partial augmentations of a torsion unit of $\mathbb{Z}G$ are non-zero only for conjugacy classes of G whose representative has order dividing that one of u. It is clear that this and modifications of the arguments for Proposition 2.3 will lead to stronger results on the possible orders of torsion units in integral group rings. However in this article we restrict ourselves on (PQ).
b) A typical example for an application of Proposition 2.3 is the automorphism group of the smallest Suzuki group Sz(8). Because 3 does not divide $|\mathrm{Sz}(8)|$ but $|\mathrm{Out}(\mathrm{Sz}(8)| = 3$ it follows from Proposition 2.3 that $V(\mathbb{Z}\mathrm{Aut}(\mathrm{Sz}(8))$ has elements of order $3 \cdot p$ if, and only if, this is the case in Aut(Sz(8)).

Proof of Theorem 2.1. Clearly Theorem 2.1 holds when G itself is almost simple.
Thus we may apply induction on the length of a chief series. If G has a minimal normal subgroup N which is abelian we see by Proposition 2.2 that the Theorem is valid provided it holds for G/N.
Suppose that all minimal normal subgroups of G are perfect. Let $p, q \in \pi(G)$ be different primes and suppose that $V(\mathbb{Z}G)$ has an element of order $p \cdot q$. By Proposition 2.3 it follows that G has an element of order $p\cdot q$ provided $p \in \pi(G) \setminus \pi(G/N)$ or $q \in \pi(G) \setminus \pi(G/N)$ for at least one minimal normal subgroup N. So assume that p and q divide the order of each minimal normal subgroups of G.
If G has at least two different minimal normal subgroups N_1, N_2 then obviously G has elements of order $p \cdot q$ because $N_1 \times N_2$ is a subgroup of G. The same argument applies if G has a minimal normal perfect subgroup which is not simple because such a subgroup is a direct product of at least two copies of isomorphic simple groups.
Thus the only case which remains is that G has a unique minimal normal subgroup which is a non-abelian simple group S and p and q divide the order of S. But this case holds by assumption. $\square$

3. Groups of orders divisible by three primes only

The goal of this section is the following result.

Theorem 3.1. *Let G be a finite group of order $p^aq^br^c$, where p, q and r are primes. Then $\Pi(V(\mathbb{Z}G) = \Pi(G)$ except possibly the case that M_{10} or $\mathrm{PGL}(2,9) = A6.2_2$ are involved in G.*

Proof. By [11] the simple groups of order divisible by three primes only are

$$PSL(2,5) \cong A_5, PSL(2,7), PSL(2,8), PSL(2,9) \cong A_6$$

$$PSL(2,17), PSL(3,3), PSP(3,4) \cong U(4,2), U(3,3).$$

For some of these groups and their automorphism groups the first Zassenhaus conjecture ZC–1 holds. This has been proved for A_5 and S_5 by Luthar and Passi [15] and by Luthar and Trama [16] respectively. Hertweck verified the prime graph question for $PSL(2,p), p$ a prime [9] and for A_6 [10].
For the remaining cases the Luthar-Passi-Hertweck algorithm (also called the HeLP-method) yields the following.

Case 1. Let $G = PSL(2,8)$. Then $|G| = 504 = 2^3 \cdot 3^2 \cdot 7$ and $exp(G) = 126 = 2 \cdot 3^2 \cdot 7$. The group G has elements of orders 2, 3, 7 and 9. By [8] (Proposition 3.1), it follows immediately that torsion units of orders 2 and 3 are rationally conjugate to an element of G.

For torsion units of order 7 we obtain the system of inequalities

$$\begin{aligned}
\mu_1(u,\chi_7,*) &= \tfrac{1}{7}(5\nu_{7a} - 2\nu_{7b} - 2\nu_{7c} + 9) \geq 0;\\
\mu_2(u,\chi_7,*) &= \tfrac{1}{7}(-2\nu_{7a} + 5\nu_{7b} - 2\nu_{7c} + 9) \geq 0;\\
\mu_3(u,\chi_7,*) &= \tfrac{1}{7}(-2\nu_{7a} - 2\nu_{7b} + 5\nu_{7c} + 9) \geq 0;\\
\mu_1(u,\chi_2,2) &= \tfrac{1}{7}(5\nu_{7a} - 2\nu_{7b} - 2\nu_{7c} + 2) \geq 0;\\
\mu_2(u,\chi_2,2) &= \tfrac{1}{7}(-2\nu_{7a} + 5\nu_{7b} - 2\nu_{7c} + 2) \geq 0;\\
\mu_3(u,\chi_2,2) &= \tfrac{1}{7}(-2\nu_{7a} - 2\nu_{7b} + 5\nu_{7c} + 2) \geq 0;\\
\mu_1(u,\chi_5,2) &= \tfrac{1}{7}(3\nu_{7a} + 3\nu_{7b} - 4\nu_{7c} + 4) \geq 0;\\
\mu_2(u,\chi_5,2) &= \tfrac{1}{7}(-4\nu_{7a} + 3\nu_{7b} + 3\nu_{7c} + 4) \geq 0;\\
\mu_3(u,\chi_5,2) &= \tfrac{1}{7}(3\nu_{7a} - 4\nu_{7b} + 3\nu_{7c} + 4) \geq 0;
\end{aligned}$$

which has only three solutions:

$$(\nu_{7a},\nu_{7b},\nu_{7c}) \in \{(1,0,0),(0,0,1),(0,1,0)\}.$$

For torsion units of order 9, first in inequalities

$$\begin{aligned}
\mu_3(u,\chi_2,*) &= \tfrac{1}{9}(6\nu_{3a} - 3\nu_{9a} - 3\nu_{9b} - 3\nu_{9c} + 3) \geq 0;\\
\mu_0(u,\chi_2,*) &= \tfrac{1}{9}(-12\nu_{3a} + 6\nu_{9a} + 6\nu_{9b} + 6\nu_{9c} + 3) \geq 0;
\end{aligned}$$

put $t_1 = 2\nu_{3a} - \nu_{9a} - \nu_{9b} - \nu_{9c}$, then $t_1 = -1$.

Next, in inequalities

$$\begin{aligned}
\mu_1(u,\chi_2,2) &= \tfrac{1}{9}(+6\nu_{9a} - 3\nu_{9b} - 3\nu_{9c} + 3) \geq 0;\\
\mu_1(u,\chi_3,*) &= \tfrac{1}{9}(-6\nu_{9a} + 3\nu_{9b} + 3\nu_{9c} + 6) \geq 0;
\end{aligned}$$

put $t_2 = +2\nu_{9a} - \nu_{9b} - \nu_{9c}$, then $t_2 \in \{-1,2\}$.

Finally, in inequalities

$$\begin{aligned}
\mu_2(u,\chi_3,*) &= \tfrac{1}{9}(+3\nu_{9a} - 6\nu_{9b} + 3\nu_{9c} + 6) \geq 0;\\
\mu_2(u,\chi_2,2) &= \tfrac{1}{9}(-3\nu_{9a} + 6\nu_{9b} - 3\nu_{9c} + 3) \geq 0;
\end{aligned}$$

put $t_3 = +\nu_{9a} - 2\nu_{9b} + \nu_{9c}$, then $t_3 \in \{-2,1\}$.

After adding the condition $\nu_{3a} + \nu_{9a} + \nu_{9b} + \nu_{9c} = 1$, solving the system of linear equations for each possible combination of t_1, t_2 and t_3 and using the additional inequality

$$\mu_4(u,\chi_2,2) = \tfrac{1}{9}(-3\nu_{9a} - 3\nu_{9b} + 6\nu_{9c} + 3) \geq 0;$$

we obtain only three possible solutions for partial augmentations of torsion units of order 9:

$$(\nu_{3a},\nu_{9a},\nu_{9b},\nu_{9c}) \in \{(0,0,0,1),(0,0,1,0),(0,1,0,0)\}.$$

Thus, torsion units of orders 2, 3, 7 and 9 are rationally conjugate to group elements. It remains to show that no other orders of torsion units are possible in $\mathbb{Z}G$, and it suffices to show this for orders 6, 14 and 21.

For elements of order 6, the system of inequalities

$$\begin{aligned}
\mu_0(u,\chi_2,*) &= \tfrac{1}{6}(-2\nu_{2a} - 4\nu_{3a} + 2) \geq 0;\\
\mu_1(u,\chi_2,*) &= \tfrac{1}{6}(-\nu_{2a} - 2\nu_{3a} + 10) \geq 0;\\
\mu_3(u,\chi_2,*) &= \tfrac{1}{6}(2\nu_{2a} + 4\nu_{3a} + 4) \geq 0;\\
\mu_0(u,\chi_3,*) &= \tfrac{1}{6}(-2\nu_{2a} + 2\nu_{3a} + 8) \geq 0;
\end{aligned}$$

has no solutions.

For elements of order 14, we use $(2,7)$-constant characters approach (cf. [3]). Taking the ordinary character χ_2 with $\chi_2(1) = 7$, $\chi_2(C_2) = -1$, $\chi_2(C_7) = 0$, we obtain the system

$$\begin{aligned} \mu_0(u,\chi_2,0) &= \tfrac{1}{14}(-6\nu_2+6) \geq 0; \\ \mu_7(u,\chi_2,0) &= \tfrac{1}{14}(6\nu_2+8) \geq 0; \\ \mu_1(u,\chi_2,0) &= \tfrac{1}{14}(\nu_2+8) \geq 0; \end{aligned}$$

which has no solutions.

Similarly, for elements of order 21 we use the same character χ_2 to construct the system

$$\begin{aligned} \mu_0(u,\chi_2,0) &= \tfrac{1}{21}(-24\nu_3+3) \geq 0; \\ \mu_7(u,\chi_2,0) &= \tfrac{1}{21}(12\nu_3+9) \geq 0; \\ \mu_1(u,\chi_2,0) &= \tfrac{1}{21}(-2\nu_3+9) \geq 0; \end{aligned}$$

which has no solutions.

Therefore, the 1st Zassenhaus conjecture holds for $PSL(2,8)$.

Case 2. Let $G = PSL(3,3)$. Then $|G| = 5616 = 2^4 \cdot 3^3 \cdot 137$ and $exp(G) = 312 = 2^3 \cdot 3 \cdot 13$. The group G has elements of orders 2, 3, 4, 6, 8 and 13. By [8] (Proposition 3.1), it follows immediately that torsion units of orders 2 are rationally conjugate to an element of G.

Using the Luthar–Passi–Hertweck method, we are able to show the rational conjugacy for torsion units of orders 4, 8 and 13, but we are unable to eliminate non-trivial solutions for orders 3 and 6. However, for the prime graph question, we need only to show that $V(\mathbb{Z}G)$ has no elements of orders 26 and 39.

For elements of order 26, we use the $(2,13)$-constant ordinary character χ_3 of degree 13, with $\chi_3(C_2) = -3$ and $\chi(C_{13}) = 0$. Then the system

$$\begin{aligned} \mu_0(u,\chi_3,0) &= \tfrac{1}{26}(-36\nu_2+10) \geq 0; \\ \mu_{13}(u,\chi_3,0) &= \tfrac{1}{26}(36\nu_2+16) \geq 0; \\ \mu_1(u,\chi_3,0) &= \tfrac{1}{26}(-3\nu_2+16) \geq 0; \end{aligned}$$

has no solutions.

For elements of order 39, we use the $(3,13)$-constant ordinary character χ_8 of degree 26, with $\chi(C_3) = -1$ and $\chi(C_{13}) = 0$. It yields

$$\begin{aligned} \mu_0(u,\chi_8,0) &= \tfrac{1}{39}(-24\nu_3+24) \geq 0; \\ \mu_{13}(u,\chi_8,0) &= \tfrac{1}{39}(12\nu_3+27) \geq 0; \\ \mu_1(u,\chi_8,0) &= \tfrac{1}{39}(-1\nu_3+27) \geq 0; \end{aligned}$$

and this system has no solutions either.

Case 3. Let $G = PSP(3,4) \cong U(4,2)$. Then $|G| = 25920 = 2^6 \cdot 3^4 \cdot 5$ and $exp(G) = 180 = 2^2 \cdot 3^2 \cdot 5$. The group G has elements of orders 2, 3, 4, 5, 6, 9 and 12. By [8] (Proposition 3.1), it follows immediately that torsion units of order 5 are rationally conjugate to an element of G. To give the positive answer to the (PQ) we need to show that $V(\mathbb{Z}G)$ has no elements of orders 10 and 15.

For elements of order 10, we use ordinary characters χ_9 of degree 20 with $\chi_9(C_2) = 4$ and $\chi_9(C_5) = 0$, and χ_{17} of degree 45 with $\chi_{17}(C_2) = -3$ and $\chi_{17}(C_5) = 0$. Then the system

$$\begin{aligned} \mu_0(u,\chi_9,0) &= \tfrac{1}{10}(16\nu_2+24) \geq 0; \\ \mu_5(u,\chi_9,0) &= \tfrac{1}{10}(-16\nu_2+16) \geq 0; \\ \mu_1(u,\chi_{17},0) &= \tfrac{1}{10}(-3\nu_2+48) \geq 0; \end{aligned}$$

has no solutions.

For elements of order 15, first we use the ordinary characters χ_{11} of degree 30 with $\chi_{11}(C_3) = 3$ and $\chi_{11}(C_5) = 0$. Then from

$$\mu_0(u, \chi_{11}, 0) = \tfrac{1}{15}(24\nu_3 + 36) \geq 0;$$
$$\mu_5(u, \chi_{11}, 0) = \tfrac{1}{15}(-12\nu_3 + 27) \geq 0;$$

it follows that $\nu_3 = 1$, so $\nu_5 = 0$. Now using the ordinary character χ_{20} of degree 81 such that $\chi_{20}(C_3) = 0$ and $\chi_{20}(C_5) = 1$, we obtain that $\mu_0(u, \chi_{20}, 0) = \frac{1}{15}(8\nu_5 + 85) \geq 0$ is not an integer when $\nu_5 = 0$.

Case 4. Let $G = U(3,3)$. Then $|G| = 6048 = 2^5 \cdot 3^3 \cdot 7$ and $exp(G) = 168 = 2^3 \cdot 3 \cdot 7$. The group G has elements of orders 2, 3, 4, 6, 7, 8 and 12. By [8] (Proposition 3.1), it follows immediately that torsion units of order 2 are rationally conjugate to an element of G. To give the positive answer to the (PQ) we need to show that $V(\mathbb{Z}G)$ has no elements of orders 14 and 21.

For elements of order 14, consider the ordinary character χ_4 of degree 7 with $\chi(C_2) = 3$, $\chi(C_7) = 0$. Then the system of constraints

$$\mu_0(u, \chi_4, 0) = \tfrac{1}{14}(18\nu_2 + 10) \geq 0;$$
$$\mu_1(u, \chi_4, 0) = \tfrac{1}{14}(3\nu_2 + 4) \geq 0;$$
$$\mu_7(u, \chi_4, 0) = \tfrac{1}{14}(-18\nu_2 + 4) \geq 0;$$

has no solutions.

For elements of order 21, consider the ordinary character χ_{10} of degree 27 with $\chi_{10}(C_3) = 0$, $\chi_{10}(C_7) = -1$. Then the system

$$\mu_0(u, \chi_{10}, 0) = \tfrac{1}{21}(-12\nu_7 + 21) \geq 0;$$
$$\mu_1(u, \chi_{10}, 0) = \tfrac{1}{21}(-1\nu_7 + 28) \geq 0;$$
$$\mu_7(u, \chi_{10}, 0) = \tfrac{1}{21}(6\nu_7 + 21) \geq 0;$$

has no solutions.

Now we need to look at automorphism groups, as specified in the next table (we use the same notations for groups as in the GAP Character Table Library [5]). Only the case of S_5 is covered by Luthar and Trama in [16]. For the remaining groups, we will investigate the prime graph question (PQ) below. Note that for some groups the results are stronger.

G	$Aut(G)$	*Remark*
$PSL(2,5) \cong A_5$	S_5	
$PSL(2,7)$	$PSL(3,2):C2$	
$PSL(2,8)$	$PSL(2,8).3$	
$PSL(2,9) \cong A_6$	$(A6.C2):C2$	i.e. to check: $S_6, A6.2_2, M10, A6.2^2$
$PSL(2,17)$	$PSL(2,17).2$	
$PSL(3,3)$	$PSL(3,3).2$	
$PSP(3,4) \cong U(4,2)$	$U(4,2).2$	
$U(3,3)$	$U(3,3).2$	

• Let $G = Aut(PSL(2,7)) = PSL(3,2):C2$. Then $|G| = 336 = 2^4 \cdot 3 \cdot 7$ and $exp(G) = 168 = 2^3 \cdot 3 \cdot 7$. The group G has elements of orders 2, 3, 4, 6, 7 and 8. By [8] (Proposition 3.1), it follows immediately that torsion units of orders 3 and 7 are rationally conjugate to an element of G. For this automorphism group we are able to show that the first Zassenhaus conjecture holds, and to do this it remains to prove rational conjugacy for torsion units of orders 2, 4, 6 and 8, and then show that $V(\mathbb{Z}G)$ has no elements of orders 12, 14 and 21.

For units of order 2, the system of inequalities

$$\mu_0(u, \chi_2, *) = \tfrac{1}{2}(\nu_{2a} - \nu_{2b} + 1) \geq 0;$$
$$\mu_1(u, \chi_2, *) = \tfrac{1}{2}(-\nu_{2a} + \nu_{2b} + 1) \geq 0;$$

has only two trivial solutions $(\nu_{2a}, \nu_{2b}) \in \{(1,0),(0,1)\}$.

For units of order 4, the system

$$\begin{aligned}
\mu_0(u,\chi_2,*) &= \tfrac{1}{4}(2\nu_{2a}+2\nu_{4a}-2\nu_{2b}+2) \geq 0;\\
\mu_2(u,\chi_2,*) &= \tfrac{1}{4}(-2\nu_{2a}-2\nu_{4a}+2\nu_{2b}+2) \geq 0;\\
\mu_0(u,\chi_3,*) &= \tfrac{1}{4}(-4\nu_{2a}+4\nu_{4a}+4) \geq 0;\\
\mu_2(u,\chi_3,*) &= \tfrac{1}{4}(4\nu_{2a}-4\nu_{4a}+4) \geq 0;
\end{aligned}$$

has only three solutions $(\nu_{2a},\nu_{4a},\nu_{2b}) \in \{(0,0,1),(0,1,0),(1,0,0)\}$. After the Cohn-Livingstone test (Theorem 4.1, [6]) it remains only one solution $(\nu_{2a},\nu_{4a},\nu_{2b}) = (0,1,0)$.

For units of order 6 we need to consider two cases dependently on whether $\chi(u^3)=\chi(2a)$ or $\chi(u^3)=\chi(2b)$.

When $\chi(u^3)=\chi(2a)$, we obtain the system

$$\begin{aligned}
\mu_0(u,\chi_3,*) &= \tfrac{1}{6}(-4\nu_{2a}+4) \geq 0;\\
\mu_2(u,\chi_3,*) &= \tfrac{1}{6}(2\nu_{2a}+4) \geq 0;\\
\mu_0(u,\chi_8,*) &= \tfrac{1}{6}(-2\nu_{3a}+4\nu_{2b}-2\nu_{6a}+6) \geq 0;\\
\mu_1(u,\chi_8,*) &= \tfrac{1}{6}(-\nu_{3a}+2\nu_{2b}-\nu_{6a}+9) \geq 0;\\
\mu_3(u,\chi_8,*) &= \tfrac{1}{6}(+2\nu_{3a}-4\nu_{2b}+2\nu_{6a}+6) \geq 0;\\
\mu_0(u,\chi_3,7) &= \tfrac{1}{6}(-2\nu_{2a}-2\nu_{2b}+4\nu_{6a}+2) \geq 0;\\
\mu_0(u,\chi_4,7) &= \tfrac{1}{6}(-2\nu_{2a}+2\nu_{2b}-4\nu_{6a}+2) \geq 0;\\
\mu_0(u,\chi_5,7) &= \tfrac{1}{6}(2\nu_{2a}-2\nu_{3a}+2\nu_{2b}+2\nu_{6a}+4) \geq 0;
\end{aligned}$$

which has no solutions.

When $\chi(u^3)=\chi(2b)$, we get the system

$$\begin{aligned}
\mu_0(u,\chi_3,*) &= \tfrac{1}{6}(-4\nu_{2a}+6) \geq 0;\\
\mu_3(u,\chi_3,*) &= \tfrac{1}{6}(4\nu_{2a}+6) \geq 0;\\
\mu_0(u,\chi_8,*) &= \tfrac{1}{6}(-2\nu_{3a}+4\nu_{2b}-2\nu_{6a}+8) \geq 0;\\
\mu_1(u,\chi_8,*) &= \tfrac{1}{6}(-\nu_{3a}+2\nu_{2b}-\nu_{6a}+7) \geq 0;\\
\mu_3(u,\chi_8,*) &= \tfrac{1}{6}(+2\nu_{3a}-4\nu_{2b}+2\nu_{6a}+4) \geq 0;
\end{aligned}$$

which has only one trivial solution $(\nu_{2a},\nu_{3a},\nu_{2b},\nu_{6a}) = (0,0,0,1)$.

For elements of order 8, we have to consider two cases: $\chi(u^4)=\chi(2a)$ and $\chi(u^4)=\chi(2b)$.

When $\chi(u^4)=\chi(2a)$, we get the system of inequalities

$$\begin{aligned}
\mu_0(u,\chi_2,*) &= \tfrac{1}{8}(4\nu_{2a}+4\nu_{4a}-4\nu_{2b}-4\nu_{8a}-4\nu_{8b}+4) \geq 0;\\
\mu_4(u,\chi_2,*) &= \tfrac{1}{8}(-4\nu_{2a}-4\nu_{4a}+4\nu_{2b}+4\nu_{8a}+4\nu_{8b}+4) \geq 0;\\
\mu_0(u,\chi_3,*) &= \tfrac{1}{8}(-8\nu_{2a}+8\nu_{4a}+8) \geq 0;\\
\mu_4(u,\chi_3,*) &= \tfrac{1}{8}(8\nu_{2a}-8\nu_{4a}+8) \geq 0;\\
\mu_1(u,\chi_4,*) &= \tfrac{1}{8}(+4\nu_{8a}-4\nu_{8b}+4) \geq 0;\\
\mu_3(u,\chi_4,*) &= \tfrac{1}{8}(-4\nu_{8a}+4\nu_{8b}+4) \geq 0;\\
\mu_0(u,\chi_6,*) &= \tfrac{1}{8}(-4\nu_{2a}-4\nu_{4a}+4\nu_{2b}-4\nu_{8a}-4\nu_{8b}+4) \geq 0;\\
\mu_4(u,\chi_6,*) &= \tfrac{1}{8}(4\nu_{2a}+4\nu_{4a}-4\nu_{2b}+4\nu_{8a}+4\nu_{8b}+4) \geq 0;\\
\mu_0(u,\chi_7,*) &= \tfrac{1}{8}(-4\nu_{2a}-4\nu_{4a}-4\nu_{2b}+4\nu_{8a}+4\nu_{8b}+4) \geq 0;
\end{aligned}$$

which has only two trivial solutions $(\nu_{2a},\nu_{4a},\nu_{2b},\nu_{8a},\nu_{8b}) \in \{(0,0,0,0,1),(0,0,0,1,0)\}$.

When $\chi(u^4)=\chi(2b)$, we get that $\mu_1(u,\chi_2,0)=1/4$ is not an integer, so this case is not possible.

Thus, now we proved rational conjugacy for elements of all orders that appear in G, and it remains only to show that $V(\mathbb{Z}G)$ has no elements of orders 12, 14 and 21.

For elements of order 12, we need to consider two cases. If $\chi(u^6) = \chi(2a)$, then $\mu_1(u, \chi_2, 0) = -1/6$ is not an integer. If $\chi(u^6) = \chi(2b)$, then $\mu_3(u, \chi_2, 0) = 1/2$ is not an integer. Therefore, no elements of order 12 can appear in $V(\mathbb{Z}G)$.

For the case of units of order 14, we use the ordinary character χ_7 of degree 7 such that $\chi(C_2) = -1$, $\chi(C_7) = 0$ to construct the system

$$\begin{aligned}
\mu_0(u, \chi_7, 0) &= \tfrac{1}{14}(-6\nu_2 + 6) \geq 0;\\
\mu_1(u, \chi_7, 0) &= \tfrac{1}{14}(\nu_2 + 8) \geq 0;\\
\mu_7(u, \chi_7, 0) &= \tfrac{1}{14}(6\nu_2 + 8) \geq 0;
\end{aligned}$$

which has no solutions.

For the case of units of order 21, we use the ordinary character χ_6 of degree 7 such that $\chi(C_3) = 1$, $\chi(C_7) = 0$ to construct the system

$$\begin{aligned}
\mu_0(u, \chi_6, 0) &= \tfrac{1}{21}(12\nu_3 + 9) \geq 0;\\
\mu_1(u, \chi_6, 0) &= \tfrac{1}{21}(\nu_3 + 6) \geq 0;\\
\mu_7(u, \chi_6, 0) &= \tfrac{1}{21}(-6\nu_3 + 6) \geq 0;
\end{aligned}$$

which has no solutions.

Thus, the Zassenhaus conjecture ZC–1 holds for G.

• Let $G = Aut(PSL(2, 8)) = PSL(2, 8).3$. Then $|G| = 1512 = 2^3 \cdot 3^3 \cdot 7$ and $exp(G) = 126 = 2 \cdot 3^2 \cdot 7$. The group G has elements of orders 2, 3, 6, 7 and 9. By [8] (Proposition 3.1), it follows immediately that torsion units of orders 2 and 7 are rationally conjugate to an element of G. For this automorphism group we are able to prove the rational conjugacy for all orders that appear in G except the order 6, and then confirm the (IP-C) conjecture, showing that $V(\mathbb{Z}G)$ has no elements of orders 14, 18 and 21.

For elements of order 3, the system

$$\begin{aligned}
\mu_0(u, \chi_2, *) &= \tfrac{1}{3}(2\nu_{3a} - \nu_{3b} - \nu_{3c} + 1) \geq 0;\\
\mu_1(u, \chi_2, *) &= \tfrac{1}{3}(-\nu_{3a} + 2\nu_{3b} - \nu_{3c} + 1) \geq 0;\\
\mu_2(u, \chi_2, *) &= \tfrac{1}{3}(-\nu_{3a} - \nu_{3b} + 2\nu_{3c} + 1) \geq 0;\\
\mu_0(u, \chi_4, *) &= \tfrac{1}{3}(-4\nu_{3a} + 2\nu_{3b} + 2\nu_{3c} + 7) \geq 0;\\
\mu_1(u, \chi_4, *) &= \tfrac{1}{3}(2\nu_{3a} - \nu_{3b} - \nu_{3c} + 7) \geq 0;\\
\mu_0(u, \chi_5, *) &= \tfrac{1}{3}(-4\nu_{3a} - \nu_{3b} - \nu_{3c} + 7) \geq 0;\\
\mu_1(u, \chi_5, *) &= \tfrac{1}{3}(2\nu_{3a} + 2\nu_{3b} - \nu_{3c} + 7) \geq 0;\\
\mu_2(u, \chi_5, *) &= \tfrac{1}{3}(2\nu_{3a} - \nu_{3b} + 2\nu_{3c} + 7) \geq 0;\\
\mu_0(u, \chi_7, *) &= \tfrac{1}{3}(6\nu_{3a} + 21) \geq 0;\\
\mu_1(u, \chi_7, *) &= \tfrac{1}{3}(-3\nu_{3a} + 21) \geq 0;\\
\mu_0(u, \chi_8, *) &= \tfrac{1}{3}(-2\nu_{3a} + 4\nu_{3b} + 4\nu_{3c} + 8) \geq 0;\\
\mu_1(u, \chi_8, *) &= \tfrac{1}{3}(\nu_{3a} - 2\nu_{3b} - 2\nu_{3c} + 8) \geq 0;\\
\mu_1(u, \chi_9, *) &= \tfrac{1}{3}(\nu_{3a} + 4\nu_{3b} - 2\nu_{3c} + 8) \geq 0;\\
\mu_2(u, \chi_9, *) &= \tfrac{1}{3}(\nu_{3a} - 2\nu_{3b} + 4\nu_{3c} + 8) \geq 0;\\
\mu_0(u, \chi_4, 2) &= \tfrac{1}{3}(-6\nu_{3a} + 6) \geq 0;\\
\mu_1(u, \chi_4, 2) &= \tfrac{1}{3}(3\nu_{3a} + 6) \geq 0;\\
\mu_0(u, \chi_5, 2) &= \tfrac{1}{3}(6\nu_{3a} + 12) \geq 0;\\
\mu_1(u, \chi_5, 2) &= \tfrac{1}{3}(-3\nu_{3a} + 12) \geq 0;
\end{aligned}$$

has only three trivial solutions $(\nu_{3a}, \nu_{3b}, \nu_{3c}) \in \{(1, 0, 0), (0, 0, 1), (0, 1, 0)\}$.

For units of order 6, we have to consider three cases for $\chi(u^2) \in \{\chi(3a), \chi(3b), \chi(3c)\}$.

For $\chi(u^2) = \chi(3a)$, we have the system

$$\begin{aligned}
\mu_0(u, \chi_4, *) &= \tfrac{1}{6}(-2\nu_{2a} - 4\nu_{3a} + 2\nu_{3b} + 2\nu_{3c} - 2\nu_{6a} - 2\nu_{6b} + 2) \geq 0;\\
\mu_1(u, \chi_4, *) &= \tfrac{1}{6}(-\nu_{2a} - 2\nu_{3a} + \nu_{3b} + \nu_{3c} - \nu_{6a} - \nu_{6b} + 10) \geq 0;\\
\mu_3(u, \chi_4, *) &= \tfrac{1}{6}(2\nu_{2a} + 4\nu_{3a} - 2\nu_{3b} - 2\nu_{3c} + 2\nu_{6a} + 2\nu_{6b} + 4) \geq 0;\\
\mu_0(u, \chi_5, *) &= \tfrac{1}{6}(-2\nu_{2a} - 4\nu_{3a} - \nu_{3b} - \nu_{3c} + \nu_{6a} + \nu_{6b} + 2) \geq 0;\\
\mu_1(u, \chi_5, *) &= \tfrac{1}{6}(-\nu_{2a} - 2\nu_{3a} + \nu_{3b} - 2\nu_{3c} - \nu_{6a} + 2\nu_{6b} + 10) \geq 0;\\
\mu_3(u, \chi_5, *) &= \tfrac{1}{6}(2\nu_{2a} + 4\nu_{3a} + \nu_{3b} + \nu_{3c} - \nu_{6a} - \nu_{6b} + 4) \geq 0;\\
\mu_4(u, \chi_5, *) &= \tfrac{1}{6}(\nu_{2a} + 2\nu_{3a} - \nu_{3b} + 2\nu_{3c} + \nu_{6a} - 2\nu_{6b} + 8) \geq 0;\\
\mu_0(u, \chi_7, *) &= \tfrac{1}{6}(-6\nu_{2a} + 6\nu_{3a} + 24) \geq 0;
\end{aligned}$$

which has 4 solutions

$$\begin{aligned}
(\nu_{2a}, \nu_{3a}, \nu_{3b}, \nu_{3c}, \nu_{6a}, \nu_{6b}) \in \{&(2, -1, -1, 2, 1, -2), (2, -1, 0, 1, 0, -1),\\
&(2, -1, 1, 0, -1, 0), (2, -1, 2, -1, -2, 1)\}.
\end{aligned}$$

When $\chi(u^2) = \chi(3b)$, the system

$$\begin{aligned}
\mu_0(u, \chi_4, *) &= \tfrac{1}{6}(-2\nu_{2a} - 4\nu_{3a} + 2\nu_{3b} + 2\nu_{3c} - 2\nu_{6a} - 2\nu_{6b} + 8) \geq 0;\\
\mu_2(u, \chi_4, *) &= \tfrac{1}{6}(\nu_{2a} + 2\nu_{3a} - \nu_{3b} - \nu_{3c} + \nu_{6a} + \nu_{6b} + 5) \geq 0;\\
\mu_0(u, \chi_5, *) &= \tfrac{1}{6}(-2\nu_{2a} - 4\nu_{3a} - \nu_{3b} - \nu_{3c} + \nu_{6a} + \nu_{6b} + 5) \geq 0;\\
\mu_1(u, \chi_5, *) &= \tfrac{1}{6}(-\nu_{2a} - 2\nu_{3a} + \nu_{3b} - 2\nu_{3c} - \nu_{6a} + 2\nu_{6b} + 10) \geq 0;\\
\mu_2(u, \chi_5, *) &= \tfrac{1}{6}(\nu_{2a} + 2\nu_{3a} + 2\nu_{3b} - \nu_{3c} - 2\nu_{6a} + \nu_{6b} + 5) \geq 0;\\
\mu_3(u, \chi_5, *) &= \tfrac{1}{6}(2\nu_{2a} + 4\nu_{3a} + \nu_{3b} + \nu_{3c} - \nu_{6a} - \nu_{6b} + 7) \geq 0;\\
\mu_4(u, \chi_5, *) &= \tfrac{1}{6}(\nu_{2a} + 2\nu_{3a} - \nu_{3b} + 2\nu_{3c} + \nu_{6a} - 2\nu_{6b} + 8) \geq 0;\\
\mu_5(u, \chi_5, *) &= \tfrac{1}{6}(-\nu_{2a} - 2\nu_{3a} - 2\nu_{3b} + \nu_{3c} + 2\nu_{6a} - \nu_{6b} + 7) \geq 0;\\
\mu_0(u, \chi_7, *) &= \tfrac{1}{6}(-6\nu_{2a} + 6\nu_{3a} + 18) \geq 0;
\end{aligned}$$

has 9 solutions:

$$\begin{aligned}
(\nu_{2a}, \nu_{3a}, \nu_{3b}, \nu_{3c}, \nu_{6a}, \nu_{6b}) \in \{&(-2, 2, -1, 2, 1, -1), (-2, 2, 0, 1, 0, 0),\\
&(-2, 2, 1, 0, -1, 1), (0, 0, -1, 1, 1, 0), (0, 0, 0, 0, 0, 1), (0, 0, 0, 3, 0, -2),\\
&(0, 0, 1, -1, -1, 2), (0, 0, 1, 2, -1, -1), (0, 0, 2, 1, -2, 0)\}
\end{aligned}$$

Finally, when $\chi(u^2) = \chi(3c)$, we obtain the system

$$\begin{aligned}
\mu_0(u, \chi_4, *) &= \tfrac{1}{6}(-2\nu_{2a} - 4\nu_{3a} + 2\nu_{3b} + 2\nu_{3c} - 2\nu_{6a} - 2\nu_{6b} + 8) \geq 0;\\
\mu_2(u, \chi_4, *) &= \tfrac{1}{6}(\nu_{2a} + 2\nu_{3a} - \nu_{3b} - \nu_{3c} + \nu_{6a} + \nu_{6b} + 5) \geq 0;\\
\mu_0(u, \chi_5, *) &= \tfrac{1}{6}(-2\nu_{2a} - 4\nu_{3a} - \nu_{3b} - \nu_{3c} + \nu_{6a} + \nu_{6b} + 5) \geq 0;\\
\mu_1(u, \chi_5, *) &= \tfrac{1}{6}(-\nu_{2a} - 2\nu_{3a} + \nu_{3b} - 2\nu_{3c} - \nu_{6a} + 2\nu_{6b} + 7) \geq 0;\\
\mu_3(u, \chi_5, *) &= \tfrac{1}{6}(2\nu_{2a} + 4\nu_{3a} + \nu_{3b} + \nu_{3c} - \nu_{6a} - \nu_{6b} + 7) \geq 0;\\
\mu_4(u, \chi_5, *) &= \tfrac{1}{6}(\nu_{2a} + 2\nu_{3a} - \nu_{3b} + 2\nu_{3c} + \nu_{6a} - 2\nu_{6b} + 5) \geq 0;\\
\mu_0(u, \chi_7, *) &= \tfrac{1}{6}(-6\nu_{2a} + 6\nu_{3a} + 18) \geq 0;
\end{aligned}$$

which has 9 solutions

$$\begin{aligned}
(\nu_{2a}, \nu_{3a}, \nu_{3b}, \nu_{3c}, \nu_{6a}, \nu_{6b}) \in \{&(-2, 2, 0, 1, 1, -1), (-2, 2, 1, 0, 0, 0),\\
&(-2, 2, 2, -1, -1, 1), (0, 0, -1, 1, 2, -1), (0, 0, 0, 0, 1, 0), (0, 0, 1, -1, 0, 1),\\
&(0, 0, 1, 2, 0, -2), (0, 0, 2, 1, -1, -1), (0, 0, 3, 0, -2, 0)\}
\end{aligned}$$

For units of order 9, again we have to consider three cases for $\chi(u^2) \in \{\chi(3a), \chi(3b), \chi(3c)\}$.

For $\chi(u^2) = \chi(3a)$, we have the system

$$
\begin{aligned}
\mu_0(u, \chi_2, *) &= \tfrac{1}{9}(6\nu_{3a} + 6\nu_{9a} - 3\nu_{3b} - 3\nu_{3c} - 3\nu_{9b} - 3\nu_{9c} + 3) \geq 0;\\
\mu_3(u, \chi_2, *) &= \tfrac{1}{9}(-3\nu_{3a} - 3\nu_{9a} + 6\nu_{3b} - 3\nu_{3c} + 6\nu_{9b} - 3\nu_{9c} + 3) \geq 0;\\
\mu_6(u, \chi_2, *) &= \tfrac{1}{9}(-3\nu_{3a} - 3\nu_{9a} - 3\nu_{3b} + 6\nu_{3c} - 3\nu_{9b} + 6\nu_{9c} + 3) \geq 0;\\
\mu_0(u, \chi_4, *) &= \tfrac{1}{9}(-12\nu_{3a} + 6\nu_{9a} + 6\nu_{3b} + 6\nu_{3c} + 6\nu_{9b} + 6\nu_{9c} + 3) \geq 0;\\
\mu_3(u, \chi_4, *) &= \tfrac{1}{9}(6\nu_{3a} - 3\nu_{9a} - 3\nu_{3b} - 3\nu_{3c} - 3\nu_{9b} - 3\nu_{9c} + 3) \geq 0;\\
\mu_0(u, \chi_5, *) &= \tfrac{1}{9}(-12\nu_{3a} + 6\nu_{9a} - 3\nu_{3b} - 3\nu_{3c} - 3\nu_{9b} - 3\nu_{9c} + 3) \geq 0;\\
\mu_3(u, \chi_5, *) &= \tfrac{1}{9}(6\nu_{3a} - 3\nu_{9a} + 6\nu_{3b} - 3\nu_{3c} + 6\nu_{9b} - 3\nu_{9c} + 3) \geq 0;\\
\mu_6(u, \chi_5, *) &= \tfrac{1}{9}(6\nu_{3a} - 3\nu_{9a} - 3\nu_{3b} + 6\nu_{3c} - 3\nu_{9b} + 6\nu_{9c} + 3) \geq 0;\\
\mu_0(u, \chi_7, *) &= \tfrac{1}{9}(18\nu_{3a} + 27) \geq 0;\\
\mu_3(u, \chi_7, *) &= \tfrac{1}{9}(-9\nu_{3a} + 27) \geq 0;\\
\mu_0(u, \chi_8, *) &= \tfrac{1}{9}(-6\nu_{3a} - 6\nu_{9a} + 12\nu_{3b} + 12\nu_{3c} - 6\nu_{9b} - 6\nu_{9c} + 6) \geq 0;\\
\mu_3(u, \chi_8, *) &= \tfrac{1}{9}(3\nu_{3a} + 3\nu_{9a} - 6\nu_{3b} - 6\nu_{3c} + 3\nu_{9b} + 3\nu_{9c} + 6) \geq 0;\\
\mu_0(u, \chi_9, *) &= \tfrac{1}{9}(-6\nu_{3a} - 6\nu_{9a} - 6\nu_{3b} - 6\nu_{3c} + 3\nu_{9b} + 3\nu_{9c} + 6) \geq 0;\\
\mu_3(u, \chi_9, *) &= \tfrac{1}{9}(3\nu_{3a} + 3\nu_{9a} + 12\nu_{3b} - 6\nu_{3c} - 6\nu_{9b} + 3\nu_{9c} + 6) \geq 0;\\
\mu_6(u, \chi_9, *) &= \tfrac{1}{9}(3\nu_{3a} + 3\nu_{9a} - 6\nu_{3b} + 12\nu_{3c} + 3\nu_{9b} - 6\nu_{9c} + 6) \geq 0;\\
\mu_0(u, \chi_4, 2) &= \tfrac{1}{9}(-18\nu_{3a}) \geq 0;\\
\mu_3(u, \chi_4, 2) &= \tfrac{1}{9}(9\nu_{3a}) \geq 0;\\
\mu_0(u, \chi_5, 2) &= \tfrac{1}{9}(18\nu_{3a} - 18\nu_{9a} + 18) \geq 0;\\
\mu_3(u, \chi_5, 2) &= \tfrac{1}{9}(-9\nu_{3a} + 9\nu_{9a} + 18) \geq 0;
\end{aligned}
$$

which has 5 solutions, only three of which remain after the Cohn-Livingstone test [6]:

$$(\nu_{3a}, \nu_{9a}, \nu_{3b}, \nu_{3c}, \nu_{9b}, \nu_{9c}) \in \{(0,0,0,0,0,1), (0,0,0,0,1,0), (0,1,0,0,0,0)\}.$$

If $\chi(u^3) = \chi(3b)$, then $\mu_1(u, \chi_2, 0) = 1/3$ is not an integer, so this case is impossible. Similarly, when $\chi(u^3) = \chi(3c)$, then $\mu_2(u, \chi_2, 0) = 1/3$, so this case is not possible as well.

Now, after we have covered orders that appear in the group G, we need to show that $V(\mathbb{Z}G)$ has no elements of orders 14, 18 and 21.

For elements of order 14, we use the ordinary character χ_4 of degree 7 with $\chi(C_2) = -1$, $\chi(C_7) = 0$. Then the system

$$
\begin{aligned}
\mu_0(u, \chi_4, 0) &= \tfrac{1}{14}(-6\nu_2 + 6) \geq 0;\\
\mu_1(u, \chi_4, 0) &= \tfrac{1}{14}(-1\nu_2 + 8) \geq 0;\\
\mu_7(u, \chi_4, 0) &= \tfrac{1}{14}(6\nu_2 + 8) \geq 0;
\end{aligned}
$$

has no solutions.

For elements of order 18, we need to consider $22 \times 3 \times 3 = 198$ cases determined by combinations of $\chi(u^6)$, $\chi(u^3)$ and $\chi(u^2)$. Using the development version of the GAP package LAGUNA [4] we verified that each case leads to a contradiction since there is certain $\mu_i(u, \chi_j, 0)$ which is not an integer. We omit full details here (but may provide a table summarising the results, if need be).

For elements of order 21, we use the ordinary character χ_{11} of degree 27 with $\chi(C_3) = 0$, $\chi(C_7) = -1$. Then the system

$$
\begin{aligned}
\mu_0(u, \chi_{11}, 0) &= \tfrac{1}{21}(-12\nu_7 + 21) \geq 0;\\
\mu_1(u, \chi_{11}, 0) &= \tfrac{1}{21}(-\nu_7 + 28) \geq 0;\\
\mu_7(u, \chi_{11}, 0) &= \tfrac{1}{21}(6\nu_7 + 21) \geq 0;
\end{aligned}
$$

has no solutions.

• Let $G = Aut(PSL(2,9)) = (A6.C2) : C2$. For this group we need to consider four possible cases. Here, as well as in the rest of the proof, we give only the outline, specifying which orders of torsion units we considered. The detailed proof will be included in the full version of the paper.

Case 1. Let $G = S_6$. Then $|G| = 720 = 2^4 \cdot 3^2 \cdot 5$ and $exp(G) = 60 = 2^2 \cdot 3 \cdot 5$. By [8] (Proposition 3.1), it follows immediately that torsion units of order 5 are rationally conjugate to an element of G. For this group we are able to confirm the (IP-C) conjecture, showing that there are no elements of orders 10, 12 and 15 in $V(\mathbb{Z}G)$.

Case 2. Let $G = A6.2_2$. Then $|G| = 720 = 2^4 \cdot 3^2 \cdot 5$ and $exp(G) = 120 = 2^3 \cdot 3 \cdot 5$. By [8] (Proposition 3.1), it follows immediately that torsion units of order 3 are rationally conjugate to an element of G. For the remaining orders, we are able to prove the rational conjugacy for all orders that appear in G (2,3,4,5,8 and 10), and also to show that there are no elements of orders 15 and 20 in $V(\mathbb{Z}G)$. However, we are not yet able to eliminate the tuple $(\nu_{2a}, \nu_{3a}, \nu_{2b}) = (-2, 3, 0)$ for torsion units of order 6, so (PQ) still remains open for this group.

Case 3. Let $G = M_{10}$. Then $|G| = 720 = 2^4 \cdot 3^2 \cdot 5$ and $exp(G) = 120 = 2^3 \cdot 3 \cdot 5$. By [8] (Proposition 3.1), it follows immediately that torsion units of orders 2,3 and 5 are rationally conjugate to an element of G. We can also prove rational conjugacy for the order 4, but not for the order 8. For orders that do not appear in G, we are able to show that there are no elements of orders 10 and 15 in $V(\mathbb{Z}G)$. However, we are not yet able to eliminate the tuple $(\nu_{2a}, \nu_{3a}) = (-2, 3)$ for torsion units of order 6, so (PQ) still remains open for this group.

Case 4. Let $G = A6.2^2$. In this case using the HeLP - method we are able to give positive answer to (PQ) eliminating the order 15.

For the remaining four automorphism groups the HeLP - method answers (PQ) positively, eliminating the following orders of torsion units of $V(\mathbb{Z}G)$: orders 34 and 51 for $Aut(PSL(2,17)) = PSL(2,17).2$; orders 26 and 39 for $Aut(PSL(3,3)) = PSL(3,3).2$; order 15 for $Aut(PSP(3,4)) = U(4,2).2$; and finally, orders 14 and 21 for $Aut(U(3,3)) = U(3,3).2$. This completes the proof.

References

[1] S. D. Berman, On certain properties of integral group rings (Russian), Dokl. Akad. Nauk SSSR (N.S.) **91** (1953), 7 – 9.

[2] V. Bovdi, E. Jespers, A. Konovalov, Torsion units in integral group rings of Janko simple groups, Math.Comp. **80** (2011), no.273, 593 – 615.

[3] V. Bovdi, A. Konovalov, S. Linton, Torsion units in integral group rings of Conway simple groups, Int. J. Algebra Comput. 21 (2011), no.4, 615 – 634.

[4] V. Bovdi, A. Konovalov, R. Rossmanith and Cs. Schneider, *LAGUNA — Lie AlGebras and UNits of group Algebras, Version 3.6.1*; GAP package, 2012, (`http://www.cs.st-andrews.ac.uk/~alexk/laguna/`).

[5] T. Breuer, *The GAP Character Table Library, Version 1.2.1*; GAP package, 2012, (`http://www.math.rwth-aachen.de/~Thomas.Breuer/ctbllib`).

[6] J. A. Cohn and D. Livingstone, On the structure of group algebras I, Canad. J. Math. **17** (1965), 583 – 593.

[7] The GAP Group, *GAP — Groups, Algorithms, and Programming, Version 4.5.6*; 2012, (`http://www.gap-system.org`).

[8] M. Hertweck, On the torsion units of some integral group rings, Algebra Colloq. **13**(2) (2006) 329–348.

[9] M. Hertweck, Partial augmentations and Brauer character values of torsion units in group rings, e-print arXiv:math.RA/0612429v2.

[10] M. Hertweck, Zassenhaus conjecture for A_6. Proc. Indian Acad. Sci. Math. Sci. 118 (2008), no. 2, 189–195.

[11] M. Herzog, On finite simple groups divisible by three primes only, J.Algebra **10** (1968) 383 - 388.

[12] G. Higman, Units in group rings, D.Phil.thesis, Oxford Univ. 1940.

[13] E.Jespers, W.Kimmerle, G.Nebe and Z.Marciciak, Miniworkshop Arithmetik von Gruppenringen, vol.4 Oberwolfach Reports No.55, pp. 3149 – 3179, EMS, Zürich 2007.

[14] W. Kimmerle, On the prime graph of the unit group of integral group rings of finite groups, Contemporary Mathematics Vol.**420** 2006, 215 – 228.

[15] I. S. Luthar and I. B. S. Passi, Zassenhaus conjecture for A_5, J. Nat. Acad. Math. India **99** (1989), 1 – 5.

[16] I. S. Luthar and Poonam Trama, Zassenhaus conjecture for S_5, Comm. Alg. **19(8)** (1991), 2353 – 2362

[17] Z. S. Marciniak, J. Ritter, S. K. Sehgal and A. Weiss, Torsion Units in Integral Group Rings of Some Metabelian Groups, II, J. Number Theory **25** (1987), 340 – 352.

[18] R. Sandling Graham Higman's thesis "Units in Group Rings" in Integral Representations and Applications ed. K.W. Roggenkamp, Springer Lecture Notes **882** 93–116, 1981.

[19] S. K. Sehgal, Units in integral group rings, Pitman Monographs and Surveys in Pure and Applied Mathematics, Longman Scientific & Technical, Essex, 1993.

Wolfgang Kimmerle
Universität Stuttgart
Institut für Geometrie und Topologie
Lehrstuhl für Differentialgeometrie
Pfaffenwaldring 57
D-70569 Stuttgart
Germany
WWW: http://www.igt.uni-stuttgart.de/LstDiffgeo/Kimmerle/

Alexander Konovalov
School of Computer Science
University of St Andrews
Jack Cole Building, North Haugh
St Andrews
Fife, Scotland
KY16 9SX
UK

Erschienene Preprints ab Nummer 2007/2007-001

Komplette Liste:
`http://www.mathematik.uni-stuttgart.de/preprints`

2012-018 *Kimmerle, W.; Konovalov, A.:* On the Prime Graph of the Unit Group of Integral Group Rings of Finite Groups II
2012-017 *Stroppel, B.; Stroppel, M.:* Desargues, Doily, Dualities, and Exceptional Isomorphisms
2012-016 *Moroianu, A.; Pilca, M.; Semmelmann, U.:* Homogeneous almost quaternion-Hermitian manifolds
2012-015 *Steinke, G.F.; Stroppel, M.J.:* Simple groups acting two-transitively on the set of generators of a finite elation Laguerre plane
2012-014 *Steinke, G.F.; Stroppel, M.J.:* Finite elation Laguerre planes admitting a two-transitive group on their set of generators
2012-013 *Diaz Ramos, J.C.; Dominguez Vázquez, M.; Kollross, A.:* Polar actions on complex hyperbolic spaces
2012-012 *Moroianu; A.; Semmelmann, U.:* Weakly complex homogeneous spaces
2012-011 *Moroianu; A.; Semmelmann, U.:* Invariant four-forms and symmetric pairs
2012-010 *Hamilton, M.J.D.:* The closure of the symplectic cone of elliptic surfaces
2012-009 *Hamilton, M.J.D.:* Iterated fibre sums of algebraic Lefschetz fibrations
2012-008 *Hamilton, M.J.D.:* The minimal genus problem for elliptic surfaces
2012-007 *Ferrario, P.:* Partitioning estimation of local variance based on nearest neighbors under censoring
2012-006 *Stroppel, M.:* Buttons, Holes and Loops of String: Lacing the Doily
2012-005 *Hantsch, F.:* Existence of Minimizers in Restricted Hartree-Fock Theory
2012-004 *Grundhöfer, T.; Stroppel, M.; Van Maldeghem, H.:* Unitals admitting all translations
2012-003 *Hamilton, M.J.D.:* Representing homology classes by symplectic surfaces
2012-002 *Hamilton, M.J.D.:* On certain exotic 4-manifolds of Akhmedov and Park
2012-001 *Jentsch, T.:* Parallel submanifolds of the real 2-Grassmannian
2011-028 *Spreer, J.:* Combinatorial 3-manifolds with cyclic automorphism group
2011-027 *Griesemer, M.; Hantsch, F.; Wellig, D.:* On the Magnetic Pekar Functional and the Existence of Bipolarons
2011-026 *Müller, S.:* Bootstrapping for Bandwidth Selection in Functional Data Regression
2011-025 *Felber, T.; Jones, D.; Kohler, M.; Walk, H.:* Weakly universally consistent static forecasting of stationary and ergodic time series via local averaging and least squares estimates
2011-024 *Jones, D.; Kohler, M.; Walk, H.:* Weakly universally consistent forecasting of stationary and ergodic time series
2011-023 *Györfi, L.; Walk, H.:* Strongly consistent nonparametric tests of conditional independence
2011-022 *Ferrario, P.G.; Walk, H.:* Nonparametric partitioning estimation of residual and local variance based on first and second nearest neighbors
2011-021 *Eberts, M.; Steinwart, I.:* Optimal regression rates for SVMs using Gaussian kernels
2011-020 *Frank, R.L.; Geisinger, L.:* Refined Semiclassical Asymptotics for Fractional Powers of the Laplace Operator
2011-019 *Frank, R.L.; Geisinger, L.:* Two-term spectral asymptotics for the Dirichlet Laplacian on a bounded domain
2011-018 *Hänel, A.; Schulz, C.; Wirth, J.:* Embedded eigenvalues for the elastic strip with cracks
2011-017 *Wirth, J.:* Thermo-elasticity for anisotropic media in higher dimensions
2011-016 *Höllig, K.; Hörner, J.:* Programming Multigrid Methods with B-Splines
2011-015 *Ferrario, P.:* Nonparametric Local Averaging Estimation of the Local Variance Function

2011-014 *Müller, S.; Dippon, J.:* k-NN Kernel Estimate for Nonparametric Functional Regression in Time Series Analysis
2011-013 *Knarr, N.; Stroppel, M.:* Unitals over composition algebras
2011-012 *Knarr, N.; Stroppel, M.:* Baer involutions and polarities in Moufang planes of characteristic two
2011-011 *Knarr, N.; Stroppel, M.:* Polarities and planar collineations of Moufang planes
2011-010 *Jentsch, T.; Moroianu, A.; Semmelmann, U.:* Extrinsic hyperspheres in manifolds with special holonomy
2011-009 *Wirth, J.:* Asymptotic Behaviour of Solutions to Hyperbolic Partial Differential Equations
2011-008 *Stroppel, M.:* Orthogonal polar spaces and unitals
2011-007 *Nagl, M.:* Charakterisierung der Symmetrischen Gruppen durch ihre komplexe Gruppenalgebra
2011-006 *Solanes, G.; Teufel, E.:* Horo-tightness and total (absolute) curvatures in hyperbolic spaces
2011-005 *Ginoux, N.; Semmelmann, U.:* Imaginary Kählerian Killing spinors I
2011-004 *Scherer, C.W.; Köse, I.E.:* Control Synthesis using Dynamic D-Scales: Part II — Gain-Scheduled Control
2011-003 *Scherer, C.W.; Köse, I.E.:* Control Synthesis using Dynamic D-Scales: Part I — Robust Control
2011-002 *Alexandrov, B.; Semmelmann, U.:* Deformations of nearly parallel G_2-structures
2011-001 *Geisinger, L.; Weidl, T.:* Sharp spectral estimates in domains of infinite volume
2010-018 *Kimmerle, W.; Konovalov, A.:* On integral-like units of modular group rings
2010-017 *Gauduchon, P.; Moroianu, A.; Semmelmann, U.:* Almost complex structures on quaternion-Kähler manifolds and inner symmetric spaces
2010-016 *Moroianu, A.; Semmelmann,U.:* Clifford structures on Riemannian manifolds
2010-015 *Grafarend, E.W.; Kühnel, W.:* A minimal atlas for the rotation group $SO(3)$
2010-014 *Weidl, T.:* Semiclassical Spectral Bounds and Beyond
2010-013 *Stroppel, M.:* Early explicit examples of non-desarguesian plane geometries
2010-012 *Effenberger, F.:* Stacked polytopes and tight triangulations of manifolds
2010-011 *Györfi, L.; Walk, H.:* Empirical portfolio selection strategies with proportional transaction costs
2010-010 *Kohler, M.; Krzyżak, A.; Walk, H.:* Estimation of the essential supremum of a regression function
2010-009 *Geisinger, L.; Laptev, A.; Weidl, T.:* Geometrical Versions of improved Berezin-Li-Yau Inequalities
2010-008 *Poppitz, S.; Stroppel, M.:* Polarities of Schellhammer Planes
2010-007 *Grundhöfer, T.; Krinn, B.; Stroppel, M.:* Non-existence of isomorphisms between certain unitals
2010-006 *Höllig, K.; Hörner, J.; Hoffacker, A.:* Finite Element Analysis with B-Splines: Weighted and Isogeometric Methods
2010-005 *Kaltenbacher, B.; Walk, H.:* On convergence of local averaging regression function estimates for the regularization of inverse problems
2010-004 *Kühnel, W.; Solanes, G.:* Tight surfaces with boundary
2010-003 *Kohler, M; Walk, H.:* On optimal exercising of American options in discrete time for stationary and ergodic data
2010-002 *Gulde, M.; Stroppel, M.:* Stabilizers of Subspaces under Similitudes of the Klein Quadric, and Automorphisms of Heisenberg Algebras
2010-001 *Leitner, F.:* Examples of almost Einstein structures on products and in cohomogeneity one
2009-008 *Griesemer, M.; Zenk, H.:* On the atomic photoeffect in non-relativistic QED
2009-007 *Griesemer, M.; Moeller, J.S.:* Bounds on the minimal energy of translation invariant n-polaron systems

2009-006 *Demirel, S.; Harrell II, E.M.:* On semiclassical and universal inequalities for eigenvalues of quantum graphs
2009-005 *Bächle, A, Kimmerle, W.:* Torsion subgroups in integral group rings of finite groups
2009-004 *Geisinger, L.; Weidl, T.:* Universal bounds for traces of the Dirichlet Laplace operator
2009-003 *Walk, H.:* Strong laws of large numbers and nonparametric estimation
2009-002 *Leitner, F.:* The collapsing sphere product of Poincaré-Einstein spaces
2009-001 *Brehm, U.; Kühnel, W.:* Lattice triangulations of E^3 and of the 3-torus
2008-006 *Kohler, M.; Krzyżak, A.; Walk, H.:* Upper bounds for Bermudan options on Markovian data using nonparametric regression and a reduced number of nested Monte Carlo steps
2008-005 *Kaltenbacher, B.; Schöpfer, F.; Schuster, T.:* Iterative methods for nonlinear ill-posed problems in Banach spaces: convergence and applications to parameter identification problems
2008-004 *Leitner, F.:* Conformally closed Poincaré-Einstein metrics with intersecting scale singularities
2008-003 *Effenberger, F.; Kühnel, W.:* Hamiltonian submanifolds of regular polytope
2008-002 *Hertweck, M.; Höfert, C.R.; Kimmerle, W.:* Finite groups of units and their composition factors in the integral group rings of the groups $PSL(2, q)$
2008-001 *Kovarik, H.; Vugalter, S.; Weidl, T.:* Two dimensional Berezin-Li-Yau inequalities with a correction term
2007-006 *Weidl, T.:* Improved Berezin-Li-Yau inequalities with a remainder term
2007-005 *Frank, R.L.; Loss, M.; Weidl, T.:* Polya's conjecture in the presence of a constant magnetic field
2007-004 *Ekholm, T.; Frank, R.L.; Kovarik, H.:* Eigenvalue estimates for Schrödinger operators on metric trees
2007-003 *Lesky, P.H.; Racke, R.:* Elastic and electro-magnetic waves in infinite waveguides
2007-002 *Teufel, E.:* Spherical transforms and Radon transforms in Moebius geometry
2007-001 *Meister, A.:* Deconvolution from Fourier-oscillating error densities under decay and smoothness restrictions