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Norbert Knarr, Markus J. Stroppel

Abstract

For Heisenberg groups constructed over semifields (i.e., not neccessarily associative division rings), we solve the isomorphism problem and determine the automorphism groups. We show that two Heisenberg groups over semifields are isomorphic precisely if the semifields are isotopic or anti-isotopic to each other. This condition means that the corresponding translation are isomorphic or dual to each other.
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In [7], Heisenberg's example of a step 2 nilpotent group has been generalized, replacing the ground field by an arbitrary associative ring S with $2 \in S^\times$. The main focus of [7] was on the group of automorphisms of such a Heisenberg group. In the present note we extend this further (see 1.2 below), dropping the restriction on invertibility of 2 and allowing the multiplication in S to be non-associative. The main results of the present paper require that S is a semifield. In fact, it has already become clear in [7, 11.2] that there are rings (containing zero divisors) where the corresponding Heisenberg group does not determine the ring. However, in [14] we have obtained positive results for the case where S is a split composition algebra.

The groups that we consider here have already been studied by Cronheim [3] who characterized those that arise from semifields (and actually reconstructed the semifield from the group, up to isotopism). However, in Cronheim's work a pair of commutative subgroups with special properties is distinguished (this pair occurs as (X_S, Y_S) below), and this pair is kept fixed by assumption. We clarify in the present paper that these subgroups are not invariant in general (although they are in certain cases, cf. 4.5, 5.4, and [14]). However, in each of the cases that we study here, it turns out that the collection of pairs in question forms a single orbit under automorphisms (see 5.4 and 5.5).

Theorem 5.7 below extends the result (implicit in Cronheim's paper [3]) that the isomorphism types of groups in the class considered here correspond to isomorphism types of incidence graphs of projective planes of Lenz type V. Our results extend Hiramine's paper [9] who has proved versions of 5.4 and 5.7 for finite semifields.

We note that semifields exist in abundance; see [11] and [15] for the finite case (and [6] for a recent addition to that copious supply), [17, 64.16, 82.1, 82.16, 82.21] for topological semifields (where the Heisenberg group is a real Lie group), and [5] for a general theory. The notion of Heisenberg group used in the present paper is more general than that used elsewhere (e.g., in [8], [7], or [18]); we include characteristic two cases.

1 Heisenberg groups defined by semifields

1.1 Definitions. Let $(S, +, \cdot)$ be a (not necessarily associative) unitary algebra, i.e., a set S with two binary operations $+$ and $\cdot$ called addition and multiplication, and two distinguished elements 0 and 1 such that

- $(S, +)$ is a group with neutral element 0,
- the distributive laws $a \cdot (b + c) = a \cdot b + a \cdot c$ and $(a + b) \cdot c = a \cdot c + b \cdot c$ hold,
- the element 1 is a neutral element for the multiplication.

Note that these conditions imply that addition is commutative. For the sake of readability, we will often suppress the symbol for the multiplication, writing ab for $a \cdot b$.

Mostly, we are interested in the case of a *semifield*[1], i.e., a unitary algebra such that

- the equations $a \cdot x = b$ or $y \cdot a = b$ have unique solutions x and y, respectively, whenever $a \in S \setminus \{0\}$.

The *middle nucleus* $S_{\mathrm{m}} := \{m \in S \mid \forall\, a, c \in S \colon a(mc) = (am)c\}$ of a semifield S is, in general, a skew field. If (the multiplication of) S is commutative then S_{m} is a field. The sets $S_{\mathrm{r}} := \{x \in S \mid \forall\, a, c \in S \colon a(cx) = (ac)x\}$ and $S_{\mathrm{l}} := \{x \in S \mid \forall\, a, c \in S \colon (xa)c = x(ac)\}$ are, respectively, called the *right nucleus* and the *left nucleus* of S; these form skew fields in general. If S is commutative, we observe $a(rc) = a(cr) = (ac)r = (ca)r = c(ar) = (ar)c$. In that case, we have that $S_{\mathrm{l}} = S_{\mathrm{r}}$ is a subfield of the middle nucleus S_{m}.

1.2 Definition. Let S be a unitary algebra. We define two binary operations on the set S^3, as follows.

$$\begin{aligned} (a, s, x) \odot (b, t, y) &:= (a + b, s + t, x + y + sb)\,, \\ (a, s, x) \mathbin{\#} (b, t, y) &:= (a + b, s + t, x + y + sb - ta)\,. \end{aligned}$$

Straightforward verification shows that both $\mathrm{h}_S := (S^3, \odot)$ and $\mathrm{H}_S := (S^3, \#)$ are groups.

We identify $S^2 \times S = \{(v, x) \mid v \in S^2, x \in S\}$ with S^3 whenever this seems convenient. In order to keep notation simple, we write $\langle (a, s) \underset{S}{|} (b, t) \rangle := sb - ta$.

1.3 Proposition. **1.** *The group h_S is nilpotent of class 2, more explicitly, the subset $Z_S := \{(0, 0)\} \times S$ equals both the center and the commutator subgroup of h_S.*

2. *Commutators in the group h_S are given by $\big[(u, x), (v, y)\big]_\odot = \big(0, \langle u \underset{S}{|} v \rangle\big)$; those in H_S by $\big[(u, x), (v, y)\big]_\# = \big(0, 2\langle u \underset{S}{|} v \rangle\big)$.*

3. *If $2\,\mathrm{id}$ is invertible in $\mathrm{End}(S, +)$ then $\eta_S \colon \mathrm{h}_S \to \mathrm{H}_S \colon (a, s, x) \mapsto (a, s, 2x - sa)$ is an isomorphism.*

4. *If $2\,\mathrm{id} = 0$ (i.e., if $\mathrm{char}\, S = 2$) then H_S is elementary abelian while h_S is not abelian.*

5. *If S is a semifield then every non-trivial element of H_S has the same order, namely, the characteristic of S if this is finite, and infinite order otherwise.*

6. *If $\mathrm{char}\, S = 2$ then every element of $\{(a, s, x) \in \mathrm{h}_S \mid as \neq 0\}$ has order 4, and every element of $\{(a, s, x) \in \mathrm{h}_S \mid as = 0\} \setminus \{(0, 0, 0)\}$ has order 2. The latter set contains $Z_S \setminus \{(0, 0, 0)\}$.*

[1] Other sources call these structures *division algebras*.

Proof. We compute the inverse of (a, s, x) in h_S as $(-a, -s, sa - x)$ and the commutator of (a, s, x) and (b, t, y) then as $(a, s, x)(b, t, y)(-a, -s, sa-x)(-b, -t, tb-y) = (0, 0, sb-ta)$. This gives the assertions about center and commutator subgroup of h_S.

Regarding assertion 3 we note that the map $\eta_S \colon \mathrm{h}_S \to \mathrm{H}_S \colon (a, s, x) \mapsto (a, s, 2x - sa)$ is a homomorphism. It is bijective if 2 is invertible in S.

The remaining assertions are well known facts about generalized Heisenberg groups (see the general treatment in [7, 2.4]); we do not need the special definition of sb via the semifield multiplication but only the fact that mapping $((a, s), (b, t))$ to sb is bi-additive. □

1.4 Lemma. *The centralizer of* (a, s, x) *in* h_S *is* $\mathrm{C}_{\mathrm{h}_S}(a, s, x) = \{(b, t, y) \mid sb = ta\}$. □

1.5 Definition. For $(a, s) \in S^2$ put $\mathrm{C}_{(a,s)} := \{(b, t) \in S^2 \mid at = bs\}$; this means $\mathrm{C}_{\mathrm{h}_S}(a, s, x) = \mathrm{C}_{(a,s)} \times S$. We call $\mathrm{C}_{(a,s)}$ *abelian* if the subgroup $\mathrm{C}_{(a,s)} \times S$ is commutative.

1.6 Remark. For each semifield S, the group h_S acts on the affine plane over S, and thus on the projective closure of that plane. To make this explicit, we define more generally the incidence structure $\mathrm{DAP}(S)$ for any algebra S as $\mathrm{DAP}(S) := (S^2, \mathcal{L})$ with $\mathcal{L} := \{[m, b] \mid m, b \in S\}$, where $[m, b] := \{(x, mx + b \mid x \in S\}$. (In [3], our $\mathrm{DAP}(S)$ is studied systematically, as the P-system corresponding to h_S, considered as a T-group with distinguished subgroups $\mathrm{C}_{(1,0)} \times S$ and $\mathrm{C}_{(0,1)} \times S$.)

If S is a semifield, we recall that the that affine plane over S is an extension of $\mathrm{DAP}(S)$; we add the set $\mathcal{L}_\infty := \{[v] \mid v \in S\}$ of *vertical lines* of the form $[v] := \{v\} \times S$ to obtain the line set $\mathcal{L} \cup \mathcal{L}_\infty$.

Returning to the case of a general algebra S we note that

$$\omega_P \colon \mathrm{h}_S \times S^2 \to S^2 \colon ((a, s, z), (x, y)) \mapsto (a + x, z + sx + y)$$

is an action of the group h_S on the point set of $\mathrm{DAP}(S)$ such that each line is mapped onto a line (here we use both distributive laws). The corresponding action on the set of lines is

$$\omega_{\mathcal{L}} \colon \mathrm{h}_S \times \mathcal{L} \to \mathcal{L} \colon ((a, s, z), [m, b]) \mapsto [s + m, z - sa - ma + b] .$$

If S is a semifield then $((a, s, z), [c]) \mapsto [c + a]$ extends this action to the affine plane; and we also obtain an action on the projective closure of the affine plane.

The subgroup $\mathrm{C}_{(1,0)} \times S$ consists of all *translations* of the plane, the subgroup $\mathrm{C}_{(0,1)} \times \{0\}$ is the group of all shears with axis $[0]$ and center $\mathcal{L}_\infty$ (this parallel class is regarded as a point at infinity, as usual).

A commutative group of automorphisms of a projective plane is called a *shift group* (cf. [12]) if there is an incident point-line pair (p, L) such that the group acts (sharply) transitively on the set of points not on L and on the set of lines not through p. In 5.1 below, the subgroups of the form $\mathrm{C}_u \times S$ with $u \in S^2$ such that C_u is abelian are characterized as the shift groups in h_S.

2 Isotopisms

2.1 Definition. Let $(S, +, \cdot)$ and $(T, +, *)$ be algebras (not necessarily associative). An *isotopism*[2] from $(S, +, \cdot)$ onto $(T, +, *)$ is a triplet (A, B, C) of additive bijections from S onto T

[2] Since we are going to interpret autotopisms as collineations as in [4, 3.1.32] (cf. also [13]), we use notation that differs from the more algebraically oriented sources, such as [2], [16], or [15].

such that $B(x \cdot y) = C(x) * A(y)$ holds for all $x, y \in S$. Note that (A, B, C) is an isomorphism of algebras precisely if $A = B = C$. If S and T are unitary algebras, this is also equivalent to $A(1) = 1 = C(1)$; in fact, evaluating $B(x \cdot 1) = C(x) * 1$ and $B(1 \cdot y) = 1 * C(y)$ we find $A = B = C$.

A triplet (D, E, F) of additive bijections from S onto T is called an *anti-isotopism* from $(S, +, \cdot)$ onto $(T, +, *)$ if $E(x \cdot y) = D(y) * F(x)$ holds for all $x, y \in S$.

For any isotopism (A, B, C) of an algebra S onto an algebra T we obtain an isomorphism (i.e., a collineation) $\alpha_{A|B|C}$ from $\mathrm{DAP}(S)$ onto $\mathrm{DAP}(T)$ (which extends to isomorphisms between the affine and between the projective planes, respectively, if S and T are semifields), mapping (x, y) to $(A(x), B(y))$ and $[m, b] \to [C(m), B(b)]$; cf. [4, 3.1.32].

Each anti-isotopism (D, E, F) from S onto T yields an isomorphism $\delta_{D|E|F}$ from $\mathrm{DAP}(S)$ onto the dual of $\mathrm{DAP}(T)$, mapping the point (x, y) to the line $[D(x), -E(y)]$ and the line $[m, b]$ onto $(F(m), -E(b))$.

For isotopisms (A', B', C'), (A, B, C) and anti-isotopisms (D', E', F'), (D, E, F), respectively, the compositions

$$\alpha_{A'|B'|C'}\,\alpha_{A|B|C} = \alpha_{A'A|B'B|C'C} \quad \text{and} \quad \delta_{D'|E'|F'}\,\delta_{D|E|F} = \alpha_{F'D|E'E|D'F}$$

are collineations, while

$$\alpha_{A'|B'|C'}\,\delta_{D|E|F} = \delta_{C'D|-B'E|A'F} \quad \text{and} \quad \delta_{D'|E'|F'}\,\alpha_{A|B|C} = \delta_{D'A|-E'B|F'C}$$

are dualities. We obtain the subgroup

$$\begin{aligned} \nabla_S \;:=\; & \{\alpha_{A|B|C} \mid (A, B, C) \text{ is an autotopism of } S\} \\ & \cup \{\delta_{D|E|F} \mid (D, E, F) \text{ is an anti-autotopism of } S\} \end{aligned}$$

in the group of all automorphisms and dualities of $\mathrm{DAP}(S)$.

2.2 Proposition. *Let S and T be unitary algebras.*

1. *If (A, B, C) is an isotopism from S onto T then*

$$\lfloor A|B|C \rfloor : (a, s, x) \mapsto (A(a), C(s), B(x))$$

is an isomorphism from h_S onto h_T, and an isomorphism from H_S onto H_T, as well.

2. *If (D, E, F) is an anti-isotopism from S onto T then*

$$\lceil D|E|F \rceil : (a, s, x) \mapsto (F(s), D(a), E(sa - x))$$

is an isomorphism from h_S onto h_T, and

$$\lceil D|E|F \rceil_{\#} : (a, s, x) \mapsto (F(s), D(a), -E(x))$$

is an isomorphism from H_S onto H_T.

3. *Now assume $S = T$. Then the map*

$$\omega\colon \begin{cases} \alpha_{X|Y|Z} \mapsto \lfloor X|Y|Z \rfloor & \text{if } (X, Y, Z) \text{ is an autotopism of } S \\ \delta_{X|Y|Z} \mapsto \lceil X|Y|Z \rceil & \text{if } (X, Y, Z) \text{ is an anti-autotopism of } S \end{cases}$$

is a group homomorphism from the group ∇_S into $\mathrm{Aut}(\mathrm{h}_S)$.

Proof. Clearly $\lfloor A|B|C \rfloor$ and $\lceil D|E|F \rceil$ are bijective. It remains to compute

$$\begin{aligned}\lfloor A|B|C \rfloor(a,s,x) \odot \lfloor A|B|C \rfloor(b,t,y) &= \big(A(a),C(s),B(x)\big) \odot \big(A(b),C(t),B(y)\big) \\ &= \big(A(a+b),C(s+t),B(x+y)+C(s)A(b)\big) \\ &= \big(A(a+b),C(s+t),B(x+y+sb)\big) \\ &= \lfloor A|B|C \rfloor \big((a,s,x)\odot(b,t,y)\big)\end{aligned}$$

and

$$\begin{aligned}\lceil D|E|F \rceil(a,s,x) \odot \lceil D|E|F \rceil(b,t,y) &= \big(F(s),D(a),E(sa-x)\big) \odot \big(F(t),D(b),E(tb-y)\big) \\ &= \big(F(s+t),D(a+b),E(sa+tb-x-y)+D(a)F(t)\big) \\ &= \big(F(s+t),D(a+b),E(sa+tb-x-y+ta)\big) \\ &= \lceil D|E|F \rceil(a+b,s+t,x+y+sb) \\ &= \lceil D|E|F \rceil \big((a,s,x)\odot(b,t,y)\big)\ .\end{aligned}$$

An analogous computation shows that $\lfloor A|B|C \rfloor$ is an isomorphism from H_S onto H_T, but $\lceil D|E|F \rceil$ is not. If 2 is invertible in S then $\eta_T \lfloor A|B|C \rfloor \eta_S^{-1} = \lfloor A|B|C \rfloor$ and $\eta_T \lceil D|E|F \rceil \eta_S^{-1} \neq \lceil D|E|F \rceil$ are isomorphisms from H_S onto H_T; here η_S and η_T are as in 1.3.3. □

2.3 Remark. If 2 is invertible in S then the isomorphisms

$$\begin{aligned} \eta_S\colon \mathrm{h}_S \to \mathrm{H}_S\colon &\quad (a,s,x) \mapsto (a,s,2x-sa) \\ \text{and } \eta_T\colon \mathrm{h}_T \to \mathrm{H}_T\colon &\quad (b,t,y) \mapsto (b,t,2y-tb)\end{aligned}$$

(cf. 1.3.3) yield $\eta_T \lfloor A|B|C \rfloor \eta_S^{-1} = \lfloor A|B|C \rfloor$ and $\eta_T \lceil D|E|F \rceil \eta_S^{-1} = \lceil D|E|F \rceil_{\#}$.

2.4 Definition. For any algebra S, we write $\mathrm{Atp}(S) \le \mathrm{Aut}(\mathrm{h}_S)$ for the group of all automorphisms $\lfloor A|B|C \rfloor$ induced by autotopisms (A,B,C) of S, and $\mathrm{AntiAtp}(S)$ for the group of automorphisms induced by autotopisms and anti-autotopisms.

3 Semifields that are isotopic to commutative ones

For any algebra $(S,+,\cdot)$ and $a \in S$, we consider the endomorphisms $\lambda_a\colon S \to S\colon x \mapsto a\cdot x$ and $\rho_a\colon S \to S\colon x \mapsto x \cdot a$ of the additive group $(S,+)$.

3.1 Lemma. *Let $(S,+,\cdot)$ be a unitary algebra. Assume that there exists $d \in S$ such that λ_d is invertible and $(d\cdot x)\cdot y = (d\cdot y)\cdot x$ holds for all $x,y \in S$. Then $(S,+,\cdot)$ is isotopic to a commutative algebra.*

If $(S,+,\cdot)$ is a semifield with this property then it is isotopic to a commutative semifield.

Proof. As we will change the multiplication on S, it is necessary to make the binary operations explicit. Assume that the given semifield is $(S,+,\cdot)$.

We define a new multiplication $*$ on S by $d\cdot(x*y) = (d\cdot x)\cdot y$. Our assumption yields that this multiplication is commutative. Now $(\mathrm{id},\lambda_d,\lambda_d)$ is an isotopism from $(S,+,\cdot)$ onto $(S,+,*)$. The neutral element of the new multiplication is the same as for the old multiplication.

In order to distinguish it from λ_x, we write $\lambda^*_x\colon s \mapsto x*s$ for the left multiplication map in $(S,+,*)$. We have $x*y = \lambda_d^{-1}\left(\lambda_d(x*y)\right) = \lambda_d^{-1}\left(\lambda_{d\cdot x}(y)\right) = \lambda_d^{-1}\left(\rho_y(\lambda_d(x))\right)$. Therefore, the left multiplication $\lambda^*_x = \lambda_d^{-1}\lambda_{d\cdot x}$ is invertible precisely if $\lambda_{d\cdot x}$ is invertible, and $\rho^*_y = \lambda_d^{-1}\rho_y\lambda_d$ is invertible precisely if ρ_y is invertible. In particular, the algebra $(S,+,*)$ is a semifield if, and only if, the algebra $(S,+,\cdot)$ is a semifield. □

3.2 Lemma. *Let S be an unitary algebra, and consider $a \in S \setminus \{0\}$.*

1. *If $\mathrm{C}_{(a,0)}$ is abelian then $\mathrm{C}_{(a,0)} = S\times\{0\}$ and the endomorphism $\rho_a\colon x \mapsto xa$ of the additive group $(S,+)$ is injective.*
2. *If $\mathrm{C}_{(0,a)}$ is abelian then $\mathrm{C}_{(0,a)} = \{0\}\times S$ and the endomorphism $\lambda_a\colon x \mapsto ax$ of $(S,+)$ is injective.*

Proof. In any case, we have $S\times\{0\} \subseteq \mathrm{C}_{(a,0)} = \{(b,t)\in S^2 \mid ta = 0\}$. If $\mathrm{C}_{(a,0)}$ is abelian then $(1,0)\in \mathrm{C}_{(a,0)}$ yields $0 = \langle (1,0) \underset{S}{|} (b,t)\rangle = t$. Thus $\mathrm{C}_{(a,0)} \subseteq S\times\{0\}$, and equality is proved. Injectivity of ρ_a now follows from $S\times(\ker\rho_a) \subseteq \mathrm{C}_{(a,0)}$. The proof for $(0,a)$ is completely analogous. □

For any semifield S and any $a\in S\setminus\{0\}$ the sets $\mathrm{C}_{(a,0)} = S\times\{0\}$ and $\mathrm{C}_{(0,a)} = \{0\}\times S$ are abelian. In fact, for any $x\in S$ the centralizers of $(a,0,x)$ and $(0,a,x)$, respectively, are vector spaces over the middle nucleus S_{m}; in particular, they are elementary abelian groups if $\operatorname{char} S > 0$.

In general, however, the set $\mathrm{C}_{(a,s)}$ will not be abelian if $sa \neq 0$:

3.3 Lemma. *Assume that S is a semifield and that there exists $(a,s)\in S^2$ with $sa\neq 0$ such that $\mathrm{C}_{(a,s)}$ is abelian. Let d be the solution for $da = s$.*

1. *We have $\mathrm{C}_{(a,s)} = \mathrm{C}_{(1,d)} = \{(c,dc) \mid c\in S\}$.*
2. *The semifield S is isotopic to a commutative one.*
3. *If S is commutative then d belongs to the middle nucleus S_{m}.*

Conversely, if S is commutative then $\mathrm{C}_{(1,d)}$ is abelian for each $d\in S_{\mathrm{m}}$.

Proof. For each $b\in S$ let t_b be the unique solution of the equation $t_b\cdot a = s\cdot b$. Then $\mathrm{C}_{(a,x)} = \{(b,t_b) \mid b\in S\}$. Our assumption that $\mathrm{C}_{(a,x)}$ is abelian yields $\mathrm{C}_{(a,x)} \subseteq \mathrm{C}_{(1,d)} = \{(c, d\cdot c \mid c\in S\}$ for $d := t_1$. As both sets contain precisely one pair for each possible left entry, they coincide. As $\mathrm{C}_{(1,d)}$ is abelian, we find $(d\cdot x)\cdot y - (d\cdot y)\cdot x = \langle (x, d\cdot x) \underset{S}{|} (y, d\cdot y)\rangle = 0$. Then 3.3 yields that S is isotopic to a commutative algebra.

If $(S,+,\cdot)$ is commutative then $(d\cdot x)\cdot y - (d\cdot y)\cdot x = 0$ is equivalent to $(x\cdot d)\cdot y = x\cdot(d\cdot y)$. This means $d\in S_{\mathrm{m}}$, as claimed. □

Assertion 3.3.2 is taken from [12, 9.3]. The reader may wonder how a non-commutative semifield may be isotopic to a commutative one. In fact, this is a quite common feature in the absence of associativity; every non-associative semifield is isotopic to a non-commutative one, see [1, (2.4), p. 110], cf. also [10, 2.4].

Lemma 3.3.2 cannot be extended easily to algebras that are not semifields; see [7, 5.9].

3.4 Proposition. *If S is a semifield with $\operatorname{char} S = 2$ then the only elements in h_S with elementary abelian centralizers are those in $(S\times\{0\}\times S)\cup(\{0\}\times S\times S)$.*

Proof. Let $(a,s,x) \in S^3$ with $as \neq 0$. If the centralizer of (a,s,x) in h_S is commutative then $\mathrm{C}_{(a,s)} = \mathrm{C}_{(1,d)} = \{(c,cd) \mid c \in S\}$ for some $d \in S \smallsetminus \{0\}$ by 3.3. For each $c \in S \smallsetminus \{0\}$ we now obtain that the square $(c,cd,x) \odot (c,cd,x) = (0,0,c(cd))$ is not trivial, and the centralizer $\mathrm{C}_{(a,s)} \times S$ is not elementary abelian. □

4 Isomorphisms between Heisenberg groups

If S is a unitary algebra then $Z_S = \{(0,0)\} \times S = \mathrm{h}_S'$ is characteristic in h_S. It is quite natural to study the following map:

4.1 Definition. Let S be an algebra. The *commutator map* of h_S is

$$\gamma_S \colon\ \mathrm{h}_S/Z_S \times \mathrm{h}_S/Z_S \to Z_S \colon\ (Z_S + (u,x), Z_S + (v,y)) \mapsto \big[(u,x),(v,y)\big]_\odot = (0,0,\langle u \underset{S}{|} v\rangle)\,.$$

4.2 Definition. Let S be a (not necessarily associative) algebra and consider an arbitrary additive map $N \colon S^2 \to S$. Then $\xi_N \colon S^3 \to S^3 \colon (a,x,u) \mapsto (a,x,u+N(a,x))$ is an automorphism both of h_S and of H_S. We call ξ_N a *nil-automorphism* and write $\Xi_S := \{\xi_N \mid N \in \mathrm{Hom}(S^2,S)\}$.

4.3 Lemma. *The group Ξ_S of nil-automorphisms consists of those automorphisms that act trivially both on Z_S and on the quotient modulo Z_S. If Z_S is characteristic (in particular, if the algebra S is unitary) then Ξ_S is a normal subgroup of $\mathrm{Aut}(\mathrm{h}_S)$ (and also of $\mathrm{Aut}(\mathrm{H}_S)$, if 2 is invertible in S).* □

4.4 Definition. For any algebra S we put $X_S := S \times \{0\} \times S$ and $Y_S := \{0\} \times S \times S$.

4.5 Theorem. *Assume that S and T are unitary algebras, and that $\varphi \colon \mathrm{h}_S \to \mathrm{h}_T$ is an isomorphism mapping $\{X_S, Y_S\}$ to $\{X_T, Y_T\}$.*

1. *If $\varphi(X_S) = X_T$ then there exists an isotopism η from S onto T such that $\varphi \in \Xi_T \circ \lfloor \eta \rfloor$.*
2. *If $\varphi(X_S) = Y_T$ then there is an anti-isotopism α from S onto T with $\varphi \in \Xi_T \circ \lceil \alpha \rceil$.*

If S is a semifield with $\mathrm{char}\, S = 2$ or if S is a semifield not isotopic to a commutative one then every isomorphism $\varphi \colon \mathrm{h}_S \to \mathrm{h}_T$ maps $\{X_S, Y_S\}$ to $\{X_T, Y_T\}$.

Proof. The isomorphism φ maps the center Z_S of h_S onto Z_T. Let $U \colon S^2 \to T^2$ be the map induced between the quotients modulo the centers, i.e., such that $\varphi(Z_S + (u,0)) = Z_T + (U(u),0)$ holds for each $u \in S^2$. Moreover, let $U' \colon S \to T$ be defined by $\varphi(0,0,x) = (0,0,U'(x))$. As φ translates the commutator map γ_S into γ_T we have

$$\forall\, u,v \in S \colon \langle U(u) \underset{T}{|} U(v)\rangle = U'\langle u \underset{S}{|} v\rangle\,. \qquad (\lozenge)$$

If $\varphi(X_S) = X_T$ then $\varphi(Y_S) = Y_T$ and there are additive bijections A and C from S onto T such that $U(a,s) = (A(a), C(s))$. Equation $(\lozenge)$ now implies

$$U'(s \cdot b) = \langle U(0,s) \underset{T}{|} U(b,0)\rangle = \langle (0,C(s)) \underset{T}{|} (A(b),0)\rangle = C(s) * A(b)\,.$$

Thus (A, U', C) is an isotopism, and $\varphi \circ \lfloor A|U'|C \rfloor^{-1}$ belongs to Ξ_T.

Now assume $U(S \times \{0\}) = \{0\} \times T$. Then $U(\{0\} \times S) = T \times \{0\}$, there are additive bijections D and F from S onto T such that $U(a, s) = (F(s), D(a))$, and $(\lozenge)$ implies $U'(s \cdot b) = -D(b) * F(s)$. Thus $(D, -U', F)$ is an anti-isotopism, and $\varphi \circ \lceil D| - U'|F\rceil^{-1}$ belongs to Ξ_T.

If $\operatorname{char} S = 2$ then 3.4 says that X_S and Y_S are the only abelian centralizers. If $\operatorname{char} S \neq 2$ then our assumptions on S again imply that X_S and Y_S are the only abelian centralizers. In any one of these cases, we have that φ maps the two-element set $\{X_S, Y_S\}$ onto $\{X_T, Y_T\}$. □

For any field R of characteristic two, the Heisenberg group $\mathrm{h}_{R\times R}$ contains many commutative centralizers; cf. [14, 4.2]. Also, according to [7, 5.9], the direct product A of a commutative ring with a non-commutative one yields a Heisenberg group h_A with commutative centralizers apart from those in $\{X_A, Y_A\}$. These examples show that the extra assumption (i.e., absence of zero divisors) in the last assertion of 4.5 is not superfluous.

If $\operatorname{char} S \neq 2$ then H_S belongs to the class of reduced Heisenberg groups, where isomorphisms are well understood, cf. [19] and [8]:

4.6 Lemma. *Let S and T be unitary algebras such that $2\,\mathrm{id}$ is invertible in $\operatorname{End}(S, +)$, and let $\varphi\colon \mathrm{H}_S \to \mathrm{H}_T$ be an isomorphism. Then there are uniquely determined additive bijections $U\colon S^2 \to T^2$ and $U'\colon S \to T$ together with an additive map $N\colon S^2 \to T$ such that*

$$\varphi(u, x) = (U(u), U'(x) + N(u))$$

holds for all $(u, x) \in S^2 \times S$. The maps U and U' satisfy $(\lozenge)$.

Conversely, if $U\colon (S^2, +) \to (T^2, +)$ and $U'\colon (S, +) \to (T, +)$ are isomorphisms satisfying equation $(\lozenge)$ then U' is uniquely determined by U, and

$$\psi_U\colon S^2 \times S \to T^2 \times T\colon (u, x) \mapsto (U(u), U'(x))$$

is an isomorphism from H_S onto H_T. We obtain $\varphi = \xi_{N\circ U^{-1}} \circ \psi_U$. □

4.7 Definition. If $S = T$ we write Ψ_S for the set of all ψ_U where $U \in \operatorname{Aut}(S^2, +)$ satisfies equation $(\lozenge)$. Thus $\operatorname{Aut}(\mathrm{H}_S) = \Xi_S \circ \Psi_S$.

In order to show that 4.6 does not easily extend to the characteristic two case, we study the smallest example. Here we use the field $\mathbb{F}_2$ for the semifield, and consider $\mathrm{h}_{\mathbb{F}_2}$:

4.8 Example. The group $\mathrm{h}_{\mathbb{F}_2}$ has order 8. The element $(1, 1, 0)$ and its inverse $(1, 1, 1)$ have order 4; they both have the square $(0, 0, 1)$. Each element $(a, s, x) \in \mathbb{F}_2^3$ with $sa = 0$ satisfies $(a, s, x) \odot (a, s, x) = (0, 0, sa) = (0, 0, 0)$. Thus the group $\mathrm{h}_{\mathbb{F}_2}$ contains precisely 5 involutions, namely, the central involution $\zeta := (0, 0, 1)$, and the non-central elements $\sigma := (1, 0, 0)$, $\tau := (0, 1, 0)$, $\sigma\tau\sigma = \tau\zeta = (0, 1, 1)$, and $\tau\sigma\tau = \sigma\zeta = (1, 0, 1)$. Among the groups of order 8, this characterizes those isomorphic to the dihedral group D_4.

The group $\operatorname{Aut}(\mathrm{h}_{\mathbb{F}_2})$ acts on the set of all involutions in $\mathrm{h}_{\mathbb{F}_2}$, permuting the four involutions different from ζ. The set $\{(\sigma, \sigma\zeta), (\tau, \tau\zeta)\}$ of commuting pairs is invariant under that action. This gives an embedding of $\operatorname{Aut}(\mathrm{h}_{\mathbb{F}_2})$ into D_4. On the other hand, we have that the dihedral group D_8 of order 16 acts by automorphisms via conjugation on D_4 (which forms a normal subgroup in D_8). The kernel of that action has order 2, and we see that $\operatorname{Aut}(\mathrm{h}_{\mathbb{F}_2}) \cong \operatorname{Aut}(\mathrm{D}_4)$ is isomorphic to D_4.

The normal subgroup $\Xi_{\mathbb{F}_2} \leq \operatorname{Aut}(\mathrm{h}_{\mathbb{F}_2})$ is elementary abelian of order 4. The unique involution interchanging σ with τ in $\operatorname{Aut}(\mathrm{h}_{\mathbb{F}_2})$ generates a complement to $\Xi_{\mathbb{F}_2}$; so the extension $1 \to \Xi_{\mathbb{F}_2} \to \operatorname{Aut}(\mathrm{h}_{\mathbb{F}_2}) \to \mathrm{C}_2 \to 1$ splits.

The stabilizer of the set $\mathbb{F}_2^2 \times \{0\}$ is trivial because it fixes $\tau\sigma = (1, 1, 0)$ and interchanging σ with τ would also interchange $\tau\sigma$ with its inverse $\sigma\tau$.

4.9 Remark. Our results 4.5 and 5.5 yield that the group extension $\Xi_S \to \mathrm{Aut}(\mathrm{h}_S)$ splits whenever either W is a semifield, or associative (i.e., in all those cases there exists a subgroup $K \le \mathrm{Aut}(\mathrm{h}_S)$ with trivial intersection $\Xi_S \cap K$ and such that $\mathrm{Aut}(\mathrm{h}_S) = \Xi_S K$ is a semidirect product). In [14] we obtain the same result for the case where S is a composition algebra.

However, example 4.8 shows that we can not, in general, use the stabilizer of the set $S^2 \times \{0\}$ (which forms a vector subspace of the Lie algebra associated with h_S) for the complement K. This is due to the fact that this set is not invariant under inversion in h_S, and thus not invariant under the action of $\lceil D|E|F \rceil$ if (D, E, F) is an anti-autotopism of S (cf. 2.2).

The problem disappears if $\mathrm{char}\, S \neq 2$ because then we can use $\mathrm{H}_S \cong \mathrm{h}_S$ — and $S^2 \times \{0\}$ is invariant under inversion in H_S.

5 Automorphisms of Heisenberg groups over semifields

We characterize the shift groups (cf. 1.6) inside h_S:

5.1 Lemma. *Let S be a commutative semifield. For a subgroup $\Delta \le \mathrm{H}_S$ the following are equivalent:*

1. *There is $d \in S_\mathrm{m} \smallsetminus \{0\}$ such that $\Delta = \mathrm{C}_{(1,d)} \times S$.*
2. *There is $d \in S \smallsetminus \{0\}$ such that $\Delta = \mathrm{C}_{(1,d)} \times S$, and Δ is commutative.*
3. *The actions $\omega_P|_{P\times\Delta}$ and $\omega_\mathcal{L}|_{(\mathcal{L}\smallsetminus\mathcal{L}_\infty)\times\Delta}$ introduced in 1.6 are both transitive, and the group Δ is commutative.*

Proof. The first two assertions are equivalent by 3.3. Straightforward computations show that $\mathrm{C}_{(1,d)} \times S$ satisfies the transitivity assumptions imposed by the third assertion. It remains to prove that the third assertion implies the first one; this has been done in [12, 9.4, 9.6]. □

5.2 Remarks. The subgroups $X_S = \mathrm{C}_{(1,0)} \times S$ and $Y_X := \mathrm{C}_{(0,1)} \times S$ are commutative subgroups of h_S but each of them satisfies only one of the two conditions on transitivity imposed in 5.1. In fact, the group X_S is transitive on the point set but not on $\mathcal{L}$, while Y_S is transitive on $\mathcal{L}$ but not on the point set.

5.3 Definition. Let S be a commutative semifield. We define

$$\Sigma_S := \{A \in \mathrm{GL}(2, S_\mathrm{m}) \,|\, \det A \in S_\mathrm{r}\}\,.$$

The set of all commutative centralizers is

$$\begin{aligned} \mathcal{A}_S &:= \{\mathrm{C}_{\mathrm{h}_S}(g) \,|\, g \in \mathrm{h}_S,\ \mathrm{C}_{\mathrm{h}_S}(g) \text{ is commutative}\} \\ &= \{\mathrm{C}_u \times S \,|\, u \in S^2,\ \mathrm{C}_u \text{ is abelian}\}\,. \end{aligned}$$

5.4 Proposition. *Let S be a commutative semifield.*

1. *The set Σ_S forms a group (with respect to ordinary matrix multiplication).*
2. *The map $\omega\colon \Sigma_S \times \mathrm{H}_S \to \mathrm{H}_S\colon \big(A, (u, x)\big) \mapsto (A(u), x \det A)$ is an action of Σ_S by automorphisms of H_S.*

3. *If* $\operatorname{char} S \neq 2$ *then this action induces a two-transitive action on the set* $\mathcal{A}_S$ *of all commutative centralizers in* h_S.

Proof. The set Σ_S is the pre-image of the subgroup $S_\mathrm{r} \smallsetminus \{0\}$ of the multiplicative group of S_m under the determinant homomorphism. Thus Σ_S is a subgroup of $\mathrm{GL}(2, S_\mathrm{m})$.

Clearly ω is an action. In order to see that it is an action by automorphisms of H_S, we have to verify that the action of $A \in \Sigma_S$ satisfies condition $(\Diamond)$, see 4.6. To this end, we write $u = (a, s)$ and $v = (b, t)$ and consider the following generators of Σ_S:

$A = \left(\begin{smallmatrix} 1 & 0 \\ m & 1 \end{smallmatrix}\right)$ for $m \in S_\mathrm{m}$: Here $\det A = 1$. Using commutativity of S and the defining property of the middle nucleus S_m we compute

$$\begin{aligned} \langle A(u) \underset{S}{|} A(v) \rangle \;=\; \langle (a, ma+s) \underset{S}{|} (b, mb+t) \rangle \;&=\; (ma+s)b - (mb+t)a \\ &=\; (ma)b - (mb)a + sb - ta \\ &=\; (am)b - a(bm) + \langle u \underset{S}{|} v \rangle \;=\; \langle u \underset{S}{|} v \rangle , \end{aligned}$$

as required for $(\Diamond)$.

$A = \left(\begin{smallmatrix} 1 & m \\ 0 & 1 \end{smallmatrix}\right)$ for $m \in S_\mathrm{m}$: This case is analogous to the previous one.

$A = \left(\begin{smallmatrix} 1 & 0 \\ 0 & r \end{smallmatrix}\right)$ for $r \in S_\mathrm{r}$: Here $\det A = r$, and

$$\begin{aligned} \langle A(u) \underset{S}{|} A(v) \rangle \;=\; \langle (a, rs) \underset{S}{|} (b, rt) \rangle \;&=\; a(rt) - b(rs) \\ &=\; a(tr) - b(sr) \;=\; (at - bs)r \;=\; \langle u \underset{S}{|} v \rangle \det A \end{aligned}$$

shows that $(\Diamond)$ is satisfied.

Thus the action of Σ_S is an action by automorphisms of H_S, and assertion 2 is established.

From 3.3 we know that $\mathrm{C}_{(1,d)}$ is abelian precisely if $d \in S_\mathrm{m}$, that each abelian set C_u is of the form $\mathrm{C}_{(1,d)}$ for a suitable $d \in S_\mathrm{m}$, and that $S(1,d) = \mathrm{C}_u$ in that case. In order to prove assertion 3 it remains to observe that $\left(\begin{smallmatrix} 1 & 0 \\ d & 1 \end{smallmatrix}\right) \in \Sigma_S$ maps $\mathrm{C}_{(1,0)} \times S$ to $\mathrm{C}_{(1,d)} \times S$, and that $\left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right) \in \Sigma_S$ interchanges $\mathrm{C}_{(1,0)} \times S$ with $\mathrm{C}_{(0,1)} \times S$. □

5.5 Theorem. *Assume that* S *is a semifield.*

1. *If* S *is isotopic to a commutative semifield* T *with* $\operatorname{char} T \neq 2$ *then* $\mathrm{Aut}(\mathrm{h}_S) \cong \mathrm{Aut}(\mathrm{H}_S) \cong \mathrm{Aut}(\mathrm{H}_T) = \Xi_T \circ \mathrm{SL}(2, T_\mathrm{m}) \circ \mathrm{Atp}(T) = \Xi_T \circ \Sigma_T \circ \mathrm{Atp}(T)$.

2. *If* S *is not isotopic to a commutative semifield then* $\mathrm{Aut}(\mathrm{h}_S) = \Xi_S \circ \mathrm{AntiAtp}(S)$.

3. *If* $\operatorname{char} S = 2$ *then* $\mathrm{Aut}(\mathrm{h}_S) = \Xi_S \circ \mathrm{AntiAtp}(S)$.

Proof. Under any automorphism of H_S the subgroups X_S and Y_S are mapped to elements of $\mathcal{A}_S$. If S is commutative with $\operatorname{char} S \neq 2$ we know from 5.4 that $\mathrm{SL}(2, R_\mathrm{m}) \leq \Sigma_S$ acts two-transitively on $\mathcal{A}_S$. If S is not isotopic to a commutative semifield then the set $\{X_S, Y_S\}$ is invariant under all automorphisms of h_S. Invariance of that set also holds if $\operatorname{char} S = 2$, see 4.5.

The stabilizer of the set $\{X_S, Y_S\}$ has been determined in 4.5. Note that anti-autotopisms of S are autotopisms if S is commutative. □

5.6 Remark. Our result 4.5 also shows that we do not have an action of $\mathrm{SL}(2, S_{\mathrm{m}})$ or of Σ_S as in 5.4 if $\operatorname{char} S = 2$.

5.7 Theorem. *Let S and T be semifields. The following are equivalent:*

1. *S and T are isotopic or anti-isotopic.*
2. *The projective planes over S and T are isomorphic or dual to each other.*
3. *The incidence graphs of the projective planes over S and T are isomorphic to each other.*
4. *There is an isomorphism between h_S and h_T.*

Proof. The equivalence of the first three assertions is well known; see 2.1.

The first assertion implies the last one, see 2.2. So assume that there is an isomorphism $\varphi\colon \mathrm{h}_S \to \mathrm{h}_T$. From 4.5 we know that the existence of φ yields the existence of an isotopism or an anti-isotopism if $\operatorname{char} S = 2$ or if S is not isotopic to a commutative semifield. It remains to treat the case where S and T are (isotopic to) commutative semifields, and $\operatorname{char} S \neq 2$. According to 5.5, we may then assume that φ maps $\{X_S, Y_S\}$ to $\{X_T, Y_T\}$. Now 4.5 yields the existence of an isotopism or an anti-isotopism, as required. □

5.8 Remark. For the equivalence of assertions 5.7.1 and 5.7.4 we only need to assume that one of the algebras in question is a semifield; for the other one this then follows from the existence of an (anti-)isotopism.

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