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# De Casteljau's Algorithm on Manifolds

Esfandiar Nava-Yazdani, Konrad Polthier

## Abstract

This paper proposes a generalization of the ordinary de Casteljau algorithm to manifold-valued data including an important special case which uses the exponential map of a symmetric space or Riemannian manifold. We investigate some basic properties of the corresponding Bézier curves and present applications to curve design on polyhedra and implicit surfaces as well as motion of rigid body and positive definite matrices. Moreover, we apply our approach to construct canal and developable surfaces.

**Keywords.** Bézier curve, Canal surface, Ruled surface, Symmetric spaces, Rigid body motion, Euclidean group, Positive definite matrix, Nonlinear subdivision, Lie group, De Casteljau's algorithm

## 1 Introduction

In various applications like medical imaging, elasticity, array signal processing and dynamics, one has to deal with data living in a manifold, Lie group or more generally symmetric space. In several works including [19], [20], [23], [21], [5], [6], [10], [25], [11], [7] and [3] subdivision schemes have been generalized and applied to nonlinear settings. Besides subdivision schemes, the de Casteljau algorithm is of significant importance in modeling and computer aided geometric design. There are several publications on Bézier curves in the sphere including the early work [17]. Moreover, [9] generalizes the classical de Casteljau algorithm for constructing spherical Bézier based on corner cutting. For further approaches and modifications concerning the de Casteljau algorithm in certain nonlinear cases we refer to [12], [2] and [8]. In the present work, we introduce a framework generalizing previous results to manifold-valued data and study some of main properties of the resulting Bézier curve. For many practical purposes geodesics are computationally too expensive or time-consuming. We show how to use alternative approaches preserving main desired properties of the construction. Moreover, we present applications to the geodesic Bézier approach for polyhedral surfaces, Euclidean group of motion and diffusion tensors and also use the produced curves to construct canal and ruled surfaces. Throughout this work  $M$  denotes a  $C^k$  manifold, smooth stands for  $C^k$  and we refer to  $p_0, \dots, p_n \in M$  as control points.

**Definition 1.** Suppose that for each  $x \in M$  there is a neighborhood  $U_x$  and a map

$$[0, 1] \times U_x \ni (t, y) \mapsto \Phi_t(x, y) \in U_x$$

such that  $\Phi_t$  is smooth in  $x$  and  $y$ . We call

$$L(x, y) := \{\Phi_t(x, y) : 0 \leq t \leq 1\}$$

the segment from  $x$  to  $y$  and the function  $\Phi$  basic iff the following holds.  $x \neq y$  implies that  $L(x, y)$  is a simple and regular curve, i.e.,  $t \mapsto \Phi_t(x, y)$  is an injective immersion and

$$\Phi_0(x, y) = x \tag{1}$$

$$\Phi_1(x, y) = y \tag{2}$$

$$L(x, x) = \{x\}. \tag{3}$$

In particular, if the map  $\Phi$  is globally defined ( $U_x = M$ ), then it is a smooth homotopy between the first and second coordinate projection of  $M \times M$  relative to the diagonal  $diag(M) := \{(x, x) : x \in M\}$ .

Let  $g(x, y) := \dot{\Phi}_0(x, y)$ . We call  $\Phi$  locally rigid if for each  $x \in M$  the identity  $D_2g(x, x) = Id$  holds. As the map  $x \mapsto \Phi(x, \cdot)$  is in general only locally defined, in the following we assume that the control points are close enough, i.e., there is a neighborhood  $U$  of  $\{p_0, \dots, p_n\}$  such that

$$x, y \in U \implies L(x, y) \subset U.$$

Next, we define the  $(\Phi)$ -de Casteljau algorithm.

## 2 The Algorithm

**Definition 2.**

$$\begin{aligned} p_i^0(t) &:= p_i, \quad r = 1, \dots, n, \quad i = 0, \dots, n-r, \\ p_i^r(t) &:= \Phi_t(p_i^{r-1}(t), p_{i+1}^{r-1}(t)), \quad 0 \leq t \leq 1. \end{aligned}$$

Let  $p(t) := p_0^n(t)$ . Let us call  $B(p_0, \dots, p_n) := p([0, 1])$  the Bézier curve with control points  $p_0, \dots, p_n$ . Obviously, this curve is invariant under affine parameter transformations and satisfies  $p(0) = p_0$  and  $p(1) = p_n$ .

**Definition 3.** We call

$$L(p_0, \dots, p_n) := \cup_{i=0}^{n-1} L(p_i, p_{i+1})$$

the polygon of  $p_0, \dots, p_n$ .

In our framework many properties of the Bézier curve and its parametrization  $p$  from the linear case remain valid. Smoothness of  $p$  is an immediate consequence of the fact that  $p$  is constructed as a finite composition of smooth operations and  $L(x, y)$  is a smooth curve for all  $x, y \in U$ , provided  $M$  and  $\Phi$  are smooth. Moreover, reversing the order of control points does not affect their Bézier curve:

$$B(p_n, \dots, p_0) = B(p_0, \dots, p_n).$$

Suppose now  $0 = t_0 < t_1 < \dots < t_n = 1$ . Our algorithm can simply be modified to produce a solution to the interpolation problem  $p(t_i) = p_i$  by setting

$$p_i^r(t) = \Phi_{\frac{t-t_i}{t_{i+r}-t_i}}(p_i^{r-1}(t), p_{i+1}^{r-1}(t)) \quad (\text{Aitken's algorithm}).$$

*Example 4.* Suppose that  $M$  is a convex subset of the general linear group  $GL_n$ . Then

$$\Phi_t(x, y) = x((1-t)y + tx)^{-1}y$$

defines a rational basic function on  $M$ . In many applications a natural symmetry condition to impose on a basic function is the following

$$L(x, y) = L(y, x).$$

In this example a rational basic function meeting the symmetry condition is given by

$$\Phi_t(x, y) = \frac{1}{2}(x((1-t)y + tx)^{-1}y + y((1-t)y + tx)^{-1}x).$$

*Example 5.* Let  $x, y \in \mathbb{R}^2$ . Then the circle with the diameter  $\|x - y\|$  and center  $\frac{x+y}{2}$  given by

$$\Phi_t(x, y) = \frac{1}{2}((x + y) + \begin{bmatrix} \cos(\pi t) & \sin(\pi t) \\ -\sin(\pi t) & \cos(\pi t) \end{bmatrix} (x - y))$$

defines a basic function on  $\mathbb{R}^2$ .

In many applications, using a point wise linear structure associated to  $M$  and having a corresponding geometric or physical interpretation in mind, we may construct a basic function  $\Phi$  as follows.

**Definition 6.** Let  $T_x M \ni g(x, y) := \dot{\Phi}_0(x, y)$ . We call  $\Phi$  dynamical iff there is a map  $f : TM \rightarrow M$  such that for each  $x$  there exists a neighborhood  $U$  of  $x$  with

$$\Phi_t(x, y) = f(x, tg(x, y)) \text{ for each } 0 \leq t \leq 1, y \in U. \quad (4)$$

Note that in this case

$$\begin{aligned} f(x, 0) &= x, f(x, g(x, y)) = y \\ f(x, (s + t)g(x, y)) &= f(f(x, tg(x, y)), sg(x, y)) \text{ (local flow property)}. \end{aligned}$$

A good example to have in mind is the following. Let  $M$  be Riemannian and  $f = \exp$  the Riemannian exponential map. In this case, if  $x, y \in U$  with a distance within the injectivity radius of  $f$  at  $x$ , then  $g(x, y)$  is the velocity and  $\Phi(x, y)$  is the geodesic from  $x$  to  $y$ . Moreover, if  $M$  is geodesically complete, then  $f$  is defined on whole  $TM$  and any two points can be joined by a (not necessarily unique) geodesic. For instance, see example 1. Another class of important applications is given, if  $M$  is a Lie group or more generally a symmetric space with  $f$  the exponential map. Note that if  $M$  is a Lie group admitting a bi-invariant metric (e.g., if  $M$  is compact), then the Lie group exponential map coincides with the Riemannian one.

If  $M$  is just the  $m$ -dimensional Euclidean space  $\mathbb{R}^m$ , then  $TM = \mathbb{R}^m \times \mathbb{R}^m$ ,  $f(x, v) = x + v$  and  $g(x, y) = y - x$ . In this case, definition (2) gives the ordinary linear de Casteljau algorithm. Other examples are provided by  $g$  being the nearest point projection in the ambient Euclidean space onto  $M$  as well as stereographic projection in case of a sphere. In this examples  $TM = M \times \mathbb{R}^N$  and the maps  $f$  and  $g$  are independent of the base point  $x$ . Another example is given by  $g(x, y)$  the orthogonal projection of  $y$  to  $T_x M$  the tangent space at  $x$ . Compared to the geodesic construction, this approach has the advantages that it is computationally less time-consuming ( $f$  and  $g$  are simpler) and one needs only to deal with the resp. linearization of the manifold  $M$  at the control points. Next, consider any chart around  $x \in U$  and replace  $f$  and  $g$  by their local representation. Let us denote derivatives of  $f$  and  $g$  in the first resp. second argument by  $D_1 f$  resp.  $D_2 f$  and  $D_1 g$  resp.  $D_2 g$ . Due to  $f(x, g(x, y)) = y$  we have

$$D_1 f(x, g(x, y)) + D_2 f(x, g(x, y)) D_1 g(x, y) = 0, \quad (5)$$

$$D_2 f(x, g(x, y)) D_2 g(x, y) = Id. \quad (6)$$

We say that  $f$  is locally rigid iff

$$D_2 f(x, 0) = Id \text{ for all } x \in U. \quad (7)$$

We remark that this property is independent of the chosen chart. Moreover, due to  $f(x, 0) = x$  we have  $D_1 f(x, 0) = Id$  on  $U$ . Note that (3) implies  $g(x, x) = 0$ . Hence, in view of (6) local rigidity of  $\Phi$  is equivalent to (7).

**Definition 7.** Suppose that the basic function is given by the pair  $(f, g)$  as defined above. We call

$$C(p_0, \dots, p_n) := \cup_{i=0}^n \{f(p_i, \sum_{j=0}^n t_j g(p_i, p_j)) : \sum_{j=0}^n t_j = 1, t_j \in \mathbb{R}_{\geq 0}, j = 0, \dots, n\}$$

the convex hull of  $p_0, \dots, p_n$ .

Note that the Bézier curve  $B(p_0, \dots, p_n)$  is entirely contained in the convex hull of  $p_0, \dots, p_n$ . Furthermore, if control points lie on the segment between endpoints, then the corresponding Bézier curve coincides with that segment:

$$p_1, \dots, p_{n-1} \in L(p_0, p_n) \Rightarrow B(p_0, \dots, p_n) = L(p_0, \dots, p_n).$$

The following theorem summarizes further properties of the de Casteljau's algorithm.

**Theorem 8.** *Consider control points  $P := (p_0, \dots, p_n)$  and  $Q := (q_0, \dots, q_n)$  in  $U$ . Then the following holds.*

a) *Transformation invariance: Suppose a Lie group  $H$  acts on  $M$  by*

$$H \times M \ni (h, x) \mapsto hx \in M$$

*leaving  $U$  invariant, i.e.,  $HU \subset U$ . If the action is segment-equivariant, i.e., for every  $h \in G$*

$$hL(x, y) = L(hx, hy) \text{ for all } x, y \in U.$$

*Then*

$$hB(p_0, \dots, p_n) = B(hp_0, \dots, hp_n) \text{ for all } h \in H.$$

b) *Local control: Consider (w.l.o.g.)  $M$  as embedded in an Euclidean space with any norm  $\|\cdot\|$ . Then*

$$\|p - q\|_\infty \leq C\|P - Q\|_\infty$$

*where  $C$  denotes a positive constant depending only on  $n$ ,  $U$  and  $\Phi$ .*

c) *Endpoint velocity: Suppose that  $\Phi$  is given by  $(f, g)$ . Denote  $\dot{p}(t) := \frac{d}{dt}p(t)$  and  $\Delta p_i := g(p_i, p_{i+1})$ . If  $f$  meets the local rigid condition (7), then:*

$$\begin{aligned} \dot{p}(0) &= n\Delta p_0, \\ \dot{p}(1) &= n\Delta p_{n-1}. \end{aligned}$$

*Proof.* For a) note that the segments  $L(hp_i, hp_{i+1})$  and  $hL(p_i, p_{i+1})$  have the same endpoints:

$$\begin{aligned} \Phi_0(hp_i, hp_{i+1}) &= hp_i = h\Phi_0(p_i, p_{i+1}), \\ \Phi_1(hp_i, hp_{i+1}) &= hp_{i+1} = h\Phi_1(p_i, p_{i+1}). \end{aligned}$$

To show b) note that by finiteness of  $n$  there is a positive constant  $K$  determined by  $U$  and Lipschitz constants of  $\Phi$  on  $U$  such that

$$\begin{aligned} \|p_i^r(t) - q_i^r(t)\| &= \|\Phi_t(p_i^{r-1}(t), p_{i+1}^{r-1}(t)) - \Phi_t(q_i^{r-1}(t), q_{i+1}^{r-1}(t))\| \\ &\leq K\|(p_i^{r-1}(t), p_{i+1}^{r-1}(t)) - (q_i^{r-1}(t), q_{i+1}^{r-1}(t))\|. \end{aligned}$$

Iteration yields

$$\|p(t) - q(t)\| = \|p_0^n(t) - q_0^n(t)\| \leq K^n\|P - Q\|_\infty \text{ for all } 0 \leq t \leq 1$$

which immediately results in the desired inequality. To prove c) define  $p'_i = p_{n-i}$  for  $i = 0, \dots, n$  and note that the Bézier curve with control points  $p'_n, \dots, p'_0$  coincides with the one corresponding to  $p_0, \dots, p_n$ . Hence the velocity vector of the curve  $B(p'_n, \dots, p'_0)$  at  $p'_n$  is the same as the velocity vector of  $B(p_0, \dots, p_n)$  at  $p_0$ . On the other hand  $p(1-t) = p'(t)$  implies  $\dot{p}(0) = -\dot{p}'(1)$ . Therefore, the second equation in c) follows from the first one. Now, applying (5) and (6) the de Casteljau algorithm yields

$$\begin{aligned} \dot{p}_i^r &= (1-t)D_1f(p_i^{r-1}, tg(p_i^{r-1}, p_{i+1}^{r-1}))\dot{p}_i^{r-1} + t\dot{p}_{i+1}^{r-1} \\ &\quad + D_2f(p_i^{r-1}, tg(p_i^{r-1}, p_{i+1}^{r-1}))g(p_i^{r-1}, p_{i+1}^{r-1}). \end{aligned}$$



Iterating this formula yields for  $i = 0$  and  $r = n$

$$\dot{p}(t) = \sum_{k=0}^{n-1} \sum_{j=0}^k \binom{k}{j} (1-t)^j t^{k-j} \underbrace{D_1 f(\cdot, \cdot) \dots D_1 f(\cdot, \cdot)}_{j \text{ times}} D_2 f(\cdot, \cdot) \cdot g(p_{k-j}^{n-1-k}, p_{k-j+1}^{n-1-k})$$

where all second arguments of  $D_1 f(\cdot, \cdot)$  and  $D_2 f(\cdot, \cdot)$  evaluate to zero at  $t = 0$ . Setting  $t = 0$  in this formula and using the assumptions  $D_2 f(p_i, 0) = Id$  and  $D_1 f(p_i, 0) = Id$  we arrive at  $\dot{p}(0) = n\Delta p_0$ .  $\square$

Given a basic function  $\Phi$  on a manifold  $M$ , in many applications there is a canonical way to construct another basic function  $\Phi'$  and the corresponding Bézier curves/patches etc. on a manifold  $M'$ . To proceed, we first present the following

**Definition 9.** Suppose that  $\Phi$  resp.  $\Phi'$  are basic functions on manifolds  $M$  resp.  $M'$ . We call  $\Phi$  and  $\Phi'$  conjugate iff there is a diffeomorphism  $H : M \rightarrow M'$  such that for each  $0 \leq t \leq 1$  and  $x, y \in M$

$$H(\Phi_t(x, y)) := \Phi'_t(H(x), H(y))$$

and write  $\Phi \simeq_H \Phi'$ .

Obviously  $\simeq$  defines an equivalence relation. Next we show how to construct new basic functions from a given one and provide flexibility in design of new Bézier curves.

**Theorem 10.** Suppose that  $\Phi$  resp.  $\Phi'$  are basic functions on manifolds  $M$  resp.  $M'$  and  $\Phi \simeq_H \Phi'$ . Then the following holds

- a) If  $\Phi$  is dynamical, then  $\Phi'$  is also dynamical.
- b) Let  $p(\cdot)$  resp.  $p'(\cdot)$  denote the parametrization of Bézier curves corresponding to  $p_0, \dots, p_n$  resp.  $H(p_0), \dots, H(p_n)$ . Then  $H(p(\cdot)) = p'(\cdot)$ . Particularly  $H(B(p_0, \dots, p_n)) = B(H(p_0), \dots, H(p_n))$ .
- c) Let  $\tilde{\Phi}$  be a basic functions on  $M$ . Then

$$\Psi_t(x, y) := \tilde{\Phi}_t(\Phi_t(x, y), \Phi_t(x, y))$$

is also a basic function on  $M$

*Proof.* a) Define the functions  $f' : TM' \mapsto M'$  and  $g' : M' \times M' \mapsto TM'$  by

$$f'(x', v') = H(f(x, v)), \quad g'(x', y') = d_x H g(x, y)$$

where  $x', y' \in M'$ ,  $v' \in T_{x'} M'$ ,  $x' = H(x)$  and  $y' = H(y)$ . We can write

$$\begin{aligned} \Phi'_t(x', y') &= H(\Phi_t(x, y)) = H(f(x, tg(x, y))) = f'(x', d_x H(tg(x, y))) \\ &= f'(x', td_x H(g(x, y))) = f'(x', tg'(x', y')). \end{aligned}$$

b) and c) are straightforward.  $\square$

Next, we present two applications of the preceding theorem.

*Example 11.* Suppose that computing a Bézier curve in a manifold  $M'$  in terms of a basic function  $\Phi'$  is more manageable than in  $M$  and  $H : M \rightarrow M'$  is a diffeomorphism. Then, we may use the basic function  $\Phi_t(x, y) := H^{-1}(\Phi'_t(H(x), H(y)))$  on  $M$ . As example suppose that  $M$  is implicitly given by a submersion  $h : \mathbb{R}^3 \rightarrow \mathbb{R}$  as  $M = h^{-1}(0)$  and  $H_3 = h$ . Then choosing  $\Phi'$  as the ordinary affine linear function gives  $\Phi_t(x, y) = H^{-1}((1-t)H(x) + H(y))$  on  $M$ .

Besides geometric and design issues, computational aspects are of great importance in most applications. Particularly, a polynomial or at least rational basic function is desirable. Note that the corresponding Bézier curve enjoys the same property.

*Example 12.* Let  $\pi$  denote the stereographic projection from the north pole of  $M = S^m$  onto  $\mathbb{R}^m$ . Then

$$\Phi_t(x, y) := \pi^{-1}((1 - t)\pi(x) + t\pi(y))$$

defines a rational basic function on  $S^m$ .

From a geometric view point the following is a natural property for applications in design and geometric approximation.

**Definition 13.** We call a basic function  $\Phi$  on  $M$  segmenting iff for each  $x, z \in M$

$$y \in L(x, z) \Rightarrow L(x, y) \cup L(y, z) = L(x, z). \quad (8)$$

Note that the basic functions given in examples 4 and 5 are not segmenting. Moreover, every dynamical function is segmenting. Next, we characterize dynamical functions in terms of above geometric intuitive property.

**Theorem 14.** Suppose that  $\Phi$  is a locally rigid basic segmenting function on  $M$ . Then  $\Phi$  is dynamical.

*Proof.* Fix  $x \in M$ . Due to local rigidity there is a neighborhood  $U$  of  $x$  such that  $U \ni y \mapsto g(x, y) := \dot{\Phi}_0(x, y)$  is injective. Let  $f(x, \cdot) := g(x, \cdot)|_U^{-1}$  and  $v \in g(x, U)$ . Then there exists  $z \in U$  with  $g(x, z) = v$ . Let  $y \in L(x, z)$ , i.e.,  $y = \Phi_t(x, z)$  for some  $0 \leq t \leq 1$ . Since  $[0, 1] \ni s \mapsto \Phi_{st}(x, z)$  parametrizes  $L(x, y)$  and due to segmenting property  $L(x, y) \subset L(x, z)$ , we have  $g(x, y) = \frac{d}{ds}\big|_{s=0} \Phi_{st}(x, z) = t\dot{\Phi}_0(x, z) = tg(x, z)$ . Hence  $\Phi_t(x, z) = y = f(x, g(x, y)) = f(x, tg(x, z))$ .  $\square$

### 3 Bézier curves

*Example 15.* The choice  $f = \exp$  where  $\exp$  denotes the Riemannian exponential map on  $M$  leads to the geodesic de Casteljau's algorithm. The following figure illustrates the corresponding Bézier curve on a torus. This approach requires solving the geodesic boundary value problem. Another

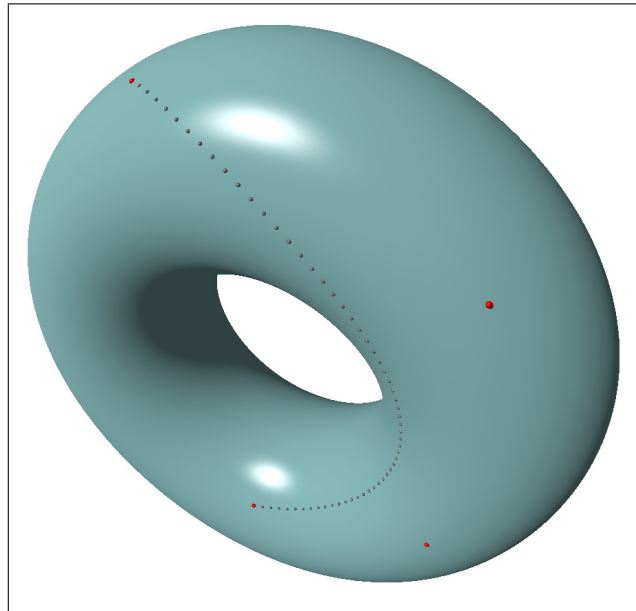


Figure 1: Geodesic Bézier curve on torus

choice is the computationally more suitable map  $g(x, y)$  given by the orthogonal projection of  $y$  to the tangent plane of the surface at  $x$

$$g(x, y) = y - x - \langle y - x, n \rangle n$$

and  $f(x, \cdot)$  the local inverse of  $g$ . Here  $\langle \cdot, \cdot \rangle$  denotes the Euclidean scalar product and  $n$  stands for the normal at  $x$ . For few examples geodesics are explicitly known. For instance consider the 2-sphere  $M = S^2$  with  $f$  the Riemannian exponential map

$$f(x, v) = \cos(\|v\|)x + \sin(\|v\|)\frac{v}{\|v\|}$$

$$g(x, y) = \arccos(\langle x, y \rangle)\frac{y - \langle x, y \rangle x}{\|y - \langle x, y \rangle x\|}, \quad x, y \text{ not antipodal.}$$

The resulting segment between  $x$  and  $y$  is just the the geodesic

$$\Phi_t(x, y) = \frac{\sin((1-t)\varphi)}{\sin \varphi}x + \frac{\sin(t\varphi)}{\sin \varphi}y$$

where  $\cos \varphi = \langle x, y \rangle$ . The orthogonal projection of  $y$  to the tangent plane of the sphere at  $x$  and its local inverse are given by

$$g(x, y) = y - \langle x, y \rangle x,$$

$$f(x, v) = v + \sqrt{1 - \|v\|^2} x.$$

resulting in the segment

$$\tilde{\Phi}_t(x, y) = (\sqrt{1 - t^2 \sin^2 \varphi} - t \cos \varphi)x + ty.$$

Note that in this example  $\tilde{\Phi}(x, y)$  is just a reparametrization of  $\Phi(x, y)$  as for the sphere both curves coincide with the arc of the great circle joining  $x$  and  $y$ .

Next, we consider Lie groups. We denote the Lie group of special orthogonal matrices by  $\text{SO}_n$ , i.e.,

$$\text{SO}_n := \{x \in \mathbb{R}^{n \times n} : xx^t = \text{id}, \det(x) = 1\},$$

where  $\text{id}$  denotes the identity matrix in the general linear group  $\text{GL}_n$ . In general, if  $M$  is a Lie group, then the exponential map is given by

$$f(x, v) = x \exp(v) = x \sum_{k=0}^{\infty} \frac{1}{k!} v^k$$

where  $x \in M$  and  $v$  is a vector in the Lie algebra  $\text{Lie}(M)$  of  $M$ . As our consideration is local, using Ado's theorem ([18], Chapter 10) we may and do assume that  $M$  is a subgroup of the general linear group  $\text{GL}_n$ . Now, if  $h \in \text{GL}_n$  satisfies  $\|I - h\| < 1$  (for any matrix norm  $\|\cdot\|$ ), then  $\log$  is well-defined and given by

$$\log(h) = \sum_{k=1}^{\infty} \frac{1}{k} (I - h)^k.$$

Considering  $M$  as a subgroup of  $\text{GL}_n$ , we also may take the orthogonal projection. In the following example we compare this two choices of  $f$ .

*Example 16.* Let  $x \in \text{SE}_3$  where  $\text{SE}_3$  denotes the special group of Euclidean motion. Then we have

$$x = \begin{bmatrix} q & b \\ 0 & 1 \end{bmatrix},$$

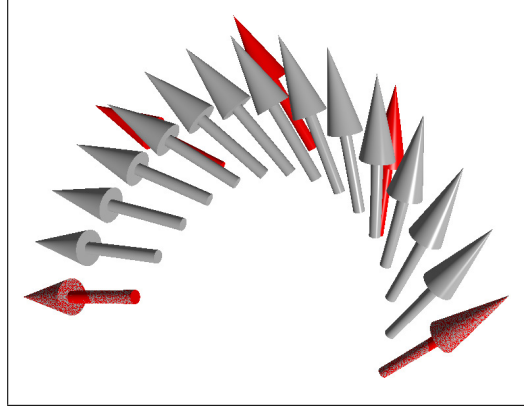


Figure 2: Application of projection onto  $SE_3$  to refine poses of a rigid body

with  $q$  being a rotation in  $\mathbb{R}^3$ , i.e.,  $q \in SO_3$  and  $b \in \mathbb{R}^3$  (translation). The standard scalar product for  $x, y \in \mathbb{R}^{n \times n}$  is given by

$$\langle x, y \rangle = \text{tr}(xy^t)$$

and the induced norm is the Frobenius norm

$$\|x\| = \left( \sum_{j=1}^n \sum_{i=1}^n x_{ij}^2 \right)^{1/2}.$$

The closest point projection  $f_\pi$  onto  $SE_3$  can be computed efficiently using singular value decomposition (see [GL96]). Let  $q = usv^t$  be a singular decomposition of  $q$  where  $q^t$  stands for the transpose of  $q$ . It is well-known that

$$\text{argmin}_{z \in SO_3} \|q - z\| = uv^t = (qq^t)^{-1/2}q$$

provided  $\det(q) > 0$ . Hence we have

$$f_\pi(x) = \begin{bmatrix} (qq^t)^{-1/2}q & b \\ 0 & 1 \end{bmatrix}.$$

Let us denote the Bernstein polynomials of degree  $n$  by  $B_0, \dots, B_n$ . For control points  $p_0, \dots, p_n \in SE_3$  we define the Bézier curve in  $SE_3$  as the closest point projection of the corresponding Bézier curve  $\sum_{i=0}^n B_i(t)p_i$  in the ambient Euclidean space  $\mathbb{R}^{4 \times 4}$  onto  $SE_3$

$$p(t) = f_\pi\left(\sum_{i=0}^n B_i(t)p_i\right) = f_\pi\left(\begin{bmatrix} \sum_{i=0}^n B_i(t)q_i & \sum_{i=0}^n B_i(t)b_i \\ 0 & 1 \end{bmatrix}\right)$$

For well-definedness we have to show that for all  $0 \leq t \leq 1$  the condition  $\det(\sum_{i=0}^n B_i(t)q_i) > 0$  holds. Since  $\sum_{j=0}^n B_j(t) = 1$  for all  $0 \leq t \leq 1$ , we have

$$\left\| \sum_{i=0}^n B_i(t)q_i - q_1 \right\| = \left\| \sum_{i=0}^n B_i(t)(q_i - q_1) \right\| \leq \sum_{i=0}^n B_i(t) \max_i \|q_i - q_1\| = \max_i \|q_i - q_1\|.$$

In view of  $\det(q_1) = 1$  and continuity of  $\det$  it follows that for sufficiently dense control points the desired condition  $\det(\sum_{i=0}^n B_i(t)q_i) > 0$  is satisfied. Note that another advantage of this choice of  $f$  is independency of base point ( $g(x, y) = y$ ,  $f(x, v) = f_\pi(v)$  and  $\Lambda = \mathbb{R}^{4 \times 4}$ ), resulting in a faster approach. A vector  $v$  in the Lie algebra  $\mathfrak{se}_3$  of  $SE_3$  has the form

$$v = \begin{bmatrix} h & u \\ 0 & 0 \end{bmatrix}.$$

Here  $h \in \mathfrak{so}_3$  where  $\mathfrak{so}_3$  consisting of 3 by 3 skew-symmetric matrices stands for the Lie algebra of  $SO_3$  and  $u \in \mathbb{R}^3$ . The exponential map of  $SE_3$  reads now

$$\text{Exp}(x, v) = x \begin{bmatrix} \exp(h) & Hu \\ 0 & 1 \end{bmatrix},$$

$H := 1 + \frac{1 - \cos(\|h\|)}{\|h\|^2} h + \frac{\|h\| - \sin(\|h\|)}{\|h\|^3} h^2$ . The logarithm in this case is given by

$$\text{Log}(x, x') = x^{-1} \begin{bmatrix} \log(q') & H^{-1}b' \\ 0 & 0 \end{bmatrix}$$

where

$$x' = \begin{bmatrix} q' & b' \\ 0 & 1 \end{bmatrix}.$$

*Example 17.* Another map useful in computations for  $SO_n$  is given by the Cayley map

$$f(x, v) := x(I + \frac{1}{2}v)(I - \frac{1}{2}v)^{-1} \text{ with } x \in SO_3, v \in \mathfrak{so}_3.$$

Its point wise inverse is

$$g(x, y) = -2(I - x^{-1}y)(I + x^{-1}y)^{-1} \text{ with } x, y \in SO_3.$$

The Cayley map provides a suitable linearization of the exponential map near identity. In contrast to the exponential map it is rational. This fact is useful in numerical applications, since evaluating transcendental functions can be time-consuming. Cayley map is a first order Padé approximation. In our example any higher order Padé approximation can be used to increase accuracy.

Next, we consider the generalization of Lie groups to symmetric spaces. There are various applications of symmetric spaces in many areas and their structure admits the notion of exponential map. Positive definite symmetric matrices provide a prominent example of a symmetric space and arise in many applications like elasticity and medical imaging. Affine subspaces of Euclidean space provide further examples. For convenient of the reader, we recall some basic facts and refer to [22] and [11] for details. Suppose that  $M$  is a homogenous space, i.e.,  $M = G/K$  where  $K$  is a Lie subgroup of a Lie group  $G$ . Then  $M$  is called a symmetric space iff there is an involutive automorphism  $s \in \text{Aut}(G) \setminus \{\text{id}\}$ ,  $s^2 = \text{id}$  such that the fixpoint set of  $s$  denoted by  $\text{Fix}(s)$  is closed and contained in  $K$ , and  $K$  and  $\text{Fix}(s)$  have the same identity component.

*Example 18.* Let us consider the space of positive definite symmetric  $n \times n$  matrices  $\text{Pos}_n = \text{GL}_n/\text{O}_n$  with its exponential map

$$\text{Exp}(x, v) = x^{\frac{1}{2}} \exp(x^{-\frac{1}{2}} v x^{-\frac{1}{2}}) x^{\frac{1}{2}}$$

and the logarithm which is globally defined

$$\text{Log}(x, y) = x^{\frac{1}{2}} \log(x^{-\frac{1}{2}} y x^{-\frac{1}{2}}) x^{\frac{1}{2}}.$$

Here  $x, y \in \text{Pos}_n$  and  $v \in \text{Sym}_n$  (symmetric  $n \times n$  matrices). The following figure shows an application of de Casteljau's algorithm to four control points in  $\text{Pos}_3$ . Note that  $\text{Pos}_n$  is also a Riemannian symmetric space with the geodesic distance given by

$$d(x, y) = \|\log(x^{-\frac{1}{2}} y x^{-\frac{1}{2}})\|.$$

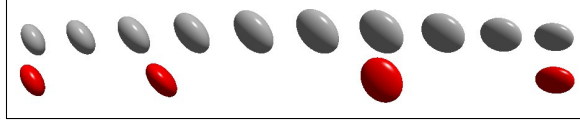


Figure 3: Diffusion tensors visualized as ellipsoids

Moreover, the corresponding geodesic joining  $x$  and  $y$  is given by

$$\Phi_t(x, y) = x^{1/2}(x^{-1/2}y^{1/2}x^{-1/2})^t x^{1/2}.$$

Note that the corresponding Bézier curve in  $\text{Pos}_n$  coincides with the one inherited as a subset of  $\text{GL}_n$ , since

$$x^{\frac{1}{2}} \exp(x^{-\frac{1}{2}} v x^{-\frac{1}{2}}) x^{\frac{1}{2}} = x \exp(x^{-1} v).$$

*Example 19.* This example addresses construction of Bézier curves on polyhedral surfaces. In polyhedra, one can consider shortest as well as straightest geodesics. The concept of straightest

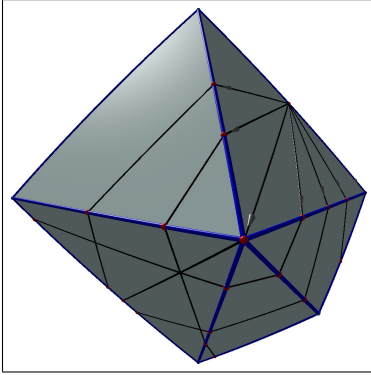


Figure 4: exp: straightest geodesics

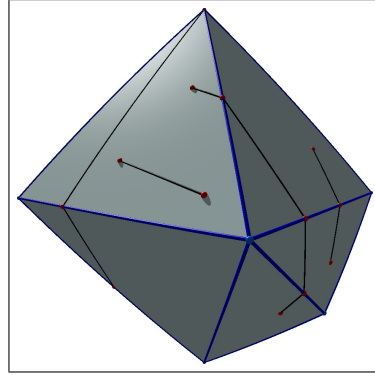


Figure 5: log: shortest geodesics

geodesics has been introduced by Polthier and Schmiess in [14]. Straightest geodesics in polyhedral surfaces are characterized by the property that at inner point of an edge outbound and inbound angles coincide and at a vertex left and right angles sum up to half of total vertex-angle. In contrast to shortest, straightest geodesics uniquely solve the geodesic initial value problem (see [14]). Moreover a straightest geodesic coincides with the shortest, provided the geodesic does not pass through a spherical vertex. Hence, for an initial point  $x \in M$  and initial velocity  $v$  the Riemannian exponential map  $\exp(x, v)$  is the endpoint of the corresponding straightest geodesic. As for  $n$  vertices computation of the endpoint  $\exp(x, v)$  is done by a loop using at most all  $n$  vertices, the underlying time complexity is just  $O(n)$ . The optimal algorithm for the computation of  $\log(x, y)$ , i.e., the initial velocity of the shortest path joining  $x$  and  $y$  in  $M$  can be found in [1] resp. [16] and has time complexity  $O(n^2)$  resp.  $O(n \log n)$  for convex polyhedra.

## 4 Construction of surfaces

Many important surfaces can be constructed as a 1-parameter family of simple curves. The most important examples are canal and ruled surfaces (1-parameter family of circles resp. straight lines). Envelope of a family of spheres whose centers lie on a space curve is a canal surface. Particularly, if the sphere centers lie on a straight line, then the canal surface is a surface of revolution. A surface  $S$  is ruled if through every point of  $S$  there is a straight line that lies on  $S$ . Hence it can be parametrized as a one-parameter family of straight lines. For applications of canal and ruled surfaces in architecture we refer to [15].

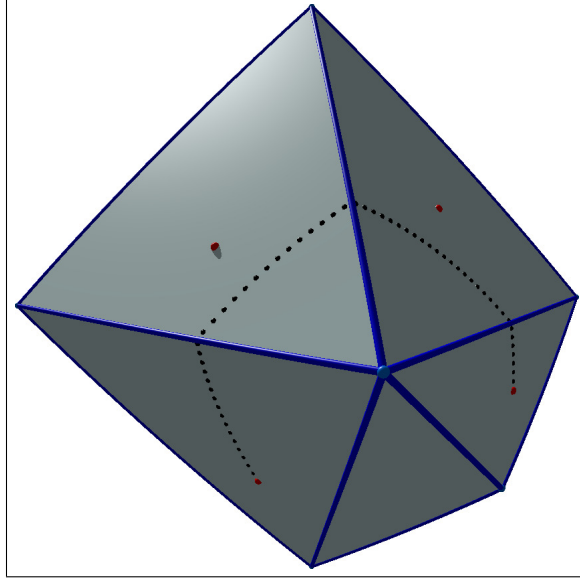


Figure 6: Bézier curve on a polyhedron

*Example 20.* Consider a canal surface  $S$  generated by a regular curve  $m$ , i.e.,  $S : x(s, t) = m(t) + r(t)(\cos(s)N(t) + \sin(s)B(t))$  with  $N$  the normal and  $B$  the binormal of the curve  $m$  and  $0 \leq t \leq 1, 0 \leq s < 2\pi$ . Denote the unit tangent map of the curve  $m$  by  $n$ . We may write  $S$  as  $S = \cup_{0 \leq t \leq 1} S_t$  where  $S_t$  denotes the circle with center  $m(t)$  and radius  $r(t)$  in the plane through the point  $m(t)$  with the normal  $n(t)$ . Now, given control points  $(m_i, r_i, n_i) \in \mathbb{R}^3 \times \mathbb{R}_+ \times S^2$  we may apply our approach to construct an approximation of the surface  $S$ .

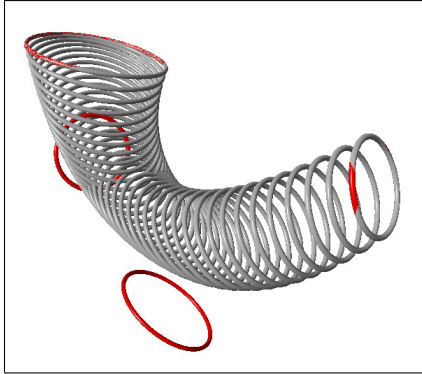


Figure 7: Canal surface

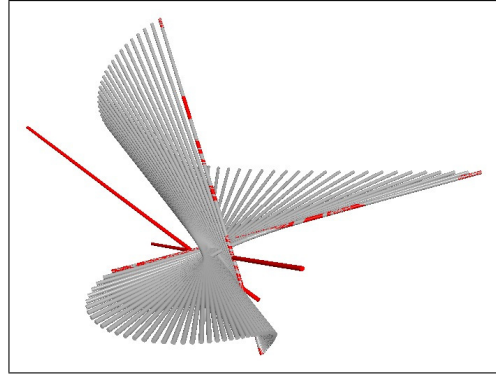


Figure 8: Ruled surface

*Example 21.* For control points  $p_0, \dots, p_n$  in a ruled surface given by  $S : x(s, t) = m(t) + sv(t)$  we may write  $p_i = (m_i, v_i) \in \mathbb{R}^3 \times S^2$ .

Our approach can also be applied to produce further classical surfaces. For instance, [13] gives a constructive approach to design cylinders with constant negative curvature using their characterization via Gauß map. Therein the task is reduced to refine a discrete Chebyshev net in the 2-sphere. The Chebyshev net is completely determined by initial points  $N_{i,i}$  and  $N_{i+1,i}$ . Due to theorem 3 applying (2) parallelograms refine to parallelograms. Hence we may apply (2) with  $M = S^2$  and  $f$  and  $g$  from our first example to construct the net. Using the method described in [13] the surface can then be reconstructed.

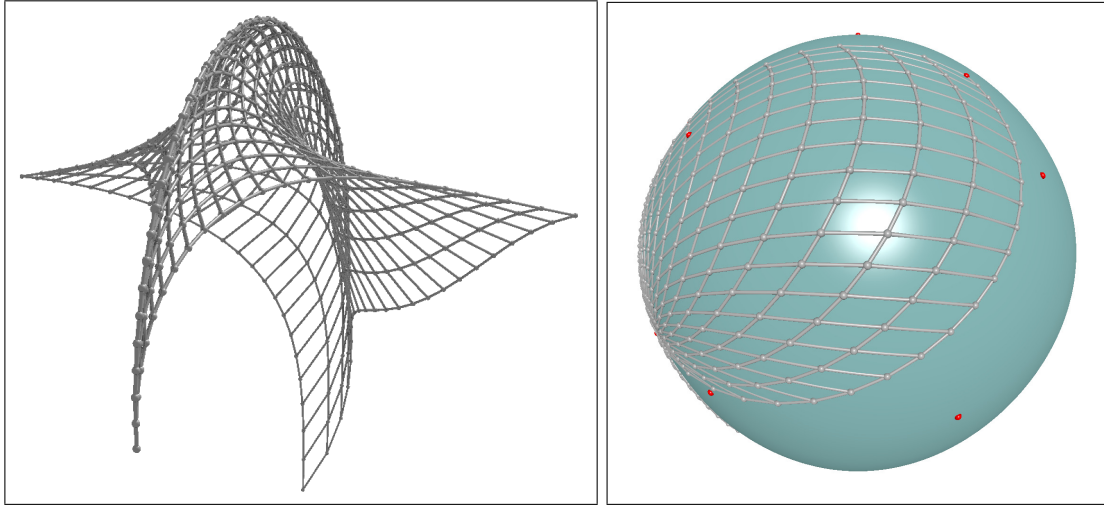


Figure 9: Bi-cubic Chebyshev net on the 2-sphere and corresponding  $k$ -surface

## 5 Conclusion

Considering computational and geometric aspects with focus on applications in CAGD and approximation tasks, we have presented a generalization of de Casteljau's algorithm to manifold-valued data. The canonical choice for Riemannian manifolds as well as Lie groups and more generally symmetric spaces uses an accurate approximation of the exponential map. The Cayley map and orthogonal projection on the tangent space provide further examples. Also, using certain Bézier curves on the 2-sphere we can efficiently construct canal resp. ruled surfaces as 1-parameter family of circles resp. straight lines.

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