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Abstract

We solve the isomorphism problem for Heisenberg groups constructed over composition algebras, including the split case and characteristic two. Such groups are isomorphic if, and only if, the corresponding composition algebras are isomorphic as $\mathbb{Z}$-algebras.

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1 Introduction

In [6] we have solved the isomorphism problem for Heisenberg groups over semifields; showing that the Heisenberg groups over two semifields S and T are isomorphic if, and only if, the semifields are either isotopic or anti-isotopic (meaning that the projective translation planes over these semifields are either isomorphic or dual to each other). In the present notes, we solve the isomorphism problem for (not necessarily associative) algebras that may contain divisors of zero, under the extra assumption that at least one of the algebras is a composition algebra.

If we allow divisors of zero in our algebra S, the structure of the group H_S depends heavily on the structure of S, even in the case where S is associative with $2 \in S^\times$ (this case has been studied thoroughly in [3], where $V_S \cong \mathrm{H}_S \cong \mathrm{h}_S$). Aiming at a temperate amount of generalization we will thus retain a modest amount of associativity; we will study *composition* algebras in the sequel.

1.1 Definition. Let S be a unitary algebra. We define two binary operations on the set S^3, as follows.

$$\begin{aligned}(a,s,x)\odot(b,t,y) &:= (a+b,s+t,x+y+sb)\,,\\ (a,s,x)\,\#\,(b,t,y) &:= (a+b,s+t,x+y+sb-ta)\,.\end{aligned}$$

Straightforward verification shows that both $\mathrm{h}_S := (S^3,\odot)$ and $\mathrm{H}_S := (S^3,\#)$ are groups.

In order to keep notation simple, we write $\langle (a,s) \underset{S}{|} (b,t)\rangle := sb - ta$.

Recall from [6, 1.2] that h_S is nilpotent of class 2 (the subset $Z_S := \{(0,0)\}\times S$ equals both the center and the commutator subgroup of h_S) but H_S is elementary abelian if $\operatorname{char} S = 2$. However, we have:

1.2 Lemma. *Let S be a unitary algebra. Then $\eta_S\colon\ \mathrm{h}_S \to \mathrm{H}_S\colon (a,s,x) \mapsto (a,s,2x-sa)$ is a homomorphism; it is an isomorphism if 2 is invertible in S.* □

2 Isotopisms of composition algebras

An important source of unitary non-associative algebras is the class of octonion algebras, i.e., composition algebras of dimension 8 (also known as Cayley algebras). We treat these algebras in the context of alternative algebras here, i.e., algebras satisfying the *alternative identities* $x(xy) = (xx)y$ and $y(xx) = (xy)x$, cf. [7, Sect. III]. In each alternative algebra, we also have the *Moufang identities* (see [7, (3.4)–(3.6)] or [8, 1.4.1]):

$$(ax)(ya) = a((xy)a)\,, \qquad a(x(ay)) = (a(xa))y\,, \qquad x(a(ya)) = ((xa)y)a\,.$$

Among the consequences of these identities is Artin's result (first published in Zorn's paper [10], cf. [7, Thm. 3.1, p. 29]) asserting that any subalgebra generated by two elements is associative. In particular, we have $a(xa) = (ax)a$.

Assume now that A is an alternative unitary algebra, and consider $a \in A$. If the equations $ax = 1$ and $ya = 1$ both have solutions in A then their solutions are unique and coincide; we will denote the solution by a^{-1} and call a *invertible* in that case. The set $A^\times$ of all invertible elements is closed under multiplication; it forms a so-called Moufang loop.

Recall that a *composition algebra* A over a field R has a multiplicative quadratic form $N\colon A \to R$ with non-degenerate polar form $f_N\colon A \times A \to R\colon (x, y) \mapsto N(x+y)-N(x)-N(y)$. The *standard involution* $\kappa\colon A \to A\colon x \mapsto \bar{x} := f_N(x, 1) - x$ is an anti-automorphism, and $N(x) = x\bar{x} = \bar{x}x$. A good reference for the basic theory of composition algebras is [8, Ch. 1].

Composition algebras occur in dimensions $d \in \{1, 2, 4, 8\}$. While each composition algebra with $d \le 4$ is associative, the octonion algebras are not associative. However, they are still alternative. In any composition algebra A, we have $a \in A^\times \iff N(a) \neq 0$; in fact $a^{-1} := N(a)^{-1}\bar{a}$.

We remark that every element of an octonion algebra A is contained in some quaternion subalgebra of A (cf. [8, 1.6.4]). The set R of scalar multiples of 1 forms the center of A, see [8, 1.9.1].

2.1 Lemma. *If A is a composition algebra of dimension 8 then every invertible element can be written as a product of two elements of the set $P := \{p \in A \mid \bar{p} = -p\}$ of pure elements in A.*

Proof. Consider $a \in A^\times$. The set P is just the space of all elements orthogonal to 1 with respect to the norm form N. Thus we have $\dim(P) = 7 = \dim(Pa)$ and $\dim(Pa \cap P) \ge 6$. As the quadratic form N has non-degenerate polar form, its Witt index is at most $\frac{1}{2}\dim(A) = 4$, and the subspace $Pa \cap P$ cannot be contained $A \smallsetminus A^\times = \{x \in A \mid N(x) = 0\}$. Pick any $p \in P$ such that $q := pa$ lies in $P \cap A^\times$. Then $N(p) \neq 0$, we have $p^{-1} = \frac{1}{N(p)}\bar{p} = -\frac{1}{N(p)}p \in P$, and $a = p^{-1}q \in PP$, as required. □

We remark that the result in 2.1 remains true for composition algebras of dimension 4 (but needs a different proof for split quaternion algebras); it becomes false for 2-dimensional algebras (where PP is one-dimensional).

2.2 Definition. Let $(S, +, \cdot)$ and $(T, +, *)$ be algebras (not necessarily associative). An *isotopism*[1] from $(S, +, \cdot)$ onto $(T, +, *)$ is a triplet (A, B, C) of additive bijections from S onto T such that $B(x \cdot y) = C(x) * A(y)$ holds for all $x, y \in S$. Note that (A, B, C) is an isomorphism

[1] Our notation follows [6] and thus [2, 3.1.32] (cf. also [5]), where geometrical aspects lead to an assignment of roles for the three bijections that may appear confusing to a more algebraically bent reader.

of algebras precisely if $A = B = C$. If S and T are unitary algebras, this is also equivalent to $A(1) = 1 = C(1)$; in fact, evaluating $B(x \cdot 1) = C(x) * 1$ and $B(1 \cdot y) = 1 * C(y)$ we find $A = B = C$.

A triplet (D, E, F) of additive bijections from S onto T is called an *anti-isotopism* from $(S, +, \cdot)$ onto $(T, +, *)$ if $E(x \cdot y) = D(y) * F(x)$ holds for all $x, y \in S$.

As usual, an (anti-)isotopism from S onto S itself is called an *(anti-)autotopism*.

See [5] for an application of autotopisms of octonion fields (i.e., semifields that are octonion algebras, viz. octonion algebras with anisotropic norm form) to polarities and Baer involutions in the corresponding projective planes.

2.3 Examples. The Moufang identities yield that the following triplets are autotopisms, for each $a \in A^\times$:

$$(\rho_a, \lambda_a \circ \rho_a, \lambda_a)\,, \qquad (\lambda_a \circ \rho_a, \rho_a, \rho_a^{-1})\,, \qquad (\lambda_a^{-1}, \lambda_a, \lambda_a \circ \rho_a)\,,$$

where $\lambda_a \colon x \mapsto ax$ and $\rho_a \colon x \mapsto xa$.

For each $z \in \mathrm{Z}(A) \cap A^\times$ we also have the autotopisms $(\mathrm{id}, \lambda_z, \lambda_z)$ and $(\lambda_z, \lambda_z, \mathrm{id})$.

If A is a composition algebra then the standard involution κ is an anti-automorphism of A, and gives an anti-autotopism (κ, κ, κ).

2.4 Proposition ([6, 2.2]). *Let S and T be unitary algebras.*

1. *If (A, B, C) is an isotopism from S onto T then*

$$\lfloor A|B|C \rfloor : (a, s, x) \mapsto (A(a), C(s), B(x))$$

is an isomorphism from h_S onto h_T, and an isomorphism from H_S onto H_T, as well.

2. *If (D, E, F) is an anti-isotopism from S onto T then*

$$\lceil D|E|F \rceil : (a, s, x) \mapsto (F(s), D(a), E(sa - x))$$

is an isomorphism from h_S onto h_T, and

$$\lceil D|E|F \rceil_\# : (a, s, x) \mapsto (F(s), D(a), -E(x))$$

is an isomorphism from H_S onto H_T. □

2.5 Definition. For any algebra S, we write $\mathrm{Atp}(S) \le \mathrm{Aut}(\mathrm{h}_S)$ for the group of all automorphisms $\lfloor A|B|C \rfloor$ induced by autotopisms (A, B, C) of S, and $\mathrm{AntiAtp}(S)$ for the group of automorphisms induced by autotopisms and anti-autotopisms.

Note that isotopisms need not preserve the neutral element of multiplication; there are even isotopisms between unitary algebras and algebras without neutral element for the multiplication. Also, a commutative algebra may be isotopic to a non-commutative one. However, associativity is preserved; see 2.6.

For each element a of an algebra S, we consider the endomorphisms $\lambda_a^S \colon S \to S \colon s \mapsto as$ and $\rho_a^S \colon S \to S \colon s \mapsto sa$ of the additive group of S.

2.6 Lemma. *Let $(S, +, \cdot)$ and $(T, +, *)$ be algebras, and let (A, B, C) be an isotopism from $(S, +, \cdot)$ onto $(T, +, *)$.*

1. *For each $a \in S$ we have:*

a. λ_a^S *is injective* $\iff$ $\lambda_{C(a)}^T$ *is injective, and* λ_a^S *is surjective* $\iff$ $\lambda_{C(a)}^T$ *is surjective.*

b. ρ_a^S *is injective* $\iff$ $\rho_{A(a)}^T$ *is injective, and* ρ_a^S *is surjective* $\iff$ $\rho_{A(a)}^T$ *is surjective.*

2. *If S is unitary and T is a composition algebra then each $X \in \{A, C\}$ maps 1 into $T^\times$.*

3. *If the algebras are unitary and at least one of them is associative or a composition algebra then the algebras are (anti-)isomorphic (as $\mathbb{Z}$-algebras) if, and only if, they are (anti-)isotopic. In particular, (anti-)isotopic composition algebras are isomorphic.*

Proof. The observations $C(a) * t = 0 \iff a \cdot A^{-1}(t) = 0$, $t * A(a) = 0 \iff C^{-1}(t) \cdot a = 0$, $C(a) * T = B(a \cdot S)$, and $T * A(a) = B(S \cdot a)$ yield the equivalences stated in the first assertion.

Now assume that S is unitary, and put $a := A(1)$ and $c := C(1)$. Then λ_c^T and ρ_a^T are bijections. If T is a composition algebra then $T^\times = \{t \in T \mid N(t) \neq 0\} = \{t \in T \mid \lambda_t \text{ is bijective}\} = \{t \in T \mid \rho_t \text{ is bijective}\}$. This yields the second assertion.

If the algebra T is associative, it admits the autotopism $(\rho_a, \lambda_c\rho_a, \lambda_c)$. The composition $(\rho_a, \lambda_c\rho_a, \lambda_c)^{-1}(A, B, C)$ is then an isomorphism from S onto T.

Now assume that T is a composition algebra, write P for the set of pure elements in T, and put $d := a^{-1} * c$. Without loss, we may assume that T is not associative; then $\dim(T) = 8$. From 2.1 we know that there exist invertible elements $p, q \in T^\times \cap P$ with $d = p * q$. Recall that $u \in T^\times \cap P$ means $u^2 = -N(u) \in \mathrm{Z}(T)$. We have the autotopisms $\mu_b := (\lambda_b\rho_b, \rho_b, \rho_b^{-1})$ for $b \in T^\times$ and $\zeta := (\lambda_{N(d)}, \lambda_{N(d)}, \mathrm{id})$, cf. 2.3. Now the composition

$$(A', B', C') = \zeta^{-1}\, \mu_q\, \mu_p\, (\rho_a, \lambda_a\rho_a, \lambda_a)^{-1}(A, B, C)$$

is an isotopism from S onto T, with $A'(1) = 1 = C'(1)$. Thus S and T are isomorphic.

If S (instead of T) is associative or a composition algebra), we consider the inverse of the given isotopism.

If (A, B, C) is an anti-isotopism, we use the opposite algebra $(T, +, \S)$, where $x \,\S\, y := y * x$. Our arguments above show that $(S, +, \cdot)$ is isomorphic to $(T, +, \S)$. □

2.7 Remarks. For the associative case, the result from 2.6.3 seems to date back to [1, Thm. 2]. For split composition algebras of dimension at least 4, said result can also be deduced from the fact that split composition algebras are determined, up to isomorphism, by the ground field and the dimension (cf. [8, 1.8.1]). The present arguments for composition algebras have been adapted from [5, 1.6, 1.7] where octonion fields were treated.

It remains as an open problem whether 2.6.3 can be extended to the general case of alternative algebras.

3 Isomorphisms between Heisenberg groups

3.1 Definitions. Let S be a (not necessarily associative) unitary algebra. Then the center $Z_S = \{(0,0)\} \times S = \mathrm{h}'_S$ is characteristic in h_S. The *commutator map* of h_S is

$$\gamma_S\colon\ \mathrm{h}_S / Z_S \times \mathrm{h}_S / Z_S \to Z_S\colon\ (Z_S + (u, x), Z_S + (v, y)) \mapsto [(u, x), (v, y)]_\odot = (0, 0, \langle u \underset{S}{|} v\rangle)\,.$$

For $(a,s) \in S^2$ put $\mathrm{C}_{(a,s)} := \{(b,t) \in S^2 \mid at = bs\}$; this means $\mathrm{C}_{\mathrm{h}_S}(a,s,x) = \mathrm{C}_{(a,s)} \times S$. We call $\mathrm{C}_{(a,s)}$ *abelian* if the subgroup $\mathrm{C}_{(a,s)} \times S$ is commutative. We abbreviate $X_S := \mathrm{C}_{\mathrm{h}_S}(1,0,0) = S \times \{0\} \times S$ and $Y_S := \mathrm{C}_{\mathrm{h}_S}(0,1,0) = \{0\} \times S \times S$.

Let $N \colon S^2 \to S$ be an additive map. Then $\xi_N \colon S^3 \to S^3 \colon (a,x,u) \mapsto (a,x,u+N(a,x))$ is an automorphism both of h_S and of H_S. We call ξ_N a *nil-automorphism* and write $\Xi_S := \{\xi_N \mid N \in \mathrm{Hom}(S^2,S)\}$.

The group Ξ_S of nil-automorphisms consists of those automorphisms that act trivially both on Z_S and on the quotient modulo Z_S; therefore, it is a normal subgroup of $\mathrm{Aut}(\mathrm{h}_S)$ (and also of $\mathrm{Aut}(\mathrm{H}_S)$ if $2 \in S^\times$).

3.2 Theorem ([6, 4.5]). *Assume that S and T are unitary algebras, and that $\varphi\colon \mathrm{h}_S \to \mathrm{h}_T$ is an isomorphism mapping $\{X_S, Y_S\}$ to $\{X_T, Y_T\}$.*

1. *If $\varphi(X_S) = X_T$ then there exists an isotopism η from S onto T such that $\varphi \in \Xi_T \circ \lfloor \eta \rfloor$.*

2. *If $\varphi(X_S) = Y_T$ then there is an anti-isotopism α from S onto T with $\varphi \in \Xi_T \circ \lceil \alpha \rceil$.*

If S is a semifield with $\mathrm{char}\, S = 2$ or if S is a semifield not isotopic to a commutative one then every isomorphism $\varphi\colon \mathrm{h}_S \to \mathrm{h}_T$ maps $\{X_S, Y_S\}$ to $\{X_T, Y_T\}$. □

See 4.2 below for an example of an associative algebra A of characteristic 2 where h_A has commutative centralizers apart from those in $\{X_A, Y_A\}$. This shows that the extra assumption ("semifield") in the last assertion of 3.2 is not superfluous.

If $\mathrm{char}\, S \neq 2$ then general results about isomorphisms between reduced Heisenberg groups (cf. [9] and [4]) can be applied:

3.3 Lemma. *Let S and T be unitary algebras such that $2\,\mathrm{id}$ is invertible in $\mathrm{End}(S,+)$, and let $\varphi\colon \mathrm{H}_S \to \mathrm{H}_T$ be an isomorphism. Then there are uniquely determined additive bijections $U\colon S^2 \to T^2$ and $U'\colon S \to T$ together with an additive map $N\colon S^2 \to T$ such that*

$$\varphi(u,x) = (U(u), U'(x) + N(u))$$

holds for all $(u,x) \in S^2 \times S$. The maps U and U' satisfy ($\Diamond$).

Conversely, if $U\colon (S^2,+) \to (T^2,+)$ and $U'\colon (S,+) \to (T,+)$ are isomorphisms satisfying equation ($\Diamond$) then U' is uniquely determined by U, and

$$\psi_U\colon S^2 \times S \to T^2 \times T\colon (u,x) \mapsto (U(u), U'(x))$$

is an isomorphism from H_S onto H_T. We obtain $\varphi = \xi_{N \circ U^{-1}} \circ \psi_U$. □

3.4 Definition. If $S = T$ we write Ψ_S for the set of all ψ_U where $U \in \mathrm{Aut}(S^2,+)$ satisfies equation ($\Diamond$). Thus $\mathrm{Aut}(\mathrm{H}_S) = \Xi_S \circ \Psi_S$.

4 Heisenberg groups over composition algebras

If A is a composition algebra with divisors of zero then there is some commutative field R such that either $A = R \times R$, or $A = R^{2\times 2}$ is a split quaternion algebra, or A is a split Cayley algebra over R. Under the extra assumption $\mathrm{char}\, R \neq 2$ the associative cases have been studied in [3]. We recall the results about the group Ψ_A introduced in 3.4:

4.1 Theorem ([3, 7.2, 7.5, 8.4]). *Let R be a commutative field with $\operatorname{char} R \neq 2$.*

1. *If $A = R \times R$ then $\Psi_A = \{\psi_U \mid U \in \Gamma\mathrm{L}(2,A)\} \cong \Gamma\mathrm{L}(2,A) \cong \langle\kappa\rangle \ltimes (\Gamma\mathrm{L}(2,R) \times \Gamma\mathrm{L}(2,R))$.*
2. *If $A = R^{2\times 2}$ then C_u is abelian precisely if $u \in (\mathrm{GL}(2,R) \times \{0\}) \cup (\{0\} \times \mathrm{GL}(2,R))$.* □

We remark that the automorphisms of $\mathrm{H}_{R\times R}$ and those of $\mathrm{H}_{R^{2\times 2}}$ have been determined completely under the additional assumption $\operatorname{char} R \neq 2$, see [3, 7.2–7.7, 8.5–8.10]. The aim of the present notes is to get rid of this additional assumption.

Recall that $\mathcal{A}_S$ denotes the set of all commutative centralizers in h_S.

4.2 Lemma. *Let R be a commutative field, and abbreviate $A := R \times R$.*

1. *In any case, the map*

$$\iota\colon A^3 \to A^3\colon\ ((a_1,a_2),(s_1,s_2),(x_1,x_2)) \mapsto ((a_1,s_2),(s_1,-a_2),(x_1,x_2 - s_2 a_2))$$

is an automorphism of h_A.

2. *If $\operatorname{char} R \neq 2$ then*

$$\mathcal{A}_A = \{\mathrm{C}_{\mathrm{h}_A}(a,s,x) \mid x \in A, \{a,s\} \subset A \smallsetminus ((R \times \{0\}) \cup (\{0\} \times R))\},$$

and Ψ_A acts transitively both on $\mathcal{A}_A$ and on the set

$$\mathcal{DA}_A := \left\{(B,C) \in \mathcal{A}_A \,\middle|\, [B,C]_\odot = A\right\}.$$

Therefore, we have[2] $\mathrm{Aut}(\mathrm{h}_A) \cong \mathrm{Aut}(\mathrm{H}_A) = \Xi_A \circ \Psi_A$.

3. *If $\operatorname{char} R = 2$ then the set of elementary abelian centralizers is*

$$\mathcal{EA}_A := \left\{ \mathrm{C}_{\mathrm{h}_A}(a,s,x) \,\middle|\, x \in A, (a,s) \in \begin{pmatrix} (A^\times \times \{0\}) \cup (\{0\} \times A^\times) \\ \cup\, (R^\times \times \{0\}) \times (\{0\} \times R^\times) \\ \cup\, (\{0\} \times R^\times) \times (R^\times \times \{0\}) \end{pmatrix} \right\}.$$

The set of all commutative centralizers is obtained as

$$\mathcal{A}_A = \mathcal{EA}_A \cup \{\mathrm{C}_{\mathrm{h}_A}(a,s,x) \mid x \in A, (a,s) \in (A^\times \times A) \cup (A \times A^\times)\}.$$

The group $\langle\{\iota\} \cup \mathrm{Atp}(A)\rangle \le \mathrm{Aut}(\mathrm{h}_A)$ acts transitively on $\mathcal{EA}_A$, and also transitively on

$$\mathcal{DEA}_A := \left\{(B,C) \in \mathcal{A}_A \,\middle|\, [B,C]_\odot = A\right\}.$$

Therefore, we have $\mathrm{Aut}(\mathrm{h}_A) = \Xi_A \circ \langle\{\iota\} \cup \mathrm{Atp}(A)\rangle$ in this case.

4. *In any case, the group h_S (for any algebra S) is isomorphic to h_A precisely if S is isomorphic to A.*

[2] Note that $\mathrm{Atp}(A) \le \Psi_A$ in this case.

Proof. We use the standard involution $\kappa\colon A \to A\colon (a_1,a_2) \mapsto (a_2,a_1)$. In order to determine $\mathcal{A}_A$ we consider C_u for $u=(a,s)\in A^2$. If $a\in A^\times$ then $\mathrm{C}_{(a,s)} = \{(b, sba^{-1}) \mid b\in A\}$ is abelian because A is commutative. If $s\neq 0$ then $(1, sa^{-1}, 0)^2 = (2, 2sa^{-1}, sa^{-1}) \neq (0,0,0)$ shows that the centralizer $\mathrm{C}_{\mathrm{h}_A}(a,s,x)$ is not elementary abelian if $\operatorname{char} R = 2$. Application of κ reduces the case $s\in A^\times$ to $a\in A^\times$.

Now assume that both a and s are not invertible. We may (after possible application of κ) assume $a\in R^\times\times\{0\}$. If $a+s$ is invertible then $s\in\{0\}\times R^\times$, and $\iota(a,s,x) = (a+s,0,x)$ belongs to the orbit of $(1,0,x)$ under $\operatorname{Aut}(\mathrm{h}_A)$. Thus this case is reduced to the one considered above. If none of the elements a, s, and $a+s$ is invertible then they all belong to $R\times\{0\}$. Now $\mathrm{C}_{a,s}$ contains $(\{0\}\times R)^2$, and is not abelian.

From now on, we have to distinguish the cases according to the characteristic. Assume first that $\operatorname{char} R = 2$. For any $B\in\mathcal{EA}_A$ we have seen that there is an element of $\langle \iota, \lceil\kappa|\kappa|\kappa\rceil\rangle \leq \operatorname{Aut}(\mathrm{h}_A)$ mapping B to $\mathrm{C}_{\mathrm{h}_A}(1,0,0) = A\times\{0\}\times A$, and transitivity on $\mathcal{EA}_A$ is established. For $(B,C)\in\mathcal{DEA}_A$ we may thus assume $B=\mathrm{C}_{\mathrm{h}_A}(1,0,0)$. Then $[B,C]_\odot = A$ yields that there exist $a,x\in A$ such that $(a,1,x)\in C$. As C belongs to $\mathcal{EA}_A$ we obtain $C=\mathrm{C}_{\mathrm{h}_A}(a,1,x) = \{0\}\times A\times A$.

If $\operatorname{char} R\neq 2$ then we consider the group H_A; the isomorphism η_A in 1.2 leaves both $\mathcal{A}_A$ and $\mathcal{DA}_A$ invariant. For $(a,s)\in A^2$ and $x\in A$ we note that $\mathrm{C}_{\mathrm{H}_A}(a,s,x)$ is commutative if, and only if, there is $(b,t)\in A^2$ such that the matrix $\left(\begin{smallmatrix} a & b\\ s & t\end{smallmatrix}\right)$ is invertible. Moreover, the pair $(\mathrm{C}_{\mathrm{H}_A}(a,s,x), \mathrm{C}_{\mathrm{H}_A}(b,t,y))$ belongs to $\mathcal{DA}_A$ precisely if $\det\left(\begin{smallmatrix} a & b\\ s & t\end{smallmatrix}\right)\in A^\times$. Therefore, the obvious subgroup isomorphic to $\mathrm{GL}(2,A)$ in Ψ_A acts transitively both on $\mathcal{A}_A$ and on $\mathcal{DA}_A$.

If $\varphi\colon \mathrm{h}_S\to\mathrm{h}_A$ is an isomorphism then our observations so far imply that $(\varphi(X_S),\varphi(Y_S))$ belongs to $\mathcal{DA}_A$ if $\operatorname{char} R\neq 2$, and to $\mathcal{DEA}_A$ if $\operatorname{char} R = 2$. The transitivity properties established above then yield the existence of an isomorphism mapping X_S to X_A and Y_S to Y_A. From 3.2 we then know that there exists an isotopism from S onto A, and 2.6.3 yields that S is isomorphic to A. □

4.3 Lemma. *Let R be a commutative field, and abbreviate $B=R^{2\times 2}$.*

1. *We have $\mathcal{A}_B = \{X_B, Y_B\}$.*
2. *The full group of automorphisms is $\operatorname{Aut}(\mathrm{h}_B) = \Xi_A\circ\operatorname{AntiAtp}(B)$.*
3. *The group h_S (for any algebra S) is isomorphic to h_B precisely if S is isomorphic to B.*

Proof. As B is associative, each triplet (u,v,w) of invertible elements yields an autotopism $(\lambda_u\rho_v, \lambda_w\rho_v, \lambda_w\rho_u^{-1})$. Recall from 2.4 that such an autotopism induces an automorphism $\lfloor\lambda_u\rho_v|\lambda_w\rho_v|\lambda_w\rho_u^{-1}\rfloor$ on h_B mapping (a,s,x) to (uav, wsu^{-1}, wxv). We will also use the standard involution

$$\kappa\colon B\to B\colon \begin{pmatrix} a_{11} & a_{12}\\ a_{21} & a_{22}\end{pmatrix} \mapsto \begin{pmatrix} a_{22} & -a_{12}\\ -a_{21} & a_{11}\end{pmatrix}\ ;$$

this is an anti-automorphism of B, leading to the automorphism

$$\lceil\kappa|\kappa|\kappa\rceil : (a,s,x)\mapsto (\kappa(s), -\kappa(a), \kappa(x-sa))\ .$$

If a is invertible then $(u,v,w) = (1,a^{-1},1)$ yields an automorphism mapping (a,s,x) to $(1,s,xa^{-1})$. As the commutator $\{xy-yx \mid x,y\in B\}$ contains invertible elements, we find that $\mathrm{C}_{1,s} = \{(b,sb)\mid b\in B\}$ is abelian precisely if $s=0$. If s is invertible, we apply $\lceil\kappa|\kappa|\kappa\rceil$ for a reduction to the previous case.

If $a \neq 0$ is not invertible then there exist $u, v \in B^\times$ such that $uav = p := \left(\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right)$. We note that $\mathrm{C}_{p,0} = B \times B(1-p)$ is not abelian. Again, we apply $\lceil\kappa|\kappa|\kappa\rceil$ to see that $\mathrm{C}_{0,p}$ is not abelian.

It remains to study $\mathrm{C}_{a,s}$ if $\{a, s\} \subset B \smallsetminus (B^\times \cup \{0\})$. We may assume (up to an automorphism of h_B) that $a = p$. Now $upv = p$ holds whenever $u = \left(\begin{smallmatrix} u_{11} & u_{12} \\ 0 & u_{22} \end{smallmatrix}\right)$ and $v = \left(\begin{smallmatrix} v_{11} & 0 \\ v_{21} & v_{22} \end{smallmatrix}\right)$ with $u_{22}\, v_{22} \neq 0$ and $u_{11}v_{11} = 1$.

As $s \neq 0$ has linearly dependent rows, there exists $w \in B^\times$ such that the second row of ws is zero. Using suitable u, v, we may achieve $upv = p$ and $wsu^{-1} \in \{p, n\}$ with $n = \left(\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right)$. The transpose n' of n satisfies $nn' = p$. Using $(1, p), (p, 1) \in \mathrm{C}_{p,p}$ and $(p, 0), (n', p) \in \mathrm{C}_{p,n}$ we now see that $\mathrm{C}_{p,p}$ and $\mathrm{C}_{p,n}$ are not abelian.

Thus we have shown that the elements of $(B^\times \times \{0\} \times B) \cup (\{0\} \times B^\times \times B$ are just those with commutative centralizers, and $\mathcal{A}_B = \{X_B, Y_B\}$ follows.

Every isomorphism from h_S to h_B will thus map $\{X_S, Y_S\}$ to $\{X_B, Y_B\}$, and the last two assertions follow with 3.2 and 2.6.3. □

4.4 Lemma. *Let A be a composition algebra over R. For each $a \in A \smallsetminus \{0\}$ with $N(a) = 0$ we have $\ker(\lambda_a) = \bar{a}A$ and $\ker(\rho_a) = A\bar{a}$. These subspaces are maximal totally singular ones, of dimension $\frac{1}{2}\dim(A)$.*

Proof. We show first that $\ker(\lambda_a) = \{y \in A \mid ay = 0\}$ is totally singular. In fact, for $y \in \ker(\lambda_a)$ we have $0 = (ay)\bar{y} = a(y\bar{y})$ by alternativity (see [8, 1.3.3]), and $y\bar{y} = 0$ follows. We have $\dim(\ker(\lambda_a)) \leq \frac{1}{2}\dim(A)$ because the polar form is not degenerate.

For any $b \in A$ and $y \in \ker(\lambda_a)$ we use [8, 1.3.2] to compute the polar form $f_N(\bar{a}b, y) = f_N(b, ay) = f_N(b, 0) = 0$. This shows $\bar{a}A \leq \ker(\lambda_a)$. The previous paragraph yields $\dim(\bar{a}A) = \dim(A) - \dim(\ker(\lambda_{\bar{a}})) \geq \frac{1}{2}\dim(A)$, and $\ker(\lambda_a) = \bar{a}A$ follows. This means that $\ker(\lambda_a)$ is a totally singular subspace of dimension $\frac{1}{2}\dim(A)$, and thus a maximal one. □

4.5 Lemma. *Let A be a octonion algebra, and consider $a, s \in A$. Then $\mathrm{C}_{(a,s)}$ is abelian precisely if $(a, s) \in (A^\times \times \{0\}) \cup (\{0\} \times A^\times)$; i.e., if $\mathrm{C}_{(a,s)} \in \{\mathrm{C}_{(1,0)}, \mathrm{C}_{(0,1)}\}$.*

Proof. Clearly $\mathrm{C}_{(1,0)}$ and $\mathrm{C}_{(0,1)}$ are abelian. Conversely, consider $(a, s) \in A^2$ such that $\mathrm{C}_{(a,s)}$ is abelian.

Assume first that $N(a) \neq 0 \neq N(s)$. From 2.3 we then know that $(\lambda_a^{-1}, \lambda_a, \lambda_a \circ \rho_a)$ is an autotopism of A. The automorphism $\lfloor\lambda_a^{-1}|\lambda_a|\lambda_a \circ \rho_a\rfloor$ maps (a, s) to $(1, d)$ with $d := asa \in A^\times$. If $\mathrm{C}_{(1,d)} = \{(b, db) \mid b \in A\}$ were abelian then $(dc)b = (db)c$ would hold for all $b, c \in A$.

Now [6, 3.1] and 2.6 yield that A is isotopic and then even isomorphic to a commutative algebra. This is impossible.

Now assume $N(s) = 0 \neq s$, and consider a quaternion subalgebra B of A with $s \in B$ (see [8, 1.6.4] for the existence of such a subalgebra). Then B is a split quaternion algebra because s is not invertible, and $B \cong R^{2\times 2}$ follows. Under the isomorphism of algebras, our element s corresponds to a matrix of rank 1 that annihilates each element of $\{xy - yx \mid x, y \in R^{2\times 2}\}$. The latter set contains invertible elements; and we have reached a contradiction.

Thus we have proved $s = 0$ if $a \in A^\times$ and $\mathrm{C}_{(a,s)}$ is abelian. The case $s \in A^\times$ is reduced to the previous one by the automorphism $\lceil\kappa|\kappa|\kappa\rceil$.

From now on, we assume $N(a) = 0 = N(s)$. If $s = 0$ then $\mathrm{C}_{(a,0)} = A \times A\bar{a}$ by 4.4, and $\langle(b, 0)|(c, \bar{a})\rangle = \bar{a}b$ shows that $\mathrm{C}_{(a,0)}$ is not abelian. The case $a = 0$ is reduced to the case $s = 0$ via an application of $\lceil\kappa|\kappa|\kappa\rceil$.

It remains to treat the case where $\{a,s\} \subset A \setminus (A^\times \cup \{0\}$. According to 4.4, we have $\bar{s}A \times A\bar{a} \leq \mathrm{C}_{(a,s)}$. Our assumption that $\mathrm{C}_{(a,s)}$ is abelian entails $(y\bar{a})(\bar{s}c) = (d\bar{a})(\bar{s}x)$ for all $c,d,x,y \in A$. Specializing $d = a$ and $c = 1 = y$ we find $\bar{a}\bar{s} = 0$. Thus $sa = 0$, and $s \in A\bar{a}$ by 4.4.

If $\bar{a} = -a$ we consider $s = d\bar{a}$ with $d \in A$. Then $\{1\} \times (-d + Aa) \subseteq \mathrm{C}_{(a,s)}$, and $\langle (1,-d) | (1,-d-a) \rangle = a \neq 0$ shows that $\mathrm{C}_{(a,s)}$ is not abelian.

If $\bar{a} \neq -a$ we consider a quaternion subalgebra B with $a \in B$: in that algebra (isomorphic to $R^{2\times 2}$) there exists an invertible element b such that ab has trace 0, i.e., such that $\overline{ab} = -ab$. Now the automorphism $\lfloor \rho_b | \lambda_b \circ \rho_b | \lambda_b \rfloor$ maps $\mathrm{C}_{(a,s)}$ to $\mathrm{C}_{(ab,bs)}$ which is not abelian by the previous paragraph. This contradiction finally shows that $a \neq 0 \neq s$ implies that $\mathrm{C}_{(a,s)}$ is not abelian. □

4.6 Theorem. *Let A be a composition algebra (possibly with divisors of zero), and let S be an arbitrary algebra.*

1. *If A is not commutative then every isomorphism between h_S and h_A maps the set $\{X_S, Y_S\}$ to the set $\{X_A, Y_A\}$. In those cases, the full group of automorphisms is $\mathrm{Aut}(\mathrm{h}_A) = \Xi_A \circ \mathrm{AntiAtp}(A)$.*

2. *In any case, the algebras S and A are isomorphic (as $\mathbb{Z}$-algebras) if, and only if, the groups h_S and h_A are isomorphic.*

Proof. If A is associative then A is a quaternion algebra, and not isotopic to any commutative algebra (cf. 2.6.3). If A is not associative then A is an octonion algebra. From 4.3 and 4.6 we now know that the set $\{X_A, Y_A\}$ is characteristic in h_A in any case, and 3.2 applies as in the proof of [6, 5.6].

For commutative composition algebras, the second assertion has been proved in 4.2 (for the two-dimensional case with zero divisors) and in [6, 5.6].

If S and A are isotopic or anti-isotopic then they are in fact isomorphic as $\mathbb{Z}$-algebras, see 2.6. □

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